

Accurate optimal quantum error correction thresholds from coherent information

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Quantum error correcting (QEC) codes protect quantum information from decoherence as long as error rates fall below critical error thresholds. In general, obtaining thresholds implies simulating the QEC procedure using, in general, suboptimal decoding strategies. In a few cases and for sufficiently simple noise models, optimal decoding of QEC codes can be framed as a phase transition in disordered classical spin models. In both situations, accurate estimation of thresholds demands intensive computational resources. Here we use the coherent information of the mixed state of noisy QEC codes to accurately estimate the associated optimal QEC thresholds already from small-distance codes at moderate computational cost. We show the effectiveness and versatility of our method by applying it first to the topological surface and color code under bit-flip and depolarizing noise. We then extend the coherent information based methodology to phenomenological and quantum circuit level noise settings. For all examples considered, we obtain highly accurate estimates of optimal error thresholds from small, low-distance instances of the codes, in close agreement with threshold values reported in the literature. Our findings establish the coherent information as a reliable competitive *practical* tool for the calculation of optimal thresholds of state-of-the-art QEC codes under realistic noise models.

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Successful quantum error correction (QEC) is a fundamental milestone on the route to fault-tolerant quantum computing [1–3]. QEC codes offer protection against decoherence by encoding quantum information redundantly in entangled multiqubit states. The detrimental effect of noise can then be counteracted by measuring quantum correlations. These measurement outcomes, forming the error syndrome, can be processed by decoding procedures to correct errors and thereby restore the encoded quantum states. It is known that there are fundamental limits to the amount of noise that QEC codes can tolerate [4]. There is a *threshold* error rate beyond which the quantum information cannot be retrieved with certainty [5–8]. To be concrete, below threshold the originally prepared state can be recovered with fidelity approaching 1 [9,10]. Therefore computing and/or estimating QEC thresholds is essential for developing and improving QEC codes. Recent QEC experiments, e.g., with trapped ions [11–13], superconducting qubits [14–19], and neutral atoms [20], have demonstrated concepts of fault-tolerant QEC, with the performance of logical qubits approaching or reaching a break-even point with physical qubits.

In general, the determination of error thresholds is a computationally hard problem [21]. The standard approach involves simulating the QEC procedure by sampling errors from the error model and estimating logical failure rates. The latter requires us to choose a decoder [22] that detects and corrects errors. For stabilizer QEC codes and Clifford noise models, one can simulate very large code instances, allowing for good extrapolation of the threshold. Regarding the choice of decoder, maximum likelihood decoding (MLD) is optimal and thus achieves optimal thresholds [21,23]. Therefore, thresholds from QEC simulations are optimal if and only if the decoding strategy is equivalent to MLD. One downside of MLD, in general, is its exponential scaling in the number of physical qubits for finding the optimal recovery operation given a syndrome measurement [21], which often severely limits its practicality. In contrast, suboptimal decoders may be efficient but do not reach optimal thresholds because they cannot make use of the whole information provided by the syndrome measurements.

For certain QEC codes and error models, the optimal decoding problem acquires enough structure to allow for a mapping to partition functions of statistical mechanics problems. In the seminal work of Ref. [24], it was shown that the threshold problem of the two-dimensional (2D) toric code [25] is equivalent to the phase transition of the classical random bond Ising model (RBIM) in 2D along the Nishimori line [26]. Since then, statistical mechanics mappings for other topological codes and error models have been developed [27–36]. This approach has the advantage of being tractable via Monte Carlo sampling of the partition function,

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providing estimates of QEC thresholds as critical points of phase transitions. The main challenge lies in the significant amount of computational resources needed to accurately locate critical points of disordered interacting spin models. In summary there are two ways to estimate optimal thresholds of QEC codes: (i) a general QEC simulation and use of MLD [21,23], and (ii) constructing statistical mechanics mappings for specific codes and error models [24,27–37].

