

Quenched disorder-induced chaotic resonance: Optimal stimulus response near the edge of chaos

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We study the collective stimulus response in two classical nonlinear dynamical models, the globally coupled bistable oscillators and the excitable FitzHugh-Nagumo neurons, where the coupling strength among the units are randomly distributed with fixed mean and varying variance. We find theoretically and numerically that in both models the ensemble response to a weak periodic stimulus exhibits a bell-shaped amplification manifesting a different type of resonance mechanism, namely, coupling-disorder-induced resonance, as the quenched disorder of coupling increases. Of particular interest is that the optimal collective response emerges near the order-chaos transition point, i.e., at the edge of chaos. As a promising application, we discuss briefly the potential reasonability for exploiting such a type of chaotic resonance to explain the optimal mutual information in the realistic inferior olive. Our results provide explicitly a clue that the disorderly coupled elements may utilize the coupling-disorder-induced chaos to optimize their stimulus response.

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Introduction. Heterogeneous couplings among the entities of either natural or manmade complex systems play a significant role in producing emergent collective behaviors [1–4]. In rat neuronal tissues, the degree of coupling disorder is closely related to the transitions between physiological and pathological behaviors [1,2]. The free-running period of neurons in the suprachiasmatic nucleus is highly correlated with the heterogeneous couplings among them in sustaining the biological rhythm [3]. Noteworthy, biodiversity in alpine plant communities is mainly guaranteed by the co-occurrence of positive and negative associations among species [4]. Thus, taking into account explicitly the coupling disorder among the ingredients of a complex system seems quite essential and necessary in understanding its functionality.

In recent decades, a great deal of effort has been devoted to revealing the fascinating ensemble behaviors and the underlying mechanisms induced by the coupling disorder [5–8]. For instance, Edwards and Anderson proposed the spin glass model wherein the couplings between every pair of neighboring spins are i.i.d. (independent and identically distributed) sampled from a Gaussian distribution, and found that spins become frozen in random directions whereas the total magnetization is vanished when the temperature is below the critical value [6]. Strikingly, the spin glass model can well explain the strange susceptibility cusp in the experimental observation of dilute magnetic alloys [5]. Also, inspired by the spin glass,

Daido put forward the conjecture of “oscillator glass” where the interacting limit cycle oscillator population can exhibit a glass state as the standard deviation of the evenly distributed coupling increases [7]. Moreover, May obtained the theoretical boundary of stable coexistence of multispecies on the basis of the random matrix theory, which triggered the famous diversity-stability debate [8].

It is worth noting that the coupling disorder in neural elements also lead to the emergence of novel phenomena at the system level. Sompolinsky *et al.* considered a network of firing-rate neurons with random asymmetrical couplings and found that an order-chaos transition emerges as the disorder of random couplings goes beyond a critical value [9]. Since then, a few subsequent studies showed that the order-chaos transition is closely related to the optimal performance of message processing, computation capability, and memory storage in neural networks and machine learning [10–14]. Encouragingly, such an order-chaos transition can be regulated by external time-varying stimulus, i.e., noise or periodical signal, to achieve specific functions [12,15,16]. For instance, external noise can shift the transition point to a larger degree of disorder and result in a coexistent regime with locally expansive but nonchaotic dynamics, which optimizes the capacity of storing sequential input [12]. Nevertheless, little is known about how such an order-chaos transition affects the collective stimulus response despite the fact that periodical stimulus usually plays a pivotal role in our daily lives, such as the periodical solar energy influx on the earth surface [17], temporal signals among neuronal cells [18], alternating current input in nanomechanical resonator devices [19], etc.

In this Research Letter, we study the collective stimulus response in two classical nonlinear dynamical models, namely, the globally coupled bistable oscillators and the excitable FitzHugh-Nagumo neurons, where the coupling

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strength among the units are randomly distributed with fixed mean and varying variance. Using both analytical treatment and numerical simulations, we reveal that the collective stimulus response behaves like a bell-shaped amplification with the increase of the coupling disorder among the units, signifying the occurrence of a resonant phenomenon. Interestingly, we find that the optimal response emerges at the edge of chaos, which hints at the existence of a different type of mechanism giving rise to chaotic resonance. In particular, such a resonant collective response can be substantially affirmed by the dynamical mean-field equation based on the dynamical cavity method [20]. As a potential application, we finally discuss the rationality for utilizing such a resonance to explain the optimal mutual information in the realistic inferior olive.

