Letter

Constructing quantum dark solitons with stable scattering properties

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A soliton is a class of solitary-wave solutions having specific mathematical properties, which universally appears in integrable systems. A quantum dark soliton is an emerging concept having theoretical and experimental relevance. Although recent studies have proposed quantum states with density profiles similar to those of classical dark solitons, the concept of quantum dark solitons is not yet fully established in that their collision dynamics have not been examined. Here, we first construct dark soliton states that possess stable scattering properties that are unique to solitons using the Lieb-Liniger model, establishing a direct link between quantum and classical dark soliton theories.

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Introduction. A soliton is a mathematical concept of solitary-wave solutions that are universally found in classical integrable systems [1,2]. Not only for its physical relevance, but also for its rich mathematical structure, its properties have been very well documented. In addition, the soliton has been also studied in the context of quantum physics theoretically [3–21] and experimentally [22–27]. The dark soliton [28] is an important class of solitons that have received significant attention in recent years, especially in the fields of quantum information processing [29–33] and matter wave interferometry [34–37]. Despite their potential for vast applications, our theoretical understanding of the dark solitons in quantum systems (quantum dark solitons) is very limited. Crucially, even the way to construct the quantum states of dark solitons has not been fully established. Because most previous studies have considered the mean-field approximation to investigate the dark solitons, their nature in the quantum regime has not been fully understood. This is in stark contrast with the situation of another class of solitons, the bright soliton [38], whose quantum states can be constructed from the string-type Bethe eigenstates corresponding to the bound states [3–5].

We consider quantum dark solitons in the Lieb-Liniger (LL) model that describes bosons interacting with a δ-function potential in one dimension, which is one of the most common quantum integrable systems to study the quantum-classical correspondences of solitons [39,40]. The time-evolution equation of the field operator in the LL model corresponds to the nonlinear Schrödinger (NS) equation [28,38], and type-II excitations (i.e., one-hole excitations) in the LL model are believed to be related to the dark solitons in the NS equation because they have the same dispersion relations in the weak coupling limit [6].

The past theoretical studies have constructed quantum states whose density profiles at the initial time correspond to the classical dark one-soliton [7–10] and two-soliton [12] by superposing the Bethe eigenstates. (We call such states "quantum soliton states" hereafter.) For the time evolution of a one-soliton state, it has been confirmed that a solitary wave of the density profile moves at a constant velocity while dispersing due to the quantum effect. Meanwhile, the dynamics of more than one quantum dark soliton has not been examined. In general, the classical solitons are stable against collisions; that is, the shape and velocity of the colliding solitons are preserved before and after the collision, but with a shift in position. This is one of the key criteria for a state to be defined as a soliton. However, such collision properties have not been confirmed in the quantum dark solitons.

In this sense, the realization of the soliton-like colliding behavior is a missing link towards establishing a theoretical concept of the quantum dark soliton with full correspondence to classical dark soliton theory. Here we provide a construction of such multi-soliton states that exhibit stable scattering properties in the quantum regime. Our method allows us to easily manipulate the velocity of the solitons, thereby allowing clear observation of the scattering properties. In this way, we show that quantum dark solitons with stable scattering properties do exist beyond the mean-field regime and contribute to bridging the quantum and classical soliton theories.

Model. We consider the LL model for N bosons with repulsive interaction ($c > 0$):

$$\hat{\mathcal{H}}_{\mathrm{LL}} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j<k}^{N} \delta(x_j - x_k). \tag{1}$$

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Here, we impose the periodic boundary condition with the system size L and use units with $2m = \hbar = 1$, where m is the mass of the particles. By introducing the Bose field operator $\hat{\psi}(x,t)$, the Hamiltonian in the second quantization formalism is given by

$$\hat{\mathcal{H}}_{\rm LL} = \int_0^L dx[\partial_x\hat{\psi}^\dagger\partial_x\hat{\psi} + c\hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi}]. \tag{2}$$

The Heisenberg equation of motion for the Bose field operator $\hat{\psi}(x,t)$ is

$$i\partial_t\hat{\psi} = -\partial_x^2\hat{\psi} + 2c\hat{\psi}^\dagger\hat{\psi}\hat{\psi}, \tag{3}$$

which corresponds to the NS equation.

