

Quantum resources for assisted optimal state discrimination

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We revisit the discussion of the quantum resources required to perform optimal unambiguous discrimination between two nonorthogonal quantum states when the protocol is assisted by an auxiliary system (ancilla). Initially, the formulation of the problem led to the assumption that entanglement between the main system and the ancilla was the key resource for the task. However, it was later shown that entanglement is not necessary at all, while quantum dissonance may or may not be required, depending on the relationship between the overlap and the *a priori* probabilities of the inputs [Zhang, *Sci. Rep.* **3**, 2134 (2013)]. Here, we show that, regardless of this relationship, coherence and purity are required resources for the task and thus more fundamental than dissonance. Coherence is analyzed based on its generalized resource theory, which allowed us to find the scenario that generates the maximal cryptographic randomness gain in the protocol. Purity, on the other hand, is shown to be greater than coherence when both measures are defined via the same distance, including the extreme cases of orthogonal and parallel states, where the latter is zero. Therefore, we conclude that purity is the most fundamental resource for assisted optimal state discrimination.

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I. INTRODUCTION

The problem of optimal unambiguous discrimination between two nonorthogonal pure states of a qubit was introduced mainly to address the fundamental aspects of quantum measurement theory. The goal was to show a quantum task for which it is advantageous to use a positive operator-valued measure (POVM) instead of a projective measurement [1–4]. Later, this task became a powerful tool for quantum information processing: For instance, it may be applied in quantum key distribution [5], quantum teleportation [6], quantum tomography [7], and quantum random number generation [8].

The discrimination between the two states is carried out with a three-outcome POVM: Two of the outcomes identify each of the states with certainty, and the third outcome leads to an inconclusive answer. The protocol is optimized by the POVM that minimizes the probability of inconclusive results. A typical way to implement the optimal POVM is by introducing an auxiliary qubit (ancilla) in a pure state, coupling it with the main qubit and performing a projective measurement on both [1–3]. This type of implementation is usually referred to as assisted optimal state discrimination (AOSD).

In recent years, AOSD has also served as a platform for studying the nature of the quantum resources required for its implementation [9–13]. The formulation of the problem led to the intuitive perception that entanglement should be such

a resource. However, the pioneering work by Roa *et al.* [9] showed that for equally likely states, AOSD does not require entanglement, only quantum dissonance, a nonclassical correlation of separable states [14]. Subsequently, Zhang *et al.* [11] generalized this result, showing that entanglement is not necessary at all for AOSD, whatever the *a priori* probabilities of the input states. They also showed that dissonance may be required or not depending on the relationship between the overlap and the *a priori* probabilities of the inputs. This relationship divides the protocol into two ranges: In the first, dissonance is a key resource, while in the second, it is not present.

The gap left by this result reopened the quest for more elementary quantum resources than dissonance for performing AOSD. A catalytic factor for this was the understanding that quantum coherence is a resource that can be used to create quantum correlations such as entanglement, discord, and dissonance and therefore is more fundamental than them [15–18]. In this context, recent works showed that coherence is indeed a key resource for AOSD. Namkung and Kwon [19] investigated how the coherence is distributed between the system-ancilla state; they showed that, in the first range described above, it is distributed to both subsystems, while, in the second range, it is localized in the ancilla. Kim *et al.* [20] also explored the role of coherence in unambiguous discrimination but considered a quite different assisted protocol compared to that from Refs. [9–11], which is addressed here.¹

¹The POVM is performed by coupling an ancilla *qutrit* to the qubit, followed by a projective measurement only on the former.

The quest mentioned above, however, does not end with coherence since it was recently discovered that purity is a more elementary resource than any of the others mentioned here [21]. Although purity, coherence, discord, and entanglement represent distinct measures of quantumness with different physical interpretations, they can be quantitatively compared by evaluating them with a consistent distance metric. Doing so, Streltsov *et al.* [21] demonstrated that, under unitary operations, purity corresponds to the maximal achievable coherence and bounds the maximal amount of entanglement and discord. Therefore, it will also be important to establish its role as a resource for AOSD.

In this work, we revisit the discussion of the required quantum resources for AOSD, focusing on the roles of coherence and purity. Regarding coherence, unlike [19,20], we resort to the generalized resource theory developed by Bischof *et al.* [22], which established the notion of coherence with respect to POVMs, in contrast to the standard theory that is formulated on the grounds of projective measurements [23]. The POVM-based coherence is the coherence resource required to implement a POVM on the state of a given system by extending its Hilbert space, which is exactly the way AOSD is implemented. Within this framework, we provide a clearer view of the key role of coherence in the protocol, showing that, unlike dissonance, it is required in both ranges (except at the boundaries), making it a more fundamental resource. In addition, according to the operational interpretation of the POVM coherence [24], we also find the scenario of AOSD that generates the maximal cryptographic randomness gain, which accounts for the unpredictability of the measurement outcomes relative to an eavesdropper holding maximal side information about the measured state.

To study the role of purity in AOSD, we consider two scenarios. In the first, in which the ancilla is used for optimal discrimination, we show that purity is required across both ranges of the protocol, including its boundaries, and it is always greater than coherence. In the second scenario, in which an ancilla is not effectively required (valid in the second range and at the boundaries of the protocol), we show that purity is greater than coherence, except at a single point where they are equal. Therefore, we conclude that purity is the most fundamental quantum resource for AOSD.

This article is organized as follows: In Sec. II we describe the AOSD protocol outlined above and review the main results of Zhang *et al.* [11] concerning the roles of entanglement and dissonance for its implementation. In Sec. III A we describe how the generalized theory of coherence fits into AOSD and compute the POVM coherence using a relative-entropy-based measure. The more fundamental role of this resource for the protocol is established in comparison with dissonance, and the implications of this result are discussed. In Sec. IV we study the role of purity in AOSD and show that it is the most elementary quantum resource for the protocol. Finally, in Sec. V we present our concluding remarks.

II. QUANTUM CORRELATIONS AS RESOURCES FOR AOSD

A. Assisted optimal state discrimination

Consider a qubit randomly prepared in one of two pure states $|\psi_0\rangle$ and $|\psi_1\rangle$ with *a priori* probabilities η_0 and $\eta_1 =$

