

Coupled calibration-control method for complex systems

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We formulate the calibration-control dilemma which occurs in most interesting out-of-equilibrium dynamical systems in the natural and social sciences. Despite a lack of knowledge of the main control parameters, standard control methodology assumes that calibration can determine them. They are, however, obfuscated by the existence of control, leading to large undesired intermittent bursts, crises, and instabilities. By merging parameter calibration and control in a data assimilation framework, our joint calibration-control method nearly matches the performance achieved with complete knowledge of the system parameters, as demonstrated by both analytical and numerical simulations. This paves the way toward more versatile control of complex systems.

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Control theory provides a principled framework combining a wide range of mathematical and engineering concepts and tools to analyze and influence system behaviors for achieving specific goals [1]. Applications are manifold and include robotics, aerospace flight control, chemical reactors, biomedical engineering, and power grids, just to name a few. However, traditional control theory, primarily designed for well understood systems, finds it challenging to control complex systems as it typically presupposes well-defined system equations and parameters. Conversely, complex systems such as transportation networks, ecosystems, climate, or social systems often encompass uncertain and obscure foundational structures. They require continuous supervision to avoid instabilities, making their inherent dynamics inaccessible. Attempts to control complex systems may compromise a deep understanding of their inherent dynamics because they are obfuscated by the action of control, with severe unintended consequences [2,3]. History is fraught with instances of overzealous human control. The introduction of the cane toad to Australia as a pest control measure turned the toad into a harmful invasive species that threatens native wildlife by poisoning predators and competing for food [4]. Fire suppression policies inadvertently led to the accumulation of dry vegetation, thus setting the stage for larger and more intense wildfires [5]. Masked under the guise of risk control, financial institutions engaged in the trade of ill-understood financial instruments which triggered the 2008 financial meltdown [6,7]. Instances like these illustrate the necessity for a better understanding

of the consequences of control on poorly specified complex systems.

We generalize a previous observation that concurrent analysis of the system dynamics to determine its parameters and control actions generate unanticipated harmful side effects [2,3,8]. The conflict between control and parameter calibration originates from a paradoxical reality. Effective control necessitates a precise determination of the parameters of the system. Simultaneously, successful control can obscure the genuine system dynamics by directing it toward behaviors that largely deviate from its natural course. This dilemma provides a mechanism for power law distributions and self-organized criticality [8]. Rather than accepting this dilemma, our main contribution is a strategy to resolve it. Allowing for simultaneous noise filtering, parameter calibration, and control optimization provides stable control with mild fluctuations on par with the cases of perfect knowledge of the system parameters.

Consider a system which, when uncontrolled, is described at time t by its state y_t and follows the dynamics $y_{t+1} = F(Y_t; \theta) + \sigma \epsilon_t$ with $\epsilon_t \stackrel{\text{iid}}{\sim} \mathcal{N}(0, 1)$, and F can be stochastic or deterministic and depends on past states $Y_t \equiv \{y_s\}_{s \leq t}$ and a set of parameters θ . The goal of control is to constrain the system by applying a control $C_T = \{c_t\}_{t=0}^T$ at each time t from 0 to T such as to minimize the following value function:

$$\mathcal{V}_t(Z_t) = \min_{C_T} \mathbb{E}_t \left[\sum_{s=t}^T (g(z_s) + \gamma(c_s)) \right], \quad (1)$$

where $Z_t \equiv \{z_i\}_{i \leq t}$ is the state of the system modified by the presence of control. The expectation value $\mathbb{E}_t[\cdot]$ taken at time t extends over future expectations up to time T based on knowledge of θ which governs the controlled dynamics

$$z_{t+1} = F_t(Z_t; \theta) - c_t + \sigma \epsilon_t. \quad (2)$$

The term $\sum_{s=t}^T g(z_s)$ in (1) is the objective to be minimized by the control action. If the objective g and cost of control

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function γ are path independent, (1) is solved recursively with dynamic programming (see the Supplemental Material [9]).