The LL model is exactly solvable by the Bethe ansatz [41]. Each eigenstate of the Hamiltonian is determined by a set of Bethe quantum numbers, $\{I\}_N = \{I_1, I_2, \ldots, I_N\}$. Here, I_j's are integers (half-integers) for odd (even) particle number N. For a given set of $\{I\}_N$ chosen as $I_1 < I_2 < \cdots < I_N$, the Bethe equations

$$k_jL = 2\pi I_j - \sum_{l\neq j}^{N}\arctan\left(\frac{k_j-k_l}{c}\right), \quad \text{for} \quad j = 1,2,\ldots,N, \tag{4}$$

have a unique real solution $\{k\}_N = \{k_1, k_2, \ldots, k_N\}$ such that $k_1 < k_2 < \cdots < k_N$. The total momentum P and the energy eigenvalue E are expressed in terms of the Bethe roots $\{k\}_N$ as

$$P = \sum_{j=1}^{N}k_j = \frac{2\pi}{L}\sum_{j=1}^{N}I_j, \quad E = \sum_{j=1}^{N}k_j^2. \tag{5}$$

The set of Bethe quantum numbers corresponding to the ground state is given by

$$\{I\}_{N,0} := \left\{-\frac{N-1}{2} + j \middle| 0 \leqslant j \leqslant N-1\right\}. \tag{6}$$

The type-II excitations are defined as one-hole excitations, the Bethe quantum numbers of which are given by

$$\{I\}_{N,h} := \left\{-\frac{N-1}{2} + j \middle| 0 \leqslant j \leqslant N\right\} \setminus \left\{\frac{N+1}{2} - h\right\}, \tag{7}$$

where h $(0 \leqslant h \leqslant N-1)$ denotes the position of a hole. The total momentum of $\{I\}_{N,h}$ is $P_h = \frac{2\pi h}{L}$.

Construction of a quantum dark multi-soliton state. Before going into details, we briefly explain our strategy to construct a quantum dark multi-soliton state. Our approach is inspired by the analogy of the construction of classical soliton solutions. Generally, classical n-soliton solutions are constructed from n vertex operators [2]. We can regard each vertex operator as a unit corresponding to a soliton, and a multi-soliton solution is constructed by combining them. As an analogy, we consider a set of type-II excitations as a unit corresponding to a quantum soliton, and by combining them, we construct a quantum dark multi-soliton state.

The construction is twofold. First, we extend the type-II excitation (7) by introducing the "shift parameter." This has two aims. (i) By introducing this parameter, the type-II excitations can be treated as a unit, as we explain later. (ii) This parameter also controls the velocity of each soliton, by which we can realize the collisons of quantum dark solitons. Second, we define a new family of sets of Bethe quantum numbers by considering pairs of units (the sets of the extended type-II excitations). A multi-soliton state is constructed from the superposition of the eigenstates corresponding to these Bethe quantum numbers.

