

Higher-order Poincaré spheres and spatio-spectral Poincaré beams

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Light with a complex structure has become a powerful tool in many studies in classical and quantum optics. Here, we introduce a visual representation of light that is structured simultaneously in space, frequency, and polarization in the form of a higher-order Poincaré sphere and describe interesting links to its fundamental counterpart. We leverage the description to show and experimentally generate light possessing all possible polarization states across its spatio-spectral shape, which we term spatio-spectral Poincaré beams. By extending the powerful description of higher-order Poincaré spheres to the spatio-spectral domain, our work will help to gain a deeper understanding and to develop beneficial applications of structured light.

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I. INTRODUCTION

Increasing the complexity of light fields by shaping their amplitude in space, time, and polarization, i.e., all their degrees of freedom (DOFs), is advancing research in various different fields. Driven by increasingly advanced technologies to control light, ideas from different optics fields have been adapted, combined, and extended leading to growing interest in fundamental studies of linear, nonlinear, and quantum optics, as well as applications in imaging, sensing, computing, and communications [1,2]. A lot of attention has been attributed to form complex spatial polarization states by combining polarization with different spatial mode structures, leading to so-called spatial vector beams [3]. Spatial vector beams are used in applications such as tight focusing [4], the study of complex entanglement pattern [5], high-speed imaging schemes [6], and the observation of three-dimensional polarization structures like knots and links [7]. Mathematically, they have been described through a generalization of the fundamental Poincaré sphere (PS), the so-called higher-order PS for spatial vector beams [8,9]. The representation of vector fields by high-order Poincaré spheres helps in describing their complex structure, their evolution, and their innate geometric effects, such as higher-order Berry or Pancharatnam phases [10]. Additionally, the concept of higher-order PS can be linked to different applications in communications [11], metamaterial design [12], and extensions to more complex spatially structured light [13].

A particularly interesting set of spatial vector beams describes light fields containing all possible polarization states, so-called Poincaré beams [14]. For such beams, the transverse

polarization pattern can be mapped to the entire fundamental PS through a stereographic projection. The patterns can further be linked to interesting topological features, such as complex polarization singularities [15], polarization Möbius stripes [16], and optical skyrmions in the classical [17] and quantum domain [18] to highlight a few.

In this article, we extend the idea of high-order PSs to states of light with a complex polarization structure over its spatial as well as spectral DOF, and present interesting links to its fundamental counterpart. We first demonstrate that single points on higher-order PSs of spatial as well as spectral vector beams can be linked to latitudes and meridians on the fundamental PS, respectively. We then introduce a more complex higher-order PS describing spatio-spectral vector beams, i.e., light with a spatially and spectrally inhomogeneous polarization state. We show that such light fields can exhibit all possible polarization states over the spatio-spectral domain and how the patterns can be mapped onto the entire surface of the fundamental PS. As such, we term these states of light spatio-spectral Poincaré beams (SSPBs). We then generate SSPBs using a simple experimental setup and verify that all polarization states of the PS are found using spatio-spectrally resolving polarization tomography.

II. HIGHER-ORDER POINCARÉ SPHERES

The vectorial nature of light, i.e., its polarization, can be described as a two-dimensional state using the circular polarization basis $\{|R\rangle, |L\rangle\}$, so that any state can be constructed as a superposition with

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|R\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|L\rangle, \quad (1)$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. These states are conveniently represented through the surface of the PS, where $|R\rangle$ and $|L\rangle$ are located on the north and south pole of the PS, respectively (see the Supplemental Material [19]). With a similar argument, one can also describe other DOFs of a paraxial light field, i.e., the transverse spatial [20–22] and the spectral

