

Fracton topological order at finite temperature

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As new kinds of stabilizer code models, fracton models have been promising in realizing quantum memory or quantum hard drives. However, it has been shown that the fracton topological order of 3D fracton models occurs only at zero temperature. In this Letter, we show that higher dimensional fracton models can support a fracton topological order below a nonzero critical temperature T_c . Focusing on a typical four-dimensional (4D) X-cube model, we show that there is a finite critical temperature T_c by analyzing its free energy from duality. We also obtained the expectation value of the 't Hooft loops in the 4D X-cube model, which directly shows a confinement-deconfinement phase transition at finite temperature. This finite-temperature phase transition can be understood as spontaneously breaking the $\mathbb{Z}_2$ one-form subsystem symmetry. Moreover, we propose an alternative no-go theorem for finite-temperature quantum fracton topological order.

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Introduction. Reliable quantum information in realistic quantum computers requires the ability of error correction [1–4]. A theoretical proposal is the self-correcting quantum memory (SCQM) [5–9], whose memory time are expected to be made arbitrarily long with the increase of the system size even at finite temperature [7,10,11]. Promising candidates for SCQM are the stabilizer code models with topological order [5,12–14]. They have degenerate ground states which cannot be distinguished by any local operators, and are thus robust quantum memory at zero temperature [13,15,16]. However, topological order is required to exist at a finite temperature to be SCQM [17], whose expectation values of Wilson and 't Hooft loops are both nonzero. But there are no-go theorems excluding two-dimensional (2D) and 3D stabilizer code with scale and translation symmetries (STS models) to have finite temperature quantum topological order [17,18]. These models include the 2D and 3D toric code models, and only the toric code model in 4D has finite temperature quantum topological order, thus a candidate for SCQM [5]. What is more, if only one kind of operators have nonzero expectation value at finite temperature, we may define this order as finite-temperature classical topological order [19,20], such as 3D toric code model, which can be viewed as candidates for self-correcting classical memory. Recently, there are proposals of SCQM using 2D symmetry enriched topological order on the edge of 3D systems, if 1-form symmetries are enforced in the Hamiltonian and the dynamics [10,21].

Fracton models [22–32] are new kinds of stabilizer code models proposed recently, which are beyond the above no-go theorems. The 3D X-cube model, which is a prototypical fracton model, has very different properties with conventional topological order. Its ground state degeneracy on the torus grows exponentially with the linear system size, while that of conventional topological order is a constant and independent of the system size. This is deeply rooted in the foliated structure of the fracton models [33–36]. Thus, this kind of model is expected to encode substantially more quantum information compared with conventional topologically ordered models [11,24,25,37]. It is then highly desired to search for fracton models which can be employed as SCQM. For now, studies of possible finite-temperature fracton topological order (FTFTO) mainly focus on 3D fracton models, including the 3D X-cube model and the Haah's code. Although the Haah code and Chamon's models are proposed to be partially self-correcting in finite system size [7,11,25,38], their topological order are thermally fragile in the thermodynamic limit [39–41]. It is then natural to ask the following question: whether there is an exactly solvable model with classical/quantum FTFTO in higher dimensions? Meanwhile, the interesting physics of the 3D X-cube model also motivates the study of higher dimensional generalizations. The anyon excitations, different from conventional topological order, have restricted motion directions without the input of energy. For example, the fracton excitations are totally immobile, which is due to the dipole conservation and captured by their exotic effective tensor gauge theories [34,42–54].

In this Letter, we investigate a series of higher dimensional fracton models with classical FTFTO. We first focus on a 4D generalization of X-cube model with string-like excitations in one sector, which is similar to the 3D toric code model [55]. As a result, this part of partition function is regular near zero temperature which implies there is no phase transition at zero temperature in this sector. We show this part of partition

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$$H_{4DX\text{-cube}} = \sum_{hc} \left(\text{hypercube term} \right) + \sum_l \left(\text{link term} + \text{link term} + \text{link term} \right)$$

