

Entanglement – nonstabilizerness separation in hybrid quantum circuits

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Nonstabilizerness describes the distance of a quantum state to its closest stabilizer state. It is—like entanglement—a necessary resource for a quantum advantage over classical computing. We study nonstabilizerness, quantified by stabilizer entropy, in a hybrid quantum circuit with projective measurements and a controlled injection of non-Clifford resources. We discover a phase transition between a power law and constant scaling of nonstabilizerness with system size controlled by the rate of measurements. The same circuit also exhibits a phase transition in entanglement that appears, however, at a different critical measurement rate. This mechanism shows how, from the viewpoint of a quantum advantage, hybrid circuits can host multiple distinct transitions where not only entanglement, but also other nonlinear properties of the density matrix come into play.

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The resources of quantum physics allow one to achieve an advantage over classical computing. It is well known that entanglement is a necessary resource to achieve that goal, but it is, however, not sufficient. Clifford circuits may lead to highly entangled states but can nonetheless be simulated classically in polynomial time [1–3]. They are generated by the nonuniversal gate set of the Hadamard, $\pi/4$ phase, and controlled-NOT gates. The states that can be reached with Clifford circuits starting from states in the computational basis are called *stabilizer states* and they play an important role in the field of error correction as they quite naturally lead to a large and powerful class of error correcting codes [4–7]. Adding any non-Clifford gate, such as the T gate ($\pi/8$ phase gate) to the Clifford gate set makes it universal and thus allows one to reach any state. Because the stabilizer states can be efficiently simulated with a classical computer despite their potentially highly entangled nature, any quantum circuit that should achieve a quantum advantage must therefore not only include highly entangled states but also include non-Clifford gates. The amount of non-Clifford resources necessary to create a state is called *nonstabilizerness* or *magic* [8–11]. Furthermore, a deepened understanding of nonstabilizerness is particularly relevant in the development of fault-tolerant universal quantum computing, where Clifford gates may be implemented straightforwardly in a fault-tolerant way, but non-Clifford gates are typically more challenging and require magic state distillation or similar procedures [12].

The importance of entanglement has stimulated intense interest in its role not only in quantum information but

also in other fields reaching from condensed matter to the quantum mechanics of black holes [13–19]. In some many body systems entanglement may display critical behavior even when all state observables are featureless. One example of this are hybrid quantum circuits that consist of unitary gates interspersed with measurements [20–23]. These circuits can exhibit so-called measurement induced phase transitions (MIPTs) in entanglement. There are multiple different circuits which show such MIPTs in entanglement, such as random Clifford [22,23] or Haar-random circuits [21] interspersed with projective measurements. Both of these examples show the same qualitative behavior of entanglement, namely a volume law below a critical measurement rate and an area law above [24]. However, from the perspective of quantum advantage these two examples are very different. This is because even in the entanglement volume law phase the random Clifford circuit can be efficiently simulated classically using the stabilizer formalism, while the same is not true for the Haar-random circuit. Furthermore, MIPTs in entanglement have been related to the performance of quantum error correction codes, for which it has been found that hybrid circuits that show a volume law in entanglement encode quantum information, while circuits that show an area law destroy it irreversibly [25]. In this context, nonstabilizerness could further differentiate between quantum error correction codes that require non-Clifford resources (which are typically hard to implement in a fault-tolerant way) and codes that do not. It is thus natural to ask whether hybrid quantum circuits can also show MIPTs in nonstabilizerness and, if yes, how those phase transitions relate to the known MIPTs in entanglement.

In this Letter we present a new kind of phase transition between a power law scaling and constant nonstabilizerness in a hybrid quantum circuit controlled by the rate of measurement. Most notably, we find that this phase transition appears at a different critical measurement rate compared to the critical

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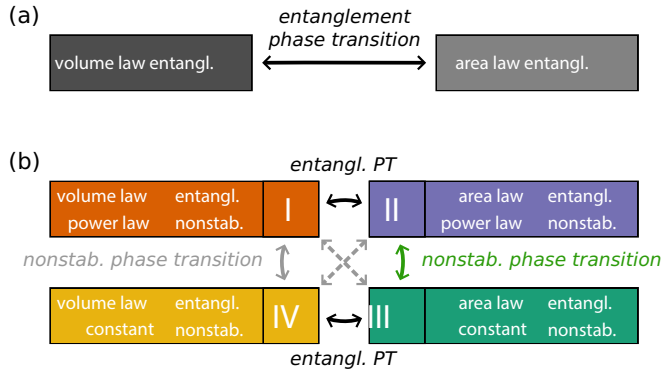