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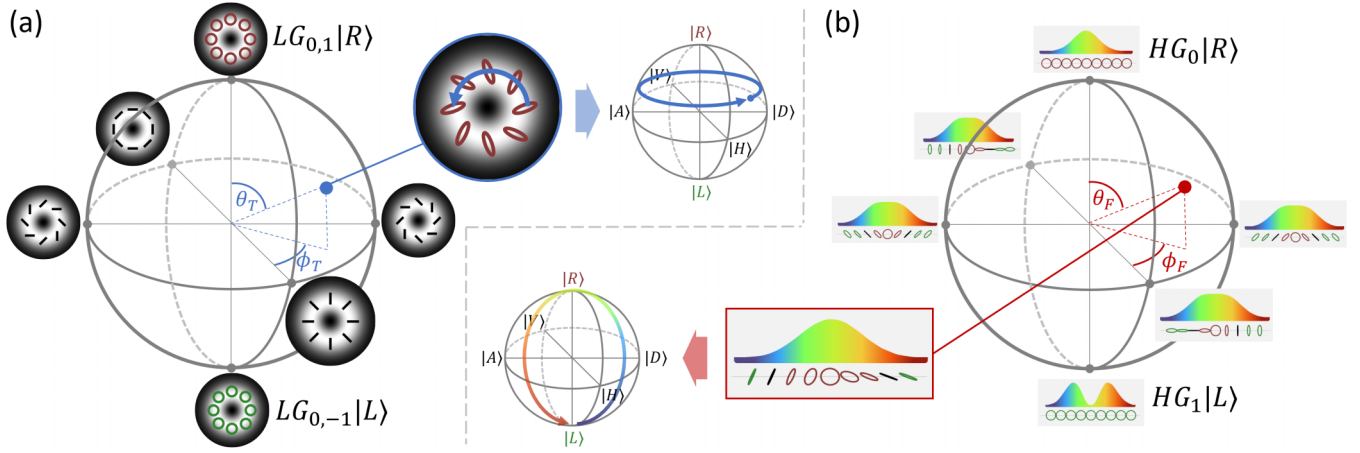


FIG. 1. Higher-order Poincaré spheres (PS) in the space and frequency domain. (a) Higher-order spatial PS constructed by two states at the poles with orthogonal circular polarization and two spatial Laguerre-Gaussian modes $LG_{0,\ell}$ with $\ell = \pm 1$. A single point on the sphere describes a beam with a transverse spatial polarization pattern, which itself can be visualized as a line along a latitude on the fundamental PS (see inset). The states on opposing sides are graphically depicted by their transverse intensity in gray scale overlaid by the polarization ellipses. (b) Higher-order spectral PS built by the two spectral Hermite-Gaussian modes HG_n with $n = 0, 1$, each with the opposite circular polarization. Here, a single point corresponds to a beam with a spectrally varying polarization pattern, whose distribution follows a meridian of the fundamental PS (see inset). Graphically, the states are depicted by their spectral intensity in rainbow colors with the polarization ellipses underneath. $|H\rangle$, $|V\rangle$, $|D\rangle$, $|A\rangle$, $|R\rangle$, and $|L\rangle$ stand for horizontal, vertical, diagonal, antidiagonal, and left- and right-handed circular polarization states, respectively. Both inset examples, (a) and (b), are shown for $\theta = 2\pi/3$ and $\phi = \pi/4$. For all polarization ellipses, linear polarization is depicted by black lines, and right- and left-handed polarization by red and green ellipses, respectively.

(temporal) [23] DOF, or, in a more general framework, two-dimensional quantum states in the form of the Bloch sphere [24].

A. Spatial vector beams

When polarization is correlated with one of the other DOFs, a higher-order PS can be defined [8,9]. In the spatial domain, any set of two orthogonal transverse spatial modes $T_n(\vec{r})$ is sufficient, where n denotes the mode number, and $\vec{r}$ the transverse spatial coordinate. Thus, Eq. (1) is transformed to

$$|\Psi\rangle_T = T_1(\vec{r}) \cos\left(\frac{\theta_T}{2}\right) |R\rangle + T_2(\vec{r}) \sin\left(\frac{\theta_T}{2}\right) e^{i\phi_T} |L\rangle, \quad (2)$$

with $\theta_T \in [0, \pi]$ and $\phi_T \in [0, 2\pi]$, as before. The complex-valued spatial mode functions $T_n(\vec{r})$ introduce a spatially varying polarization pattern in the transverse plane, such that these beams are termed spatial vector beams. For the spatial higher-order PS, the two poles are described by two orthogonal polarization states, where each state has an orthogonal spatial mode function and where θ_T and ϕ_T span the whole surface of the sphere. Note, that for $T_1 = T_2$, the higher-order PS reduces to the fundamental PS.