FIG. 1. The schematic representation of the 4D X-cube model in a pictorial language. On each plaquette there is a spin-1/2 degree of freedom. The red part is the hypercube terms, each of which is a product of σ^z 's on the 24 plaquettes of a hypercube, and the blue parts are the link terms, each of which is a product of σ^x 's on the four plaquettes sharing the link l .

function is partially dual to that of 3D toric code model. The other partition function sector is dual to Ising chain with zero temperature phase transition. These results together imply a classical FTFTO. Further, we show the physics can be understood by the generalized Elitzur's theorem which prohibits the spontaneous symmetry breaking (SSB) of higher-form subsystem symmetry. The above discussions can be generalized to a new no-go theorem excluding quantum FTFTO in fracton models with one-form subsystem symmetry.

The 4D x-cube model. The 4D X-cube model on a cubic lattice with length L illustrated in Fig. 1 is defined as:

$$H = -J_A \sum_{hc} \mathcal{A}_{hc} - J_B \sum_{l\mu} \mathcal{B}_l^\mu, \quad (1)$$

where $\mathcal{A}_{hc}$ and $\mathcal{B}_l^\mu$ are referred to as the hypercube and link terms, respectively. The index l labels link [56] and μ represents one direction perpendicular to link l , where $\mu = x, y, z, w$ (w labeling the fourth direction in 4D). Both the hypercube and link terms are tensor product of Pauli matrices living on plaquettes of the hypercubic lattice:

$$\mathcal{A}_{hc} = \prod_{i \in hc} \sigma_i^z, \quad \mathcal{B}_l^\mu = \prod_{i \in \tilde{S}_l^\mu} \sigma_i^x, \quad (2)$$

where the set $\{\tilde{S}_l^\mu\}$ consists of four plaquettes which share the link l and are perpendicular to the surface S_l^μ , where S_l^μ is the plaquette expanded by the directions l and μ . It is clear that every term in the Hamiltonian commutes with each other:

$$[\mathcal{A}_{hc_1}, \mathcal{A}_{hc_2}] = 0, \quad [\mathcal{B}_{l_1}^{\mu_1}, \mathcal{B}_{l_2}^{\mu_2}] = 0, \quad [\mathcal{A}_{hc}, \mathcal{B}_l^\mu] = 0. \quad (3)$$

Consequently, the 4D X-cube model defined in Eq. (1) is exactly solvable. Hereafter we set $J_A = J_B = 1$ as energy unit and our results obtained below apply to generic values of J_A and J_B .

The effective field theory. Effective field theory plays an important role in understanding the physics of the lattice models with (fracton) topological order [49] and gauge structure [57], such as the 3D X-cube model, 3D toric code model, etc. All the eigenstates and gauge invariant operators of the lattice model can be constructed from the effective field theory. The effective field theory here is especially useful to construct gauge invariant Wilson/'t Hooft operators directly without referring to the complicated four dimensional spatial geometry. The nonzero expectation value of noncontractible Wilson and 't Hooft operators is required for the X-cube phase. The canonical coordinate and momenta of the field theory are

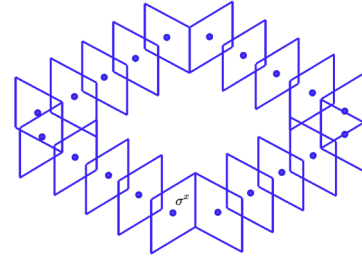


FIG. 2. The schematic representation of a contractible 't Hooft loop operator $T_\gamma = \prod_{r \in \gamma} \sigma^x(r)$, where γ labels the loop formed by the centers of plaquettes along the closed contour.