We first consider the control of a linearly stable or unstable system subjected to additive noise:

$$y_{t+1} = \alpha y_t + \sigma \epsilon_t. \quad (3)$$

This equation corresponds to a linear expansion around a fixed point and y_t is the distance to the fixed point. This expansion is common in the analytical treatment of physical systems in the neighborhood of a fixed point. For $\alpha \leq 1$, Eq. (3) defines an AR(1) system. For $\alpha > 1$, Eq. (3) describes an exponentially growing instability in the presence of noise. Assuming that y_t is to be kept close to the fixed point, we set $g(z_t) = \frac{1}{2}qz_t^2$ which is minimal when $z_t = 0$. We further assume a quadratic cost of control, $\gamma(c_s) = \frac{1}{2}rc_s^2$, which recovers the well-known linear-quadratic-Gaussian (LQG) control problem. The optimal control, in the limit of large T , reads (see the Supplemental Material [9])

$$c_t = \alpha \frac{p}{r+p} z_t, \quad (4)$$

where p is a constant that depends on q , r , and α . For a myopic time horizon ($T = 1$), p is just equal to q . This solution holds both when the uncontrolled system y_t is stable ($|\alpha| < 1$) or unstable ($|\alpha| > 1$). In the later case, control is required for the observed system z_t to be stable. Crucially, the control depends on knowledge of the system dynamics via α (while σ factors out due to symmetry). This assumption is typical in engineering where the system is well understood and one can determine the key parameters before control is activated. By contrast, in complex natural and socioeconomic systems, it is in general not possible to observe the uncontrolled state but only its controlled counterpart (2). It may either be practically difficult to turn off control or the system might be unstable, making it too costly to observe the unperturbed system. Typical problems of this type include economic time series (see the Supplemental Material [9] for an example of inflation), climate variables, or traffic flow. Hence, in the control dynamics (4), α has to be replaced by its best estimation based on the n last observations $z_t, z_{t-1}, \dots, z_{t-n+1}$ obtained by maximum likelihood (see the Supplemental Material [9]):

$$\hat{\alpha}_t^{(n)} = \frac{\sum_{i=0}^{n-1} (c_{t-i-1} z_{t-i-1} + z_{t-i-1} z_{t-i})}{\sum_{j=0}^{n-1} z_{t-j-1}^2}. \quad (5)$$

Similarly, the noise term is estimated as

$$\hat{\sigma}_t^2 = \frac{1}{n} \sum_{i=0}^{n-1} (z_{t-i} + c_{t-i-1} - \hat{\alpha}_t^{(n)} z_{t-i-1})^2. \quad (6)$$

For large n , it holds that $\alpha z_t - c_t \simeq \mathcal{O}(\sigma/\sqrt{n})$ and z_t becomes asymptotically $z_t \simeq \sigma \epsilon_t$. However, in complex systems, expression (3) typically serves as a phenomenological, simplified depiction of more intricate dynamics, with its parameters subject to drift and noise. It is thus unrealistic to assume that the system is stationary. One must consider the nonasymptotic regime where n is smaller than the time-scale over which the parameters are expected to drift. While this

control with (5) does prevent the systems from diverging for $|\alpha| > 1$, it leads to large transient fluctuations distributed as $P(z_t) \propto 1/z_t^{1+\mu^{(n)}}$, with $\mu^{(n)} \sim n$ [2,3]. A similar power law is observed even for intrinsically stable systems ($|\alpha| < 1$). The underlying cause for such large fluctuations is visible from the denominator of (5). As the control works well, realizations of z_t become close to zero, leading to an explosion of the estimated $\hat{\alpha}_{t+1}^{(1)}$ and therefore of the corresponding control (4) at the next time step. In other words, when control works “too well” it obfuscates information needed to estimate the system parameters (α), which then cascades into a misinformed control action. This is visualized in Fig. 1 for a system that is stable if left uncontrolled ($\alpha = 0.5 < 1$). Subplot (a) shows the resulting evolution of $|z_t|$ over 100 time steps. The controlled variable is punctuated by large bursts, illustrating that control with the empirically estimated α is, at times, much worse than no control. Results are similar for other values of α and even much accentuated for $|\alpha| > 1$. To contrast, another trajectory shows $|z_t|$ under the condition of “engineering” control in which α is known and set to its true value. Subplot (b) quantifies the amplitude of the bursts by plotting the distribution of $|z_t|$ for simulations over 10 000 time steps and for different values of n . The smaller n is, the heavier is the tail of the distribution, in agreement with the prediction that the tail exponent $\mu^{(n)}$ is proportional to n (see the Supplemental Material [9]). For sufficiently large n under the stationary condition that the true α is constant, near-optimal control is achieved. However, so long as the system has to be assumed nonstationary, n cannot be chosen arbitrarily but needs to be less than the time scale over which the system changes. Henceforth, we set $n = 3$.