Arguably, the most popular spatial vector beams are realized when the spatial mode functions are implemented by higher-order Laguerre-Gaussian (LG) modes, i.e., $T_n(\vec{r}) = LG_{p,\ell} = G(r)L_p^{|\ell|}(\rho)e^{-i\ell\varphi}$, with r being the radius, $\rho = 2r^2/w_0^2$ with w_0 being the beam waist, $G(r)$ describing a Gaussian profile, $L_p^{|\ell|}(\rho)$ are the generalized Laguerre polynomials, and φ the azimuthal angle (simplified description of the transverse field for clarity). In the case of no radial structure ($L_0^{|\ell|}(\rho) = 1$), we can choose the two first order LG modes carrying opposite orbital angular momentum [25] ($\ell = \pm 1$),

such that $T_{\pm 1}(\vec{r}) = G(r)e^{\pm i\varphi}$. Then, Eq. (2) becomes

$$|\psi\rangle_T = G(r)e^{i\varphi} \left[\cos\left(\frac{\theta_T}{2}\right) |R\rangle + \sin\left(\frac{\theta_T}{2}\right) e^{i(\phi_T - 2\varphi)} |L\rangle \right]. \quad (3)$$

The resulting higher-order PS is shown in Fig. 1(a), where the two poles of the sphere are described by two donut-shaped light fields with two orthogonal transverse uniform circular polarizations. Any point on the higher-order PS, except the two poles, corresponds to one combination of $\{\theta_T, \phi_T\}$, forming a unique spatial vector beam. It contains a transverse varying polarization state, where the angular coordinate φ relates to different phases between the two circular polarization components and cycles through all possible phases twice. Thus, the higher-order PS and the fundamental PS are linked through $\theta = \theta_T$ and $\phi = \phi_T + 2\phi$. Interestingly, this results in each spatial vector beam containing all polarization states of a latitude (horizontal line) on the fundamental PS (see the example and inset in Fig. 1(a) and the Supplemental Material [19]).

B. Spectral vector beams

Although less studied, a similar vector beam can be constructed by combining polarization and the spectral or temporal DOF of light [26]. As before, we can extend Eq. (1) by including two orthogonal spectral mode functions $F_n(\omega)$ such that we obtain a so-called spectral vector beam [27] of light described by the state

$$|\Psi\rangle_F = F_1(\omega) \cos\left(\frac{\theta_F}{2}\right) |R\rangle + F_2(\omega) \sin\left(\frac{\theta_F}{2}\right) e^{i\phi_F} |L\rangle, \quad (4)$$

with ω being the angular frequency of the light field, $\theta_F \in [0, \pi]$, and $\phi_F \in [0, 2\pi]$. The states are forming the surface of

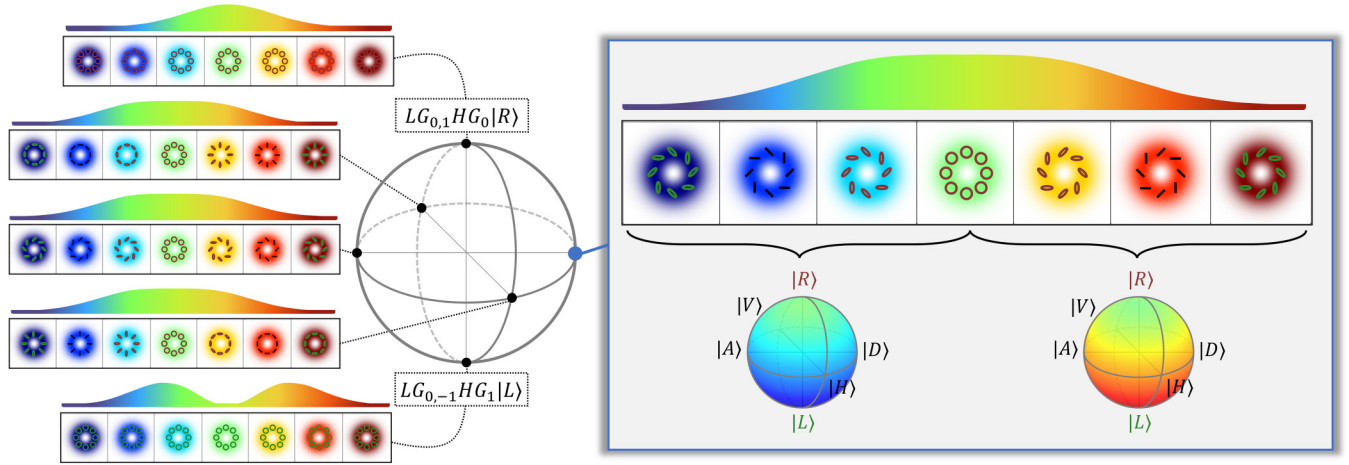


FIG. 2. Spatospectral Poincaré sphere with the north (south) pole describing a right-circular (left-circular) polarization state with a Gaussian (first order Hermite-Gaussian) spectral mode and a transverse spatial Laguerre-Gaussian mode with $\ell = 1$ ($\ell = -1$). The inset on the right shows a single spatospectral Poincaré beam, where half of the spectrum covers the whole surface of a fundamental PS. Linear polarization is depicted by black lines, and right- and left-handed circular polarization by red and green ellipses, respectively.