related to the lattice operators as:

$$\begin{aligned} \hat{Z}_i(t) &\sim \exp \left(i \int_S Z_{\alpha\beta}(\mathbf{x}, t) \right) = \exp \left(i \int_S A_{\gamma\delta} | \epsilon^{0\alpha\beta\gamma\delta} | \right), \\ \hat{X}_i(t) &\sim \exp \left(i \int_{\perp S} X_{\alpha\beta}(\mathbf{x}, t) \right) = \exp \left(i \int_{\perp S} B_{\gamma\delta} | \epsilon^{0\alpha\beta\gamma\delta} | \right), \end{aligned} \quad (4)$$

where the integration regime S is the plaquette of the lattice operators and $\perp S$ is its Poincare dual. The label $\alpha\beta$ are the spatial directions of the plaquette. The hypercube terms and link terms are conserved charges and generators of the gauge transformation of the field theory, which means the Lagrangian density is:

$$\begin{aligned} \mathcal{L}_{X\text{-cube}} &= \frac{1}{\pi} A_{\alpha\beta} \partial_0 B_{\alpha\beta} + B_0 \left(\frac{1}{\pi} \partial_\alpha \partial_\beta A_{\alpha\beta} - i^0 \right) \\ &\quad + A_{0;\alpha\beta} \left(\frac{1}{\pi} \epsilon^{0\alpha\beta\gamma\delta} \partial_\gamma B_{\delta} - j^{0;\alpha\beta} \right) \\ &\quad - A_{\alpha\beta} j^{\alpha\beta} - B_{\alpha\beta} i^{\alpha\beta}, \end{aligned} \quad (5)$$

where A and B are rank-2 tensor gauge fields, and the time and spatial components of i, j correspond to the excitation density and current configuration. The time components of the gauge fields are denoted as $A_{0;\alpha\beta}$ and B_0 , and the spatial components are denoted as $A_{\alpha\beta}$ and $B_{\alpha\beta}$. The gauge invariant contractible (noncontractible) Wilson/'t Hooft operators can be constructed from the field theory and are illustrated in Figs. 2 and 3, respectively [58].

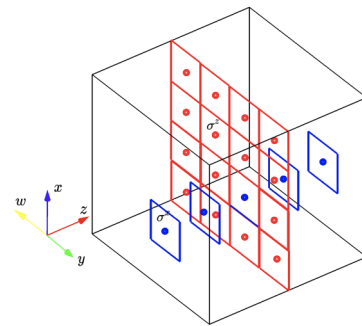


FIG. 3. The red and blue operators are noncontractible Wilson loop and 't Hooft loop operators, respectively.

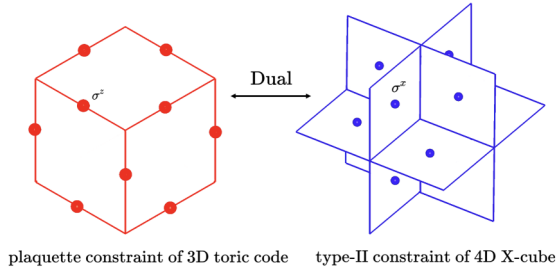


FIG. 4. An illustration of the duality between the type-II constraints in the 4D X-cube model and the plaquette constraints in the 3D toric code model.

The partition function duality. For the partition function of the 4D X-cube model $\mathcal{Z} = \text{Tr}(e^{-\beta H})$, the link part and hypercube part are decoupled due to the facts that $\mathcal{A}_{hc}$ and $\mathcal{B}_l^\mu$ are products of traceless matrices σ^z, σ^x and the product of $\mathcal{A}_{hc}$ and $\mathcal{B}_l^\mu$ is thus traceless. Consequently, these two parts can be calculated independently: $\mathcal{Z} = \frac{1}{d_h} \mathcal{Z}_{\text{hypercube}} \mathcal{Z}_{\text{link}}$, where $d_h = \text{Tr}[\mathbb{I}]$ is the dimension of the Hilbert space, $\mathcal{Z}_{\text{hypercube}} = \text{Tr}(e^{\beta \sum_{hc} \mathcal{A}_{hc}})$, $\mathcal{Z}_{\text{link}} = \text{Tr}(e^{\beta \sum_{l\mu} \mathcal{B}_l^\mu})$. Here we use the open boundary condition (OBC) [40,59].