Bistable oscillators. Since the bistable oscillator model is a conventional paradigm for portraying the ensemble signal response in physics and neurodynamics [21–23], we first focus on an all-to-all network of bistable oscillators of size N , whose dynamics are governed by the following differential equations:

$$\dot{x}_i = x_i - x_i^3 + \sum_{j=1}^N c_{ij}(x_j - x_i) + A \sin(\omega t), \quad i = 1, \dots, N, \quad (1)$$

where x_i is the state variable, and the coefficient c_{ij} stands for the coupling strength of j on i . Without loss of generality, we sample c_{ij} from a Gaussian distribution with $c_m = 1.0/N$ and $\langle (c_i - c_m)(c_j - c_m) \rangle = \sigma^2 \delta_{ij}/N$. Here c_m roughly measures the ratio of positive couplings in the distribution and σ evaluates the disorder of the random couplings among the oscillators. The N^{-1} dependence of the moments on the oscillator size is to ensure that there is a well-defined thermodynamic limit [9,24]. Since the fraction of positive (excitatory) neurons is dominated in the mammalian brain networks as compared to negative (inhibitory) ones [24,25], we here reasonably consider that the mean of the random couplings is equal to unity. Additionally, the external weak periodical stimulus is modeled by a sinusoidal function with amplitude $A = 0.3$ and period $T = 50$. Under such a setting, the external signal is weak and cannot trigger the isolated oscillators to overcome the barrier to evoke the large-amplitude oscillation [26]. As a result, the enlarged oscillations of elements are, if anything, purely evoked by the random couplings.

The ensemble stimulus response can be measured by the spectral amplification factor η_S , which is defined as [22,23]

$$\eta_S = \frac{4}{A^2} |\langle e^{i\omega t} X(t) \rangle|^2, \quad (2)$$

where the average activity of the ensemble oscillators is described by $X(t) = N^{-1} \sum_{j=1}^N x_j(t)$, and the symbol $\langle \cdot \rangle$ denotes a temporal average. Unless otherwise specified, the collective signal response is averaged over 100 independent realizations throughout our work. The initial states of the oscillator i are chosen randomly from ± 1 . The main results shown in the present work are robust to diverse initial conditions.

As shown in Fig. 1(a), for a sufficiently low degree of disorder $\sigma = 0.2$, all the oscillators synchronously joggle around the fixed point $x^* = 1.0$. Noticeably, for an