In this work, we introduce a third approach to the estimation of optimal QEC thresholds based on the *coherent information* (CI) [9,38]. This quantity is the theoretical upper limit to the amount of quantum information that can be recovered from a noisy mixed state, hence reflecting the capability of the given QEC code to protect the logical qubits under a particular error model. As such, the CI signals the threshold error rate beyond which the quantum information cannot be recovered by any means. Thus only optimal decoders can attain the threshold obtained from the CI, while thresholds reached by suboptimal decoders are always smaller than the optimal thresholds we study in this work. To illustrate the use and power of the CI, we estimate optimal thresholds of topological QEC codes in the *code capacity* setting, namely when the state becomes corrupted after a single round of noise on each data qubit. We show that the CI, evaluated for small instances of topological codes, provides threshold values that come very close to known optimal thresholds, exhibiting strikingly small finite-size effects. Moreover, in real devices all components of the circuit are faulty (e.g., measurement and gates), which compels the inclusion of gate errors and stabilizer measurements in the error model. Therefore, we adapt the CI as a tool to determine critical error thresholds to the *phenomenological and circuit level noise* settings. Formally, the latter setting includes gate and measurement errors alike, whereas the former only considers measurement errors. We then apply this approach to the quantum repetition code in the aforementioned settings and show that this again yields very accurate threshold estimates. Importantly, in all cases of study, the finite distance crossings are *almost exactly* at the location of the known thresholds (discrepancies start at the third digit), suggesting little finite-size corrections in the CI.

Before introducing formal definitions, we briefly compare the approach introduced here with the other two main approaches for obtaining optimal thresholds. On the one hand, statistical mechanics mappings must be crafted on a case-by-case basis, and they are limited mostly to the code capacity and phenomenological noise settings—with few exceptions [27,34]. In contrast, the CI approach is easily adapted to different codes and error models. On the other hand, MLD and computing the CI both suffer from exponential overhead in the number of physical qubits. However, the CI does not require error sampling, thus getting rid of statistical errors in the threshold estimation, and is thus purely limited by minimal finite size-effects.

The CI was first introduced as a measure of the quantum information transmitted through a noisy channel [38]. Indeed maximizing the CI over all possible encodings yields the quantum capacity of a noisy quantum channel [39]. At the same time, it was established as a key ingredient for the existence of a successful quantum error correction protocol [9,40]. Following the setting shown in Fig. 1, the coherent

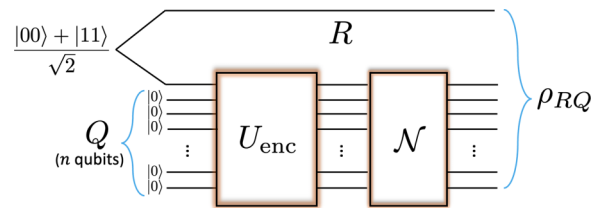


FIG. 1. Setup for the coherent information for a QEC code encoding a single logical qubit. We start with a Bell pair between two physical qubits, one of which is kept as a (noise-free) reference qubit (R). The other qubit is used for preparing the logical state on the register of n qubits (Q). The encoding unitary U_{enc} on the data qubits creates a superposition of logical states, such that in the RQ system we have $(|0_R 0_L\rangle + |1_R 1_L\rangle)/\sqrt{2}$. After encoding, the Q system is exposed to noise $\mathcal{N}$ according to a noise model. The coherent information of the noisy state ρ_{RQ} allows one to extract optimal thresholds.

information is defined as

$$I = S(\rho_Q) - S(\rho_{RQ}). \quad (1)$$

Here, $S(\rho) = -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy of the state, and $\rho_Q = \text{Tr}_R(\rho_{RQ})$ is the reduced density matrix after tracing out the reference qubit. It was shown in Ref. [9] that I can only decrease or stay constant after successively applying CPTP maps to the Q system. In this setup, the CI before application of the noise is $I_0 = \log 2$, therefore $I \leq \log 2$. The equality $I = \log 2$ ensures the existence of a quantum error correction procedure that recovers the state with unit fidelity. In Ref. [41], the CI has been used in finding thresholds of concatenated repetition codes, although there the coherent information was identified as the quantum code capacity. Generalizing the setup in Fig. 1 to codes with $k > 1$ logical qubits is straightforward; see the supplemental material (SM) [42] for details.

