

Origin of anticlastic curvature in a cellular metaplate

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Generating a three-dimensional curved surface from flat sheets using simple mechanical actuation is a crucial step in developing functional shape-shifting materials. Among the methodologies relying on emergent concepts, the traditional cellular solid-based plate, with its highly porous structures, provides a versatile tool for creating doubly curved surfaces through planar bending actuation. Leveraging a combination of digital fabrication, physical experiments, finite element simulation (FES), and linear elasticity theory, we demonstrate how such a cellular metaplate can exhibit doubly curved shapes. By extending Lamb's classical theory of the anticlastic effect in thin elastic plates to our cellular metaplate, we develop a scaling law for the crossover length below which the doubly curved surface appears as $b_c \sim b_{\text{Lamb}}/\sqrt{\rho}$, where $b_{\text{Lamb}} \sim \sqrt{Rw}$, with R being the externally imposed radius of curvature, w being the plate thickness, and ρ being the relative density of a given cellular geometry. The prediction is validated by our experiment and FES. The proposed framework is versatile and emphasizes the fundamental physical principles governing the mechanics of shape-morphing surfaces.

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Introduction. Shape-shifting surfaces can be found in a variety of systems, including biological morphogenesis, plant organs, smart textiles, and space technology. In addition, the creation of a curved surface from a flat sheet is one of the central themes of our industrial technology, from flexible electronics to architecture. However, owing to several fundamental physical and geometric constraints, it is generally impossible to transform a flat surface into a smooth one with arbitrary curvature. To overcome this difficulty, several novel ideas have recently been developed, such as the use of non-Euclidean deformation due to nonuniform growth [1], origami or kirigami [2–11], swelling pressure [12,13], and other computer-assisted shape-programming methods [14,15]. These emerging technologies are pushing the limits of the shape-morphing abilities of functional surfaces and expanding the range of their applications. However, it is well known that traditional sandwich panels with honeycomb cores, which are the key components of lightweight structures, often spontaneously form an unfavorable, doubly curved shape upon planar bending [16]. This calls into question the extent to which a flat plate with a given internal structure exhibits a doubly curved shape when it is bent into one direction. This question should be of fundamental importance toward establishing a predictive framework of shape programming and engineering [8,17,18].

According to the classical Kirchhoff theory of elastic plates, bending in one direction always creates a doubly curved (mostly saddle) shape, which is essentially induced

by a positive Poisson's effect [19]. However, because such a deformation involves a change in the surface metric that requires large stretching energy, a sufficiently wide plate prefers to bend into a cylindrical, and thus isometric, surface. What is the critical system size b_c that separates the geometrically distinct bending behaviors? This classical problem was solved by Lamb more than a century ago [20], when he gave $b_c \sim \sqrt{Rt}$, where t and R are the thickness of the plate and the imposed radius of curvature, respectively. Lamb's theory explains why the anticlastic effect is observed only in sufficiently narrow plates or beams [Figs. 1(d) and 1(e)] [21–25]. However, in examples of sandwich panels, the bending behavior of a wide cellular plate, which is a highly porous, lightweight plate, does not seem to obey the classical theory. Rather, cellular plates exhibit a remarkable secondary curvature induced by the imposed planar bending [Figs. 1(a)–1(c)] [16,26–30]. This

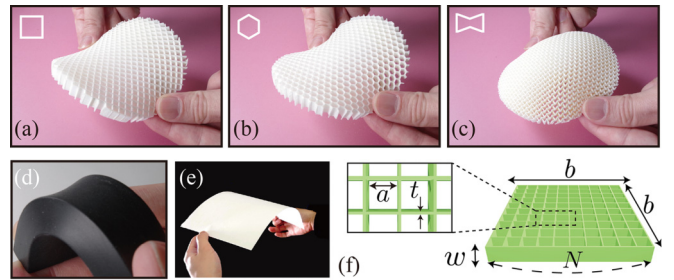


FIG. 1. Doubly curved cellular-solid-based plates. (a)–(c) Photographs of 3D-printed flexible cellular metaplates. The unit cell is (a) square, (b) hexagon, or (c) inverted hexagon. The bending induces the anticlastic [(a) and (b)] or synclastic [(c)] curvature. Note also that the inverted hexagonal cell in (c) shows the negative Poisson's (auxetic) effect. [(d) and (e)] Bending of a thick rubber beam and a sheet of A4 paper, showing the hyperbolic and cylindrical surfaces, respectively. (f) Geometric parameters characterizing a square lattice cellular plate.