$1 - \eta_0$, respectively, and let $\langle\psi_0|\psi_1\rangle = \alpha = |\alpha|e^{i\theta}$ be the *a priori* fixed overlap between them. If $0 < |\alpha| < 1$, an error-free identification of the qubit state is only possible with a certain probability of success, and the optimal process is carried out through a POVM that maximizes it [1–4]. To implement this POVM, one can extend the Hilbert space of the qubit system by attaching and coupling an ancilla qubit and carrying out a projective measurement on both [9]. Let $|k\rangle_a$ be the known initial state of the ancilla; its coupling with the system will be accomplished by a joint unitary transformation U , acting as

$$U|\psi_0\rangle|k\rangle_a = \sqrt{1 - |\alpha_0|^2}|0\rangle_a + \alpha_0|\Phi\rangle|1\rangle_a, \quad (1a)$$

$$U|\psi_1\rangle|k\rangle_a = \sqrt{1 - |\alpha_1|^2}|1\rangle_a + \alpha_1|\Phi\rangle|1\rangle_a, \quad (1b)$$

where $\{|0\rangle, |1\rangle\}$ and $\{|0\rangle_a, |1\rangle_a\}$ are canonical bases for the system and ancilla qubits, respectively, and

$$|\Phi\rangle = \cos\beta|0\rangle + e^{i\delta}\sin\beta|1\rangle \quad (2)$$

for $\beta \in [0, \pi/2]$ and $\delta \in [0, 2\pi)$. The explicit form of this transformation can be found in [11]; due its unitarity, the probability amplitudes α_j are constrained by $\langle\psi_0|\psi_1\rangle = \alpha = \alpha_0^*\alpha_1$. After coupling, the ancilla is measured in the canonical basis: If it is projected onto $|1\rangle_a$, the discrimination is inconclusive as the system is projected onto $|\Phi\rangle$, whatever the input state. Otherwise, if the ancilla is projected onto $|0\rangle_a$, the discrimination succeeds, as the system state can be unambiguously identified through a measurement in the canonical basis. Successful events occur with a probability that depends only on $|\alpha_0|$ (η_0 and α are fixed *a priori*) and is given by

$$P_{\text{succ}} = 1 - \eta_0|\alpha_0|^2 - \eta_1\frac{|\alpha|^2}{|\alpha_0|^2}. \quad (3)$$

Without loss of generality, let us assume that $\eta_0 \leq \eta_1$. The *a priori* probabilities set two ranges for $|\alpha|$ and, within each range, an optimal probability of success [9]. In the first range, we have

$$0 \leq |\alpha| < \sqrt{\frac{\eta_0}{\eta_1}}, \quad P_{\text{succ}} = 1 - 2\sqrt{\eta_0\eta_1}|\alpha|, \quad (4)$$

which is attained for $|\alpha_0| = \sqrt[4]{\eta_1/\eta_0}\sqrt{|\alpha|}$. For the second range,

$$\sqrt{\frac{\eta_0}{\eta_1}} \leq |\alpha| \leq 1, \quad P_{\text{succ}} = \eta_1(1 - |\alpha|^2), \quad (5)$$

attained for $|\alpha_0| = 1$. Within the first range, both states can be identified. On the other hand, in the second range, it is possible to identify only one of them, $|\psi_1\rangle$ in this case (or $|\psi_0\rangle$ if $\eta_1 \leq \eta_0$).

B. The role of quantum correlations in AOSD

1. Entanglement

A hasty look at Eq. (1), may lead one to infer that the system-ancilla entanglement is an essential requirement for AOSD. However, a more careful analysis demands a closer look at the global system-ancilla state after the unitary, i.e.,

$$\begin{aligned} \rho_{\text{sa}} &= U(\rho \otimes |k\rangle_a\langle k|)U^\dagger \\ &= \eta_0 U(|\psi_0\rangle\langle\psi_0| \otimes |k\rangle_a\langle k|)U^\dagger \\ &\quad + \eta_1 U(|\psi_1\rangle\langle\psi_1| \otimes |k\rangle_a\langle k|)U^\dagger, \end{aligned} \quad (6)$$

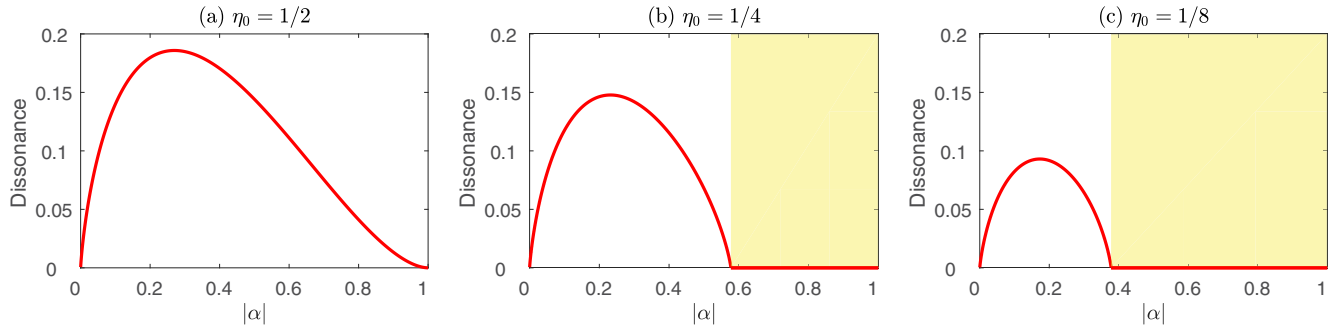


FIG. 1. Quantum dissonance in AOSD as a function of $|\alpha|$ for the *a priori* probability η_0 shown at the top of each panel. The white background in the plots corresponds to the range of $|\alpha|$ defined in Eq. (4), for which the dissonance is greater than zero, except for trivial cases of orthogonal ($|\alpha| = 0$) or parallel ($|\alpha| = 1$) states. The yellow background corresponds to the range defined in Eq. (5), for which the dissonance is always zero. These results were obtained by Zhang *et al.* [11].

and computing its concurrence [25], which is given by

$$C(\rho_{sa}) = 2|\eta_0 \sin \beta \sqrt{|\alpha_0|^2 - |\alpha_1|^4} e^{-i(\theta+\delta)} - \eta_1 \cos \beta \sqrt{|\alpha_1|^2 - |\alpha_0|^4}|, \quad (7)$$

recalling that $|\alpha_1| = |\alpha|/|\alpha_0|$. Zhang *et al.* [11] noted that while the success probability of AOSD depends only on the magnitude of the probability amplitude α_0 [see Eq. (3)], the concurrence of the bipartite state depends also on the parameters β and δ , which define the “inconclusive” state $|\Phi\rangle$ given by Eq. (2). Therefore, it is possible to set these parameters freely in the unitary transformation U without affecting the optimal P_{succ} in the protocol. From Eq. (7), it is easy to see that $C(\rho_{sa}) = 0$ for any η_0 , α , and α_0 if

$$\delta = -\theta, \quad \beta = \arctan \left(\frac{\eta_1 \sqrt{|\alpha_1|^2 - |\alpha_0|^4}}{\eta_0 \sqrt{|\alpha_0|^2 - |\alpha_1|^4}} \right). \quad (8)$$

It is also easy to check that by using these parameters and Eqs. (1) and (2), the bipartite state in Eq. (6) becomes separable, as it should (see Ref. [11]).

This result shows that entanglement can be set to zero in the arbitrary case, which encompasses both ranges in the AOSD problem defined in Eqs. (4) and (5). Therefore, it is definitely not a required resource for the protocol.