The unperturbed distribution of y_t is known to be Gaussian with a variance amplified to $\sigma^2/(1-\alpha^2)$ compared with the variance σ^2 of the additive noise $\sigma \epsilon_t$. Implementing control transforms the Gaussian distribution into a power law. On the one hand, control increases the probability weight of small values of $|z_t|$ —the so-called “long tail”—which is the desired goal. On the other hand, the unintended consequence of control is that it increases the probability weight of large values—the so-called “fat tail” [10]. Setting n larger is *a priori* not an option, since we have to account for the fact that the description of the linear stability dynamics around a targeted fixed point of a complex system is in general nonstationary and its parameters exhibit drift or noise. This reveals an apparent incompatibility between calibration and control, which is all the more relevant to complex systems with intermittent dynamics and regime switching. At the heart of the matter lies intermittent misestimations of $\hat{\alpha}_t^{(n)}$ via (5) which cascade into wild fluctuations in the control via (4). So long as the calibration and control remain separate, this issue persists because the control continues to be proportional to the estimated $\hat{\alpha}_t^{(n)}$.

We thus propose to replace objective (1) by the following coupled calibration-control value function:

$$V_t(z_t^a, c_t) = \frac{q}{2} \mathbb{E}_t[(z_{t+1}^p)^2] + \frac{r}{2} c_t^2 + \eta J(z_t^a), \quad (7)$$

where, for the sake of simplicity and to obtain manageable formulas, we consider a myopic time horizon of $T = 1$. This turns out to be sufficient to illustrate the power of our proposed method. Parameters q , r , and η quantify respectively