a higher-order spectral PS, where the poles are describing two orthogonal circular polarization states, each having a different orthogonal spectral mode. As before, for $F_1 = F_2$, the higher-order PS reduces to the fundamental one. Note, that although here the higher-order PS is discussed in the spectral domain, all results hold also for the temporal domain as both are linked via the Fourier relation.

Commonly, the spectral mode functions of choice are Hermite-Gaussian (HG) spectral modes with $F_n(\omega) = HG_n = G(\omega)H_n(\Omega)$, where $G(\omega)$ describes a Gaussian profile, H_n are the Hermite polynomials, and $\Omega = (\omega_0 - \omega)/\sigma$ with σ labeling the width of the spectrum and ω_0 its central frequency [23]. For the two lowest order modes, i.e., the first two Hermite polynomials $H_0(x) = 1$ and $H_1(x) = 2x$, we obtain $F_1(\omega) = G(\omega)$ and $F_2 = G(\omega)2\Omega$ and Eq. (4) can be rewritten as

$$|\Psi\rangle_F = G(\omega)\Omega' \left[\frac{1}{\Omega'} \cos\left(\frac{\theta_F}{2}\right)|R\rangle + \Omega' \sin\left(\frac{\theta_F}{2}\right)e^{i\phi_F}|L\rangle \right], \quad (5)$$

with $\Omega' = \sqrt{2\Omega}$. The resulting higher-order PS is shown in Fig. 1(b), where all but the two states on the poles have a polarization that varies depending on the frequency component. In contrast to the previously introduced spatial vector beams, these spectral vector beams show a different connection to the fundamental PS. Equation (5) shows that for $\Omega' = 0$, i.e., $\omega = \omega_0$, the beam is in state $|L\rangle$. For $\Omega' \rightarrow \pm\infty$ with a fixed phase $\pm\phi_F$ between the right- and left-handed component, the polarization continuously evolves to $|R\rangle$. The correct mapping of these beams onto the fundamental PS is then given by $\phi = \phi_F$ and $\theta = \arctan(\Omega' \tan \theta_F/2)$. Thus, any point $\{\theta_F, \phi_F\}$ on the higher-order PS contains all polarization states of a meridian (vertical line) on the fundamental PS (see the inset in Fig. 1(b), and the Supplemental Material [19]).

C. Spatospectral Poincaré beams

To fully describe a complex vectorial light field in space and spectrum (time), we combine all DOFs, such that one can

write

$$|\Psi\rangle_{TF} = T_1(\vec{r})F_1(\omega) \cos\left(\frac{\theta_{TF}}{2}\right)|R\rangle + T_2(\vec{r})F_2(\omega) \sin\left(\frac{\theta_{TF}}{2}\right)e^{i\phi_{TF}}|L\rangle. \quad (6)$$

Because the polarization state changes across the transverse beam profile as well as its frequency spectrum, these beams can be termed spatospectral vector beams [28]. As before, for $T_1 = T_2$ and $F_1 = F_2$ the description reduces to simple homogeneous polarization states. If only one of the two cases hold, the reduction instead leads to a spatial or spectral vector beam.

In the following, we use $T_{1,2}(\vec{r}) = LG_{0,\pm 1}$ and $F_{1,2}(\omega) = HG_{0,1}$, to align with the examples of higher-order PSs described above. As before, it is possible to define a more complex higher-order PS (see Fig. 2), where the two poles have a uniform polarization. However, all other states now contain all possible polarization states of the fundamental PS when sampled across space and spectrum, i.e., they are spatospectral Poincaré beams (SSPB). Analogue to spatial and spectral vector beams, all SSPBs states, excluding those on the poles, have a frequency dependent amplitude ratio between their $|L\rangle$ and $|R\rangle$ contributions. At the same time, the relative phase between $|L\rangle$ and $|R\rangle$ cycles from 0 to 2π twice dependent on the transverse angle φ . In other words, the polarization pattern of such a beam is a complex stereographic projection of the PS onto the three-dimensional space of the spatospectral domain (see the Supplemental Material [19]). Let's visualize this feature by analyzing the exemplary state on the equator of the higher-order PS with $\{\theta_{TF} = \pi/2, \phi_{TF} = \pi/2\}$ (inset in Fig. 2). The corresponding points P on the fundamental PS are given by

$$P_{\frac{\pi}{2}, \frac{\pi}{2}} = (\sin \arctan \Omega')\{\sin 2\varphi, \cos 2\varphi, 1\}. \quad (7)$$