2. Dissonance

Nonclassical correlations go beyond entanglement, and when it is absent in a given quantum state, dissonance will be present if its discord is nonzero [14]. Following the approach from related works [9,11], the discord of ρ_{sa} in Eq. (6) can be derived analytically by introducing a third (fictitious) qubit to purify it:

$$|\Psi\rangle = \sqrt{\eta_0}(U|\psi_0\rangle|k\rangle_a)|0\rangle_c + \sqrt{\eta_1}(U|\psi_1\rangle|k\rangle_a)|1\rangle_c, \quad (9)$$

so $\rho_{sa} = \text{Tr}_c |\Psi\rangle\langle\Psi|$. With the Koashi-Winter relation [26], quantum discord takes the form $D(\rho_{sa}) = E(\rho_{sc}) + S(\rho_a) - S(\rho_c)$, where $E(\rho_{sc})$ is the entanglement of formation between the main system and qubit c and $S(\rho)$ is the von Neumann entropy. Considering the condition of zero system-ancilla entanglement, i.e., $C(\rho_{sa}) = 0$, achieved with the parameters of

Eq. (8), this expression can be written as [11]

$$D(\rho_{sa}) = h(\tau_{sc}) + h(\tau_a) - h(\tau_c), \quad (10)$$

where $h(x) = -\sum_{j=0}^1 f_j(x) \log_2 f_j(x)$ for $f_j(x) = [1 + (-1)^j \sqrt{1-x}]/2$ and

$$\tau_a = 4\eta_0\eta_1(|\alpha_0|^2 - 2|\alpha|^2 + |\alpha|^2/|\alpha_0|^2), \quad (11a)$$

$$\tau_c = 4\eta_0\eta_1(1 - |\alpha|^2), \quad (11b)$$

$$\tau_{sc} = \tau_c - \tau_a. \quad (11c)$$

Equation (10) gives us the discord for zero system-ancilla entanglement; therefore, it is equivalent to the quantum dissonance, which can be computed as a function of $|\alpha|$ and a given *a priori* probability η_0 by replacing the value of $|\alpha_0|$ given by Eqs. (11): For the first range of $|\alpha|$, defined in Eq. (4), $|\alpha_0| = \sqrt[4]{\eta_1/\eta_0} \sqrt{|\alpha|}$; for the second one, defined in Eq. (5), $|\alpha_0| = 1$. The obtained results for $\eta_0 = 1/2$, $\eta_0 = 1/4$, and $\eta_0 = 1/8$ are shown in Figs. 1(a)–1(c), respectively. In these plots, the white (yellow) background corresponds to the first (second) range of $|\alpha|$. It can be seen that dissonance is nonzero only within the first range, except for trivial cases of orthogonal ($|\alpha| = 0$) or parallel ($|\alpha| = 1$) states; for the second range, the dissonance is always zero. Zhang *et al.* [11] showed in the latter case that the state ρ_{sa} in Eq. (6) is indeed a product state, thus possessing no correlations at all.

Therefore, unlike entanglement, quantum dissonance is a key resource for AOSD but not the most fundamental one since it is not present in all possible cases.

III. POVM-BASED COHERENCE AS A RESOURCE FOR AOSD

A. POVM-based coherence

Seeking a more fundamental resource for AOSD than quantum discord, here, we study the role of quantum coherence in the protocol. Among the several measures to quantify coherence in a quantum state ρ [15], let us consider the relative entropy of coherence $C_{\text{rel}}(\rho)$, which is defined as

$$C_{\text{rel}}(\rho) = S(\Delta[\rho]) - S(\rho), \quad (12)$$

where Δ is the dephasing map. Given an orthonormal basis $\{|i\rangle\}$ for the d -dimensional Hilbert space $\mathcal{H}$, this map yields

$$\Delta[\rho] = \sum_{i=0}^{d-1} |i\rangle\langle i| \rho |i\rangle\langle i| = \sum_{i=0}^{d-1} p_i |i\rangle\langle i| \equiv \rho', \quad (13)$$

where $p_i = \langle i|\rho|i\rangle$. The state ρ' is incoherent [i.e., $C_{\text{rel}}(\rho') = 0$] with respect to the basis $\{|i\rangle\}$. As Eq. (13) shows, it can be viewed as arising from a rank-1 projective measurement of ρ in this basis before recording the result. Thus, the coherence quantified by $C_{\text{rel}}(\rho)$ is the necessary resource to implement this measurement.

The above notion of coherence was recently extended to encompass POVMs [22], which are the formalism describing quantum measurements more general than projective ones [27]. A POVM with n outcomes on $\mathcal{H}$ is a set $\mathbf{E} = \{E_i = A_i^\dagger A_i\}_{i=0}^{n-1}$ of positive semidefinite operators satisfying $\sum_i E_i = I$, where A_i is the measurement operator associated with E_i . The measurement on ρ yields the i th outcome with probability $p_i = \text{Tr}(A_i \rho A_i^\dagger)$ and generates the postmeasurement state $\rho_i = A_i \rho A_i^\dagger / p_i$. From Naimark's theorem, a POVM on a d -dimensional space may be implemented as a projective measurement on an extended n -dimensional Hilbert space [27]. Considering the extension via a tensor product (as in AOSD), this procedure is done as follows: First, one attaches an ancilla in a fixed state, say, $|k\rangle_a |k\rangle$, to the main system. Next, the initial system-ancilla state $\rho_{\text{sa}} = \rho \otimes |k\rangle_a |k\rangle$ is coupled through a suitable global unitary V , so that $\rho_{\text{sa}} \rightarrow \rho'_{\text{sa}} = V(\rho \otimes |k\rangle_a |k\rangle) V^\dagger$. Finally, in an n -dimensional subspace of the extended Hilbert space, a projective measurement $\Pi = \{\Pi_i\}_{i=0}^{n-1}$ on ρ'_{sa} is carried out, where $\Pi_i = V^\dagger(\pi_i \otimes |i\rangle_a \langle i|) V$ and $\{\pi_i\}$ is a set of projectors in $\mathcal{H}$ which ensures that $\text{Tr}(\Pi_i \rho'_{\text{sa}}) = \text{Tr}(A_i \rho A_i^\dagger) = p_i$. Thus, the postmeasurement system-ancilla state before the result is recorded will be given by

$$\rho'_{\text{sa}} = \sum_{i=0}^{n-1} \Pi_i \rho_{\text{sa}} \Pi_i = \sum_{i=0}^{n-1} p_i \rho_i \otimes |i\rangle_a \langle i|. \quad (14)$$

The von Neumann entropies of the initial and final global states are $S(\rho_{\text{sa}}) = S(\rho)$ and $S(\rho'_{\text{sa}}) = H[\{p_i\}] + \sum_i p_i S(\rho_i)$, respectively, where $H[\{p_i\}]$ is the Shannon entropy of the probability distribution $\{p_i\}$. Following the standard notion of coherence, the relative entropy of POVM-based coherence is then given by [22]

$$C_{\text{rel}}(\rho, \mathbf{E}) = S(\rho'_{\text{sa}}) - S(\rho_{\text{sa}}) = H[\{p_i\}] + \sum_{i=0}^{n-1} p_i S(\rho_i) - S(\rho). \quad (15)$$

The state ρ'_{sa} in Eq. (14) is block diagonal and exhibits zero POVM coherence. Therefore, $C_{\text{rel}}(\rho, \mathbf{E})$ quantifies the amount of coherence that is consumed to implement the POVM $\mathbf{E}$ on the state ρ . Note that if $\mathbf{E}$ is a projective measurement, Eq. (15) reduces to Eq. (12).