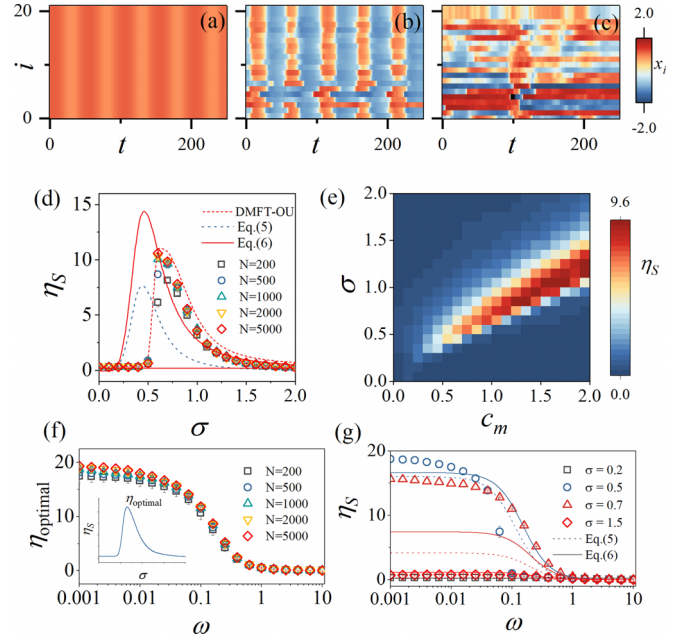


FIG. 1. Resonant stimulus response induced by quenched disorder in globally coupled bistable oscillators. (a)–(c) Activities of 20 oscillators sampled randomly from the oscillators pool for three degrees of coupling disorder: (a) $\sigma = 0.2$, (b) $\sigma = 0.7$, and (c) $\sigma = 1.5$. (d) The spectral amplification factor η_S vs the coupling disorder for different system sizes. The red short dashed line corresponds to the dynamical mean-field approximation with OU noise and numerical moment $M(t)$ for $N = 5000$. The horizontal red solid line is the theoretical signal response, $\eta_S \approx 1/(\omega^2 + 4)$, for the subthreshold disorder. (e) The dependence of the collective signal response on the mean and the standard deviation of the random couplings for $N = 200$. (f) The optimal signal response η_{optimal} vs the external driving frequency in globally coupled bistable oscillators. The definition of η_{optimal} is delineated in the inset; say, for a fixed frequency, we first measure the signal response in a wide enough quenched disorder interval, then find the optimal signal response. As a result, the dependence of η_{optimal} on the external signal frequency showcases a comprehensive view of the frequency effect. (g) The spectral amplification factor η_S vs the external signal frequency for different quenched disorders for $N = 5000$.

intermediate degree of disorder $\sigma = 0.7$, the vast majority of the bistable elements synergistically oscillate between $x^* = -1.0$ and $x^* = 1.0$, and the rest oscillate disorderly; see Fig. 1(b). For a high level of disorder $\sigma = 1.5$, the elements oscillate promiscuously between $x^* = -2.0$ and $x^* = 2.0$; see Fig. 1(c). These results indicate that the system undergoes a transition from a synchronous small-amplitude shake to a synergistic large-amplitude oscillation, and finally converges to a disordered activity as the coupling disorder increases continuously. The dependence of the collective stimulus response η_S on the coupling disorder parameter σ for a different system size is summarized in Fig. 1(d). Interestingly, we obtain a bell-shaped stimulus amplification curve of η_S vs σ , a typical characteristic of a resonance-like signal response in analogy with stochastic resonance [27–29], chaotic resonance [30,31], and diversity-induced resonance [23]. We thus conclude that the disorder of random coupling can induce a resonant signal response in coupled bistable oscillators. We remark that such

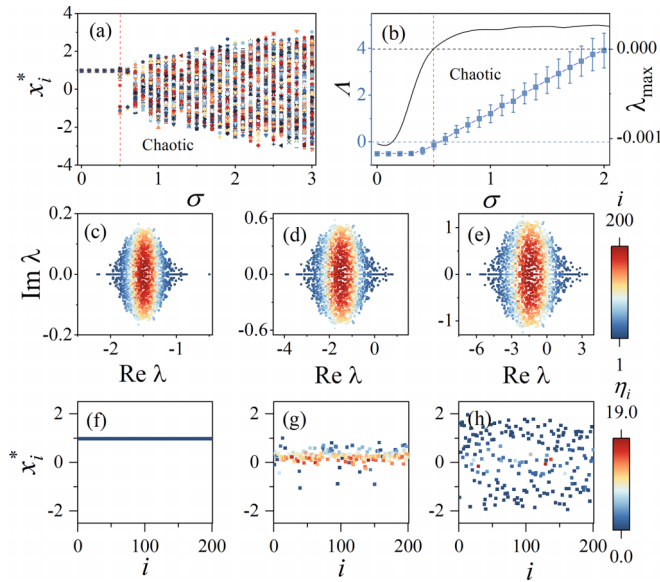


FIG. 2. Mechanism of the quenched disorder-induced chaotic resonance in bistable oscillators. (a) The fixed points distribution vs the coupling disorder. (b) The real part of the rightmost eigenvalue or the maximal Lyapunov exponent with respect to the disorder parameter. (c)–(e) The eigenvalue distribution in the complex plane for three degrees of disorder, wherein (c) $\sigma = 0.2$, (d) $\sigma = 0.7$, and (e) $\sigma = 1.5$. (f)–(h) The individual stimulus response distribution for (f) $\sigma = 0.2$, (g) $\sigma = 0.7$, and (h) $\sigma = 1.5$.