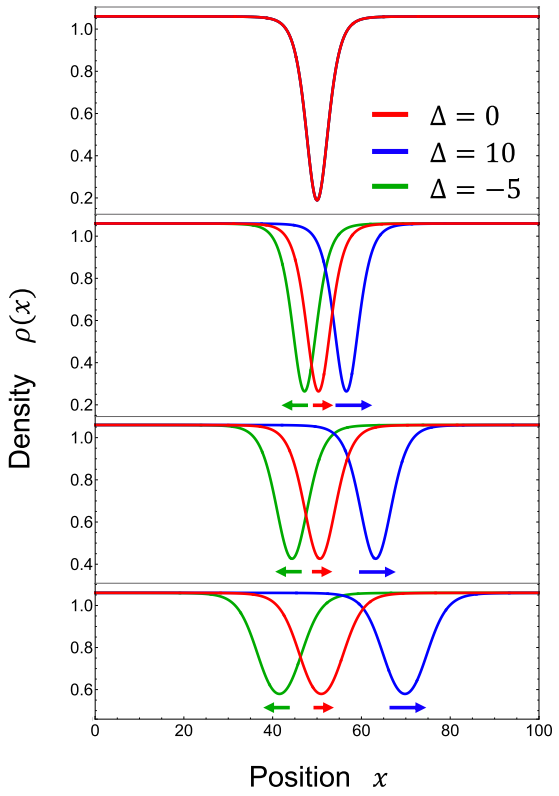


FIG. 1. Time evolution of the density profile in the one-soliton states for $\Delta = 0$, 10, and -5. We set $L = N = 100$, $X = \frac{L}{2}$, and $c = 0.1$.

The type-II excitation (7) is extended as follows. We define a Δ-shifted type-II excitation by introducing the shift parameter $\Delta(\in \mathbb{Z})$ as

$$\{I\}_{N,h,\Delta} := \{I\}_{N,h} + \Delta, \tag{8}$$

where h $(0 \leqslant h \leqslant N-1)$ indicates the hole position. The total momentum of this state is equal to $P_h = \frac{2\pi(h+N\Delta)}{L}$. Then, we define a family of sets of Δ-shifted type-II excitations with particle number N as

$$\mathcal{I}_{N,\Delta} := \{\{I\}_{N,h,\Delta} | 0 \leqslant h \leqslant N-1\}. \tag{9}$$

A quantum one-soliton state is constructed by superposing all eigenstates in $\mathcal{I}_{N,\Delta}$ as follows:

$$|X\rangle = \frac{1}{\sqrt{N}}\sum_{h=0}^{N-1}\exp(-iP_hQ)|P_h\rangle, \tag{10}$$

where $|P_h\rangle$ is the normalized Bethe eigenstate of the Δ-shifted type-II excitation (i.e., $\{I\}_{N,h,\Delta}$), and Q is a parameter that determines the position of the soliton (denoted by X) as $X = Q + \frac{L}{2}$. This is an extension of the quantum soliton states proposed in Refs. [7,8] with a parameter Δ. When $\Delta = 0$, Eq. (10) coincides with the conventional one. As we

show later (Fig. 1), the initial density profile of a one-soliton state does not depend on Δ. Thus, our extension preserves the property of the conventional soliton states that the initial density profile corresponds to a classical dark one-soliton for the periodic boundary condition in the weak coupling limit $c \to 0$ [7,8].

Considering the family of sets of Δ-shifted type-II excitations, $\mathcal{I}_{N,\Delta}$, as a unit, we construct the multi-soliton states. As shown in Eq. (10), a one-soliton state is constructed by superposing all eigenstates in $\mathcal{I}_{N,\Delta}$. In the following, we define a "connected type-II excitation" by a union of n sets of type-II excitations, $\{I\}_{N_j,h_j,\Delta_j} \in \mathcal{I}_{N_j,\Delta_j}$ $(1 \leqslant j \leqslant n)$. A quantum n-soliton state is constructed by superposing all eigenstates corresponding to the connected type-II excitation.

First, we divide N particles into n parts as $N = \sum_{j=1}^{n} N_j$. For simplicity, we assume that particle numbers N and N_j $(1 \leqslant j \leqslant n)$ are all even. Then, we consider the Δ-shifted type-II excitation $\mathcal{I}_{N_j,\Delta_j}$ $(1 \leqslant j \leqslant n)$. Here, we need to select Δ_j such that the Bethe quantum numbers do not overlap, i.e., $\{I\}_{N_j,h_j,\Delta_j} \cap \{I\}_{N_k,h_k,\Delta_k} = \emptyset$ for $1 \leqslant \forall j < \forall k \leqslant n$ and $0 \leqslant \forall h_j$(respectively, $\forall h_k$) $\leqslant N_j - 1$(respectively, $N_k - 1$). This is equivalent to the following condition:

$$|\Delta_j - \Delta_k| > \frac{N_j + N_k}{2} \quad \text{for} \quad 1 \leqslant \forall j < \forall k \leqslant n. \tag{11}$$