B. The POVM for AOSD

Unambiguous discrimination between two nonorthogonal states of a qubit requires the implementation of a three-outcome POVM on the qubit space, $\mathbf{E} = \{E_i = A_i^\dagger A_i\}_{i=0}^2$. For

the assisted protocol described in Sec. II A, the measurement operators are given by

$$A_0 = |0\rangle\langle 0|_a \langle 0|U|k\rangle_a, \quad (16a)$$

$$A_1 = |1\rangle\langle 1|_a \langle 0|U|k\rangle_a, \quad (16b)$$

$$A_2 = {}_a\langle 1|U|k\rangle_a, \quad (16c)$$

where U is the joint unitary transformation given by Eq. (1). From these definitions, we have $A_i^\dagger A_i \geq 0$ and $\sum_i A_i^\dagger A_i = I$.² The action of the measurement operators on the inputs is obtained from Eq. (1):

$$A_i |\psi_j\rangle = \delta_{ij} \sqrt{1 - |\alpha_i|^2} |i\rangle, \quad (17a)$$

$$A_2 |\psi_j\rangle = \alpha_j |\Phi\rangle \quad (17b)$$

for $i, j = 0, 1$, where $|\Phi\rangle$ is the inconclusive state defined in Eq. (2). Thus, the operator $A_0 (A_1)$ is associated with the unambiguous identification of $|\psi_0\rangle (|\psi_1\rangle)$, whereas A_2 is associated with inconclusive results.

Given the initial qubit state

$$\rho = \eta_0 |\psi_0\rangle\langle\psi_0| + \eta_1 |\psi_1\rangle\langle\psi_1| \quad (18)$$

and using Eqs. (17), the postmeasurement state before the result is recorded will be

$$\begin{aligned} \rho' &= \sum_{i=0}^2 A_i \rho A_i^\dagger = \sum_{i=0}^2 p_i \rho_i \\ &= p_0 |0\rangle\langle 0| + p_1 |1\rangle\langle 1| + p_2 |\Phi\rangle\langle\Phi|, \end{aligned} \quad (19)$$

where

$$p_0 = \eta_0 (1 - |\alpha_0|^2), \quad (20a)$$

$$p_1 = \eta_1 (1 - |\alpha_1|^2), \quad (20b)$$

$$p_2 = \eta_0 |\alpha_0|^2 + \eta_1 |\alpha_1|^2 \quad (20c)$$

are the probabilities $\{p_i = \text{Tr}(A_i \rho A_i^\dagger)\}$ of obtaining the i th outcome. Note that $p_0 + p_1 = P_{\text{succ}}$ [see Eq. (3)] and $p_2 = 1 - P_{\text{succ}}$.

C. The role of POVM-based coherence in AOSD

The initial qubit state in Eq. (18) has eigenvalues given by

$$\lambda_{\pm} = \frac{1}{2} [1 \pm \sqrt{1 - 4\eta_0\eta_1(1 - |\alpha|^2)}]. \quad (21)$$

After implementing the POVM, each postmeasurement state ρ_i associated with a given outcome is pure [see Eq. (19)], so $S(\rho_i) = 0 \forall i$. With all these elements, we can now compute the POVM coherence in the AOSD protocol. From Eq. (15), we obtain

$$C_{\text{rel}}(\rho, \mathbf{E}) = - \sum_{i=0}^2 p_i \log_2 p_i + \sum_{i=\pm} \lambda_i \log_2 \lambda_i, \quad (22)$$

with $\{p_i\}$ given by Eq. (20). The coherence depends on η_0 and $|\alpha|$ (both fixed *a priori*) and $|\alpha_0|$. Unlike entanglement

²Given an arbitrary state $|\psi\rangle$ in the main qubit space, we have, for instance, $\langle\psi|A_0^\dagger A_0|\psi\rangle = \|\langle 0|_a \langle 0|U|k\rangle|\psi\rangle\|^2 \geq 0$; similar relations hold for A_1 and A_2 . The completeness relation follows from $\sum_i A_i^\dagger A_i = {}_a\langle k|U^\dagger|0\rangle_a \langle 0|U|k\rangle_a (|0\rangle\langle 0| + |1\rangle\langle 1|) + {}_a\langle k|U^\dagger|1\rangle_a \langle 1|U|k\rangle_a = {}_a\langle k|U^\dagger(|0\rangle_a \langle 0| + |1\rangle_a \langle 1|)U|k\rangle_a = I$.

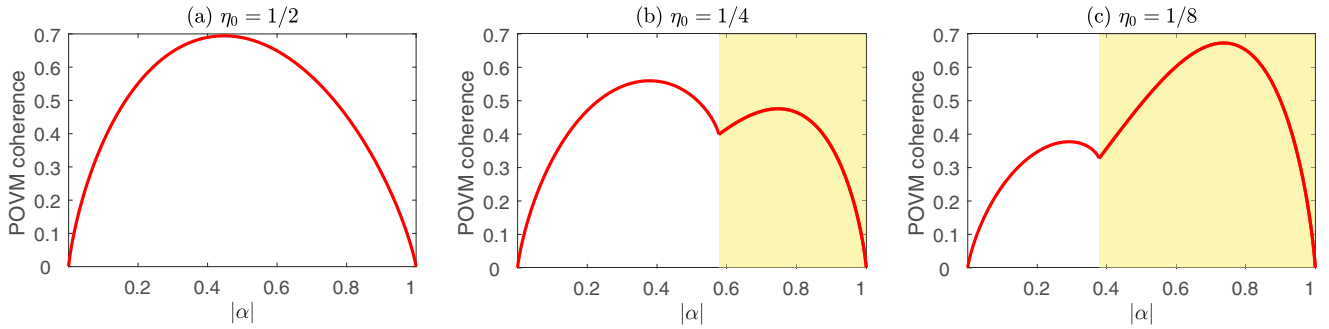


FIG. 2. POVM-based coherence in AOSD as a function of $|\alpha|$ for the *a priori* probability η_0 shown at the top of each panel. The white background in the plots corresponds to the range of $|\alpha|$ defined in Eq. (4); the yellow background corresponds to the range defined in Eq. (5). The coherence is zero only for trivial cases of orthogonal ($|\alpha| = 0$) or parallel ($|\alpha| = 1$) states. Otherwise, it is greater than zero, including the yellow range for which there is no dissonance, as can be seen from Fig. 1.

and discord, it is independent of the parameters δ and β that characterize the inconclusive state $|\Phi\rangle$, as discussed in Sec. II.