a type of coupling-disorder-induced resonant phenomenon is robust to different parameter settings. In Fig. 1(e) we show a heat map of the ensemble stimulus response η_s as functions of different average couplings c_m and coupling disorder σ , where an isolated hot island in the blue sea illustrates the amplified collective response region. In Fig. 1(f) the optimal signal response decreases monotonously as the driving frequency rises and behaves as an inverse-sigmoid-curve, demonstrating that this resonance is significant in the low-frequency region. Besides, the decreasing signal response trend can also be viewed for a slightly suprathreshold disorder; see Fig. 1(g). These results agree well with the conclusions of the frequency effect in stochastic resonance [32].

To unveil the mechanism behind the resonance, we first investigate the dependence of the fixed points of the bistable oscillators on the disorder parameter σ . As Fig. 2(a) shows, for $\sigma < \sigma_c^{\text{Bistable}} = 0.5$, all oscillators converge to the same fixed point $x_i^* = 1.0$ in the steady state. With increasing σ , these coincident fixed points go through a critical “breaking process” from the unique fixed point phase to the glasslike one. Since the latter is closely related to the resonant signal response, we further perform a stability analysis for this transition process (see [32] for details of the Jacobin matrix). As Figs. 2(b)–2(e) show, when $\sigma < \sigma_c^{\text{Bistable}}$, the real parts of all the eigenvalues are negative, which demonstrates that the unique fixed point is stable for weak disorders. Nevertheless, such a stable configuration will be broken when the disorder exceeds the critical value $\sigma_c^{\text{Bistable}}$, where the positive real parts of eigenvalues arise, revealing that the original uniform steady state loses its stability. Evidently, the stronger the disorder is, the greater the real parts of the eigenvalues become. These

results reveal that all oscillators behave persistently as one when the coupling disorder is low, and since the weak external stimulus cannot force them to oscillate tempestuously, the collective response is also weak all the time.

To disclose why the glasslike phase can induce the large-amplitude oscillation, we next check whether the phase is chaotic (see [32] for details of the maximal Lyapunov exponent) since it is well acknowledged that the disorder of random couplings can induce the order-chaotic transition [9], and the chaotic activity may contribute to the signal amplification [30,31]. As Fig. 2(b) shows, for weak disorder $\sigma < \sigma_c^{\text{Bistable}}$, the maximal Lyapunov exponent $\lambda_{\max} < 0$, which implies that the dynamics of the bistable system is ordered. On the other hand, for $\sigma > \sigma_c^{\text{Bistable}}$, the maximal Lyapunov exponent increases to a positive value, which verifies that the system enters into a chaotic state. These results are consistent with the consequences of the order-chaos transition in Ref. [9].

Now we are able to figure out the mechanism behind the resonance. The potential of Eq. (1) for a vanished coupling is read as $V(x_i) = -0.5x_i^2 + 0.25x_i^4 - Ax_i \sin(\omega t)$. The potential rises and falls periodically following the external stimulus, yet all the elements are restrained in one of the potential wells as long as the stimulus amplitude is weak enough. Whenever the oscillations become chaotic, the elements have a certain probability to escape from the original well into the other one, resulting in a interwell hopping [30–33]. Specifically, for a weak chaotic state, which corresponds to the edge of the chaos, the hopping can be entrained by the weak signal and thereby lead to a resonant-like oscillation [32,33]. Nonetheless, such an entrainment will be broken in the presence of strong chaos. As a result, the ensemble stimulus signal exhibits an chaotic resonance-like amplification. See [32] for the entrainment (frequency-matching) behavior.

Remarkably, such a chaotic resonance mechanism can be further verified from the viewpoint of the individual signal response. Inserting the individual activity x_i into the spectral amplification factor, we can obtain the individual signal response as $\eta_i = 4|(e^{i\omega t} x_i(t))|^2/A^2$. As Fig. 2(f) shows, in the ordered regime, since the positive coupling is dominated, the oscillators cohere into a single unit and the individual signal response is weak and identical. In the weak chaotic region, however, the majority of oscillators with a large signal response have comparable fixed points near the position $x^* = 0$, which suggests that the oscillations are still ordered despite the maximal Lyapunov exponent is positive; see Fig. 2(g). Note that such a result coincides with the entrainment by the stimulus [32,34]. In the strong chaotic region, the fixed points are dispersed in the region $(-2, 2)$, signifying that the influence of the external stimulus is faint, and the activities of oscillators are mainly ruled by their inherent chaotic dynamics [32].