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feature is particularly prominent in the square lattice cellular plate, which shows an anticlastic curvature upon diagonal bending, as a direct consequence of the extremely low in-plane stiffness of the underlying lattice geometry. Hence, several important questions are raised. How large can a cellular plate be to adopt a doubly curved shape upon bending? How does the density of the cellular solids influence the critical size?

To answer these questions, we combine fabrication, physical experiments, numerical simulations, and continuum elasticity theory to reveal the underlying physics of the (apparently) unusual bending behaviors of cellular metaplates.

Theory. In this study, we focus on characterizing the elasticity of a square lattice based metaplate that is characterized by a size of b and a number of unit cells N , each of which comprises identical elastic strips that possess a thickness of t , width of w , and Young's modulus of E_s [Fig. 1(f)]. In addition to the standard Cartesian coordinates (x, y) , we define the diagonal coordinates (ξ, η) , where $\xi = (x + y)/\sqrt{2}$ and $\eta = (-x + y)/\sqrt{2}$ [Fig. 2(a)]. Our objective is to develop a continuum elasticity theory for a cellular metaplate by smoothing out its underlying lattice structure, assuming a large number of unit cells, that is, $N \gg 1$. First, we introduce the concept of relative density ρ , which is a unique parameter for a cellular solid. The relative density is defined as the ratio of the area occupied by the material to the total surface area defined by the structure [31]. For a square honeycomb, $\rho = (b^2 - (aN)^2)/b^2$. The unit cell size a and thickness t change with ρ as follows:

$$a = \frac{b}{N} \sqrt{1 - \rho}, \quad t = \frac{b}{N + 1} (1 - \sqrt{1 - \rho}), \quad (1)$$

for a given b and N . Note that $\rho \approx 2t/a$ for $N \gg 1$ and $\rho \ll 1$.

The elasticity of a square lattice is strongly anisotropic [31–33], and its stiffness against shear deformation is particularly small because of the underlying mechanism based on the zero energy mode [34–37] and the slenderness of the constituent elements [36–40].

Shear induces the bending of individual beams, whereas stretching along the x and y directions necessitates the extension of beams. Given the negligible impact of Poisson's effect in the x and y directions, we estimate the stretching energy density to be as follows:

$$f_{\text{stretch}} = \frac{E}{2} (\epsilon_{xx}^2 + \epsilon_{yy}^2) + G \epsilon_{xy}^2, \quad (2)$$

where E and G represent the effective elastic moduli for pure stretching and shear deformations, respectively. For $\rho \ll 1$, E and G scale as [31]

$$E \sim (E_s \rho) w, \quad G \sim (E_s \rho^3) w, \quad (3)$$

where the different ρ dependences arise from the different t dependences of the stretching ($\sim t$) and bending ($\sim t^3$) moduli of a Kirchhoff beam. In-plane stiffness anisotropy becomes apparent for $\rho \ll 1$, where $G \ll E$. The orientation-dependent effective stiffness and Poisson's ratio can also be obtained via the classical transformation method. For additional information, refer to the Supplemental Material [41].