Let us first consider the hypercube part $\mathcal{Z}_{\text{hypercube}} = \text{Tr}[\prod_{hc} (\mathbb{I} \cosh \beta + \mathcal{A}_{hc} \sinh \beta)]$ under OBC, where the only nonzero contribution is the multiplication of all the identity matrices in the series expansion. So we have $\mathcal{Z}_{\text{hypercube}} = d_h (\cosh \beta)^{L^4} \sim (e^\beta + e^{-\beta})^{L^4}$, where L^4 is the number of lattice sites with linear size L . Thus, $\mathcal{Z}_{\text{hypercube}}$ is dual to L^4 isolated spins in a magnetic field under OBC, which is in turn dual to an open Ising chain of length $L^4 + 1$.

Next, we consider the link part $\mathcal{Z}_{\text{link}} = \text{Tr} \prod_{l\mu} (\mathbb{I} \cosh \beta + \mathcal{B}_l^\mu \sinh \beta)$. Note that not all $\mathcal{B}_l^\mu$ are independent; namely there are local constraints for $\mathcal{B}_l^\mu$. The local constraints can be divided into two types. The first type (dubbed as type-I constraints) refers to the constraint that the product of the three link terms $\mathcal{B}_l^\mu$ sharing the same link l equals to identity: $\prod_{\mu} \mathcal{B}_l^\mu = \mathbb{I}$, which is a local constraint for each l . The type-I constraints render a contribution to the partition function: $\mathcal{Z}_{\text{link(type-I)}}$, which is equivalent to $4(L-1)^4$ independent three-spin clusters $\mathcal{Z}_{3\text{-spin Curie-Weiss}}^{4(L-1)^4}$.

The second type constraints (type-II constraints) have no counterparts in the 3D X-cube model. As shown in Fig. 4, the product of six link terms sharing the same vertex equal to identity: $\prod_l \mathcal{B}_l^\mu \mathcal{B}_{\bar{l}}^\mu = \mathbb{I}$, where link l and $\bar{l}$ have opposite directions. Written explicitly, one example of type-II constraints is $\mathcal{B}_x^w \mathcal{B}_y^w \mathcal{B}_z^w \mathcal{B}_x^w \mathcal{B}_y^w \mathcal{B}_z^w = \mathbb{I}$. The type-II contribution $\mathcal{Z}_{\text{link(type-II)}}$ is dual to $(\mathcal{Z}_{\mathcal{B}_p})^{4L}$, where $\mathcal{B}_p = \prod_{i \in \text{plaquette}(p)} \sigma_i^z$ is the plaquette term in 3D toric code model. This duality is rooted in the identification of the type-II constraints with the toric code constraints under the lattice duality: $\prod_{p \in \text{cube}} \mathcal{B}_p = \mathbb{I}$. If we dual the link in the toric code model into the perpendicular surface, then the constraint on the plaquette terms in the 3D toric code model is dual to the type-II constraint, which is shown in the Fig. 4. The duality of the type-II contribution is: $\mathcal{Z}_{\text{hypercube}} \times \mathcal{Z}_{\text{link(type-II)}} \sim (\mathcal{Z}_{3\text{DTC}})^{4L} / \mathcal{Z}_{1\text{D-Ising}}$. The detailed number counting of both types contribution is in the Supplemental Material (SM). Since the 3D toric code

features a finite-temperature transition, we had proven exactly that the 4D X-cube model also has a finite-temperature transition if type-I and type-II constraints are totally independent. Nonetheless, type-I and type-II constraints can coexist locally; for instance the type-I constraint $\mathcal{B}_w^x \mathcal{B}_w^y \mathcal{B}_w^z = 1$ and type-II constraint $\mathcal{B}_w^x \mathcal{B}_y^x \mathcal{B}_z^x \mathcal{B}_w^x \mathcal{B}_y^x \mathcal{B}_z^x$ share the link term $\mathcal{B}_w^x$ such that $\mathcal{Z}_{\text{link}}$ cannot be simply written as $\mathcal{Z}_{\text{link(type-I)}} \mathcal{Z}_{\text{link(type-II)}}$. Although it is challenging to find an exact duality of the partition function, we believe that a finite-temperature phase transition should survive due to type-II constraints. Indeed, in the following sections we can prove the existence of a finite-temperature transition in the 4D X-cube model by using low/high temperature expansion and effective field theory analysis.