In Fig. 2, we plot $C_{\text{rel}}(\rho, \mathbf{E})$ as a function of $|\alpha|$ and a given *a priori* probability η_0 by replacing the value of $|\alpha_0|$ that optimizes the discrimination protocol in Eq. (20): For the first range of $|\alpha|$ defined in Eq. (4), $|\alpha_0| = \sqrt[4]{\eta_1/\eta_0}\sqrt{|\alpha|}$; for the second one, defined in Eq. (5), $|\alpha_0| = 1$. Similar to what we did for dissonance in Fig. 1, here, we set $\eta_0 = 1/2$, $\eta_0 = 1/4$, and $\eta_0 = 1/8$; the corresponding results are shown in Figs. 2(a)–2(c), respectively. In these plots, the white (yellow) background corresponds to the first (second) range of $|\alpha|$.

1. Discussion

Following the discussion in Sec. III A, the POVM for AOSD is implemented by extending the Hilbert space of the qubit with an ancilla qubit in the fixed state $|k\rangle_a\langle k|$. It is attached to the main system via a tensor product, generating the initial state $\varrho_{\text{sa}} = \rho \otimes |k\rangle_a\langle k|$. In this four-dimensional space, the projective measurement $\Pi = \{\Pi_i\}_{i=0}^2$ on a three-dimensional subspace implements the POVM on the qubit space, where $\Pi_0 = U^\dagger(|0\rangle\langle 0| \otimes |0\rangle_a\langle 0|)U$, $\Pi_1 = U^\dagger(|1\rangle\langle 1| \otimes |0\rangle_a\langle 0|)U$, and $\Pi_2 = U^\dagger(I \otimes |1\rangle_a\langle 1|)U$. Using Eqs. (16), (17), and (20), it is easy to check that $\text{Tr}(A_i \rho A_i^\dagger) = \text{Tr}(\Pi_i \varrho_{\text{sa}}) = p_i$. Bischof *et al.* [22] defined the POVM-based coherence for ρ as the block coherence of the state ϱ_{sa} with respect to the tensor product extension Π of the POVM $\mathbf{E}$, that is, $C_{\text{rel}}(\rho, \mathbf{E}) = C_{\text{rel}}(\varrho_{\text{sa}}, \Pi)$. In this sense, incoherent states are block diagonal.

With these definitions, it will be useful to explicitly show the form of the system-ancilla state in Eq. (6) to discuss the results shown in Fig. 2. Using Eqs. (1) and (20), we can write ρ_{sa} as

$$\begin{aligned} \rho_{\text{sa}} = & (p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1|) \otimes |0\rangle_a\langle 0| \\ & + p_2|\Phi\rangle\langle \Phi| \otimes |1\rangle_a\langle 1| \\ & + (\kappa_0|0\rangle\langle \Phi| + \kappa_1|1\rangle\langle \Phi|) \otimes |0\rangle_a\langle 1| + \text{H.c.}, \end{aligned} \quad (23)$$

where $\kappa_j = \eta_j \sqrt{1 - |\alpha_j|^2} \alpha_j^*$ (for $j = 0, 1$) and H.c. denotes the Hermitian conjugate of the operator in the last line. Note that $\text{Tr}_a \rho_{\text{sa}} = \rho'$ [see Eq. (19)].

For an arbitrary η_0 , the POVM coherence is zero only when the states to be discriminated are either orthogonal ($|\alpha| = 0$)

or parallel ($|\alpha| = 1$). The corresponding system-ancilla state in Eq. (23) becomes $\rho_{\text{sa}} = (\eta_0|0\rangle\langle 0| + \eta_1|1\rangle\langle 1|) \otimes |0\rangle_a\langle 0|$ and $\rho_{\text{sa}} = |\Phi\rangle\langle \Phi| \otimes |1\rangle_a\langle 1|$, respectively, which are both block diagonal states, so that $C_{\text{rel}}(\rho, \mathbf{E}) = 0$. Apart from these cases, ρ_{sa} is not block diagonal for any other value of $|\alpha|$. Thus, its coherence with respect to the POVM is greater than zero for both ranges of $|\alpha|$, as seen in Fig. 2. This means that if we set the parameters of the inconclusive state according to Eq. (8), AOSD will be performed without entanglement, while dissonance will be present only for $|\alpha| \in [0, \sqrt{\eta_0/\eta_1})$ (see Fig. 1). On the other hand, as the POVM coherence is not affected by the form of the inconclusive state, it will always be present in the protocol, showing that this is a more fundamental resource for AOSD than dissonance.

As described in Sec. II A, in the range $|\alpha| \in [0, \sqrt{\eta_0/\eta_1})$, the optimal measurement can identify both states $\{|\psi_0\rangle, |\psi_1\rangle\}$ with the optimal success probability given by Eq. (4). Within this range, we saw that both dissonance (Fig. 1) and POVM coherence (Fig. 2) are important resources for AOSD, present in all nontrivial cases. In the range $|\alpha| \in [\sqrt{\eta_0/\eta_1}, 1]$, on the other hand, the dissonance is zero, while the coherence is present. Within this range it is possible to identify only the state $|\psi_1\rangle$ (as we assumed $\eta_0 \leq \eta_1$) or to obtain an inconclusive result. This happens because we have to set $|\alpha_0| = 1$ to attain the optimal success probability, so that the system-ancilla state in Eq. (23) becomes

$$\begin{aligned} \rho_{\text{sa}} = & P_{\text{succ}}|1\rangle\langle 1| \otimes |0\rangle_a\langle 0| + (1 - P_{\text{succ}})|\Phi\rangle\langle \Phi| \otimes |1\rangle_a\langle 1| \\ & + \kappa_1|1\rangle\langle \Phi| \otimes |0\rangle_a\langle 1| + \text{H.c.}, \end{aligned} \quad (24)$$

with P_{succ} given by Eq. (5). Now, the projective measurement on the canonical basis of the ancilla suffices either to identify $|\psi_1\rangle$ (if $|0\rangle_a\langle 0|$) or to get an inconclusive answer (if $|1\rangle_a\langle 1|$). In practice, this measurement could also be performed without an ancilla using the orthonormal basis $\{|\psi_0\rangle, |\psi_0^\perp\rangle\}$ of the system: A projection onto $|\psi_0\rangle$ leads to an inconclusive result, and a projection onto $|\psi_0^\perp\rangle$ identifies $|\psi_1\rangle$. Given that $\langle \psi_0 | \psi_1 \rangle = \alpha$, we can write

$$|\psi_1\rangle = \alpha|\psi_0\rangle + \sqrt{1 - |\alpha|^2}|\psi_0^\perp\rangle, \quad (25)$$

so the qubit state in Eq. (18) can be rewritten in this basis as

$$\begin{aligned} \rho = & (1 - P_{\text{succ}})|\psi_0\rangle\langle \psi_0| + P_{\text{succ}}|\psi_0^\perp\rangle\langle \psi_0^\perp| \\ & + \eta_1\sqrt{1 - |\alpha|^2}\alpha|\psi_0\rangle\langle \psi_0^\perp| + \text{H.c.} \end{aligned} \quad (26)$$

