

Quantum phases of self-bound droplets of Bose-Bose mixtures

Junqiao Pan^{1,2}, Su Yi^{1,2,3,4,*} and Tao Shi^{1,3,†}

¹CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

²School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

³CAS Center for Excellence in Topological Quantum Computation, University of Chinese Academy of Sciences, Beijing 100049, China

⁴Shenzhen Institute for Quantum Science and Engineering, Southern University of Science and Technology, Shenzhen 518055, China



(Received 5 March 2021; accepted 13 September 2022; published 10 October 2022)

We systematically investigate the ground-state properties of self-bound droplets of quasi-two-dimensional binary Bose gases by using the Gaussian state theory. We find that quantum droplets consist of two macroscopic squeezed phases and a macroscopic coherent phase. We map out the phase diagram and determine all phase boundaries via both numerical and nearly analytical methods. In particular, we find three easily accessible signatures for the quantum phases and the stabilization mechanism of the self-bound droplets by precisely measuring their radial size. Our studies indicate that binary droplets represent an ideal platform for in-depth investigations of the quantum nature of the droplet state.

DOI: [10.1103/PhysRevResearch.4.043018](https://doi.org/10.1103/PhysRevResearch.4.043018)

I. INTRODUCTION

Quantum droplets in both dipolar [1–4] and binary condensates [5–9] represent a new state of matter emerging in the mean-field unstable regime. They have attracted great interests in studying their fascinating properties, such as self-bound feature [2,3,5,7], collective excitations [10,11], soliton to droplet transition [6], droplet-droplet collision [12], supersolid states [13–15], Goldstone mode [16], and so forth (see recent reviews [17–19] and references therein). A widely adopted treatment for droplet states is the extended Gross-Pitaevskii equation (EGPE) which perturbatively incorporates quantum fluctuation into the Gross-Pitaevskii equation in terms of the Lee-Huang-Yang (LHY) correction [20–24]. Although EGPE has provided satisfactory explanations to experimental observations, there are still discrepancies with experimental measurements [4,5,18].

Theoretical methods beyond EGPE have also been employed to study different aspects of the droplet states [25–34]. Among them, we studied the dipolar droplets using the Gaussian-state theory (GST) in which quantum fluctuation is included in a self-consistent manner [35]. Particularly, GST circumvents the difficulty in the Hartree-Fock-Bogoliubov theory (HFBT) by producing the gapless excitation spectrum of condensates [36–38]. We showed that, instead of the quantum fluctuation, dipolar droplets were stabilized by three-body repulsion. In addition, we predicted that there exist

two new quantum states of atoms, i.e., the squeezed-vacuum state (SVS) and the squeezed-coherent state (SCS). Compared to the conventional coherent state (CS), these macroscopic squeezed states were characterized by large second-order correlation and asymmetric atom-number distribution (AND), in striking contrast with those of the coherent-state-based droplets [35,36]. Whereas the asymmetric AND was experimentally observed [2,4], verifying the predication on the second-order correlation function requires further experiments beyond simple density measurements.

In the present work, we study the ground state properties of the self-bound droplets in quasi-two-dimensional (quasi-2D) binary condensates. Same as those in its dipolar counterpart, there also exist the macroscopic SVS, SCS, and CS phases. More importantly, we discover three readily accessible signatures on the radial size (σ) versus atom number (N) curve that can be used to identify the quantum states and the stabilization mechanisms. Specifically, due to the multiple quantum phases containing in the droplet states, the σ - N curve is of W shape, as opposed to the V-shaped curve expected for single quantum phase. Therefore the observation of the W-shaped σ - N curve can rule out the pure coherent explanation of the droplet state. Moreover, the *critical atom number* (CAN), i.e., the minimal number of atoms required to form a self-bound state, is determined by the quantum states of the condensate and its precise measurement of the CAN allows us to distinguish squeezed state from coherent one. Finally, the *dip atom number* (DAN), i.e., the atom number at the dip of the σ - N curve, depends on both quantum phase and stabilization mechanism, which makes it a quantitative criterion for stability mechanism.