To put our findings on a solid theoretical ground, we utilize the dynamical cavity method to derive the dynamical mean-field equations of Eq. (1) as

$$\dot{x} = (1 - c_m)x - x^3 + c_m M(t) + \sigma \zeta(t) + A \sin(\omega t), \quad (3)$$

in which $M(t) = \mathbb{E}[x(t)]$, and the self-correlation of colored noise $\zeta(t)$ is $C(t, s) = \mathbb{E}[x(t)x(s)]$ [20]. The symbol $\mathbb{E}[\cdot]$ is the average of the noise trajectories $\zeta(t)$ and the

initial conditions $x(0)$. This equation describes the statistical properties of the system under the interlaced influence of coupling disorder, stimulus, and potential initial conditions in the thermodynamic limit [20] (see [32] for details). Moreover, as the result from mean-field analysis suggests that the disordered coupling input can be equivalently represented by colored noise [9,12,16,20], we assume that the coupling input is approximately depicted by the conventional Ornstein-Uhlenbeck (OU) noise $\xi(t)$ with a proper correlation time, whose expectation and autocorrelation are $\langle \xi(t) \rangle = 0.0$ and $\langle \xi(t)\xi(\tau) \rangle = D \exp(-|t - \tau|/\tau_0)/\tau_0$, respectively. The symbols D and τ_0 are noise intensity and correlation time, respectively. As such, Eq. (3) can be rewritten as

$$\begin{aligned} \dot{x} &= (1 - c_m)x - x^3 + c_m M(t) + \sigma z + A \sin(\omega t), \\ \dot{z} &= -\frac{1}{\tau_0}z + \Gamma(t), \end{aligned} \quad (4)$$

where $\Gamma(t)$ is a Gaussian white noise with $\langle \Gamma(t) \rangle = 0$ and $\langle \Gamma(t)\Gamma(t') \rangle = 2D\delta(t - t')/\tau_0^2$ [35,36].

To validate our assumption, we compare the signal response of Eq. (4) versus the disorder of random couplings with parameters $D = 1.0$, $\tau_0 = 0.3$, and the numerical results $M(t)$ obtained from Eq. (1) (see [32] for the fit of different D and τ_0). As shown by the red short dashed line in Fig. 1(d), we find that the OU noise can well capture the numerical signal response of Eq. (1). Thus, the consideration of an OU noise is proper for the bistable oscillators. Besides, using the linear response theory [32], we successfully obtain the analytical expression of the collective stimulus response without and within the consideration of self-consistency relations as

$$\eta_S^{\text{without}} = \frac{1}{\sigma^4} \frac{4r_k^2(\tau_0)}{4r_k^2(\tau_0) + \omega^2}, \quad (5)$$

where $r_k(\tau_0) = \frac{1}{\sqrt{2\pi}}(1 + 3\tau_0)^{-\frac{1}{2}} \exp[-\frac{1}{4\sigma^2}\Delta]$ and $\Delta = [1 + 27\tau_0/16 + \tau_0^2/2]/(1 + 27\tau_0/16)$ and

$$\eta_S^{\text{within}} = \frac{(a\lambda_{\min} - c_m a^2)^2 + a^2 \omega^2}{[(\lambda_{\min} - c_m a)^2 + \omega^2]^2}, \quad (6)$$

in which $\lambda_{\min} = \sqrt{2}[1 - \frac{3}{2}\tau_0 \exp(-\frac{1}{4\sigma^2})]/\pi$, $a = -(1 + 2\tau_0)\lambda_{\min}(\langle x^2 \rangle + \tau_0\sigma^2)/\sigma^2$, respectively. As shown in Fig. 1(d), two clear bell-shaped amplification curves can be observed (shown in blue dashed and red solid lines), which proves theoretically that the disorder of random coupling can evoke a resonant stimulus response. Furthermore, these formulas can qualitatively depict the relationship between the response and driving frequency; see Fig. 1(g).