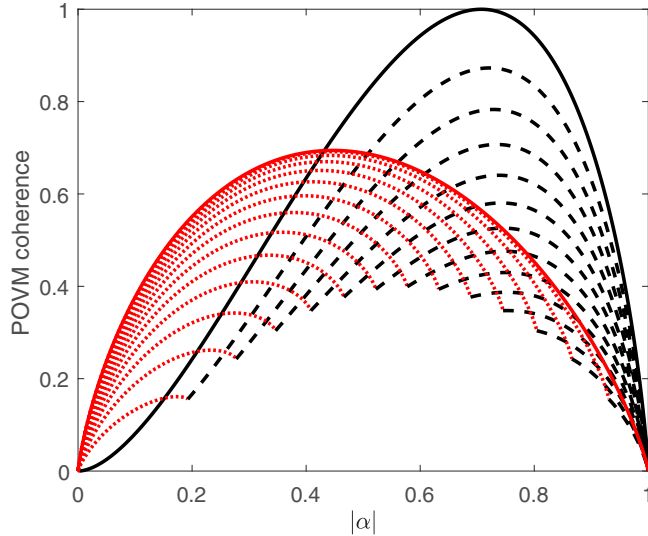


FIG. 3. POVM-based coherence in AOSD as a function of $|\alpha|$ for several *a priori* probabilities $0 \leq \eta_0 \leq 1/2$. The red (black) curves correspond to the range $0 \leq |\alpha| < \sqrt{\eta_0/\eta_1}$ ($\sqrt{\eta_0/\eta_1} \leq |\alpha| \leq 1$). The solid red and black lines correspond to $\eta_0 = 1/2$ and $\eta_0 = 0$, respectively, and show the maximal coherence within each range for a fixed value of $|\alpha|$.

After carrying out the projective measurement on ρ in the basis $\{|\psi_0\rangle, |\psi_0^\perp\rangle\}$, the state before the result is recorded will be

$$\rho' = (1 - P_{\text{succ}})|\psi_0\rangle\langle\psi_0| + P_{\text{succ}}|\psi_0^\perp\rangle\langle\psi_0^\perp|. \quad (27)$$

From Eq. (13), $\rho' = \Delta[\rho]$, so the standard relative entropy of coherence [Eq. (12)] for the above state satisfies

$$C_{\text{rel}}(\rho) = S(\Delta[\rho]) - S(\rho) = C_{\text{rel}}(\rho, \mathbf{E}). \quad (28)$$

This is not surprising since the optimal measurement within this range of $|\alpha|$ is rank 1 projective. In any case, the conclusion remains that quantum coherence is a required resource for implementing the optimal discrimination, while dissonance is not.

2. Maximal coherence in AOSD and its interpretation

The relative entropy of POVM-based coherence for an n -outcome POVM is bounded by $0 \leq C_{\text{rel}}(\rho, \mathbf{E}) \leq \log_2 n$, where the upper bound is achieved by the pure states that generate the maximal randomness of the measurement outcomes [22]. In addition, Bischof *et al.* [24] showed that this measure has an operational interpretation in the sense that it quantifies the advantages of quantum states in quantum information tasks. In light of these findings, here, we study the behavior of coherence in the AOSD protocol to extract more physical insights from the results presented above. For this purpose, in Fig. 3 we again plot the POVM coherence given by Eq. (22) as a function of $|\alpha|$, but now considering several values of η_0 from $\eta_0 = 0$ to $\eta_0 = 1/2$. The red (black) curves correspond to the first (second) range of $|\alpha|$ defined in Eq. (4) [Eq. (5)].

In the first range, where one implements a three-outcome POVM, there are no states leading coherence to reach its

upper bound $\log_2 3 \approx 1.585$. For a fixed $|\alpha|$, the highest value is achieved when $\eta_0 = 1/2$ (solid red curve), i.e., for equally likely states. In this scenario, the maximal POVM coherence is $C_{\text{rel}}(\rho, \mathbf{E}) \approx 0.6942$, attained for $|\alpha| \approx 0.4472$. This means that the implementation of the POVM for AOSD between two equally likely states with this value of $|\alpha|$ generates the maximal unpredictability of the measurement outcomes [24]. In this context, if an eavesdropper builds a purification $|\Psi\rangle_{\text{sae}}$ which holds optimal side information about $\rho_{\text{sa}} (= \text{Tr}_e |\Psi\rangle_{\text{sae}}\langle\Psi|)$, the maximal POVM coherence ensures the maximal secrecy with respect to Eve [22,24]. Therefore, looking at the POVM coherence in AOSD is an important criterion, for instance, to decide which set of equally likely nonorthogonal states is more suitable to perform quantum key distribution based on unambiguous discrimination [5].

In the second range of $|\alpha|$, one implements a projective measurement that either identifies $|\psi_1\rangle$ or gives an inconclusive result. The highest value of coherence for a fixed $|\alpha|$ is achieved for $\eta_0 = 0$ (solid black curve in Fig. 3), i.e., when only $|\psi_1\rangle$ is present. In this case, the quantum coherence is given by

$$C_{\text{rel}}(\rho) = -|\alpha|^2 \log_2 |\alpha|^2 - (1 - |\alpha|^2) \log_2 (1 - |\alpha|^2), \quad (29)$$

which is valid whether the discrimination is implemented with or without an ancilla. The maximal coherence is $C_{\text{rel}}(\rho) = 1$, which reaches the upper bound for a projective measurement. It is attained for $|\alpha| = 1/\sqrt{2}$, so that, from Eq. (25), we have $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\psi_0\rangle + |\psi_0^\perp\rangle)$. Indeed, this is the pure state that generates the maximal randomness of the measurement outcomes with respect to a projective measurement in the basis $\{|\psi_0\rangle, |\psi_0^\perp\rangle\}$. If we consider the discrimination performed with an ancilla, the system-ancilla will be in a pure state given by $|\varphi\rangle_{\text{sa}} = 1/\sqrt{2}(|1\rangle|0\rangle_a + e^{i\arg \alpha_1}|\Phi\rangle|1\rangle_a)$. It is interesting to note that if we choose the inconclusive state $|\Phi\rangle = |1\rangle$, then $|\varphi\rangle_{\text{sa}} = 1/\sqrt{2}(|1\rangle \otimes (|0\rangle_a + e^{i\arg \alpha_1}|1\rangle_a))$, which is a product state. However, if we choose $|\Phi\rangle = |0\rangle$, then $|\varphi\rangle_{\text{sa}} = 1/\sqrt{2}(|1\rangle|0\rangle_a + e^{i\arg \alpha_1}|0\rangle|1\rangle_a)$, which is a maximally entangled state. In either case, the coherence remains 1, as it does not depend on the choice of $|\Phi\rangle$.