Low temperature free energy expansion. In this section, we conduct the low-temperature free energy expansion [60,61] to analyze possible finite-temperature phase transitions of the 4D X-cube model. The decoupling of the partition function $\mathcal{Z} = \frac{1}{d_h} \mathcal{Z}_{\text{hypercube}} \mathcal{Z}_{\text{link}}$ means that the free energy can be decoupled as $F_{4\text{DX-cube}} = -\frac{1}{\beta} (\ln \mathcal{Z}_{\text{hypercube}} + \ln \mathcal{Z}_{\text{link}} - L^4 \ln 2) = F_{\text{hypercube}} + F_{\text{link}} + \frac{1}{\beta} L^4 \ln 2$. Since $F_{\text{hypercube}}$ is dual to a 1D Ising model which has no finite-temperature transition, we focus on F_{link} when considering possible finite-temperature phase transition. As we expect the gauge charges of the gauge field σ^x are deconfined at zero temperature and confined at high temperature, which can also be diagnosed using 't Hooft loops in the next section, there must be a (zero- or finite-temperature) phase transition in the link part F_{link} . Thus, if the absence of zero temperature phase transition can be proved, a finite-temperature phase transition must exist. Therefore, we perform a low-temperature series expansion of the free energy density to see whether nonanalytical behavior will emerge in the vicinity of zero temperature.

For excitations in the link part, the operator creating the lowest energy excitation is a σ^z on a specific plaquette, which flips eight link terms and the excitation energy is $16J_B$ [62]. Since there are $6L^4$ excitations of this kind, we expand $\mathcal{Z}_{\text{link}}$ in the vicinity of $T = 0$ as follows [63]:

$$\mathcal{Z}_{\text{link}}(\beta) = 2^{L^4} e^{-\beta E_0} (1 + 6L^4 e^{-16\beta} + \dots), \quad (6)$$

where $E_0 = -12L^4$ is the ground-state energy of the link Hamiltonian and $\dots$ represents contributions from higher excited states. The free-energy density (per site) $f_{\text{link}}(\beta) = -\frac{1}{\beta L^4} \ln \mathcal{Z}_{\text{link}}(\beta)$ is given as:

$$f_{\text{link}}(\beta) = -\frac{1}{\beta} (\ln 2 + 12\beta + 6e^{-16\beta} + \dots). \quad (7)$$

$\beta(f_{\text{link}} - f_{\text{link}}(T=0))$ is analytic with respect to $e^{-16\beta}$, up to the dominant terms in the low temperature regime, which indicates there is no zero temperature phase transition in the link sector.

The deconfinement-confinement transition. We now show that this finite-temperature phase transition discussed above is a deconfinement-confinement transition. This result can be seen from different behavior of the expectation value of a 't Hooft loop $T_\gamma = \prod_{r \in \gamma} \sigma^x(r)$ in high/low temperature

[64,65]:

$$\langle T_\gamma \rangle = \frac{1}{\mathcal{Z}_{\text{link}}} \text{Tr} \left[T_\gamma \prod_{l \in \mu} (\cosh \beta + \mathcal{B}_l^\mu \sinh \beta) \right], \quad (8)$$

where γ is formed by centers of the plaquettes along any closed contour as shown in Fig. 2.

In the high temperature regime $\beta \ll 1$, the leading contribution of the denominator $\mathcal{Z}_{\text{link}}$ is the product of all the $\cosh \beta$ terms. And the leading contribution to the numerator is the production of $\mathcal{B}_l^\mu \sinh \beta$ terms inside Σ , where Σ is the minimum surface with boundary γ , and $\cosh \beta$ terms elsewhere. Thus, we obtain $\langle T_\gamma \rangle$ as:

$$\langle T_\gamma \rangle \approx \tanh \beta^{S[\Sigma]} = \exp(-\ln(1/\tanh \beta)S[\Sigma]). \quad (9)$$

This is exactly the area law of the 't Hooft loop, which indicates that the phase at high temperature is confined.