Error models. We focus on two types of single-qubit error models: bit-flip noise and depolarizing noise. The bit-flip channel on qubit i with error rate p reads $\mathcal{N}_i^b(\rho) = (1-p)\rho + pX_i\rho X_i$, while the depolarizing channel is written as $\mathcal{N}_i^d(\rho) = (1-p)\rho + \frac{p}{3}X_i\rho X_i + \frac{p}{3}Y_i\rho Y_i + \frac{p}{3}Z_i\rho Z_i$. All qubits are exposed to the same error channel such that $\mathcal{N} = \prod_{i=1}^n \mathcal{N}_i^a$, where $a = b, d$ denotes the type of channel.

Quantum codes and their thresholds. We choose to benchmark our method for two well-known topological codes: (i) the rotated surface code, and (ii) the topological color code, both being leading contenders for practical QEC. (i) The rotated surface code $[[n, 1, \sqrt{n}]]$ [43,44], which is asymptotically equivalent to the toric code [25], encodes $k = 1$ logical qubit in n physical qubits and has a code distance (minimal support of logical operators) of $d = \sqrt{n}$. In Ref. [24], the threshold of the toric code under bit-flip noise was estimated from mapping the QEC code to the 2D RBIM, yielding a threshold $p_{\text{th}} = 0.109(2)$. Depolarizing noise, which also contains the occurrence of Y -type errors, introduces correlations between bit-flip and dephasing noise. Thus the corresponding statistical mechanics model becomes a stack of two RBIMs on square lattices with interlayer four-body interactions [30]. The numerically estimated threshold under depolarizing noise is $p_{\text{th}} = 0.189(3)$ [30]. (ii) The triangular 4.8.8 color code

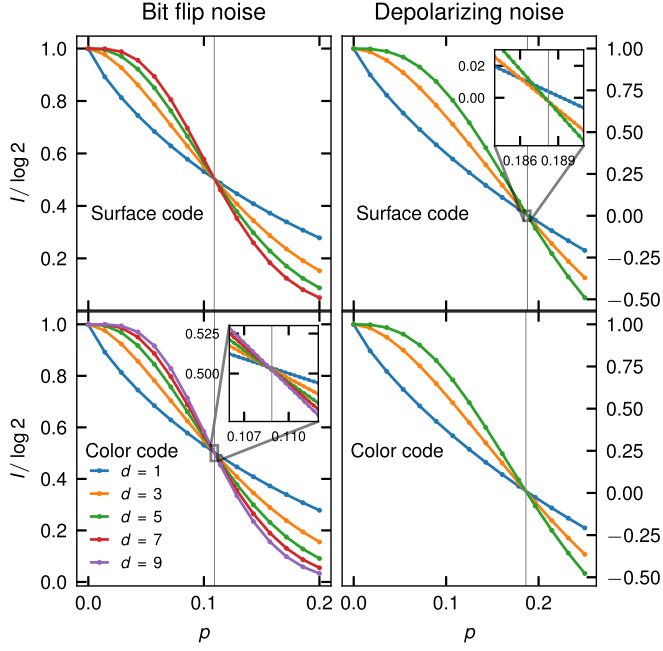


FIG. 2. Coherent information for surface (upper row), color code (lower row) under bit flip noise (left column) and depolarizing noise (right column). We show distances $d = 3$ to $9(7)$ for 4.8.8 color (rotated surface) code under bit-flip noise and $d = 3, 5$ for both codes under depolarizing noise. $d = 1$ denotes the CI of a single qubit. Gray vertical lines are the crossing between the two largest distances available (see Table I).