IV. PURITY AS A RESOURCE FOR AOSD

A. Purity

Purity has been demonstrated to be the most elementary resource for quantum information processing in the sense that, under unitary operations, it corresponds to the maximal achievable coherence and bounds the maximal amount of entanglement and discord [21]. In fact, if $\mathcal{P}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ denote measures of purity, coherence, discord, and entanglement, respectively, defined via the same distance, then, given an arbitrary multipartite quantum state ϱ , the following hierarchy of resources holds:

$$\mathcal{P}(\varrho) \geq \mathcal{C}(\varrho) \geq \mathcal{D}(\varrho) \geq \mathcal{E}(\varrho). \quad (30)$$

We will explore this hierarchy regarding the roles of purity and coherence for AOSD. In this way, we use a purity measure defined via the same distance as coherence, which is the relative entropy of purity, given by

$$\mathcal{P}_{\text{rel}}(\varrho) = \log_2 d - S(\varrho), \quad (31)$$

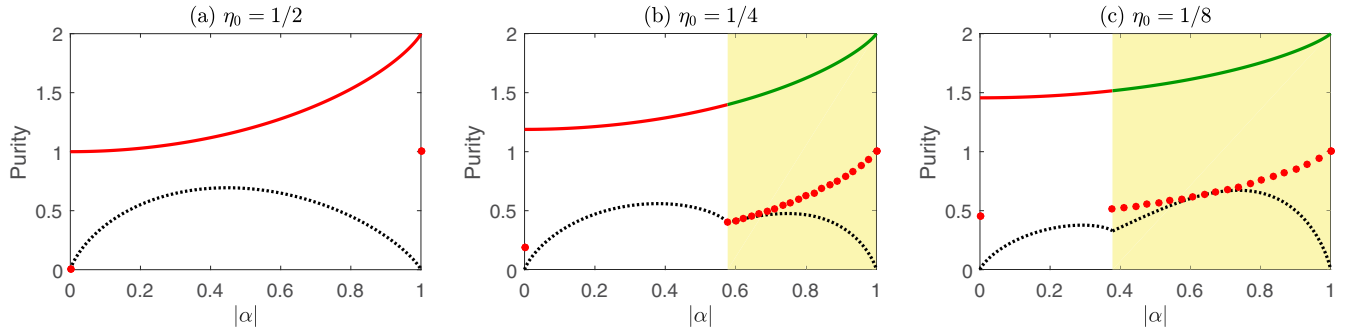


FIG. 4. Purity in AOSD as a function of $|\alpha|$ for the *a priori* probability η_0 shown at the top of each panel: $\mathcal{P}_1$ (red and green solid lines) and $\mathcal{P}_2$ (red circles). The white background in the plots corresponds to the range of $|\alpha|$ defined in Eq. (4); the yellow background corresponds to the range defined in Eq. (5). The POVM coherence (black dotted line) is shown for the sake of comparison.

where d is the dimension of state ϱ . In the context of quantum thermodynamics, this is the amount of work that can be extracted from ϱ by a party who holds the entire state [28]. This measure of purity is bounded by $0 \leq \mathcal{P}_{\text{rel}}(\varrho) \leq \log_2 d$, where the lower (upper) bound is achieved for a maximally mixed (pure) state.

B. The role of purity in AOSD

Let us now quantify the purity resource required to implement AOSD. As discussed in the previous section, the initial system-ancilla state $\varrho_{\text{sa}} = \rho \otimes |k\rangle_a \langle k|$ is transformed into $\rho_{\text{sa}} = U \varrho_{\text{sa}} U^\dagger$ by the action of the global unitary U . Since purity does not change for two states connected by a unitary transformation, we have from Eq. (31)

$$\mathcal{P}_1 \equiv \mathcal{P}_{\text{rel}}(\rho_{\text{sa}}) = \mathcal{P}_{\text{rel}}(\varrho_{\text{sa}}) = 2 - S(\rho), \quad (32)$$

which is bounded by $1 \leq \mathcal{P}_1 \leq 2$, and the lower bound is due to the ancilla qubit being introduced in a pure state.

The result above is connected to the use of an ancilla for the optimal discrimination process, and as discussed earlier, this is effectively required only in the first range of $|\alpha|$ set by Eq. (4) for $|\alpha| > 0$ (and $|\alpha| < 1$ for $\eta_0 = 1/2$). In the second range of $|\alpha|$ set by Eq. (5) or in the extremal cases of $|\alpha| = 0$ (orthogonal input states) and $|\alpha| = 1$ (parallel input states), the optimal discrimination can be implemented without an ancilla. Therefore, we have a second function to evaluate purity, which is simply the purity of the initial state ρ , i.e.,

$$\mathcal{P}_2 \equiv \mathcal{P}_{\text{rel}}(\rho) = 1 - S(\rho), \quad (33)$$

which is bounded by $0 \leq \mathcal{P}_2 \leq 1$.

In the second range of $|\alpha|$ and at its extremal values, we can compare the resource of purity for the discrimination performed with and without an ancilla. From Eqs. (32) and (33), we have $\mathcal{P}_1 - \mathcal{P}_2 = 1$, which is just the purity of the initial ancilla state that is “consumed” to implement the assisted protocol. This reinforces the notion of purity as a resource for quantum tasks.

In Fig. 4, we plot both $\mathcal{P}_1$ and $\mathcal{P}_2$ as a function of $|\alpha|$ and a given *a priori* probability η_0 (the same scenarios used for dissonance and coherence shown in Figs. 1 and 2, respectively). $\mathcal{P}_1$ is given by the red (green) solid lines in the first (second) range of $|\alpha|$, while $\mathcal{P}_2$ is given by the red circles. For the sake of comparison, we also plot the POVM coherences from Fig. 2

as the black dotted lines. In the scenario with an ancilla, we have

$$\mathcal{P}_1 > C_{\text{rel}}(\rho, \mathbf{E}) \quad (34)$$

for any value of $|\alpha|$; the lower bound $\mathcal{P}_1 = 1$ is achieved for an input of equally likely orthogonal states ($\eta_0 = 1/2$ and $|\alpha| = 0$) since in this case $\rho = I/2$, which is maximally mixed. On the other hand, the upper bound $\mathcal{P}_1 = 2$ is achieved for parallel states ($|\alpha| = 1$) and arbitrary η_0 since $\rho = |\psi_1\rangle\langle\psi_1|$. Looking only at the first range of $|\alpha|$ and excluding the extremal cases, the results from Figs. 1, 2, and 4 show that dissonance, POVM coherence, and purity are required resources for AOSD, but the role of purity clearly stands out above the rest.

In the ancillaless scenarios, Fig. 4 shows that

$$\mathcal{P}_2 \geq C_{\text{rel}}(\rho), \quad (35)$$

where the equality holds at a single point given by $|\alpha|^2 = (1/2 - \eta_0)/(1 - \eta_0)$. When the states to be discriminated are orthogonal ($|\alpha| = 0$), purity takes the form $\mathcal{P}_2 = 1 + \eta_0 \log_2 \eta_0 + \eta_1 \log_2 \eta_1$. Thus, for $\eta_0 = 1/2$, it reaches its lower bound $\mathcal{P}_2 = 0$ since $\rho = I/2$. Otherwise, $\mathcal{P}_2 > 0$ for any η_0 in this case. On the other hand, when the states are parallel ($|\alpha| = 1$), $\mathcal{P}_2$ reaches its upper bound for any η_0 . Looking only at the second range of $|\alpha|$ and excluding the extremal cases, the results from Figs. 1, 2, and 4 show that only coherence and purity are required resources for the discrimination protocol. However, when including extreme cases, we see that coherence is always zero, while purity, except in one case ($|\alpha| = 0$ and $\eta_0 = 1/2$), will always be nonzero. Therefore, following the findings of Ref. [21], this supports the role of purity as a more elementary quantum resource than any other, such as coherence and quantum correlations.