FIG. 1. Phase transitions in entanglement (a) compared to the possible phase transitions in entanglement and nonstabilizerness (b) in one-dimensional quantum circuits. Quantum circuits in phase I may lead to an advantage over classical computing, while circuits in phases II and III are amenable to simulation through tensor network methods and circuits in phases III and IV can be tackled efficiently with the stabilizer formalism. The black, green, and gray arrows indicate the previously observed, new, and not yet observed phase transitions, respectively.

rate for the volume to area law transition of entanglement. This suggests that the mechanism that drives the observed nonstabilizerness phase transition is different from the mechanism driving the entanglement phase transition. From the perspective of quantum advantage it is thus necessary to extend the picture of MIPTs in entanglement [see Fig. 1(a)] to MIPTs in entanglement and nonstabilizerness [see Fig. 1(b)]. In particular, the hybrid quantum circuit studied in this Letter shows an entanglement phase transition from a volume to area law while the nonstabilizerness remains in a power law in both phases [phase I $\rightarrow$ phase II], as well as a nonstabilizerness phase transition from a power law to a constant while the entanglement keeps an area law behavior [phase II $\rightarrow$ phase III].

While phase transitions in nonstabilizerness have been reported previously in Refs. [26] and [27] we argue that the setup and the consequent phase transitions observed in this work are qualitatively different. The work by Leone *et al.* [26] studies to what extent Clifford circuits interspersed with T gates can be transformed into circuits with all T gates only in a small subset of qubits and does not involve any measurements. In Ref. [27] Niroula *et al.* study a random Clifford error correcting protocol with a stabilizer state as an input and a noise layer that injects nonstabilizerness through phase gates at a variable angle. This setup shows a phase transition of nonstabilizerness in the output state depending on the phase gate angle. Further analysis reported by Turkeshi and Sierant [28] suggests that this phase transition stems from a phase transition in the success of error correction. Following from this, the transition presented in Ref. [27] could thus be witnessed through many different properties of a state, including nonstabilizerness, but also any other property, such as entanglement, would exhibit the same critical measurement rate set by the error correcting transition.

In Ref. [29] Bejan *et al.* study the complexity of Pauli based computations [30] of hybrid quantum circuits with

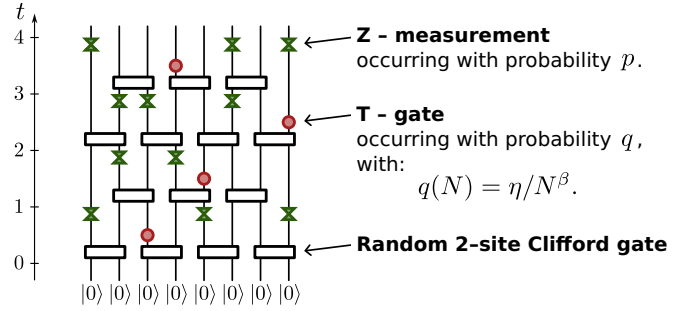


FIG. 2. Hybrid quantum circuit with a brickwork structure of random two-site Clifford gates interspersed with measurements in the computational bases and T gates for $N = 8$ qubits and four time steps. The measurements and T gates appear randomly with probability p and $q(N) = \eta/N^\beta$, respectively, where β determines the scaling of the T-gate density with prefactor η .

interspersed measurements and T gates. In this work the authors view this complexity as a proxy for nonstabilizerness and find that it exhibits a phase transition that coincides with the entanglement phase transition. In contrast to the work by Bejan *et al.* our results presented below point towards separate entanglement and nonstabilizerness phase transitions. We discuss possible reasons for this discrepancy in the Supplemental Material (SM) [31].