$[[n(d), 1, d]]$ encodes $k = 1$ logical qubit in $n(d) = (d^2 - 1)/2 + d$ physical qubits [45] and belongs to the family of 2D topological color codes [46]. The reported threshold under bit-flip noise only is $p_{\text{th}} \approx 0.109$ [31,32,47–49]. As for the surface code, depolarizing noise introduces error correlations that manifest themselves as interlayer interaction between two originally independent 2D disordered spin models. The threshold for the 4.8.8 color codes under depolarizing noise was estimated as $p_{\text{th}} = 0.189(2)$ [30]. Details about the rotated surface and 4.8.8 color code are given in the SM [42].

Coherent information in topological codes. To reach codes with a relatively large number of data qubits, we exploit the structure of the mixed state and properties of the CI and error model. This allows us to numerically compute the CI for codes with up to $n = 49$ data qubits ($d = 9$ color code) under bit-flip noise and $n = 25$ for depolarizing noise ($d = 5$ surface code). For details about the numerical calculations, see the SM [42]. It is important to note that we exploit general properties of stabilizer states and Pauli noise, thus we expect the benefits from such properties to hold regardless of other independent factors, e.g., spatial dimensionality and error correlations. In Fig. 2 we show the CI for surface and color code alongside the CI of a single qubit ($d = 1$). The CI of the code with distance $d \geq 5$ crosses the curve for distance $d - 2$ almost at the same error rate. Indeed, the crossing point of the two largest available distances is equal to the known values of the threshold up to two decimal digits (see Table I). Given the smallness of the code instances, it is quite remarkable that the finite-size crossings yield such accurate estimates of the

TABLE I. Crossings of coherent information (CI) between the two largest distances available [50] plotted in Fig. 2. For comparison, we show the known thresholds for surface and 2D color codes.

	Bit-flip (CI)	Depol. (CI)	Bit flip p_{th}	Depol. p_{th}
Surface	0.1089(2)	0.1882(2)	0.1094(2) [24]	0.189(3) [30]
Color	0.1088(2)	0.1862(2)	0.10919(7) [49]	0.189(2) [30]

QEC threshold. Furthermore, we present a pseudthreshold analysis by plotting the crossings of the CI of codes with distance $d > 1$ with the single-qubit CI (see Fig. 3). We see a slight nonmonotonous trend of the pseudthresholds that may be caused by finite-size effects. However, it is worth noting that the finite-size effects are noticeable only from the third decimal digit onwards (see the insets of Fig. 2).

Phenomenological and circuit level noise. So far we have considered the code capacity setting only. Now we allow measurements and gates to be also faulty. To this end, we consider the repetition code in the Z basis exposed to bit-flip noise. This code has distance $d = n$ but protects only against bit-flip noise. Faulty measurements of syndrome qubits (phenomenological noise) are modeled by bit-flip channels on the ancilla qubits right after each round of stabilizer measurements (see the setting and error model in Fig. 4). In this setting, the repetition code asymptotically behaves as a surface code in a space-time description [24]. When data qubit and measurement errors occur with the same error rate, the threshold is the same as that found for the surface code under bit-flip noise, $p_{\text{th}} \approx 0.109$ (see Table I). The CI crossing between $d = 3$ and 5 repetition code under phenomenological noise is located at $p \approx 0.1094(2)$ (see Fig. 5), which is *precisely* the aforementioned value [24]. We also study the more realistic case of circuit level noise, namely considering gate errors in the syndrome measurement and errors in the initialization of qubits. The extraction of fundamental thresholds for quantum