V. CONCLUSIONS

We revisited the discussion about the required resources to perform the optimal unambiguous discrimination between two pure states of a qubit assisted by an auxiliary system. Previous works showed that while entanglement is never required, quantum dissonance is a key resource when the overlap of the states α and the *a priori* probabilities $\{\eta_0, \eta_1\}$ satisfy $0 < |\alpha| < \sqrt{\eta_0/\eta_1}$ [9,11]. Otherwise, if $|\alpha| \geq \sqrt{\eta_0/\eta_1}$, even dissonance is not necessary. We found that, unlike

entanglement and dissonance, coherence and purity are resources for AOSD at both ranges of $|\alpha|$ and thereby more fundamental than those quantum correlations.

Recent works also addressed the key role of coherence in AOSD [19,20], but here, we provided a more straightforward view of it by resorting to the POVM-based coherence [22,24]. In addition, with this approach we were able to identify the optimal scenario in the protocol ($\eta_0 = \eta_1 = 1/2$ and $|\alpha| \approx 0.4472$) that maximizes the cryptographic randomness gain, which is useful, for instance, to improve security in quantum cryptography. Finally, we also showed that, although more fundamental than dissonance, coherence is not the most elementary quantum resource of AOSD. That attribute belongs to purity, which was shown to be greater than coherence in both ranges of $|\alpha|$, including at its boundaries, where coherence is zero. This result can be physically interpreted as the fact that an ancilla, initially in a pure state, enables the extension of the Hilbert space and, following the application of a global unitary transformation, allows for the possible generation of POVM

coherence, dissonance, entanglement, or other quantum correlations defined in a bipartite quantum system. Therefore, our work expands the understanding of the quantum resources necessary for the AOSD protocol, showing those which are key (dissonance, coherence, and purity) and establishing the most elementary of them (purity).

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- [1] I. Ivanovic, How to differentiate between non-orthogonal states, *Phys. Lett. A* **123**, 257 (1987).
 - [2] D. Dieks, Overlap and distinguishability of quantum states, *Phys. Lett. A* **126**, 303 (1988).
 - [3] A. Peres, How to differentiate between non-orthogonal states, *Phys. Lett. A* **128**, 19 (1988).
 - [4] G. Jaeger and A. Shimony, Optimal distinction between two non-orthogonal quantum states, *Phys. Lett. A* **197**, 83 (1995).
 - [5] C. H. Bennett, Quantum cryptography using any two nonorthogonal states, *Phys. Rev. Lett.* **68**, 3121 (1992).
 - [6] W.-L. Li, C.-F. Li, and G.-C. Guo, Probabilistic teleportation and entanglement matching, *Phys. Rev. A* **61**, 034301 (2000).
 - [7] R. Salazar and A. Delgado, Quantum tomography via unambiguous state discrimination, *Phys. Rev. A* **86**, 012118 (2012).
 - [8] J. B. Brask, A. Martin, W. Esposito, R. Houlmann, J. Bowles, H. Zbinden, and N. Brunner, Megahertz-rate semi-device-independent quantum random number generators based on unambiguous state discrimination, *Phys. Rev. Appl.* **7**, 054018 (2017).
 - [9] L. Roa, J. C. Retamal, and M. Alid-Vaccarezza, Dissonance is required for assisted optimal state discrimination, *Phys. Rev. Lett.* **107**, 080401 (2011).
 - [10] B. Li, S.-M. Fei, Z.-X. Wang, and H. Fan, Assisted state discrimination without entanglement, *Phys. Rev. A* **85**, 022328 (2012).
 - [11] F.-L. Zhang, J.-L. Chen, L. C. Kwek, and V. Vedral, Requirement of dissonance in assisted optimal state discrimination, *Sci. Rep.* **3**, 2134 (2013).
 - [12] L. F. Xu, F. L. Zhang, M. L. Liang, and J. L. Chen, Assisted optimal state discrimination without entanglement, *Europhys. Lett.* **106**, 50004 (2014).
 - [13] O. Jiménez, M. A. Solís-Prosser, L. Neves, and A. Delgado, Mutual information and quantum discord in quantum state discrimination with a fixed rate of inconclusive outcomes, *Entropy* **23**, 73 (2021).
 - [14] K. Modi, T. Paterek, W. Son, V. Vedral, and M. Williamson, Unified view of quantum and classical correlations, *Phys. Rev. Lett.* **104**, 080501 (2010).
 - [15] A. Streltsov, G. Adesso, and M. B. Plenio, Colloquium: Quantum coherence as a resource, *Rev. Mod. Phys.* **89**, 041003 (2017).
 - [16] Y. Yao, X. Xiao, L. Ge, and C. P. Sun, Quantum coherence in multipartite systems, *Phys. Rev. A* **92**, 022112 (2015).
 - [17] J. Ma, B. Yadin, D. Girolami, V. Vedral, and M. Gu, Converting coherence to quantum correlations, *Phys. Rev. Lett.* **116**, 160407 (2016).
 - [18] S. Kim, C. Xiong, A. Kumar, and J. Wu, Converting coherence based on positive-operator-valued measures into entanglement, *Phys. Rev. A* **103**, 052418 (2021).
 - [19] M. Namkung and Y. Kwon, Understanding of various type of unambiguous discrimination in view of coherence distribution, *Entropy* **22**, 1422 (2020).
 - [20] S. Kim, L. Li, A. Kumar, C. Xiong, S. Das, U. Sen, A. K. Pati, and J. Wu, Protocol for unambiguous quantum state discrimination using quantum coherence, *Quantum Inf. Comput.* **21**, 931 (2021).
 - [21] A. Streltsov, H. Kampermann, S. Wölk, M. Gessner, and D. Bruß, Maximal coherence and the resource theory of purity, *New J. Phys.* **20**, 053058 (2018).
 - [22] F. Bischof, H. Kampermann, and D. Bruß, Resource theory of coherence based on positive-operator-valued measures, *Phys. Rev. Lett.* **123**, 110402 (2019).
 - [23] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying coherence, *Phys. Rev. Lett.* **113**, 140401 (2014).
 - [24] F. Bischof, H. Kampermann, and D. Bruß, Quantifying coherence with respect to general quantum measurements, *Phys. Rev. A* **103**, 032429 (2021).
 - [25] W. K. Wootters, Entanglement of formation of an arbitrary state of two qubits, *Phys. Rev. Lett.* **80**, 2245 (1998).
 - [26] M. Koashi and A. Winter, Monogamy of quantum entanglement and other correlations, *Phys. Rev. A* **69**, 022309 (2004).
 - [27] A. Peres, *Quantum Theory: Concepts and Methods* (Springer, Dordrecht, 1995).
 - [28] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Thermodynamical approach to quantifying quantum correlations, *Phys. Rev. Lett.* **89**, 180402 (2002).