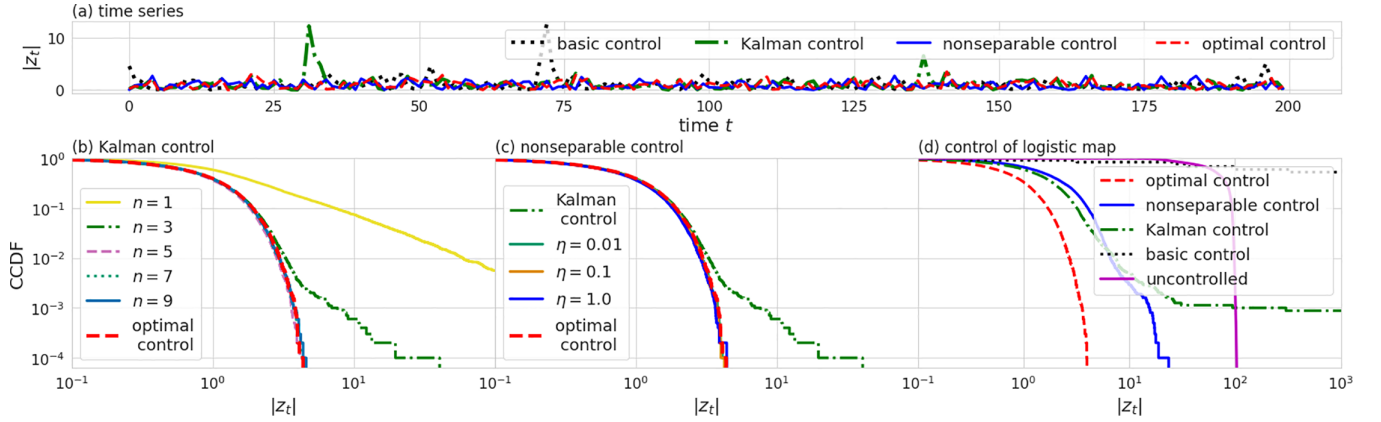


FIG. 1. Panel (a): Typical realizations of $|z_t|$ given by (2) with (3) for $\alpha = 0.5$ and $\sigma = 1$ and four types of control c_t . Red dotted-dashed line: “optimal control” when the ground truth $\alpha = 0.5$ is known and the control is given by (4) with $q = 1$, $r = 10$, and $n = 3$. Black dotted line: “basic control” $c_t = \hat{\alpha}_t^{(3)} z_t$ in (5) with $n = 3$. Green dashed line: “Kalman control,” which is the same as the “basic control” but using the filtered state z_t^k (8) to control. Blue line: “analytical coupled-control” in (10) with $q = 1$, $r = 10$, and $\eta = 0.01$. Panel (b): Complementary cumulative distribution function (CCDF) of $|z_t|$ with “Kalman control” for several values of n . Panel (c): CCDF of $|z_t|$ with the “nonseparable control” method for different values of η and $n = 3$. Panel (d): CCDF of the controlled logistic map (11) with $\kappa = 100$, $\alpha = 3.8$, and $\sigma = 1$ under different types of control. Uncontrolled dynamics (purple), optimal control if parameters are known (red), basic control via linearized LQG problem and $n = 3$ (black), Kalman filtering in two steps (green), and our nonseparable control method with $\eta = 1$ and $n = 3$ (blue). For each case, we simulated $N = 10^4$ steps.

the relative importance of (i) controlling the analyzed state z_t^a (defined below) close to zero with $\mathbb{E}_t[(z_{t+1}^p)^2] = (\hat{\alpha}_t z_t^a - c_t)^2$, (ii) the corresponding cost of control, and (iii) the significance of state estimation z_t^a to correct for the possible misattribution of information to noise in the directly observed state z_t relative to the predicted state z_t^p (defined below), i.e., $J(z_t^a) = \frac{1}{\sigma_z^2} (z_t^a - z_t)^2 + \frac{1}{\sigma_p^2} (z_t^a - z_t^p)^2$. Compared with expression (1), the new program is to minimize $V_t(z_t^a, c_t)$ both with respect to z_t^a and c_t simultaneously. This embodies the combined process of calibration and control.

There are two innovations in expression (7) revealed by the first and last term, respectively. The first term expresses that control should strive to minimize the expectation $\mathbb{E}_t[(z_{t+1}^p)^2]$ of the variance of the forecast of the future state instead of the variance of the observed state z_t . The best forecast z_{t+1}^p at time t based on the analyzed state z_t^a , on the calibrated $\hat{\alpha}_t$ and the control c_t that has been decided is $z_{t+1}^p = \hat{\alpha}_t z_t^a - c_t$, so that $\mathbb{E}_t[(z_{t+1}^p)^2] = (\hat{\alpha}_t z_t^a - c_t)^2$. The third term accounts for the fact that the sequence of observed $Z_t = \{z_s\}_{s \leq t}$ should not be fully trusted to inform the calibration of α due to the existence of noise miscalibration at the previous time step.

Minimizing the third term alone with respect to z_t^a recovers the standard Kalman filter (see the Supplemental Material [9]). We denote z_t^k the corresponding solution to distinguish it from z_t^a used below, which includes the effect of control. We have

$$z_t^k = z_t + K(z_t^p - z_t), \quad \text{where } z_t^p - z_t \sim \mathcal{N}(0, \sigma_p) \quad (8)$$

with $K = \frac{\hat{\sigma}^2}{\hat{\sigma}^2 + \hat{\sigma}_p^2}$ and $\hat{\sigma}_p^2 = \frac{\hat{\sigma}^2}{\sum_{i=0}^{n-1} z_{t-1-i}^2}$. Here, $\hat{\sigma}^2$ is the observational noise estimated by expression (6) since—in line with our main premise—we assume the true noise σ is unknown and $\hat{\sigma}_p$ estimates the prediction error of predicted state z_t^p .