Setup. To find a phase transition in nonstabilizerness we study hybrid quantum circuits that interpolate between situations found in random Clifford and Haar-random hybrid circuits in a controlled way. For this we intersperse a brickwork of random two-site Clifford gates with T gates in addition to the projective measurements. We consider open boundary conditions, but expect similar behavior for closed boundaries. Figure 2 shows an example realization of such a circuit for 8 qubits and four time steps. The measurements occur on any of the N qubits at any time step with a uniform probability p , while the T gates occur with a uniform probability q which we set to scale as $q(N) = \eta/N^\beta$. Beside the prefactor η , also the exponent β of the scaling will play an important role, as discussed further below. For any realization of such a circuit there is an ensemble of pure quantum state trajectories, where each trajectory is labeled by the measurement outcome sequence. We are interested in the long time limit of entanglement and nonstabilizerness averaged over both the quantum circuit realizations and the measurement outcomes. Both entanglement and nonstabilizerness are nonlinear in the density matrix and it is thus important to distinguish between the entanglement/nonstabilizerness of the average state versus the average of the entanglement/nonstabilizerness over the ensemble of trajectories. While the former is not even well defined (given that we use measures of entanglement/nonstabilizerness applicable only to pure states), the latter may exhibit interesting behavior.

Entanglement and stabilizer entropy. As a measure for entanglement $\mathcal{E}$ we choose the von Neumann entropy between the left and right half of the chain. Given an N qubit pure state $|\psi\rangle$ it is

$$\mathcal{E}(|\psi\rangle) = -\text{tr}[\rho_L \log_2(\rho_L)], \quad (1)$$

where $\rho_L = \text{tr}_R[|\psi\rangle\langle\psi|]$ is the reduced density matrix of the left half of the chain $L = \{1, \dots, N/2\}$ with tr_R denoting the partial trace over the right half $R = \{N/2 + 1, \dots, N\}$. While it is undisputed that the von Neumann entropy is a good measure for bipartite entanglement in pure states, the choice of a “good measure” $\mathcal{M}$ for nonstabilizeriness is under discussion in current literature [9,32–47]. In this work we choose the recently introduced stabilizer entropy [39].

The stabilizer entropy quantifies the spread of a state in the basis of Pauli string operators. In particular, the stabilizer α -Rényi entropy of an N qubit pure state $|\psi\rangle$ is

$$\mathcal{M}(|\psi\rangle) = \frac{1}{1-\alpha} \log_2 \left(\sum_{P \in \mathcal{P}_N} \Xi_P^\alpha(|\psi\rangle) \right) - N, \quad (2)$$

where $\Xi_P(|\psi\rangle) = 2^{-N} \langle \psi | P | \psi \rangle^2$ is the probability distribution over all Pauli strings $P \in \mathcal{P}_N$ with $+1$ phases. This definition is motivated by the fact that a state is a stabilizer state if and only if there are 2^N Pauli strings $P \in \mathcal{P}_N$ with an expectation value of 1 or -1 , while the expectation for all others is 0. Below we present numerical results for $\alpha = 2$, but note that we find the same critical measurement rates for the limit of $\alpha \rightarrow 1$ (which yields the stabilizer Shannon entropy). One advantage of the stabilizer entropy over many other proposed measures of nonstabilizeriness is possible to efficiently compute it even for a large number of qubits through Monte Carlo sampling of the Pauli string probability distribution [48–50].

As mentioned above, the scaling of the T-gate density with system size plays an important role for a potential phase transition in nonstabilizeriness. In particular, for a constant density of T gates ($\beta = 0$) one expects a trivial volume law scaling of nonstabilizeriness for any measurement rate $p < 1$. This is because nonstabilizeriness is a global property and—unlike entanglement—can be created locally. Because of the additivity of nonstabilizeriness [39] any finite nonstabilizeriness density will thus lead to a volume law in nonstabilizeriness for the total state. In contrast to this, entanglement occurs between at least two subsystems and may only be created through gates that lie on the boundary of those subsystems. In the setup studied here there is only one two-site Clifford gate every other time step that can increase the entanglement between the left and right half of the chain, independently of the system size. In analogy, this naturally suggests the study of nonstabilizeriness in a quantum circuit with a constant average number η of T gates per time step, also independently of the system size. This corresponds to a vanishing T-gate density $q = \eta/N^\beta$ with $\beta = 1$ and thus to a total T-gate count proportional to the circuit depth. Because the currently known lower and upper bound for the total number of T gates necessary for the preparation of any arbitrary quantum state (using only unitary Clifford and T gates) are $O(N)$ and $O(N^2)$, respectively [36,38,51], this suggests that the proposed circuit ensemble for $\beta = 1$ allows one to approximate any many-body quantum state given a circuit depth between $O(N)$ and $O(N^2)$. Furthermore, recent results have shown that Clifford circuits together with a vanishing T-gate density already suffice to approximate any arbitrary order t design [52]. In the following we thus focus on the case of $\beta = 1$ and discuss other scalings in the SM [31].