The rest of this paper is organized as follows. In Sec. II, we introduce our model and GST. In Sec. III, after determining the 3B coupling constant through fitting the experimental data, we present the results on the ground-state properties of a self-bound binary droplet, including the quantum phases

*syi@itp.ac.cn

†tshi@itp.ac.cn

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

and density profiles. The phase boundaries are determined nearly analytically. We also derive a set of simplified effective equations to describe these quantum phases. In Sec. IV, we propose to experimentally detect these quantum phases by measuring AND, the second-order correlation function, and the radial size of the droplets. Finally, we conclude in Sec. V.

II. FORMULATION

A. Model

We consider a ultracold gas of N ^{39}K atoms with $N_\uparrow$ atoms being in the hyperfine state $|\uparrow\rangle \equiv |F, m_F\rangle = |1, -1\rangle$ and $N_\downarrow$ in $|\downarrow\rangle \equiv |1, 0\rangle$. The total Hamiltonian of the system,

$$\begin{aligned} \hat{H} = & \sum_{\alpha} \int d\mathbf{r} \hat{\psi}_{\alpha}^{\dagger}(\mathbf{r}) \left(\frac{-\hbar^2 \nabla^2}{2M} + \frac{1}{2} M \omega_z^2 z^2 - \mu_{\alpha} \right) \hat{\psi}_{\alpha}(\mathbf{r}) \\ & + \sum_{\alpha\beta} \frac{g_{\alpha\beta}}{2} \int d\mathbf{r} \hat{\psi}_{\alpha}^{\dagger}(\mathbf{r}) \hat{\psi}_{\beta}^{\dagger}(\mathbf{r}) \hat{\psi}_{\beta}(\mathbf{r}) \hat{\psi}_{\alpha}(\mathbf{r}) \\ & + \frac{g_3}{3!} \sum_{\alpha\beta\gamma} \int d\mathbf{r} \hat{\psi}_{\alpha}^{\dagger}(\mathbf{r}) \hat{\psi}_{\beta}^{\dagger}(\mathbf{r}) \hat{\psi}_{\gamma}^{\dagger}(\mathbf{r}) \hat{\psi}_{\gamma}(\mathbf{r}) \hat{\psi}_{\beta}(\mathbf{r}) \hat{\psi}_{\alpha}(\mathbf{r}), \end{aligned} \quad (1)$$

where $\hat{\psi}_{\alpha}(\mathbf{r})$ ($\alpha = \uparrow, \downarrow$) are the field operators, M is the mass of an atom, ω_z is the frequency of the harmonic confinement along the z direction, and μ_{α} the chemical potential introduced to fixed the atom number in the α th spin component. The second line of Eq. (1) accounts for the two-body (2B) interaction with $g_{\alpha\beta} = 4\pi\hbar^2 a_{\alpha\beta}/M$ and $a_{\alpha\beta}$ being the scattering length between the spin components α and β . The scattering lengths are tunable through Feshbach resonance and for scenario of interest to quantum droplets, the intra- and inter-species scattering lengths satisfy $a_{\uparrow\uparrow}, a_{\downarrow\downarrow} > 0$ and $a_{\uparrow\downarrow} < 0$, respectively. Finally, the last line of Eq. (1) represents the three-body (3B) interaction with g_3 being the 3B coupling constant. For simplicity, g_3 is assumed to be independent of the spin species. In addition, since we are only interested in the ground states of the system, it is assumed that g_3 is real and positive. The value of g_3 shall be determined by fitting the experimental data.

It should be noted that the three-body interaction is not phenomenologically added in Hamiltonian Eq. (1). In fact, this many-body Hamiltonian are derived within an effective low-energy theory, which, after integrating out the high energy excitations, gives rise to interaction terms involving two particles, three particles, and so on. Therefore, for the low-energy physics of a many-body system, the three-body interaction should be considered in systems whose density is sufficiently large.