The key idea of the Kalman filter, and of data assimilation more generally, is to not fully trust observations in presence of noise. Instead, one acknowledges that past forecasts also

contain information that should be combined with the current observation to minimize the noise of the resulting analysis state z_t^k . The analyzed state z_t^k is thus a weighted sum of the observed z_t and its predicted value z_t^p at $t - 1$, as captured in (8). If the estimated noise $\hat{\sigma}$ was zero, $K = 0$ and z_t^k reduces to the pure observed state z_t . But if the estimated noise is large, one should underweight the observed state and put more emphasis on the predicted state. The resulting variance σ_k^2 of the analysis state z_t^k is given by

$$\sigma_k^2 = \frac{\sigma^2 \sigma_p^2}{\sigma^2 + \sigma_p^2}, \quad (9)$$

which is smaller than or equal to both σ^2 and σ_p^2 , thus achieving the sought reduction in variance. Returning to the full problem of minimizing $V_t(z_t^a, c_t)$ with respect to both z_t^a and c_t , we obtain (see the Supplemental Material [9])

$$z_t^a = \frac{z_t^k}{1 + \left(\frac{\hat{\alpha}_t}{\omega}\right)^2} \quad \text{and} \quad c_t = \hat{\alpha}_t \frac{q}{r + q} z_t^a \quad (10)$$

with $\omega = \sqrt{\frac{r+q}{rq} \frac{\eta}{\sigma_k^2}}$, and z_t^k is the Kalman solution (8) with the predicted state now including the control, $z_t^p = \hat{\alpha}_{t-1} z_{t-1}^k - c_{t-1}$ and σ_k^2 is given by expression (9). Note that the control c_t has the same structure as expression (4) when simplified to the myopic time horizon ($T = 1$) but with the replacement of the observed state z_t by the analyzed state z_t^a . The fundamental improvement on (4) stems from the regularizing factor $1/[1 + (\frac{\hat{\alpha}_t}{\omega})^2]$ in z_t^a that cascades to c_t . The calibrated parameter $\hat{\alpha}_t$ now enters nonlinearly such that its large fluctuations associated with misestimations due to previous over control have an attenuated impact on future control. The realized trajectory in Fig. 1(a) shows that the resulting dynamics is close the optimal one that assumes perfect knowledge of parameter α . Figure 1(d) confirms that the distribution of states $|z_t|$ is

now distributed approximately as $\mathcal{N}(0, \sigma)$ for a large range of η .

It is important to note that sequential filtering and subsequent control of the filtered state does not achieve the desired result [see the “Kalman control” trajectory in Fig. 1(a)]. This Kalman approach is obtained by first minimizing $J(z_t^a)$ with respect to z_t^a to obtain $z_t^a = z_t^k$ given by expression (8), then setting $\eta = 0$ and plugging $z_t^a = z_t^k$ into (7) (with $\eta = 0$). The minimization of V_t is now only with respect to c_t and the corresponding solution reads $c_t = \frac{q}{q+r} \hat{\alpha}_t z_t^k$.

Our framework applies to any situation in which the force around the fixed point can be linearized, but the linearization factor is unknown and stationary is not guaranteed. For instance, the balancing of a stick is approximately described by an inverted pendulum which is to first order following (3) with $\alpha > 1$, and failure to balance has been mapped to the formulation of the present article [8,11].