We define a connected type-II excitation $\{I\}_{\boldsymbol{h}}$ as follows:

$$\{I\}_{\boldsymbol{h}} := \bigcup_{j=1}^{n} \{I\}_{N_j,h_j,\Delta_j}. \tag{12}$$

A connected type-II excitation $\{I\}_{\boldsymbol{h}}$ is specified by the positions of holes $\boldsymbol{h} = (h_1, h_2, \ldots, h_n) \in H$, where we introduce a set of hole positions as the direct product $H = \prod_{j=1}^{n} \{0, 1, \ldots, N_j - 1\}$. We define a family of sets of connected type-II excitations as

$$\mathcal{I}_{\mathrm{C}} := \{\{I\}_{\boldsymbol{h}} | \boldsymbol{h} \in H\}. \tag{13}$$

$\mathcal{I}_{\mathrm{C}}$ has $C = \prod_{j=1}^{n} N_j$ elements.

A quantum n-soliton state is constructed by superposing all eigenstates in $\mathcal{I}_{\mathrm{C}}$. The total momentum of $\{I\}_{\boldsymbol{h}} \in \mathcal{I}_{\mathrm{C}}$ is given by $P_{\boldsymbol{h}} = \sum_{j=1}^{n} P_{h_j}$, where $P_{h_j} = \frac{2\pi(h_j + N_j\Delta_j)}{L}$ $(0 \leqslant h_j \leqslant N_j - 1)$. We denote the normalized Bethe eigenstate corresponding to the connected type-II excitation $\{I\}_{\boldsymbol{h}}$ as $|P_{h_1}, P_{h_2}, \ldots, P_{h_n}\rangle$. We introduce the n-soliton state $|X_1, X_2, \ldots, X_n\rangle$ as follows:

$$|X_1, X_2, \ldots, X_n\rangle = \frac{1}{\sqrt{C}} \sum_{\boldsymbol{h} \in H} \exp\left[-i \sum_{j=1}^{n} P_{h_j} Q_j\right] |P_{h_1}, P_{h_2}, \ldots, P_{h_n}\rangle, \tag{14}$$