Following the experiment, we shall focus on the quasi-2D geometry which can be realized when ω_z is sufficiently large such that the motion of atoms along the z axis is frozen to the ground state of the harmonic oscillator. As a result, the field operator can be factorized into

$$\hat{\psi}_{\alpha}(\mathbf{r}) = \hat{\psi}_{\alpha}(\boldsymbol{\rho}) \frac{e^{-z^2/(2a_z^2)}}{(\pi a_z^2)^{1/4}}, \quad (2)$$

where $\boldsymbol{\rho} \equiv (x, y)$ and $a_z = \sqrt{\hbar/(M\omega_z)}$ is the harmonic oscillator length. After integrating out the z variable, Hamiltonian

(1) reduces to

$$\begin{aligned} \hat{H} = & \sum_{\alpha} \int d\boldsymbol{\rho} \hat{\psi}_{\alpha}^{\dagger}(\boldsymbol{\rho}) \hat{h}_{\alpha} \hat{\psi}_{\alpha}(\boldsymbol{\rho}) \\ & + \sum_{\alpha\beta} \frac{g'_{\alpha\beta}}{2} \int d\boldsymbol{\rho} \hat{\psi}_{\alpha}^{\dagger}(\boldsymbol{\rho}) \hat{\psi}_{\beta}^{\dagger}(\boldsymbol{\rho}) \hat{\psi}_{\beta}(\boldsymbol{\rho}) \hat{\psi}_{\alpha}(\boldsymbol{\rho}) \\ & + \frac{g'_3}{3!} \sum_{\alpha\beta\gamma} \int d\boldsymbol{\rho} \hat{\psi}_{\alpha}^{\dagger}(\boldsymbol{\rho}) \hat{\psi}_{\beta}^{\dagger}(\boldsymbol{\rho}) \hat{\psi}_{\gamma}^{\dagger}(\boldsymbol{\rho}) \hat{\psi}_{\gamma}(\boldsymbol{\rho}) \hat{\psi}_{\beta}(\boldsymbol{\rho}) \hat{\psi}_{\alpha}(\boldsymbol{\rho}), \end{aligned} \quad (3)$$

where $\hat{h}_{\alpha} = -\hbar^2 \nabla_{\boldsymbol{\rho}}^2 / (2M) - \mu_{\alpha}$ with an unimportant constant being absorbed into the chemical potential and $g'_{\alpha\beta} = g_{\alpha\beta} / (\sqrt{2\pi} a_z)$ and $g'_3 = g_3 / (\sqrt{3\pi} a_z^2)$ are, respectively, the reduced 2B and 3B interaction strengths in the quasi-2D geometry.

B. Gaussian variational wave functions for Bose gases

We shall tackle the quantum droplets of Bose-Bose mixture using GST. Specifically, we assume that the many-body wave function of a binary gas adopts the variational ansatz [36,37,39]

$$\begin{aligned} |\Psi_{\text{GS}}\rangle = & \exp \left[\int d\boldsymbol{\rho} d\boldsymbol{\rho}' \hat{\Psi}^{\dagger}(\boldsymbol{\rho}) \Sigma^z(\boldsymbol{\rho}, \boldsymbol{\rho}') \Phi(\boldsymbol{\rho}') \right] \\ & \times \exp \left[\frac{i}{2} \int d\boldsymbol{\rho} d\boldsymbol{\rho}' \hat{\Psi}^{\dagger}(\boldsymbol{\rho}) \Xi(\boldsymbol{\rho}, \boldsymbol{\rho}') \hat{\Psi}(\boldsymbol{\rho}') \right] |0\rangle, \end{aligned} \quad (4)$$