If we fix the value of Ω' , $P_{\frac{\pi}{2}, \frac{\pi}{2}}$ evolves as the angle φ changes from 0 to 2π and creates a doubly covered great circle in the equatorial plane. If we instead fix a value of φ and vary the

frequency Ω' from $-\infty$ to $+\infty$, $P_{\frac{\pi}{2}, \frac{\pi}{2}}$ describes a great circle in the meridian plane, i.e., from the south pole to the north pole. Notice, that as $\omega \rightarrow \pm\infty$, $\arctan \Omega' \rightarrow \pm\pi/2$, which is equivalent to $\sin \arctan \Omega' \rightarrow 0, \pi$.

III. EXPERIMENTAL RESULTS

There are well-known experimental techniques generate SSPB's by using the first higher-order LG modes in space [29] and HG modes in frequency [30]. However, there is a surprisingly simple method to approximate them well by using only two optical elements. In the spectral domain, it was recently shown that a birefringence material can be used to create a polarization-dependent temporal splitting of a laser pulse and thereby producing a spectral varying polarization pattern, i.e., a spectral vector beam [27]. This technique does not generate two orthogonal spectral modes, i.e., the overlap between the two resulting spectral distributions is small but nonvanishing, however, over a certain wavelength range, the obtained state approximately translates to Eq. (5) with $\{\theta_F = \pi/2, \phi_F = \pi/2\}$. In the spatial domain, the key component to generate a polarization-dependent azimuthal phase distribution between the two LG modes of opposite handedness is a commercially available vortex retarder known as an S-vortex plate (Thorlabs, WPV10L-780). Such optical elements have a constant half-wave retardance with an azimuthally varying optical axis such that they perform the required polarization-dependent phase modulation based on the geometrical (Pancharatnam-Berry) phase [31]. More details on these approximations along with the individual experimental generation of a spectral and spatial vector beam can be found in the Supplemental Material [19].

In our experiment, we use Fourier-limited laser pulses with approximately 220 fs pulse duration centered at a wavelength of around 804.5 nm. At first, we generate a spectral vector beam by modulating the beams polarization with a half-wave plate which is aligned at 45° with respect to the optical axis of a 2 mm thick birefringence BaB_2O_4 crystal (BBO) with a cut angle of 23.4° . After propagation through the BBO crystal, the laser pulses are coherently split into two trailing pulses of half its initial amplitude, which results in the required spectral polarization pattern. Subsequently, we sent the laser through an S-vortex plate, which imprints the polarization-dependent OAM ($\ell = \pm 1$) thereby completing the generation scheme. See Fig. 3 for a simplified scheme of the setup.

To characterize the generated SSPB, we perform spatio-spectrally resolved polarization tomography. We first filter the beam for different polarizations with a set of wave plates and a polarizing beam splitter and for angular positions with a slit mask (with an opening angle of approximately 14°) and then measure its wavelength-resolved intensity with a spectrometer. From these measurements, we reconstruct the polarization ellipses for different wavelengths and angular positions of the slit mask as shown in Fig. 4(a) (see the Supplemental Material [19]). We find that the reconstructed polarizations states nicely match the expected patterns described above, despite some experimental imperfections, such as the nonvanishing overlap of the spectral modes, and small optical misalignments. To show that the spatio-spectral polarization patterns of the generated beams indeed cover the full

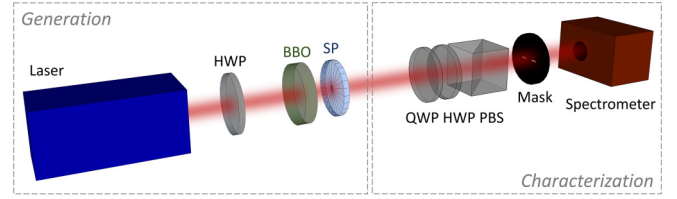


FIG. 3. Scheme to generate a spatio-spectral Poincaré beam (SSPB). By using a half-wave plate (HWP), a beam from a fs laser is diagonally polarized with respect to the optical axis of the birefringence crystal (BBO). The generated spectral vector beam is then sent through an S-vortex plate (SP) to generate a SSPB. The SSPB is characterized through spatio-spectrally resolved polarization tomography using polarization optics [quarter-wave plate (QWP), HWP, polarizing beam splitter (PBS)], a spatial slit mask, and a spectrometer.