In the low-temperature regime $\beta \gg 1$, we can assume the creation operators of lowest energy excitation in the Eq. (6) are well separated and approximately independent, which is known as the dilute limit. Thus, the low-temperature expansion of partition function is $\mathcal{Z}_{\text{link}} \approx 2^{L^4} e^{-\beta E_0} \sum_{n=0}^{\infty} C_{N_p}^n e^{-16\beta n}$, where $N_p = 6L^4$ is the number of total plaquettes. In the thermodynamic limit, the binomial coefficient can be approximated to $\frac{N_p^n}{n!}$. Thus the partition function is $\mathcal{Z}_{\text{link}} \approx 2^{L^4} e^{-\beta E_0} \sum_{n=0}^{\infty} \frac{N_p^n}{n!} e^{-16\beta n} = 2^{L^4} e^{-\beta E_0} e^{N_p e^{-16\beta}}$. As for the numerator, once a σ^z plaquette is located on the 't Hooft loop contour γ , the expectation value of the 't Hooft loop changes its sign. Thus, we obtain $\langle T_\gamma \rangle$ as:

$$\begin{aligned} \langle T_\gamma \rangle &\approx \frac{2^{L^4} \langle T_\gamma \rangle_{\text{gs}}}{\mathcal{Z}_{\text{link}}} e^{-\beta E_0} \sum_{n=0}^{\infty} \frac{(N_p - 2P)^n}{n!} e^{-16\beta n} \\ &= \langle T_\gamma \rangle_{\text{gs}} \frac{e^{-\beta E_0} e^{-16\beta(N_p - 2P)}}{e^{-\beta E_0} e^{-16\beta N_p}} = e^{-2e^{-16\beta} P}, \end{aligned} \quad (10)$$

where P is the length of the 't Hooft loop and $\langle T_\gamma \rangle_{\text{gs}}$ is the ground-state expectation value of T_γ that equals to one, which is consistent with the physical intuition that local operators cannot distinguish different ground states. The calculation of $\langle T_\gamma \rangle_{\text{gs}}$ is shown in the SM. Thus, the expectation value of the 't Hooft loop operator $\langle T_\gamma \rangle$ obeys the perimeter law in the low-temperature regime, which indicates a deconfined phase. Consequently, there must be a deconfinement-confinement phase transition at finite temperature for the 4D X-cube model.

Spontaneous symmetry breaking of $\mathbb{Z}_2$ n -form subsystem symmetry. Here we show that the finite-temperature phase transition of the 4D X-cube model can be physically understood as the SSB of $\mathbb{Z}_2$ 1-form subsystem symmetry. For an ordinary $\mathbb{Z}_2$ n -form global symmetry, the charged operators are n -dimensional objects [66], and are classified by $H^n(M, \mathbb{Z}_2)$ [67–70], where M is the spatial base manifold. Then a subsystem n -form d_s -dimensional symmetry (SNS) is defined as the union of all the n -form transformations in different d_s -dimensional subsystems of the original D -dimensional lattice with $n < d_s < D$ [71–74], and the number of generators depend on the lattice size. In this Letter, the associated subsystems are tori and the subgroup of a $\mathbb{Z}_2$ n -form d_s -dimensional subsystem symmetry in each torus $\mathbb{T}^{d_s}$ is classified by the

n -th cohomology group $H^n(\mathbb{T}^{d_s}, \mathbb{Z}_2)$. Physically, a $\mathbb{Z}_2$ n -form d_s -dimensional subsystem transformation is the same as the ordinary n -form transformation in each torus $\mathbb{T}^{d_s}$ subsystem and the generators are $d_s - n$ dimensional objects.