where $\hat{\Psi}(\boldsymbol{\rho}) \equiv \begin{pmatrix} \hat{\psi}(\boldsymbol{\rho}) \\ \hat{\psi}^{\dagger}(\boldsymbol{\rho}) \end{pmatrix}$ is the field operator expressed in the Nambu representation with $\hat{\psi}(\boldsymbol{\rho}) = \begin{pmatrix} \hat{\psi}_{\uparrow}(\boldsymbol{\rho}) \\ \hat{\psi}_{\downarrow}(\boldsymbol{\rho}) \end{pmatrix}$ and $\Sigma^z(\boldsymbol{\rho}, \boldsymbol{\rho}') = \begin{pmatrix} I_2 \otimes \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') & 0 \\ 0 & -I_2 \otimes \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \end{pmatrix}$ with I_2 being a 2×2 identity matrix. Moreover, $\Phi(\boldsymbol{\rho}) = \langle \hat{\Psi}(\boldsymbol{\rho}) \rangle = \begin{pmatrix} \phi^{(c)}(\boldsymbol{\rho}) \\ \phi^{(c)}(\boldsymbol{\rho})^* \end{pmatrix}$ and $\Xi(\boldsymbol{\rho}, \boldsymbol{\rho}') = \begin{pmatrix} X(\boldsymbol{\rho}, \boldsymbol{\rho}') & Y(\boldsymbol{\rho}, \boldsymbol{\rho}') \\ Y(\boldsymbol{\rho}, \boldsymbol{\rho}')^* & X(\boldsymbol{\rho}, \boldsymbol{\rho}')^* \end{pmatrix}$ are the variational parameters with $\phi^{(c)} = \begin{pmatrix} \phi_{\uparrow}^{(c)} \\ \phi_{\downarrow}^{(c)} \end{pmatrix}$,

$X = \begin{pmatrix} X_{\uparrow\uparrow} & X_{\uparrow\downarrow} \\ X_{\downarrow\uparrow} & X_{\downarrow\downarrow} \end{pmatrix}$, and $Y = \begin{pmatrix} Y_{\uparrow\uparrow} & Y_{\uparrow\downarrow} \\ Y_{\downarrow\uparrow} & Y_{\downarrow\downarrow} \end{pmatrix}$ in the spin space.

From the Gaussian wave function, Eq. (4), it is apparent that Φ and Ξ describes the coherent and squeezed components of a Bose gas, respectively. It should be noted that the variational parameter Ξ contain redundancy as different Ξ 's may lead to the same Gaussian state [39]. It is then more convenient to use the redundancy-free covariance matrix $\Gamma(\boldsymbol{\rho}, \boldsymbol{\rho}') \equiv \langle \{\delta\hat{\Psi}(\boldsymbol{\rho}), \delta\hat{\Psi}^{\dagger}(\boldsymbol{\rho}')\} \rangle$ as variational parameters, where $\delta\hat{\Psi}(\boldsymbol{\rho}) = \hat{\Psi}(\boldsymbol{\rho}) - \Phi(\boldsymbol{\rho})$. Writing out explicitly, the covariance matrix takes the form

$$\begin{aligned} \Gamma(\boldsymbol{\rho}, \boldsymbol{\rho}') = & 2 \begin{pmatrix} G(\boldsymbol{\rho}, \boldsymbol{\rho}') & F(\boldsymbol{\rho}, \boldsymbol{\rho}') \\ F(\boldsymbol{\rho}, \boldsymbol{\rho}')^* & G(\boldsymbol{\rho}, \boldsymbol{\rho}') \end{pmatrix} \\ & + \begin{pmatrix} I_2 \otimes \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') & 0 \\ 0 & I_2 \otimes \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \end{pmatrix}, \end{aligned} \quad (5)$$

where $G = \begin{pmatrix} G_{\uparrow\uparrow} & G_{\uparrow\downarrow} \\ G_{\downarrow\uparrow} & G_{\downarrow\downarrow} \end{pmatrix}$ and $F = \begin{pmatrix} F_{\uparrow\uparrow} & F_{\uparrow\downarrow} \\ F_{\downarrow\uparrow} & F_{\downarrow\downarrow} \end{pmatrix}$ are, respectively, the normal and anomalous correlation functions with $G_{\alpha\beta}(\rho, \rho') = \langle \delta\hat{\psi}_\beta^\dagger(\rho') \delta\hat{\psi}_\alpha(\rho) \rangle$ and $F_{\alpha\beta}(\rho, \rho') = \langle \delta\hat{\psi}_\beta(\rho') \delta\hat{\psi}_\alpha(\rho) \rangle$. It is worthwhile to mention that Γ and Ξ are connected through the symplectic matrix $S(\rho, \rho') \equiv \exp[i \int d\rho_1 \Sigma^z(\rho, \rho_1) \Xi(\rho_1, \rho')]$ as [39]

$$\Gamma(\rho, \rho') = \int d\rho_1 S(\rho, \rho_1) S^\dagger(\rho_1, \rho'), \quad (6)$$

where S satisfies

$$\int d\rho_1 d\rho'_1 S(\rho, \rho_1) \Sigma^z(\rho_1, \rho'_1) S^\dagger(\rho'_1, \rho') = \Sigma^z(\rho, \rho'). \quad (7)$$