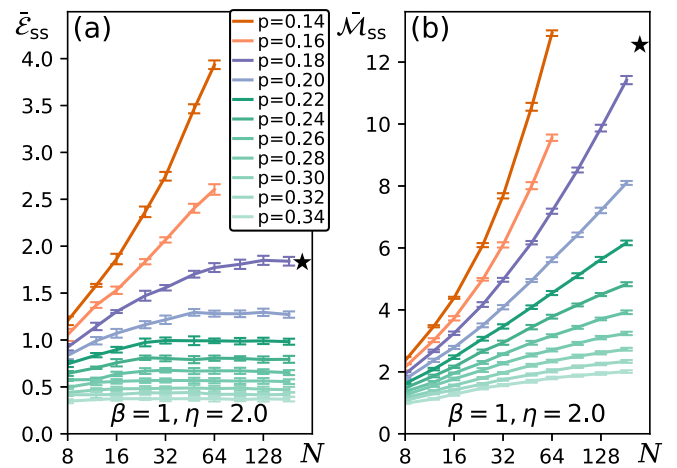


FIG. 3. Numerical results for the average steady state entanglement (a) and nonstabilizeriness (b) versus system size N for several different measurement rates p and T-gate density $q(N) = \eta/N^\beta$ with $\eta = 2.0$ and $\beta = 1$. Note that for the measurement rate $p = 0.18$ (marked with $\star$) the lines bend downwards in panel (a) but upwards in panel (b), which indicates an area law in entanglement but a power law in nonstabilizeriness.

Numerical results for the $\beta = 1$ case. Figure 3 shows the growth of the trajectory averaged entanglement and nonstabilizeriness with system size (up to 184 sites) for $\beta = 1$, $\eta = 2.0$, and several different measurement rates p . To generate this plot we simulate 512 trajectories using time evolving block decimation [53,54] for each parameter set of p and N . For each of those trajectories we periodically (every eighth time step) evaluate the von Neumann entropy $\mathcal{E}(t)$ and the stabilizer 2-Rényi entropy $\mathcal{M}(t)$ using the method introduced in [49]. Then we yield $\bar{\mathcal{E}}(t)$ and $\bar{\mathcal{M}}(t)$ from the average of those values. For the parameter sets considered here we find that $\bar{\mathcal{E}}(t)$ and $\bar{\mathcal{M}}(t)$ attain an approximate steady value within less than $2N$ time steps. To get a more accurate approximation of the steady values $\bar{\mathcal{E}}_{ss}$ and $\bar{\mathcal{M}}_{ss}$, we compute the dynamics for $4N$ time steps and average over $\bar{\mathcal{E}}(t)$ and $\bar{\mathcal{M}}(t)$ for all time steps between $2N$ and $4N$. Finally we use the spread of this distribution as an estimate for the error due to the finite number of sampled trajectories, which we display as error bars in Fig. 3. We present further details on the numerical computation and error estimation in the SM.