PS, we show the measured polarization states on the surface of the fundamental PS and, again, find a nice agreement apart from minor experimental imperfections [Fig. 4(b)].

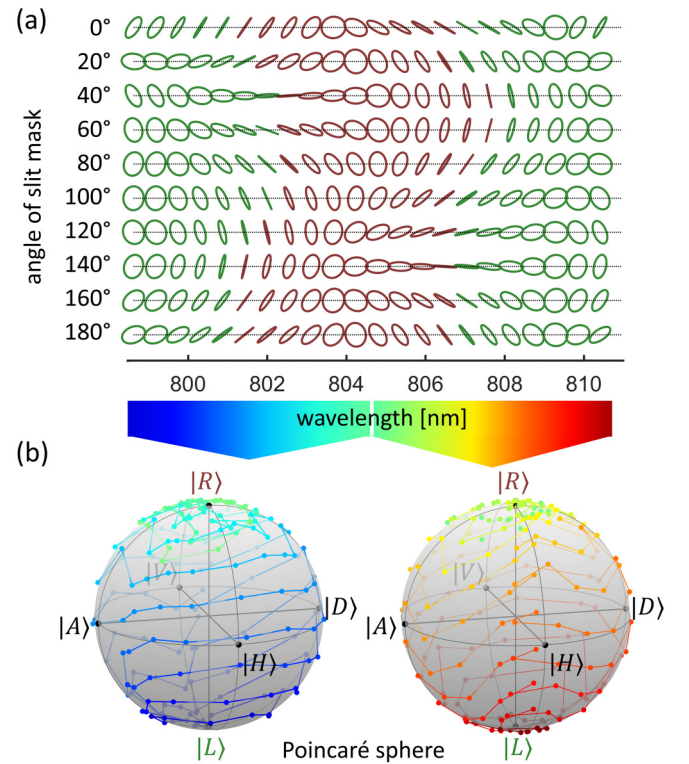


FIG. 4. Experimentally reconstructed polarization pattern of a spatio-spectral Poincaré beam. (a) Reconstructed polarization ellipses depending on the angle of the slit mask and wavelength. Right- and left-handed polarizations are depicted by red and green ellipses, respectively. (b) All measured spatio-spectral polarization states on their respective position on the fundamental PS. All data points measured at the same wavelength for different spatial mask angles are connected by a line and colored according to their wavelength. Ideally, the lines are latitudes, however, measurement errors lead to slight deformations.

IV. CONCLUSION

We introduce an extension to the concept of higher-order PSs by including all DOFs of light and discuss how light can exhibit every possible polarization state. Furthermore, we experimentally generate a SSPB and verify that its polarization ellipses indeed cover the whole PS. Following this initial study, a multitude of different research opportunities could arise. A direct next step can be to investigate the SSPBs in connection to the Pancharatnam-Berry phase, for which the introduced higher-order PS will be essential [10]. Another promising path will be to examine SSPBs in terms of complex polarization topologies, which have mostly been studied in the spatial domain [7,15,17], and see how they relate to polarization states of polychromatic light [32,33]. Moreover, having access to a higher-order PS representation for all DOFs of light fields could provide new insight into the geometrical properties of multipartite systems, as higher-order PSs allow an easier representation of such systems. This scheme, in fact, can be easily generalized to a system possessing N DOFs, thus giving access to a geometrical representation of multipartite systems beyond the present schemes based on fibrations of spheres, which are limited to tripartite states [34,35]. Finally, as all DOFs of the light field are strongly correlated in SSPBs, they can also serve as a valuable system to explore the similarities of classical and quantum nonseparable states of light [28]. Here, an extension of the recently discussed connection between spatial vector beams and entangled photon pairs [18,36] could be extended to an analogy between SSPBs and tripartite quantum entangled system leading to interesting nonlocal features.

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