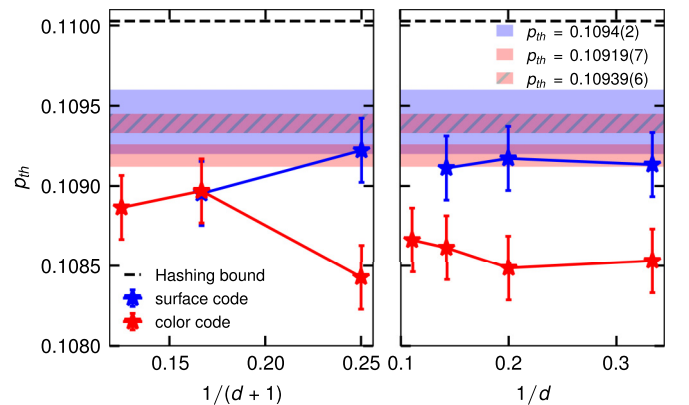


FIG. 3. Left: Finite distance crossings of CI between consecutive distances $d \geq 3$ and $d + 2$ for color and surface code under bit-flip noise. Right: Crossing between surface and color codes with distance $d \geq 3$ and the single-qubit CI. Shaded regions are previous estimates of thresholds: $p_{\text{th}} = 0.1094(2)$ from Ref. [24] for the surface code. For the color code we show $p_{\text{th}} = 0.10919(7)$ from Ref. [49] and $p_{\text{th}} = 0.10939(6)$ from Ref. [48]. Error bars are the error rate resolution prior to fitting [50].

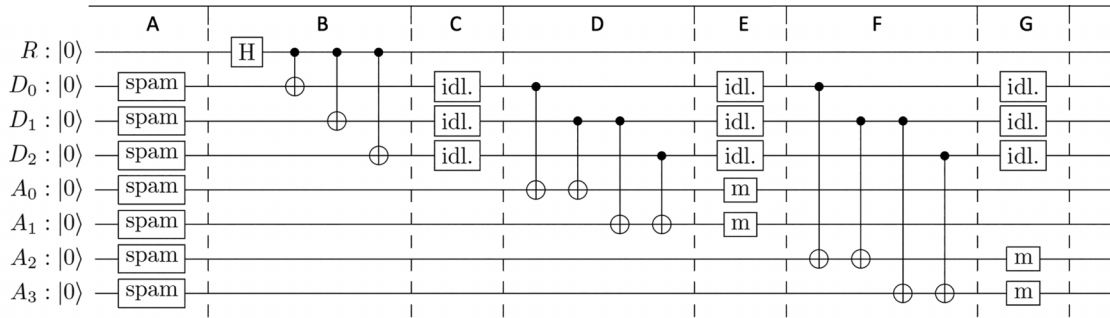


FIG. 4. Setup for computing coherent information with circuit level noise in the three-qubit repetition code. The qubits D_i and R are the data and reference qubits, respectively. The qubits A_i are ancilla qubits that store the syndrome measurement. Part A denotes state preparation and measurement (SPAM) errors with probability p_{sp} . In B the state $(|0_R 0_L\rangle + |1_R 1_L\rangle)/\sqrt{2}$ is prepared. This part is assumed to be noiseless. Parts C, E, and G contain idling noise with probability p_{id} . In parts E and G measurement errors are modeled as bit-flip channel with probability p_m . In parts D and F the stabilizer information is coherently mapped to the auxiliary qubits (without actually projectively measuring the auxiliary qubits). The CNOT gates in the stabilizer measurements fail with probability p_2 (see the SM [42] for details about the error model). Then the data qubits are exposed to bit-flip noise as usual. Here we call phenomenological noise the case $p_{\text{sp}} = p_2 = 0$ and $p_m = p_{\text{id}}$, while the setting $p_{\text{sp}} = p_m = p_{\text{id}} = p_2 = \lambda$ represents a circuit level noise realization.

circuit noise settings is challenging to tackle by mappings to disordered classical spin models [27,52], but it was considered previously in [51] for a quantum repetition code. When all circuit error rates (SPAM, idling, and gate errors) were equal to λ , the threshold was computed using MLD and found to lie at $\lambda_{\text{th}} \approx 0.033$. Later, statistical mechanics mappings for circuit level noise in repetition codes were developed [34], confirming the previously found threshold. Our CI-based analysis shows a crossing at $\lambda \approx 0.0373(5)$. The difference starts at the third digit, compatible within error bars with the previous studies.