The data shown in Fig. 3 strongly suggest the existence of two distinct MITs. Figure 3(a) shows that in this log-linear plot for $\bar{\mathcal{E}}_{ss}(N)$, the line for $p = 0.14$ tends upwards, while the line for $p = 0.16$ appears approximately straight and the lines for $p \geq 0.18$ bend downwards. This indicates a volume/log/area law in entanglement, respectively. This observation is consistent with the known measurement induced phase transition of entanglement for Clifford circuits, for which the addition of a vanishing density of T gates appears not to significantly modify the critical measurement rate $p_c^{\text{entgl}} = 0.15995(10)$ [22,55]. On the other hand, the values of nonstabilizeriness for the same trajectories plotted in Fig. 3(b) also indicate power/log/constant law, but with a significantly higher critical measurement rate $p_c \simeq 0.22$. In particular, we

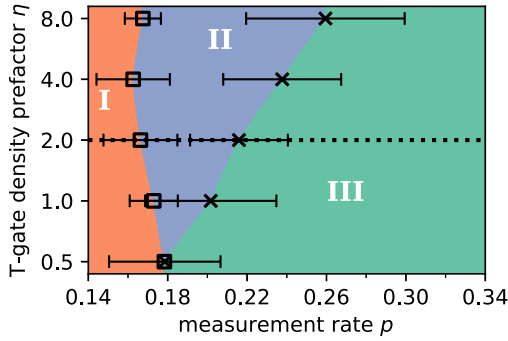


FIG. 4. Phase diagram (η vs p) extracted from the numerical results for $\beta = 1$ and $\eta = \{0.5, \dots, 8.0\}$. The squares and crosses show the position of the entanglement and nonstabilizerness phase transitions, respectively. We present the details of the numerical error estimation (displayed as error bars) in the SM [31]. The colored regions resulting from this data correspond to the phases I, II, and III shown in Fig. 1. The dotted line corresponds to the data shown in Fig. 3 ($\eta = 2.0$).

note that for $p = 0.18$ (marked with ★) the results clearly indicate an area law for entanglement but a power law for nonstabilizerness.

We analyze these transitions in two steps. First, we extract the critical measurement rate by fitting each curve with two different functions, corresponding to a power law and a constant respectively and perform a so-called F test [56]. Then, in the second step, we confirm the critical nature of the extracted critical measurement rate for the nonstabilizerness transition by performing a scaling collapse and subsequently extracting the critical exponent.

For the first step, we fit a , b , and γ in the function $f(N) = a + bN^\gamma$ with the constraint that $\gamma > 0$ for the power law and $\gamma \leq 0$ for a function that approaches a constant. For each fit we also compute the sum of the squared residuals per degree of freedom ($E = \chi^2/\text{dof}$), which quantifies how well the fit matches the data. Using a linear interpolation between the discrete set of the p values, we quote the value of p for which both fits work equally well as the critical measurement rate.

Figure 4 shows the result of this analysis for several different values of η . The data suggest that the entanglement phase transition is unaffected by the presence of a vanishing density of T gates, while the nonstabilizerness phase transition shifts depending on the prefactor η of that vanishing density. For the studied range of η between 0.5 and 8.0 this results in phase I for low measurement rates, phase II for intermediate rates, and phase III for higher rates. While the power law exponent for entanglement in phase I is known to be $\gamma_{\text{entgl}} = 1$, the exponent for nonstabilizerness appears to depend on the distance to the phase boundary, similar to the fractal entanglement scaling in dual-unitary Griffiths circuits [57]. We plot the extracted and extrapolated values for γ_{nonstab} of phase II in the SM. For $\eta = 0$ the circuit reduces to the setup studied in Ref. [23], which always gives zero nonstabilizerness and hence shows a transition from phase IV to phase III. It is unclear how this changes for small but finite values of η . This regime is, however, difficult to access numerically due to

the very sparse occurrences of T gates, which demand longer simulation times to reach the steady state.

In the second step of our analysis we confirm that the critical measurement rates extracted in the first step are indeed critical in nature, which we do by performing a scaling collapse of the data. In analogy to the known critical scaling of the entanglement phase transition we choose the scaling ansatz of the nonstabilizerness to be some universal function of $x = (p - p_c)N^{1/\nu}$. We fix p_c given by the results shown in Fig. 4 and optimize ν , which yields the expected data collapse and a value of $\nu = 1.6 \pm 0.3$ that is consistent across all values of η . We present the plots of this data collapse as well as further details on the F-test analysis and the scaling collapse in the SM.