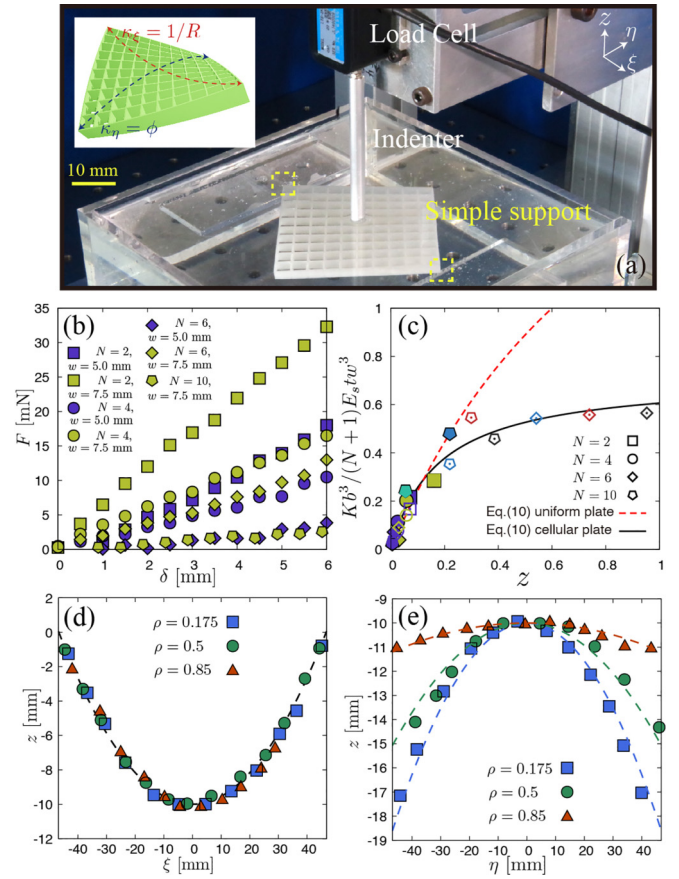


FIG. 2. Experimental setup and results. (a) Photograph of three-point bending test. Inset: Indentation-induced curvatures along the two principle directions, where $\kappa_\xi = 1/R$ is imposed, and the secondary curvature κ_η is induced. (b) Reaction force F plotted as a function of the indentation depth δ for different values of N , as indicated. (c) Rescaled indentation stiffness, $Kb^3/(N+1)E_s tw^3$, obtained from the initial linear slopes in (b), as a function of $z = (t/w)^2$, together with the dashed and solid lines representing Eq. (10) for a uniform elastic plate and our cellular plate, respectively. (d), (e) The vertical displacement profiles at $\delta = 10$ mm along the (d) ξ direction and (e) η direction for different values of ρ (indicated in the legend). Dashed lines represent the theoretically expected values (see the main text).

Notably, this in-plane stiffness property is essentially independent of the thickness of the cellular plate (or the width of the beam), w .

A closer look at the saddlelike shape of our metaplate used in the experiment and simulation suggests that each elastic beam undergoes an approximately pure twist. By resolving the bending moment in the diagonal direction into the x and y directions, it becomes evident that the moments induce the pure twisting of the beams. According to the Kirchhoff rod theory, the twist modulus scales as $C \sim E_s t^3 w$, which indicates a soft response at the plate scale. In contrast, bending about the x and y directions involves the bending of the beams in the harder directions, which significantly stiffens the structure with a bending modulus that scales as $E_s t w^3$. Assuming that the bending energy takes a quadratic form of the curvature, we

propose the following:

$$f_{\text{bend}} = \frac{B}{2}(\kappa_{xx}^2 + \kappa_{yy}^2) + S\kappa_{xy}^2, \quad (4)$$

where the cross term $\kappa_{xx}\kappa_{yy}$, which can potentially contribute to the intrinsically weak out-of-plane Poisson's effect, is of negligible significance for our objective and is omitted from further consideration. Assuming $\rho \ll 1$, we suggest the following scaling forms:

$$B \sim (E_s \rho) w^3, \quad S \sim (E_s \rho^3) w a^2. \quad (5)$$

In our Supplemental Material [41], we provide an orientation-dependent bending modulus and the out-of-plane Poisson's ratio $\nu_{\text{out}} = -\kappa_{\eta}/\kappa_{\xi}$. The analysis confirms that the plate is particularly soft against diagonal bending, where the out-of-plane Poisson's ratio approaches unity. Physically, the plate minimizes its bending energy by minimizing the square of its mean curvature $(\kappa_{\xi} + \kappa_{\eta})^2$, elucidating why the plate assumes a doubly curved (mostly saddle) shape when bent in one direction.