Excitable FitzHugh-Nagumo neurons. In order to uncover the biological relevance of such resonance, we now pay attention to globally coupled excitable FitzHugh-Nagumo (FHN) neurons subjected to a weak periodical stimulus [23,37,38]. The FHN equations can be written as

$$\begin{aligned} \epsilon \dot{x}_i &= x_i - \frac{1}{3}x_i^3 - y_i + \sum_{j=1}^N c_{ij}(x_j - x_i), \\ \dot{y}_i &= x_i + b + A \sin(\omega t), \quad i = 1, \dots, N, \end{aligned} \quad (7)$$

where x_i and y_i are an activator variable concerning the membrane potential and a recovery variable relating to the

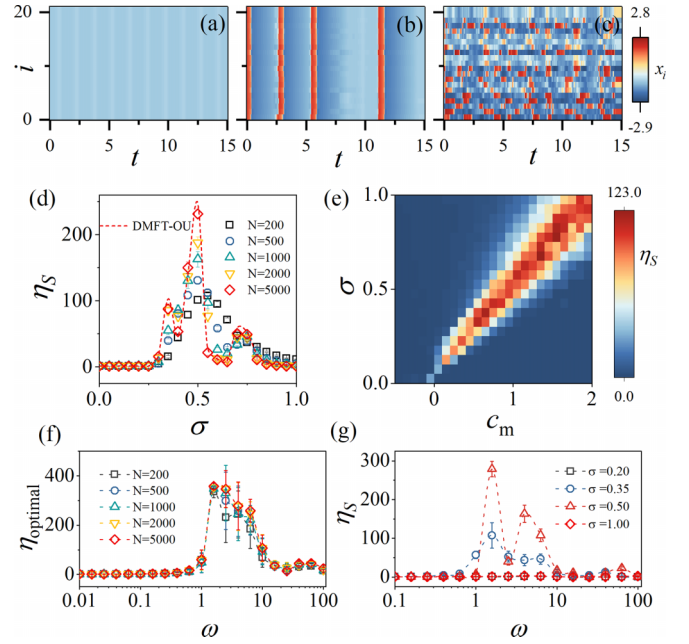


FIG. 3. Resonant stimulus response induced by quenched disorder in globally coupled excitable FitzHugh-Nagumo neurons. (a)–(c) Activities of 20 neurons sampled randomly from the neuron pool for different degrees of coupling disorder, wherein (a) $\sigma = 0.2$, (b) $\sigma = 0.5$, and (c) $\sigma = 1.0$. (d) The spectral amplification factor η_S vs the coupling disorder for different system sizes. The red short dashed line corresponds to the dynamical mean-field approximation with OU noise and numerical moment $M(t)$ for $N = 5000$. (e) The dependence of collective signal response on the mean and the standard deviation of the random couplings for $N = 200$. (f) The optimal signal response versus the frequency of the external signal in globally coupled FitzHugh-Nagumo neurons. (g) The spectral amplification factor η_S vs the external signal frequency for different quenched disorders for $N = 5000$. The dashed lines in (g) and (f) are guides for the eye.

conductivity of potassium ion channels of i th neuron, respectively. The parameter ϵ separates the timescales between the two variables, and $b = 1.05$ is the excitability parameter of the neuron [37]. For $|b| > 1$, the neuron possesses a unique stable equilibrium $(x^*, y^*) = (-b, -b + b^3/3)$, whereas for $|b| < 1$, the neuron displays a persistent oscillation [38]. The coupling coefficient c_{ij} between neurons i and j is sampled from a Gaussian distribution with a certain mean and standard deviation in the same vein as in the bistable oscillators. We choose $A = 0.05$ and $T = 1.5$ to model the weak periodical stimulus [23]. Additionally, adding such perturbation on the recovery variable captures the time-dependent conductance of the potassium channels in the membrane or the time-varying glucose level in the bloodstream [39,40]. To avoid the interference of the transient state, the initial states of these neurons are considered as quiescent states, e.g., (x^*, y^*) [38].