While specific, our analysis portrays the generic competition between magic resource injection (T gates), entanglement (CNOTs), and their destruction through local measurements. Our model most directly corresponds to an experimental scenario in between a noisy intermediate-scale quantum computer and a scalable fault-tolerant quantum computer, for which there is a scalable number of logical qubits, but only a fixed number of fault-tolerant logical qubits that allow non-Clifford operations. In order to emphasize the generality of such a scenario, motivated by a recent experiment [58], we have analyzed a simulation where the sources of magic injection are CCZ gates. The corresponding results, presented in detail in the SM, fully confirm the generality of our picture. We conclude by pointing out the possibility of an alternative scenario in which the two transitions in fact coincide. This possibility is discussed in the SM and it is compatible with the analysis presented in Ref. [29].

Conclusions. In this work we find a MIPT in nonstabilizerness that appears to be independent of the known MIPT in entanglement. A key ingredient for this result is the controlled injection of non-Clifford resources into the circuit, for which we have chosen Clifford circuits interspersed with projective measurements and T gates. We note that our results are general in the sense that also other classes of hybrid quantum circuits with different forms of non-Clifford resources will generally show the same qualitative behavior because any non-Clifford resource can be approximated by a Clifford circuit interspersed with T gates. This is particularly relevant to recent experiments with Rydberg atoms in optical lattices [58], where the circuits we propose could be immediately implemented. We believe that our results and further investigation of the questions raised in this paper are beneficial to gain a deepened understanding of MIPTs in hybrid quantum circuits and their relation to computational quantum advantage and quantum error correction.