To determine the size of a plate with such a doubly curved shape, one must consider the in-plane stretching energy in addition to the bending energy. For this analysis, we restrict our interest to the bending deflection in the diagonal directions (ξ, η) . By imposing a curvature $1/R$, we assume a secondary curvature of ϕ along the perpendicular direction. Unlike typical thin plates, while the change in the Gaussian curvature must involve the in-plate stretch, it can be dominated by the soft shear mode, as previously discussed. To quantify this observation, we revisit the following compatibility relation [25]:

$$\partial_y^2 \epsilon_{xx} + \partial_x^2 \epsilon_{yy} - 2\partial_x \partial_y \epsilon_{xy} = -K[\zeta], \quad (6)$$

where ζ is the out-of-plane deflection and $K = (\partial_x^2 \zeta)(\partial_y^2 \zeta) - (\partial_x \partial_y \zeta)^2$ is the induced Gaussian curvature [25]. In the Föppl-von Karman plate theory, the geometrically nonlinear in-plane strain $\epsilon_{\alpha\beta}$ is given by

$$\epsilon_{\alpha\beta} = \frac{1}{2}(\partial_{\alpha} u_{\beta} + \partial_{\beta} u_{\alpha}) + \frac{1}{2}\partial_{\alpha} \zeta \partial_{\beta} \zeta, \quad (7)$$

where α and β refer to the x and y coordinates and u_x and u_y denote the in-plane components of the displacement vector. It must be considered whether we can assume $\epsilon_{xx} = \epsilon_{yy} = 0$ with only the nonzero strain component ϵ_{xy} satisfying $2\partial_x \partial_y \epsilon_{xy} = K$. In the Supplemental Material, we explicitly provide the analytic expressions of u_x and u_y that satisfy $\partial_x u_x = -(1/2)(\partial_x \zeta)^2$, $\partial_y u_y = -(1/2)(\partial_y \zeta)^2$, and $2\partial_x \partial_y \epsilon_{xy} = K$ for a general hyperbolic surface $\zeta(x, y)$. In terms of the underlying lattice geometry with vanishing hinge stiffness [36,37], this amounts to the transformation of a square cell into a diamond shape while maintaining its length to ensure that only the diagonal distances change. The corresponding in-plane deformation energies per unit area Eq. (2) are $f_{\text{stretch}} = G\epsilon_{xy}^2 \sim Gb^4\phi^2/R^2$. The curvatures $\kappa_{\alpha\beta} = \partial_{\alpha} \partial_{\beta} \zeta$ are found as $\kappa_{xx} = \kappa_{yy} = (1/2)(1/R + \phi)$ and $\kappa_{xy} = (1/2)(1/R - \phi)$, which leads to the bending energy per unit area [Eq. (4)] being given by $f_{\text{bend}} \simeq B(\phi + 1/R)^2 + S(\phi - 1/R)^2$. Then, we minimize the total deformation energy $f_{\text{total}} = f_{\text{stretch}} + f_{\text{bend}}$ with respect to ϕ (assuming uniform bending across the entire plate) and obtain the resulting

out-of-plane Poisson's ratio:

$$\nu_{\text{out}}(b, \rho) = \frac{\nu_{\text{out}}^*(\rho)}{1 + (b/b_c)^4}, \quad (8)$$

where $\nu_{\text{out}}^*(\rho) = (B - S)/(B + S)$. For $b/b_c \ll 1$, we find $\nu_{\text{out}} \simeq \nu_{\text{out}}^*$, whereas for $b/b_c \gg 1$, we find $\nu_{\text{out}} \simeq 0$. The crossover length b_c is found as

$$b_c = \left(\frac{B + S}{2G} \right)^{1/4} \sqrt{R} \approx \frac{b_{\text{Lamb}}}{\sqrt{\rho}}, \quad (9)$$

where $b_{\text{Lamb}} = \sqrt{Rw}$ denotes the classical Lamb's result [20–25]. The additional factor $1/\sqrt{\rho}$ in Eq. (9) is a characteristic of cellular solids. For a dilute solid of $\rho \ll 1$, b_c can be several times greater than b_{Lamb} . In addition, our cellular plate is generally relatively thick, that is, $w \gg t$. The combined effects of Eq. (9) explain why the cellular plate exhibits a substantial doubly curved shape for a relatively large system size.