C. Imaginary-time equations of motion

To obtain the ground-state wave functions, GST introduces the imaginary-time equations of motion (ITEOM) for $\phi^{(c)}$, G ,

and F . In below, we briefly recall the procedure. To start, we make use of the Wick theorem on the Hamiltonian H , which, to quadratic order in $\delta\hat{\psi}_\alpha$, gives rise to

$$\hat{H} \approx E + \int d\rho [\delta\hat{\psi}^\dagger(\rho) \eta(\rho) + \text{H.c.}] + \frac{1}{2} \int d\rho d\rho' : \delta\hat{\Psi}^\dagger(\rho) \mathcal{H}(\rho, \rho') \delta\hat{\Psi}(\rho') :, \quad (8)$$

where $E = \langle \Psi_{\text{GS}} | \hat{H} | \Psi_{\text{GS}} \rangle = E_{\text{kin}} + E_{2B} + E_{3B}$ is the total energy consisting of the kinetic energy

$$E_{\text{kin}} = \sum_\alpha \int d\rho \left[\phi_\alpha^{(c)}(\rho)^* \hat{h}_\alpha(\rho) \phi_\alpha^{(c)}(\rho) + \lim_{\rho' \rightarrow \rho} \hat{h}_\alpha(\rho') G_{\alpha\alpha}(\rho', \rho) \right], \quad (9)$$

the two-body interaction energy

$$E_{2B} = \sum_{\alpha\beta} \frac{g'_{\alpha\beta}}{2} \int d\rho \{ n_\alpha(\rho) n_\beta(\rho) + 2\text{Re} \phi_\alpha^{(c)}(\rho) \phi_\beta^{(c)}(\rho) F_{\alpha\beta}(\rho, \rho)^* + |F_{\alpha\beta}(\rho, \rho)|^2 + [2\phi_\alpha^{(c)}(\rho)^* \phi_\beta^{(c)}(\rho) + G_{\beta\alpha}(\rho, \rho)] G_{\alpha\beta}(\rho, \rho) \}, \quad (10)$$

and the three-body interaction energy

$$E_{3B} = \frac{g'_3}{3!} \sum_{\alpha\beta\gamma} \int d\rho \{ n_\alpha(\rho) n_\beta(\rho) n_\gamma(\rho) + 6\text{Re} \phi_\alpha^{(c)}(\rho) \phi_\beta^{(c)}(\rho) [n_\gamma(\rho) F_{\alpha\beta}(\rho, \rho)^* + 2F_{\alpha\gamma}^*(\rho, \rho) G_{\gamma\beta}(\rho, \rho)] + 3[2\phi_\alpha^{(c)}(\rho)^* \phi_\beta^{(c)}(\rho) G_{\alpha\beta}(\rho, \rho) + |G_{\alpha\beta}(\rho, \rho)|^2] n_\gamma(\rho) + 2G_{\alpha\beta}(\rho, \rho) G_{\beta\gamma}(\rho, \rho) G_{\gamma\alpha}(\rho, \rho) + 6\phi_\alpha^{(c)}(\rho) \phi_\beta^{(c)}(\rho)^* [F_{\alpha\gamma}^*(\rho, \rho) F_{\beta\gamma}(\rho, \rho) + G_{\beta\gamma}(\rho, \rho) G_{\gamma\alpha}(\rho, \rho)] + 3F_{\alpha\beta}(\rho, \rho) [F_{\alpha\beta}(\rho, \rho)^* n_\gamma(\rho) + 2F_{\alpha\gamma}(\rho, \rho)^* G_{\gamma\beta}(\rho, \rho)] \}. \quad (11)$$