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- [1] D. Gottesman, The Heisenberg representation of quantum computers, [arXiv:quant-ph/9807006](#).
- [2] S. Aaronson and D. Gottesman, Improved simulation of stabilizer circuits, *Phys. Rev. A* **70**, 052328 (2004).
- [3] V. Veitch, C. Ferrie, D. Gross, and J. Emerson, Negative quasiprobability as a resource for quantum computation, *New J. Phys.* **14**, 113011 (2012).
- [4] M. A. Nielsen and I. Chuang, *Quantum Computation and Quantum Information*, 2nd ed. (Cambridge University Press, Cambridge, UK, 2010).
- [5] D. Gottesman, Class of quantum error-correcting codes saturating the quantum hamming bound, *Phys. Rev. A* **54**, 1862 (1996).
- [6] D. E. Gottesman, Stabilizer codes and quantum error correction, Ph.D. thesis, California Institute of Technology, 1997, doi:[10.7907/rzr7-dt72](#).
- [7] D. Gottesman, Theory of fault-tolerant quantum computation, *Phys. Rev. A* **57**, 127 (1998).
- [8] S. Bravyi and A. Kitaev, Universal quantum computation with ideal Clifford gates and noisy ancillas, *Phys. Rev. A* **71**, 022316 (2005).
- [9] V. Veitch, S. A. H. Mousavian, D. Gottesman, and J. Emerson, The resource theory of stabilizer quantum computation, *New J. Phys.* **16**, 013009 (2014).
- [10] E. Chitambar and G. Gour, Quantum resource theories, *Rev. Mod. Phys.* **91**, 025001 (2019).
- [11] Z.-W. Liu and A. Winter, Many-body quantum magic, *PRX Quantum* **3**, 020333 (2022).
- [12] E. T. Campbell, B. M. Terhal, and C. Vuillot, Roads towards fault-tolerant universal quantum computation, *Nature (London)* **549**, 172 (2017).
- [13] P. Calabrese and J. Cardy, Entanglement entropy and quantum field theory, *J. Stat. Mech.: Theory Exp.* (2004) P06002.
- [14] S. Popescu, A. J. Short, and A. Winter, Entanglement and the foundations of statistical mechanics, *Nat. Phys.* **2**, 754 (2006).
- [15] P. Hayden and J. Preskill, Black holes as mirrors: quantum information in random subsystems, *J. High Energy Phys.* **09** (2007) 120.
- [16] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Entanglement in many-body systems, *Rev. Mod. Phys.* **80**, 517 (2008).
- [17] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, *Rev. Mod. Phys.* **81**, 865 (2009).
- [18] N. Laflorencie, Quantum entanglement in condensed matter systems, *Phys. Rep.* **646**, 1 (2016).
- [19] A. Almheiri, N. Engelhardt, D. Marolf, and H. Maxfield, The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole, *J. High Energy Phys.* **12** (2019) 063.
- [20] A. C. Potter and R. Vasseur, Entanglement dynamics in hybrid quantum circuits, in *Entanglement in Spin Chains: From Theory to Quantum Technology Applications*, edited by A. Bayat, S. Bose, and H. Johannesson (Springer International Publishing, Cham, 2022), pp. 211–249.
- [21] B. Skinner, J. Ruhman, and A. Nahum, Measurement-induced phase transitions in the dynamics of entanglement, *Phys. Rev. X* **9**, 031009 (2019).
- [22] Y. Li, X. Chen, and M. P. A. Fisher, Quantum zeno effect and the many-body entanglement transition, *Phys. Rev. B* **98**, 205136 (2018).
- [23] Y. Li, X. Chen, and M. P. A. Fisher, Measurement-driven entanglement transition in hybrid quantum circuits, *Phys. Rev. B* **100**, 134306 (2019).
- [24] P. Sierant and X. Turkeshi, Universal behavior beyond multifractality of wave functions at measurement-induced phase transitions, *Phys. Rev. Lett.* **128**, 130605 (2022).
- [25] M. J. Gullans and D. A. Huse, Dynamical purification phase transition induced by quantum measurements, *Phys. Rev. X* **10**, 041020 (2020).
- [26] L. Leone, S. F. E. Oliviero, G. Esposito, and A. Hama, Phase transition in stabilizer entropy and efficient purity estimation, *Phys. Rev. A* **109**, 032403 (2024).
- [27] P. Niroula, C. D. White, Q. Wang, S. Johri, D. Zhu, C. Monroe, C. Noel, and M. J. Gullans, Phase transition in magic with random quantum circuits, *Nat. Phys.* (2024), doi:[10.1038/s41567-024-02637-3](#).
- [28] X. Turkeshi and P. Sierant, Error-resilience phase transitions in encoding-decoding quantum circuits, *Phys. Rev. Lett.* **132**, 140401 (2024).
- [29] M. Bejan, C. McLauchlan, and B. Béri, Dynamical magic transitions in monitored Clifford+T circuits, *PRX Quantum* **5**, 030332 (2024).
- [30] S. Bravyi, G. Smith, and J. A. Smolin, Trading classical and quantum computational resources, *Phys. Rev. X* **6**, 021043 (2016).
- [31] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevResearch.6.L042030> for additional information on (1) the properties of the stabilizer entropy, (2) the tensor network computations, (3) the extraction of the critical measurement rates, (4) the scaling collapse, (5) an analogous setup with T gates replaced by CCZ gates, (6) a conjectured phase diagram in p vs η for $\beta = 1$, (7) a conjectured phase diagram in p vs β for $\eta = 1$, and (8) an argument for coincident entanglement and nonstabilizer transitions.
- [32] M. Howard and E. Campbell, Application of a resource theory for magic states to fault-tolerant quantum computing, *Phys. Rev. Lett.* **118**, 090501 (2017).
- [33] M. Heinrich and D. Gross, Robustness of magic and symmetries of the stabiliser polytope, *Quantum* **3**, 132 (2019).
- [34] X. Wang, M. M. Wilde, and Y. Su, Quantifying the magic of quantum channels, *New J. Phys.* **21**, 103002 (2019).
- [35] K. Bu and D. E. Koh, Efficient classical simulation of Clifford circuits with nonstabilizer input states, *Phys. Rev. Lett.* **123**, 170502 (2019).
- [36] M. Beverland, E. Campbell, M. Howard, and V. Kliuchnikov, Lower bounds on the non-Clifford resources for quantum computations, *Quantum Sci. Technol.* **5**, 035009 (2020).
- [37] S. Sarkar, C. Mukhopadhyay, and A. Bayat, Characterization of an operational quantum resource in a critical many-body system, *New J. Phys.* **22**, 083077 (2020).