Experiment. To verify our theoretical predictions, we conduct a physical experiment using digitally fabricated square-lattice cellular plates, as shown in Fig. 2(a). Samples with various geometric parameters are fabricated using a silicone-based elastomer and a 3D printer (Agirista, Keyence). All the samples have regular periodic $N \times N$ square cells and are composed of two $(N + 1)$ identical elastic elements of axial length b , uniform thickness t , and width w . Note that w corresponds to the plate thickness. To compare the different samples, we fix the sample size $b = 66$ mm and change the number of cells from $N = 1$ to 10 and the relative density from $\rho = 0.175$ to 0.85. Correspondingly, the ranges of the unit cell size and thickness are $a = 2.6 - 30.0$ mm and $t = 0.6 - 3.7$ mm.

We perform a three-point bending test to quantify the out-of-plane bending properties [Fig. 2(a)]. The two diagonal edges are simply supported, and the plate center is loaded with a cylindrical indenter of radius 1.5 mm. The experiments are performed in water to ensure that the samples are neutrally buoyant. We measure the reaction force as a function of displacement and capture the instantaneous shapes using a digital camera. Then, we analyze these captured images using the image software to determine the induced surface curvatures, from which we compute the Gaussian curvature and out-of-plane Poisson's ratio.

To complement our experimental results, we also conduct finite element numerical simulations (FES) using the commercial package Abaqus (Dassault Systems). The equilibrium configuration is obtained by minimizing the elastic deformation energy of a hyperelastic neo-Hookean solid by using solid elements (C3D15H: 15-node quadratic triangular prism, hybrid with linear pressure) with geometric nonlinearity, a Young's modulus of $E_s = 3.09 \times 10^6$ Pa, and a Poisson's ratio of $\nu = 0.499$ (valid for rubberlike materials). We systematically test different sets of mesh sizes and types and confirm that the results shown below are essentially insensitive to these parameters.

The typical force-displacement curves for different plates are shown in Fig. 2(b). The indentation stiffness K is obtained from the initial slopes in Fig. 2(b). This linear response can be determined by considering only the effective bending energy

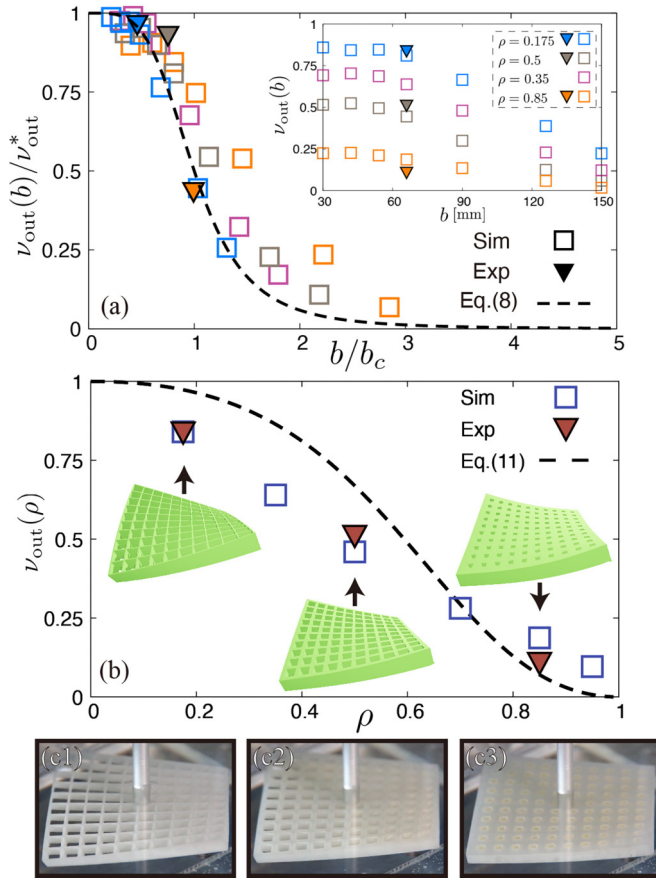


FIG. 3. Comparison between the measured and predicted out-of-plane Poisson's ratios. (a) ν_{out} plotted as a function of b/b_c , where $b_c = 2.5\sqrt{Rw/\rho}$. The dotted line represents Eq. (8). Open and filled symbols represent numerical and experimental data, respectively. Inset: Experimental and numerical data of ν_{out} as a function of the sample size b for different values of ρ . (b) The same data as shown in (a), but for $b = 66$ mm only and plotted as a function of the relative density ρ . The dotted line represents Eq. (11). The typical configurations of the plates from FES are shown as insets. (c) Photographs of the plates for $\rho = 0.175$, 0.5 , and 0.85 . In all cases, the indentation displacement δ is 10 mm.