Here, $n_\alpha(\rho) = |\phi_\alpha^{(c)}(\rho)|^2 + G_{\alpha\alpha}(\rho, \rho)$ is the density of the α th spin component. Moreover, for the linear terms of $\delta\hat{\psi}$ in Eq. (8), we have $\eta \equiv \begin{pmatrix} \eta_\uparrow(\rho) \\ \eta_\downarrow(\rho) \end{pmatrix}$ with

$$\eta_\alpha(\rho) = \hat{h}_\alpha(\rho) \phi_\alpha^{(c)}(\rho) + \sum_\beta g'_{\alpha\beta} [n_\beta(\rho) \phi_\alpha^{(c)}(\rho) + F_{\alpha\beta}(\rho, \rho) \phi_\beta^{(c)}(\rho)^* + G_{\alpha\beta}(\rho, \rho) \phi_\beta^{(c)}(\rho)] + \frac{g'_3}{2} \sum_{\beta\gamma} \{ [n_\beta(\rho) n_\gamma(\rho) + 2\text{Re} F_{\beta\gamma}(\rho, \rho)^* \phi_\beta^{(c)}(\rho) \phi_\gamma^{(c)}(\rho) + 2G_{\beta\gamma}(\rho, \rho) \phi_\beta^{(c)}(\rho)^* \phi_\gamma^{(c)}(\rho) + |F_{\beta\gamma}(\rho, \rho)|^2 + |G_{\beta\gamma}(\rho, \rho)|^2] \phi_\alpha^{(c)}(\rho) + 2[G_{\alpha\beta}(\rho, \rho) n_\gamma(\rho) + F_{\alpha\gamma}(\rho, \rho) F_{\beta\gamma}(\rho, \rho)^* + G_{\alpha\gamma}(\rho, \rho) G_{\gamma\beta}(\rho, \rho)] \phi_\beta^{(c)}(\rho) + 2[F_{\alpha\beta}(\rho, \rho) n_\gamma(\rho) + F_{\alpha\gamma}(\rho, \rho) G_{\beta\gamma}(\rho, \rho) + G_{\alpha\gamma}(\rho, \rho) F_{\gamma\beta}(\rho, \rho)] \phi_\beta^{(c)}(\rho)^* \}. \quad (12)$$

Finally, in the last line of Eq. (8), $:\hat{\mathcal{O}}:$ denotes the normal ordered operator with respect to the Gaussian state $|\text{GS}\rangle$, $\delta\hat{\Psi}(\rho) = \begin{pmatrix} \delta\hat{\psi}(\rho) \\ \delta\hat{\psi}^\dagger(\rho) \end{pmatrix}$ denotes the field operators expressed in the Nambu basis, $\mathcal{H}(\rho, \rho') = \begin{pmatrix} \mathcal{E}(\rho, \rho') & \Delta(\rho, \rho') \\ \Delta^*(\rho, \rho') & \mathcal{E}^*(\rho, \rho') \end{pmatrix}$ is the mean-field Hamiltonian with $\mathcal{E}(\rho, \rho') = \begin{pmatrix} \mathcal{E}_{\uparrow\uparrow}(\rho, \rho') & \mathcal{E}_{\uparrow\downarrow}(\rho, \rho') \\ \mathcal{E}_{\downarrow\uparrow}(\rho, \rho') & \mathcal{E}_{\downarrow\downarrow}(\rho, \rho') \end{pmatrix}$ and $\Delta(\rho, \rho') = \begin{pmatrix} \Delta_{\uparrow\uparrow}(\rho, \rho') & \Delta_{\uparrow\downarrow}(\rho, \rho') \\ \Delta_{\downarrow\uparrow}(\rho, \rho') & \Delta_{\downarrow\downarrow}(\rho, \rho') \end{pmatrix}$. Writing out explicitly, the

matrix elements of $\mathcal{E}(\boldsymbol{\rho}, \boldsymbol{\rho}')$ and $\Delta(\boldsymbol{\rho}, \boldsymbol{\rho}')$ are, respectively,