- [38] J. Jiang and X. Wang, Lower bound for the t count via unitary stabilizer nullity, *Phys. Rev. Appl.* **19**, 034052 (2023).
- [39] L. Leone, S. F. E. Oliviero, and A. Hama, Stabilizer Rényi entropy, *Phys. Rev. Lett.* **128**, 050402 (2022).
- [40] S. F. E. Oliviero, L. Leone, A. Hama, and S. Lloyd, Measuring magic on a quantum processor, *NPJ Quantum Inf.* **8**, 148 (2022).
- [41] K. Bu, R. J. Garcia, A. Jaffe, D. E. Koh, and L. Li, Complexity of quantum circuits via sensitivity, magic, and coherence, *Commun. Math. Phys.* **405**, 161 (2024).
- [42] T. Haug and L. Piroli, Stabilizer entropies and nonstabilizerness monotones, *Quantum* **7**, 1092 (2023).
- [43] K. Bu, W. Gu, and A. Jaffe, Quantum entropy and central limit theorem, *Proc. Natl. Acad. Sci. USA* **120**, e2304589120 (2023).
- [44] E. Tirrito, P. S. Tarabunga, G. Lami, T. Chanda, L. Leone, S. F. E. Oliviero, M. Dalmonte, M. Collura, and A. Hama, Quantifying non-stabilizerness through entanglement spectrum flatness, *Phys. Rev. A* **109**, L040401 (2024).
- [45] X. Turkeshi, M. Schirò, and P. Sierant, Measuring nonstabilizerness via multifractal flatness, *Phys. Rev. A* **108**, 042408 (2023).
- [46] K. Bu, W. Gu, and A. Jaffe, Stabilizer testing and magic entropy, [arXiv:2306.09292](https://arxiv.org/abs/2306.09292).
- [47] A. Gu, L. Leone, S. Ghosh, J. Eisert, S. Yelin, and Y. Quek, Pseudomagic quantum states, *Phys. Rev. Lett.* **132**, 210602 (2024).
- [48] T. Haug and L. Piroli, Quantifying nonstabilizerness of matrix product states, *Phys. Rev. B* **107**, 035148 (2023).
- [49] G. Lami and M. Collura, Quantum magic via perfect sampling of matrix product states, [arXiv:2303.05536](https://arxiv.org/abs/2303.05536).
- [50] P. S. Tarabunga, E. Tirrito, T. Chanda, and M. Dalmonte, Many-body magic via Pauli-Markov chains – from criticality to gauge theories, *PRX Quantum* **4**, 040317 (2023).
- [51] M. Amy and M. Mosca, T-count optimization and Reed–Muller codes, *IEEE Trans. Inf. Theory* **65**, 4771 (2019).
- [52] L. Leone, S. F. E. Oliviero, Y. Zhou, and A. Hama, Quantum chaos is quantum, *Quantum* **5**, 453 (2021).
- [53] G. Vidal, Efficient simulation of one-dimensional quantum many-body systems, *Phys. Rev. Lett.* **93**, 040502 (2004).
- [54] M. Fishman, S. R. White, and E. M. Stoudenmire, The ITensor Software Library for Tensor Network Calculations, *SciPost Phys. Codebases* **2022**, 4 (2022).
- [55] P. Sierant, M. Schirò, M. Lewenstein, and X. Turkeshi, Measurement-induced phase transitions in $(d + 1)$ -dimensional stabilizer circuits, *Phys. Rev. B* **106**, 214316 (2022).
- [56] D. L. Hahs-Vaughn and R. Lomax, *An Introduction to Statistical Concepts*, 4th ed. (Routledge, New York, 2020).
- [57] M. Ippoliti, T. Rakovszky, and V. Khemani, Fractal, logarithmic, and volume-law entangled nonthermal steady states via spacetime duality, *Phys. Rev. X* **12**, 011045 (2022).
- [58] D. Bluvstein, S. J. Evered, A. A. Geim, S. H. Li, H. Zhou, T. Manovitz, S. Ebadi, M. Cain, M. Kalinowski, D. Hangleiter, J. P. B. Ataiades, N. Maskara, I. Cong, X. Gao, P. S. Rodriguez, T. Karolyshyn, G. Semeghini, M. J. Gullans, M. Greiner, V. Vuletić *et al.*, Logical quantum processor based on reconfigurable atom arrays, *Nature (London)* **626**, 58 (2023).