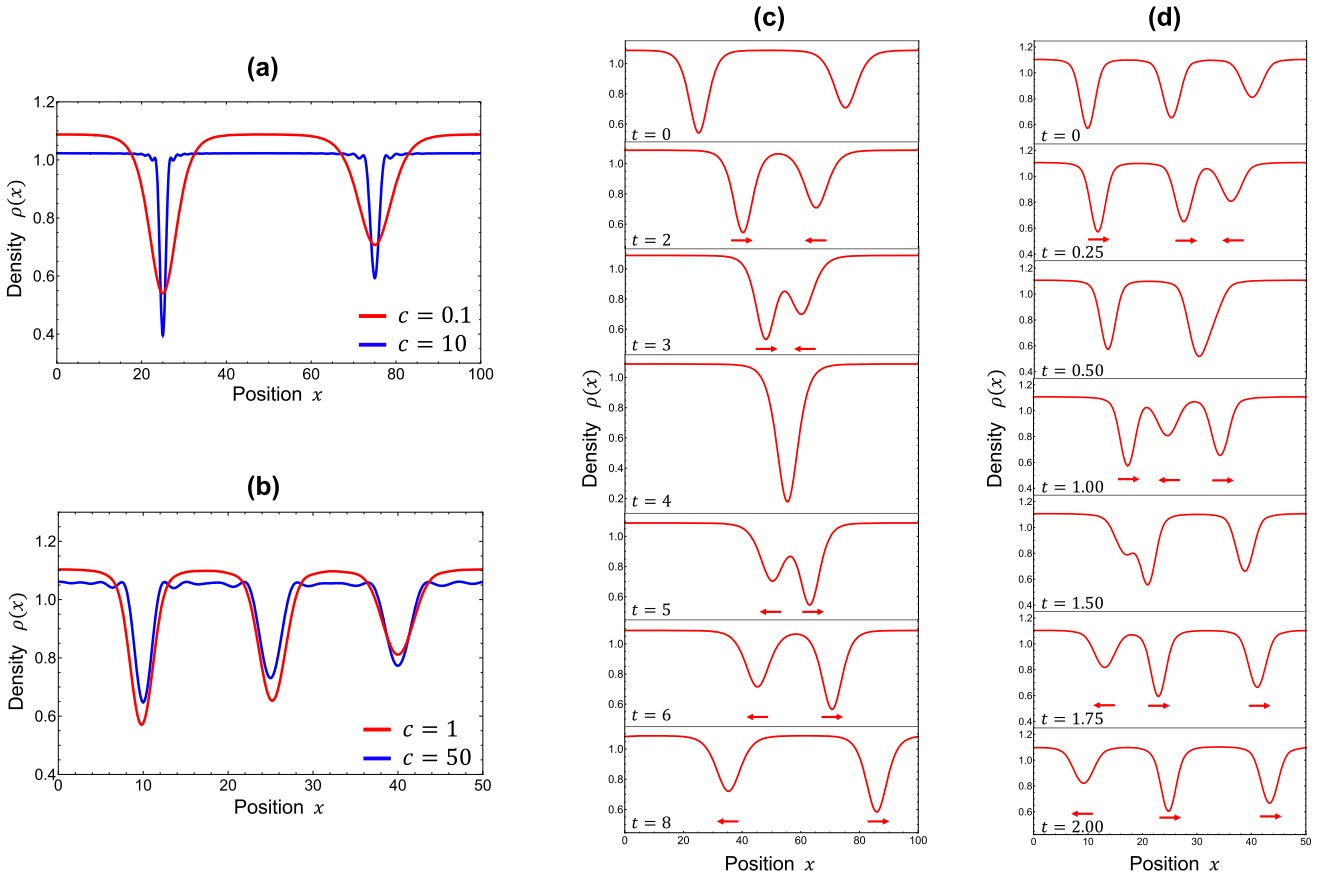


FIG. 2. Initial density profiles and collision dynamics of the multi-soliton states. (a) Initial density profiles of the two-soliton states for $c = 0.1$ and 10. (b) Initial density profiles of the three-soliton states for $c = 1$ and 50. (c) Time evolution of the density profile of the two-soliton state for $c = 0.1$. (d) Time evolution of the density profile of the three-soliton state for $c = 1$. We set $L = N = 100$, $N_1 = 60$, $N_2 = 40$, $\Delta_1 = 80$, $\Delta_2 = -70$, $X_1 = \frac{L}{4}$, and $X_2 = \frac{3L}{4}$ for the two-soliton states, and we set $L = N = 50$, $N_1 = 16$, $N_2 = 20$, $N_3 = 14$, $\Delta_1 = 50$, $\Delta_2 = 30$, $\Delta_3 = -80$, $X_1 = \frac{L}{2}$, $X_2 = \frac{L}{5}$, and $X_3 = \frac{4L}{5}$ for the three-soliton states.