The 4D X-cube model has a 1-form subsystem symmetry and a 2-form subsystem symmetry in each three-dimensional subsystem (namely $d_s = 3$). They are generated by the noncontractible Wilson loops W and 't Hooft loops T , respectively, as illustrated in Fig. 3. The Wilson loops and 't Hooft loops are order parameters of each other when they intersect and anticommute. Due to the nature of gauge-like symmetry, the expectation value of the order parameters has the so-called dimensional reduction properties, and is summarized as a generalized Elitzur's theorem in Ref. [75]. For example, the expectation value $\langle W \rangle$, is bounded from above by the expectation value of σ_z on the plaquette, where the W and T intersect, computed in a 1D system. Here the 1D system consists of the spins acted by T and in this 1D system T now becomes a global 0-form $\mathbb{Z}_2$ symmetry. Due to the Mermin-Wagner theorem, $\langle W \rangle$ must be zero at any finite temperature, which implies that the T symmetry must be restored at any finite temperature. Similarly, the expectation value of T is bounded from above by that of an order parameter of a 2D system where W symmetry becomes a global 0-form $\mathbb{Z}_2$ symmetry and it is possible to be broken at low but finite temperature. This agrees with the result discussed in the previous sections where only the link terms can support a finite-temperature phase transition. Thus, below T_c the W symmetry is broken and the 4D X-cube model has finite-temperature classical fracton topological order [19,20]. The details of the subsystem and upper bound construction are left in the Generalized Elitzur's theorem section in the SM.

Finally, the finite temperature physics of the 4D X-cube model is totally different from the 4D toric code model. The 4D toric code model has two anticommuting global 2-form symmetries and both two symmetries are SSB at finite temperature. Since the thermal phase transition of the 4D X-cube model is due to the SSB of the 1-form subsystem symmetry, its universality class is different from that of the 4D toric code model. Finally, the lower critical dimension of the 2-form subsystem and ordinary global symmetries are different, as we will show in the no-go theorem in the subsequent section.

NO-GO theorem. Similar to the no-go theorem for 2D stabilizer code and 3D STS models [17,18], we can also argue a no-go theorem for quantum FTFTO, in 4D gapped fracton models with two anticommuting discrete SNS whose charges are n_1 and n_2 dimensional objects. For 4D gapped fracton models, (n_1, n_2) can only be $(1, 1)$ or $(2, 1)$ [76]; both cases have a subsystem symmetry with $n = 1$. Using the dimensional reduction approach, it can be shown that this symmetry must be restored at any finite temperature. As a result, 4D quantum fracton topological order is absent at finite temperature under the above assumption. Quantum FTFTO can be realized with two anticommuting 2-form subsystem symmetries, and this can only be realized in at least 5D. This result can be generalized to models in any spatial dimension as long as the fracton model has a subsystem symmetry whose generators are one dimensional objects, and the charge operators of each subsystem $\mathbb{T}^{d_s}$ belonging to the cohomology group $H^{d_s-1}(\mathbb{T}^{d_s}, \mathbb{Z}_m)$. As concrete examples,

we discuss a series of fracton models in general spatial dimension with subsystem-symmetry generators of each subsystem $\mathbb{T}^3 : (H^1(\mathbb{T}^3, \mathbb{Z}_2), H^2(\mathbb{T}^3, \mathbb{Z}_2))$.

Higher-dimensional fracton models. The above discussion on the 4D X-cube model can be generalized to a series of higher-dimensional models, summarized as a family tree [62,77]. The family tree consists of fracton models in spatial D dimensions which are labeled by four indices $[d_1, d_2, d_3, D]$. Here d_2 is the dimension of the cube where spins live on. The Hamiltonian contains two terms: one is defined on the D dimensional cube, and the other is defined on the d_1 dimensional cube labeled by γ_{d_1} . Besides, the index d_3 means the second term only includes the spins in the d_3 dimensional leaf space associated with a given γ_{d_1} . The partition function of the series $[0, 1, 2, D]$ is dual to 0D and 1D systems, which has only zero temperature phase transitions. The series $[1, 2, 3, D]$ has the same SNS as the 4D X-cube model. And the free energy and 't Hooft loop calculation shows there exists a finite temperature

phase transition in this series of models. The calculation of partition function, free energy, and 't Hooft loops is left in the SM.

Conclusions. In this Letter, we have shown the existence of finite-temperature classical fracton topological order in 4D X-cube model and, in general, a series of fracton models dubbed as $[1, 2, 3, D]$ ($D \geq 4$). We also argued a no-go theorem for quantum FTFTO given the SNS which only acts on one-dimensional regions. We believe that this provides an important first step towards quantum FTFTO and SCQM using fracton models.

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