because the stretching contribution is nonlinear and negligible under this framework. For a small vertical displacement $\delta < 3$ mm $\ll b$, the plate is assumed to deflect uniformly with a curvature of $\kappa \sim \delta/b^2$. By substituting this expression into f_{bend} in Eq. (4) and minimizing it for the secondary curvature

ϕ , we arrive at the equilibrium energy of the form $(K/2)\delta^2$, where $Kb^2 \approx BS/(B+S)$ at the scaling level. We introduce the anisotropy parameter $S/B = \alpha_1 z$, where $z = (t/w)^2$ and α_1 is a numerical factor [see Eq. (5)], and we suggest the following scaling form:

$$K = \alpha_2 \times \frac{(N+1)E_s t w^3}{b^3} \frac{z}{1 + \alpha_1 z}, \quad (10)$$

where α_2 is another numerical prefactor. This method is quantitatively validated for uniform elastic plates with controlled anisotropy in FES, for which we find $\alpha_1 = 1$ and $\alpha_2 = 8/3$ analytically. See the Supplemental Material for more information [41]. For cellular plates, precise numerical factors in the out-of-plane stiffness coefficients, namely, B and S , are yet to be determined analytically [42]. To compare Eq. (10) with the experiment for the cellular plates of different geometries, we treat α_1 and α_2 as adjustable parameters and find an excellent agreement between them [Fig. 2(c)], assuming $\alpha_1 \simeq 5.6$ and $\alpha_2 \simeq 4.0$. This validates our basic assumption of the bending energy density, Eq. (4).

After characterizing the linear elasticity of our cellular plate, we investigate the doubly curved shape induced by indentation at an elevated displacement of $\delta = 10$ mm. The bent plate typically exhibits a hyperbolic paraboloid shape, and this is quantitatively confirmed through image analysis, as shown in Figs. 2(d) and 2(e). In Fig. 2(d), we predict $z(\xi) = -\delta + (2\delta/b^2)\xi^2$, which is indicated by the dashed line and is in excellent agreement with the experiment for different values of ρ . Thus, the imposed curvature is $\kappa_\xi = 4\delta/b^2 = 9.2 \times 10^{-3} \text{ mm}^{-1}$ or $R = 1/\kappa_\xi = 109$ mm. The induced curvature κ_η is obtained by fitting the data for different values of ρ with $z(\eta) = -\delta + (\kappa_\eta/2)\eta^2$ [see the dashed lines in Fig. 2(e)], from which we obtain the out-of-plane Poisson's ratio $\nu_{\text{out}} = -\kappa_\eta/\kappa_\xi$. In Fig. 3(a), we plot $\nu_{\text{out}}/\nu_{\text{out}}^*$, as obtained from the experiment and FES as a function of b/b_c , where we take $b_c = c_0\sqrt{Rw/\rho}$ with the numerical prefactor $c_0 = 2.5$. Here, the values of ν_{out}^* for different values of ρ are read from the data shown in the inset of Fig. 3(a). All the data for the different values of ρ collapse onto the predicted single master curve, $1/[1 + (b/b_c)^4]$, which confirms the main scaling predictions described in Eqs. (8) and (9). The scaling collapse for $\rho = 0.85$ is less convincing, as our scaling argument is valid for $\rho \ll 1$. Using the same data for $b = 66$ mm, we now plot $\nu_{\text{out}}(\rho)$ as a function of ρ in Fig. 3(b). By combining our analytical argument for ν_{out} with Eq. (1), we predict