As Fig. 3 shows, for a weak disorder $\sigma = 0.2$, all the neurons fire gently around the resting state $(x^*, y^*) = (-1.05, -0.66)$. For intermediate degree of coupling disorder, say, $\sigma = 0.5$, the membrane potential of neurons rises and falls within the same pulsation, indicating the synchronous discharges of the neurons. If the coupling disorder is strong

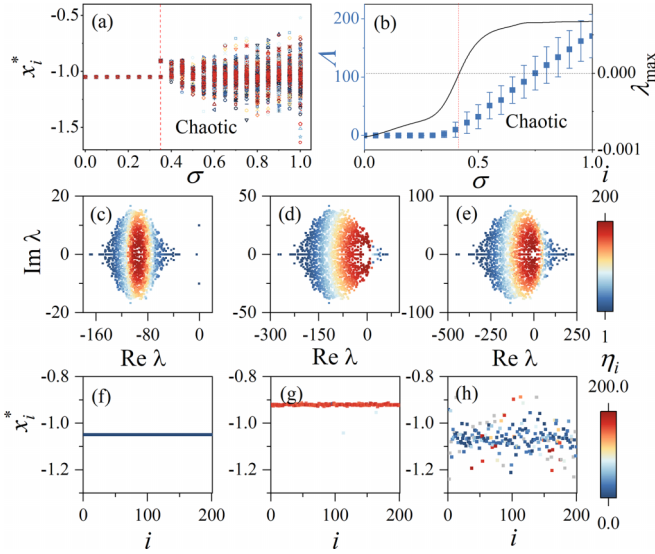


FIG. 4. Mechanism of the quenched disorder-induced chaotic resonance in excitable FHN neurons. (a) The fixed points distribution vs the quenched disorder. (b) The real part of the rightmost eigenvalue or the maximal Lyapunov exponent with respect to the disorder parameter. (c)–(e) The eigenvalue distribution in the complex plane for different degrees of disorder, wherein (c) $\sigma = 0.2$, (d) $\sigma = 0.5$, and (e) $\sigma = 1.0$. (f)–(h) The individual stimulus response distribution for (f) $\sigma = 0.2$, (g) $\sigma = 0.5$, (h) $\sigma = 1.0$.

enough, like $\sigma = 1.0$, the activities of all neurons are chaotic. These results imply definitely that an intermediate degree of disorder can induce a striking spiking activity. Once again, the collective stimulus response behaves a bell-shaped amplification with increasing disorder σ , and a resonant response island can still be observed in the heat map of η_S versus the mean and the standard deviation of the random couplings. Hence, we conclude that the resonant signal response evoked by quenched coupling disorder can emerge in FHN neurons as well. Furthermore, as the driving frequency increases, such a resonant response showcases a sensitive frequency dependence, where the optimal resonance can be obtained at an intermediate frequency; see Fig. 3(f). Besides, as shown in Fig. 3(g), for a slightly suprathreshold disorder, the signal response portrays double peaks with the rise of the driving frequency. These results demonstrate that the resonant response in FHN neurons can be regarded not only as chaotic resonance but also as mechanical resonance [32].

By performing fixed point analysis as in the bistable oscillators, we yield the critical disorder $\sigma_c^{\text{FHN}} = 0.35$ for FHN neurons, separating the unique steady state and the glasslike steady states, as plotted in Fig. 4(a). In particular, for $\sigma < \sigma_c^{\text{FHN}}$, the neurons behave as one, whereas for $\sigma > \sigma_c^{\text{FHN}}$, the unique state breaks and results in the glasslike steady states. From Figs. 4(b)–4(e), one can find that the real part of the rightmost eigenvalue Λ is maintained at a negative constant and then rises quickly with the increase of quenched disorder [32]. Note that the critical point for $\Lambda = 0$ is equal to the critical value σ_c^{FHN} . It is interesting that such a critical disorder corresponds to the order-chaos transition point as well.