In financial systems, a generalized version of the Blanchard-Watson model of rational bubbles takes the form (3) with y_t representing the log price of an asset [12]. The growth factor α is time dependent to capture ever changing economic and financial conditions with values larger than one over several successive periods corresponding to transient bubble regimes. Some scholars promote the need for competent risk control while others point out the possible unintended consequences thereof. A blocking point is the lack of understanding of what are the control actions that may work, given the many uncertainties in the equations of price motion and relevant parameters. In this context, a minimal description such as (3) is warranted, and short look backs n are required due to the strong nonstationarity of financial markets. The control c_t expresses the pressure on prices exerted for instance by the central bank intervention in the form of purchases or sales of the asset [13]. We refer to the Supplemental Material [9] for a detailed elaboration on a related problem of inflation control.

In agronomy, integrative pest control is essential to ensure that crop production and agriculture yields are not compromised by pest infestations. Equation (3) represents a simple generic model describing pest dynamics in a given area as the interplay between stochastic growth and immigration from outside [14]. Here y_t represents the pest population and α its growth factor which is influenced by changing environment, and thus is time-dependent. This temporal variability requires using a small backward-looking data length n . The noise term $\sigma \epsilon_t$ accounts for pests entering or leaving the population from external sources and other factors influencing the population. Controlling and managing pests in crop fields is a crucial aspect of agricultural management and involves diverse strategies including biological, chemical, physical, and cultural practices. Mathematically, such control enters via (2) and recovers our setup.

Let us now examine the consequences of expanding $F(z_t; \theta) = \alpha z_t + \beta z_t^2$ up to a quadratic term. For $\beta < 0$, defining the carrying capacity $\kappa := -\alpha/\beta$, $F(z_t; \theta)$ can be expressed as $F(z_t; \theta) = \alpha z_t (1 - z_t/\kappa)$. This logistic map is one of the simplest dynamical systems exhibiting deterministic chaos and has many applications ranging from population

dynamics, ecology and biology, epidemiology, economics, cryptography, and signal processing. The basic additive control strategy to bring the dynamics close to the fixed point $z^* = 0$ leads to the control equation

$$z_{t+1} = \alpha z_t \left(1 - \frac{z_t}{\kappa}\right) - c_t + \sigma \epsilon_t. \quad (11)$$

This additive control setup is different from the “control of chaos” where the goal is, for instance, to direct the dynamics toward a desired unstable periodic trajectory via actions on the control parameter α [15–17]. By assuming a much more modest understanding of the dynamics and a significantly poorer ability to control, our approach to the control problem is more realistic for complex systems. Figure 1(d) shows the result of applying our coupled calibration-control method to (11) with value function (7) leading to the control (10). The parameters are $\kappa = 100$ and $\alpha > 3.8$, for which the fixed point $z_t = 0$ is strongly unstable and, moreover, the logistic map is chaotic, as confirmed by its positive Lyapunov exponent. Figure 1(d) also shows the results obtained with other control schemes for comparison. The “basic control” employs $c_t = \hat{\alpha}_t z_t$, where $\hat{\alpha}_t$ represents the estimated linear approximation over $n = 3$ time steps via (5). As we explain in the Supplemental Material [9] and show in Fig. 1(d), this control scheme leads to unbounded growth of z_t , and induces chaos for values of α for which the logistic map alone would not be chaotic. Similarly, LQG control with Kalman filtering leads to system instability beyond typically a thousand steps due to estimation errors in α . In contrast, our analytical coupled calibration-control method limits the variance while maintaining bounded dynamics around $z^* = 0$, even when second-order effects become significant ($z_t \gtrsim 20$).

Given that most out-of-equilibrium complex systems are characterized by a lack of knowledge of the underlying equations of motions, the simplified equations that are used can only hope to be a tangent approximations of an unreachable reality, and need to be continuously re-estimated due to the lack of stability of the calibrated parameters. Applied to complex natural and socioeconomic systems, standard control methodology, which involves first calibrations and then control, leads to large undesired intermittent bursts as a result of the conflict between control and parameter calibration. By concurrently executing noise filtering, parameter calibration, and control optimization, we achieve stable control with moderate fluctuations that are comparable to scenarios with complete knowledge of the system parameters. This should pave the way toward more versatile solutions in the management of complex systems and to more scrutiny of the often hidden consequences of exerting excessive control over them.

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