Now we explain how to generalize the CI to incorporate noisy syndrome measurements. We start off by preparing the logical states and expose the data qubits to noise. Then the syndrome is measured $d - 1$ times via ancilla couplings (see the illustration of the $n = 3$ setup and explanation of the error model in Fig. 4). After the last step, we consider the whole quantum state—consisting of the reference qubit, the data, and the ancilla qubits—as the state ρ_{RQ} , and ρ_Q as the reduced state of data and ancilla qubits only, after tracing out the reference qubit. The coherent information is defined as usual,

$I = S(\rho_Q) - S(\rho_{RQ})$. The total number of qubits involved is $1 + d + (d - 1)^2$, thus our numerical calculation is limited to $d \leq 5$.

Discussion. In this work, we have established the CI in small codes as a competitive *practical* tool for reliably estimating optimal QEC thresholds with high precision on the basis of a number of different error models. To be precise, $d = 5$ codes show thresholds equal to the known optimal thresholds with two to four digits precision in all cases we consider. The efficiency of the approach is due to our observation of such tiny finite-size effects, which, in light of the variety of codes and error models investigated here, seems general, and at any rate is testable in practice.

It is worthwhile to briefly discuss the computational cost of CI. The CI is a well-defined nonlinear function of the whole mixed-state density matrices, therefore it cannot be computed using trajectory approaches, e.g., Clifford simulations, and it requires in general allocating $O(e^n)$ parameters, limiting its computation to a few tens of qubits. However, we have shown that the exponential scaling can be eased, e.g., polynomial improvement due to the block diagonal structure, by exploiting the structure of the state and error model, thus allowing us to explore code instances beyond the smallest distance despite the overall exponential scaling. Due to the small finite-size effects, the computational cost of the CI approach for obtaining optimal thresholds with high precision is then comparable to the other two main methods (see the SM for details), with the advantage of being agnostic of the code details and free of stochastic errors.

We thus envision the CI as a promising tool for estimating optimal thresholds in regimes that are currently difficult to access. One case is circuit level noise where statistical mechanics mappings are rare [34], including coherent noise [27,52,53]. Another potential application case are new codes, most prominently quantum low-density parity check (qLDPC) codes [54] (see the SM for a calculation of CI in a specific qLDPC code), whose growing number of logical qubits limits the direct use of statistical mechanics mappings [55] and MLD [56]. Hence, the CI provides an unbiased method to estimate QEC thresholds for codes with multiple logical qubits and

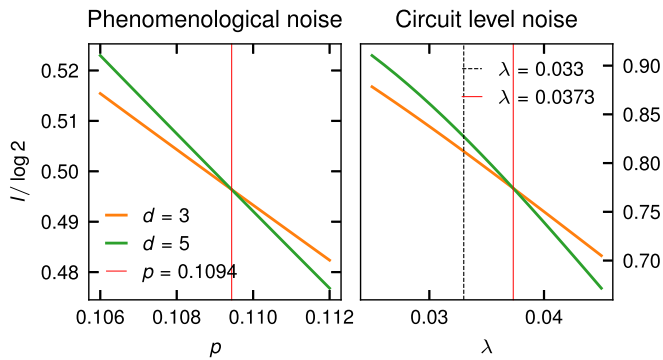


FIG. 5. Coherent information for the data and auxiliary qubit state in the repetition code for $d = 3, 5$. Left: result for phenomenological noise; the red line denotes the crossing at $p = 0.1094(2)$. Right: result for circuit level noise; the red line denotes the crossing at $\lambda = 0.0373(5)$; the black dashed line is the threshold reported in [51] using MLD.

for general noise models, ranging from code capacity and phenomenological noise to circuit level noise.

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