The mechanism behind the resonance in FHN neurons can also be discerned fairly. For a weak disorder, since the

excitatory couplings are dominating in the system, the neurons behave as one neuron sleeping in the quiescent state. The weak stimulus cannot rouse them to discharge. As a result, the collective signal response is weak; see Fig. 4(f). For an intermediate degree of disorder, the activities of neurons are chaotic, and they have a finite probability to discharge with large amplitude. Whenever the chaotic behavior is weak, the periodical external stimulus can enforce the disordered spikes into a more synchronous form, which is known as the chaotic resonance or the deterministic stochastic resonance [32–34]. A striking feature is that the fixed points of different neurons are nearly the same despite the spikes are chaotic, resulting in a strong collective stimulus response; see Fig. 4(g). For large enough disorder, the chaotic activity is much stronger than the weak force such that the spikes exhibit significant disorder with a vanishing collective response; see Fig. 4(h).

Likewise the situation in bistable oscillators, we can yield the dynamical mean field equation of the FHN model as

$$\begin{aligned} \epsilon \dot{x} &= (1 - c_m)x - \frac{1}{3}x^3 - y + c_m M(t) + \sigma z, \\ \dot{y} &= x + b + A \sin(\omega t), \dot{z} = -\frac{1}{\tau_0}z + \Gamma(t), \end{aligned} \quad (8)$$

where $D = 0.01$, $\tau_0 = 0.3$ [32]. By solving numerically these equations, we verify that the bell-shaped amplification can be well captured by Eq. (8), as shown in Fig. 3(d). See [32] for the qualitative theoretical analysis of resonant signal response.

Optimal mutual information in Inferior Olive. Our work may provide a meaningful explanation for the psychological behavior of inferior olive. The Purkinje cell [41], which is known as the sole output neuron of the cerebellar cortex, receives two major types of synaptic inputs: (1) The complex climbing fiber input within the form of low-frequency spikes, which originates from the Inferior Olive (IO) and (2) the simple multiple parallel fibers input with high discharge frequency [41,42]. Additionally, result showed that if the IOs fire synchronously and rhythmically, the Purkinje cell fails to transmit rich information [41,42]. By contrast, if the IOs discharge desynchronously and irregularly, it can restore properly the IO input content. To reveal how the electrically coupled IOs fire desynchronously and irregularly, Schweighofer *et al.* carried out a simulation research based on the realistic IO cell model [42]. They found that the activities of the electrically coupled IOs undergo the synchronization-chaos-synchronization transition as the coupling coefficient increases. Markedly, a bell-shaped mutual information with respect to the coupling coefficient is disclosed in which the optimal input-output mutual information emerges in the chaotic regime. We remark that the coupling coefficients are sampled from a uniform distribution with a limited deviation in [42]. They concluded that the electrical coupling including the balance between excitatory and inhibitory currents could provide the source of disorder that induces a chaotic resonance [42]. Combining their findings with our current work, we speculate that the chaotic firing can be also supported by the disordered couplings among the elements, and a moderate degree of quenched disorder can most efficiently evoke the chaotic resonance.

Conclusion. In summary, we have investigated the collective stimulus response with respect to the quenched coupling disorder in both bistable oscillators and excitable FHN neurons. We found that the collective stimulus response shows a bell-shaped amplification as the quenched disorder increases continuously. We argue that such a resonance is rooted in the emergence of order-chaos transition. Specifically, if the disorder is weak, the oscillators (neurons) behave as one and joggle around the fixed point (quiescent state), and the weak stimulus cannot force them to oscillate largely (discharge). However, when the disorder goes beyond a critical value, the original fixed point (quiescent state) loses its stability, and the system enters into a chaotic state. The chaotic activity supports the disordered interwell hopping (discharge) and can be entrained under the weak chaotic situation. As a consequence, the optimal signal response emerges near the order-chaos transition point, namely, the edge of chaos. Remarkably, this coupling-disorder-induced resonance can be fully verified by the dynamical mean field analysis. Our results are not only consistent with the research concerning the input-output mutual information in realistic IO model, but

also may provide a potential explanation for the relationship between the chaotic activity and the electrical coupling in coupled IOs.

Quenched random couplings are ubiquitous and known as the origin of complex emergent behavior in many-body systems [5]. Whether this type of disorder can be utilized to achieve some optimal performances has attracted a great deal of interest in recent years. Our results give a promising clue that the quenched random couplings can evoke an optimal response when the disordered system is subject to a weak stimulus.

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