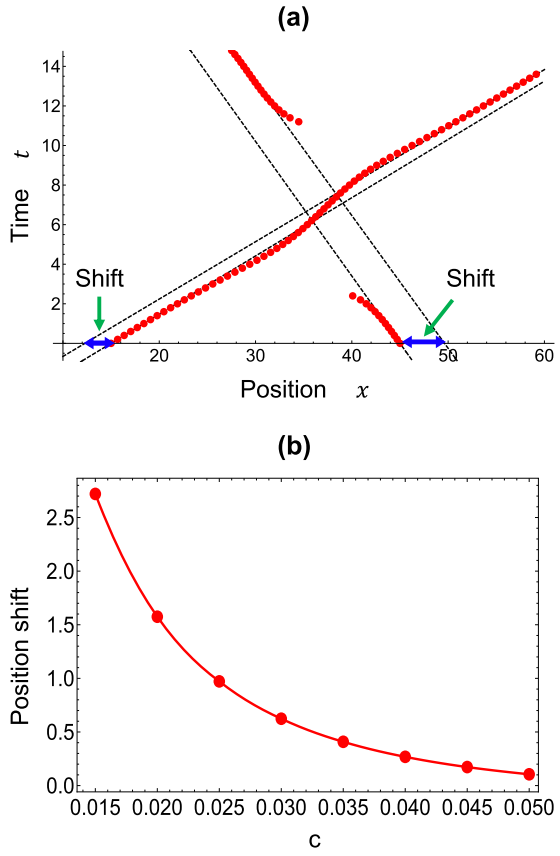


FIG. 3. Nature of the collisions between quantum solitons. We set $L = N = 60$, $N_1 = 20$, $N_2 = 40$, $\Delta_1 = -26$, $\Delta_2 = 25$, $X_1 = \frac{3L}{4}$, and $X_2 = \frac{L}{4}$. (a) Trajectories of two colliding quantum solitons for $c = 0.015$. The red dots show the positions of the local minimums in the density profile. The gray dashed lines represent the linear uniform motion extrapolated from the trajectories of the solitons before and after the collision. (b) Relationship between the position shifts of the quantum solitons and the coupling constant c. The red dots show the position shifts of the quantum soliton corresponding to $\mathcal{I}_{N_2=40,\Delta_2=25}$.

where Q_j's are parameters that determine the position of the jth soliton as $X_j = Q_j + \frac{L}{2}$. In this way, we constructed a multi-soliton state. The dark n-soliton emerges in the density profile of this state [Figs. 2(a)–2(b)]. Each $\mathcal{I}_{N_j,\Delta_j}$ corresponds to the jth soliton.

The time evolution of the density profile can be obtained as follows. We denote the n-soliton state at time t by $|X_1, X_2, \ldots, X_n; t\rangle = \exp(-i\hat{\mathcal{H}}_{\mathrm{LL}}t)|X_1, X_2, \ldots, X_n\rangle$. The expectation value of the density operator $\hat{\rho}(x) = \hat{\psi}^\dagger(x)\hat{\psi}(x)$ in this state is given by

$$\begin{aligned}&\langle X_1, X_2, \ldots, X_n; t|\hat{\rho}(x)|X_1, X_2, \ldots, X_n; t\rangle \\ &= \frac{1}{C}\sum_{\boldsymbol{h},\boldsymbol{h}'\in H}\exp\left[i\sum_{j=1}^{n}(P_{h'_j} - P_{h_j})Q_j\right. \\ &\quad \left. - i(P_{\boldsymbol{h}'} - P_{\boldsymbol{h}})x + i(E_{\boldsymbol{h}'} - E_{\boldsymbol{h}})t\right] \\ &\quad \langle P_{h'_1}, P_{h'_2}, \ldots, P_{h'_n}|\hat{\rho}(0)|P_{h_1}, P_{h_2}, \ldots, P_{h_n}\rangle,\end{aligned} \tag{15}$$

where $E_{\boldsymbol{h}}$ denotes the energy eigenvalue of the connected type-II excitation $|P_{h_1}, P_{h_2}, \ldots, P_{h_n}\rangle$.

Numerical experiments. We numerically confirm that the proposed solutions do possess the expected properties as a quantum dark n-soliton state. In numerical calculations for the density profiles (15), we use the determinant formulas [42–46] to efficiently evaluate the form factors:

$$\begin{aligned}\langle P'|\hat{\rho}(0)|P\rangle &= (-1)^{\frac{N(N+1)}{2}}(P - P')\left(\prod_{j,l=1}^{N}\frac{1}{k'_j - k_l}\right) \\ &\left(\prod_{j>l}^{N}k_{j,l}k'_{j,l}\sqrt{\frac{\hat{K}(k'_{j,l})}{\hat{K}(k_{j,l})}}\right)\frac{\det U(k,k')}{\sqrt{\det G(k)\det G(k')}},\end{aligned} \tag{16}$$