$$\begin{aligned} \mathcal{E}_{\alpha\beta}(\boldsymbol{\rho}, \boldsymbol{\rho}') &= \sum_{\gamma} [\hat{h}_{\alpha}(\boldsymbol{\rho}) + g'_{\alpha\gamma} n_{\gamma}(\boldsymbol{\rho})] \delta_{\alpha\beta} \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') + g'_{\alpha\beta} [\phi_{\alpha}^{(c)}(\boldsymbol{\rho}) \phi_{\beta}^{(c)}(\boldsymbol{\rho})^* + G_{\alpha\beta}(\boldsymbol{\rho}, \boldsymbol{\rho})] \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \\ &+ g'_3 \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \sum_{\gamma} \left\{ \sum_{\gamma'} \left[\frac{1}{2} n_{\gamma}(\boldsymbol{\rho}) n_{\gamma'}(\boldsymbol{\rho}) + \phi_{\gamma'}^{(c)}(\boldsymbol{\rho}) \phi_{\gamma}^{(c)}(\boldsymbol{\rho})^* G_{\gamma\gamma'}(\boldsymbol{\rho}, \boldsymbol{\rho}) + \text{Re} \phi_{\gamma}^{(c)}(\boldsymbol{\rho}) \phi_{\gamma'}^{(c)}(\boldsymbol{\rho}) F_{\gamma\gamma'}(\boldsymbol{\rho}, \boldsymbol{\rho})^* \right. \right. \\ &\quad \left. \left. + \frac{1}{2} |G_{\gamma\gamma'}(\boldsymbol{\rho}, \boldsymbol{\rho})|^2 + \frac{1}{2} |F_{\gamma\gamma'}(\boldsymbol{\rho}, \boldsymbol{\rho})|^2 \right] \delta_{\alpha\beta} + n_{\gamma}(\boldsymbol{\rho}) [\phi_{\alpha}^{(c)}(\boldsymbol{\rho}) \phi_{\beta}^{(c)}(\boldsymbol{\rho})^* + G_{\alpha\beta}(\boldsymbol{\rho}, \boldsymbol{\rho})] \right. \\ &\quad \left. + \phi_{\gamma}^{(c)}(\boldsymbol{\rho}) \phi_{\beta}^{(c)}(\boldsymbol{\rho})^* G_{\alpha\gamma}(\boldsymbol{\rho}, \boldsymbol{\rho}) + \phi_{\alpha}^{(c)}(\boldsymbol{\rho}) \phi_{\gamma}^{(c)}(\boldsymbol{\rho})^* G_{\gamma\beta}(\boldsymbol{\rho}, \boldsymbol{\rho}) + \phi_{\alpha}^{(c)}(\boldsymbol{\rho}) \phi_{\gamma}^{(c)}(\boldsymbol{\rho}) F_{\gamma\beta}(\boldsymbol{\rho}, \boldsymbol{\rho})^* \right. \\ &\quad \left. + \phi_{\beta}^{(c)}(\boldsymbol{\rho})^* \phi_{\gamma}^{(c)}(\boldsymbol{\rho})^* F_{\alpha\gamma}(\boldsymbol{\rho}, \boldsymbol{\rho}) + G_{\alpha\gamma}(\boldsymbol{\rho}, \boldsymbol{\rho}) G_{\gamma\beta}(\boldsymbol{\rho}, \boldsymbol{\rho}) + F_{\alpha\gamma}(\boldsymbol{\rho}, \boldsymbol{\rho}) F_{\beta\gamma}(\boldsymbol{\rho}, \boldsymbol{\rho})^* \right\} \end{aligned} \quad (13)$$