$$\nu_{\text{out}}(\rho) = \frac{1 - d_1(N+1)^{-2}h_1(\rho)b^2/w^2}{1 + d_1(N+1)^{-2}h_1(\rho)b^2/w^2 + d_2N^2(N+1)^{-2}h_2(\rho)b^4/(w^2R^2)}, \quad (11)$$

where we define $h_1(\rho) = (1 - \sqrt{1 - \rho})^2$ and $h_2(\rho) = h_1(\rho)/(1 - \rho)^{3/2}$ and introduce the numerical factors d_1 and d_2 . In Fig. 3(b), we compare Eq. (11) with the numerical and experimental data and find good agreement, assuming $d_1 = 1.0$ and $d_2 = 1.0$. As ρ increases, ν_{out} decreases significantly and approaches zero as $\rho \rightarrow 1$. This suggests that the

formation of the saddle shape is suppressed as the plate becomes denser, and the plate becomes cylindrical at $\rho = 1$. The theory captures this trend correctly, even beyond its applicable framework of $\rho \ll 1$, but this occurs only when the in-plane stretch contribution and that of the bending are correctly considered. Substituting $R = b^2/4\delta = 109$ mm [Fig. 2(d)],

$b = 66$ mm, $w = 7.5$ mm, and $\rho = 0.175$ into Eq. (9), we find $b_c \simeq c_0(Rw/\rho)^{1/2} \simeq 172$ mm, in agreement with the observations in Fig. 1(a).

In elementary elasticity theory, the secondary curvature is attributed to its in-plane Poisson's ratio. Upon bending, the upper surface is stretched, whereas the lower surface is compressed. For a positive Poisson's effect, the upper and lower ones are compressed and stretched, respectively, in the transverse direction, which generates secondary bending with a curvature of the opposite sign to that of the primary one. However, this requires the change of the Gaussian curvature and therefore induces the in-plane stretch if the system is large enough, which costs stretching energy. Thus, the energetic competition between the bending and stretching limits the maximum size of the doubly curved shape [20,24,25]. In principle, cellular plates share a common mechanism with that of classical elastic plates. However, in cellular plates, because the in-plane stretch modulus can be significantly small while the bending is relatively stiff, a significantly larger plate still exhibits the double curvatures. This general mechanism is insensitive to the unit cell geometry and may broadly apply to other nonchiral cellular plates as long as the effective elastic moduli, such as E , G , B , and S , are properly defined from unit cell architectures. Whether synclastic or anticlastic is a matter of the sign of ν_{out}^* , whereas the scaling law for b_c remains robust, independent of specific unit cell geometries. For instance, the generalization of the proposed theory to a hexagonal honeycomb plate [31,32] is straightforward. By including the cross term proportional to $\epsilon_{xx}\epsilon_{yy}$ in the in-plane energy in Eq. (2), which is responsible for a finite in-plane

Poisson's effect, and then following the same argument as discussed earlier, we find that the crossover length of the hexagonal cellular plate satisfies the scaling $b_c \sim b_{\text{Lamb}}/\sqrt{\rho}$, in agreement with our observations in Figs. 1(b) and 1(c) and those reported in recent studies [29,30].

Summary. We anticipate that our study can inspire further studies of the out-of-plane (i.e., bending and twisting) elasticity of cellular solid based plates and shells [16,28–30,43–45]. It significantly broadens the concepts of cellular solids, whose in-plane properties have been a classical topic in mechanics [31–33,37,38,46–50]. In a recent paper [51], the out-of-plane Poisson's ratio, named Mindlin's ratio in Ref. [51], has been investigated for a wide class of chiral metaplates, in the absence of an in-plane energetic penalty arising from the nonlinear geometric coupling between in-plane stretch and out-of-plane bending. Complementary to our present study, the thickness dependence, i.e., effects of multilayering, is highlighted, while keeping the system size and density fixed. As such, formulating the theoretical framework of out-of-plane mechanics of cellular plates will be a timely subject, which significantly contributes to quantifying the basic physical properties of metaplates and metabeams, whose shape programmability opens up a number of exciting applications in engineering and the applied sciences.

Note added in proof. Recently, we became aware of a recent work [52] that investigates numerically and analytically the similar problem to the present study.

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