where $|P\rangle$ is the eigenstate with the total momentum P corresponding to the Bethe roots $\{k_1, k_2, \ldots, k_N\}$. We define $k_{j,l} := k_j - k_l$, and the kernel $\hat{K}(k)$ is given by

$$\hat{K}(k) = \frac{2c}{k^2 + c^2}. \tag{17}$$

The (j, l)th element of the Gaudin matrix $G(k)$ is given by

$$\begin{aligned}G(k)_{j,l} &= \delta_{jl}\left(L + \sum_{m=1}^{N}\hat{K}(k_{j,m})\right) - \hat{K}(k_{j,l}) \\ &\text{for} \quad j, l = 1, 2, \ldots, N,\end{aligned} \tag{18}$$

and that of the $(N-1)\times(N-1)$ matrix $U(k,k')$ is expressed as

$$\begin{aligned}U(k,k')_{j,l} &= 2\delta_{jl}\mathrm{Im}\left(\prod_{a=1}^{N}\frac{k'_a - k_j + ic}{k_a - k_j + ic}\right) \\ &+ \frac{\prod_{a=1}^{N}(k'_a - k_j)}{\prod_{a\neq j}^{N}(k_a - k_j)}\left[\hat{K}(k_{j,l}) - \hat{K}(k_{N,l})\right].\end{aligned} \tag{19}$$

First, we examine the extended one-soliton state, Eq. (10). Figure 1 shows the time evolution of the density profile in the quantum one-soliton states (10) when we set $\Delta = 0$, 10, and -5. The solitary wave that corresponds to a dark soliton is observed in the density profiles. The initial density profiles are identical regardless of the value of Δ. However, the velocity of the local minimum of the density profile depends on the value of Δ. Thus, the velocity of the quantum one-soliton can be controlled by the shift parameter Δ. Note that wave packets are dispersed by quantum effects (unlike classical solitons), which is a property common to quantum solitons.

Then, we show examples for the initial density profiles of two- and three-soliton states in Figs. 2(a) and 2(b). The solitary waves that corresponds to dark two- and three-soliton are observed in the density profiles. The shape of the jth soliton depended on the particle number N_j of the jth component $\mathcal{I}_{N_j,\Delta_j}$. The depth of the jth soliton increases as the particle number N_j increases. As in the case of quantum one-solitons, the shape of the solitons also depends on the value of c. It becomes sharper as c increases.

The time evolutions of the density profiles in the two- and three-soliton states (15) are shown in Figs. 2(c) and 2(d). The collisions of the quantum dark solitons were observed. Each solitary wave was stable in terms of its shape

and velocity against the collisions. Even when more than two solitary waves existed [Fig. 2(d)], these properties were preserved. In addition, we confirmed that the scattering was accompanied by the shifts in positions [Fig. 3(a)]. The amount of positional shifts monotonically depended on the coupling constant c [Fig. 3(b)]. As c increased, the shift asymptotically approached zero. This is consistent with the conjecture proposed in Ref. [8] that the density profile of the quantum dark soliton state coincides with that of free fermions in the strong coupling limit $c \to \infty$. Taken collectively, our multi-soliton states possess the expected dynamical properties.

Conclusion. In this Letter, we established the construction of the quantum dark multi-soliton states with the stable collision properties that are expected from the correspondence to classical soliton theory. The key was to consider the family of type-II excitations as a unit by introducing the shift parameter Δ. Thereby, as the construction of a classical soliton solution, a multi-soliton state can be constructed from the union of units. Moreover, the shift parameter Δ enables us to control the velocity of a quantum soliton. This makes it possible to realize the collison of quantum solitions in the quantum regime. The stable scattering property is not trivial. In fact, we can also construct a soliton-like state that is unstable against a collision from a superposition of the Bethe eigenstates. To clarify the condition for having this property is an important task for the future.

In the present work, we consider the construction of multi-solitons in the LL model. Given the fact that classical solitons universally appear in classical systems, we may expect such ubiquitousness also in quantum solitons. In future works, our approach could be extended to general quantum integrable systems, which will lead to a general theoretical formulation of quantum solitons.

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