and

$$\begin{aligned} \Delta_{\alpha\beta}(\boldsymbol{\rho}, \boldsymbol{\rho}') &= g'_{\alpha\beta} \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') [\phi_{\alpha}^{(c)}(\boldsymbol{\rho}) \phi_{\beta}^{(c)}(\boldsymbol{\rho}) + F_{\alpha\beta}(\boldsymbol{\rho}, \boldsymbol{\rho})] \\ &+ g'_3 \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \sum_{\gamma} \{ n_{\gamma}(\boldsymbol{\rho}) [\phi_{\alpha}^{(c)}(\boldsymbol{\rho}) \phi_{\beta}^{(c)}(\boldsymbol{\rho}) + F_{\alpha\beta}(\boldsymbol{\rho}, \boldsymbol{\rho})] + \phi_{\beta}^{(c)}(\boldsymbol{\rho}) \phi_{\gamma}^{(c)}(\boldsymbol{\rho}) G_{\alpha\gamma}(\boldsymbol{\rho}, \boldsymbol{\rho}) \\ &\quad + \phi_{\alpha}^{(c)}(\boldsymbol{\rho}) \phi_{\gamma}^{(c)}(\boldsymbol{\rho}) G_{\beta\gamma}(\boldsymbol{\rho}, \boldsymbol{\rho}) + [\phi_{\alpha}^{(c)}(\boldsymbol{\rho}) \phi_{\gamma}^{(c)}(\boldsymbol{\rho})^* + G_{\alpha\gamma}(\boldsymbol{\rho}, \boldsymbol{\rho})] F_{\beta\gamma}(\boldsymbol{\rho}, \boldsymbol{\rho}) \\ &\quad + [\phi_{\beta}^{(c)}(\boldsymbol{\rho}) \phi_{\gamma}^{(c)}(\boldsymbol{\rho})^* + G_{\beta\gamma}(\boldsymbol{\rho}, \boldsymbol{\rho})] F_{\alpha\gamma}(\boldsymbol{\rho}, \boldsymbol{\rho}) \}. \end{aligned} \quad (14)$$

In GST, $\phi^{(c)}(\boldsymbol{\rho})$, $G(\boldsymbol{\rho}, \boldsymbol{\rho}')$, and $F(\boldsymbol{\rho}, \boldsymbol{\rho}')$ can be regarded as the order parameters of the system which completely specify the Gaussian state wave function $|\Psi_{\text{GS}}\rangle$. These order parameters for the ground state of a condensate can be determined by minimizing the total energy E , which, after projecting the resulting equations onto the tangential space of the Gaussian manifold, leads to the equations of motion in the imaginary time τ as

$$\partial_{\tau} \phi^{(c)} = -\eta - 2[G\eta + F\eta^*], \quad (15a)$$

$$\begin{aligned} \partial_{\tau} G &= -[\mathcal{E}G + \Delta F^{\dagger} + (\mathcal{E}G + \Delta F^{\dagger})^{\dagger}] \\ &\quad - 2(G\mathcal{E}G + F\Delta^{\dagger}G + G\Delta F^{\dagger} + F\mathcal{E}^T F^{\dagger}), \end{aligned} \quad (15b)$$

$$\begin{aligned} \partial_{\tau} F &= -\Delta - [\mathcal{E}F + \Delta G^* + (\mathcal{E}F + \Delta G^*)^T] \\ &\quad - 2(G\mathcal{E}F + F\Delta^{\dagger}F + G\Delta G^* + F\mathcal{E}^T G^*), \end{aligned} \quad (15c)$$

where for simplicity, we have introduced the matrix notations. Specifically, $\phi^{(c)}$ and η should be understood as vectors in both spin and coordinate spaces whose elements are, e.g., $\phi_{\alpha\rho}^{(c)} \equiv \phi_{\alpha}^{(c)}(\boldsymbol{\rho})$. G , F , $\mathcal{E}$, and Δ are matrices in both spin and coordinate spaces with elements, e.g. $G_{\alpha\rho, \beta\rho'} \equiv G_{\alpha\beta}(\boldsymbol{\rho}, \boldsymbol{\rho}')$. Then the products in Eqs. (15) represent the matrix products, namely summation over the repeated indices in the spin space and integrating over the repeated coordinates in the spatial space. As examples, we write out the explicit expressions for the following term:

$$F\eta^* = \begin{pmatrix} F_{\uparrow\uparrow} & F_{\uparrow\downarrow} \\ F_{\downarrow\uparrow} & F_{\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} \eta_{\uparrow}^* \\ \eta_{\downarrow}^* \end{pmatrix} = \left(\sum_{\beta} \int d\boldsymbol{\rho}' F_{\uparrow\beta}(\boldsymbol{\rho}, \boldsymbol{\rho}') \eta_{\beta}^*(\boldsymbol{\rho}') \right).$$

Alternatively, in terms of quantities defined in the Nambu basis, i.e., ϕ and Γ , the imaginary-time equations of motion

take the form [36,39]

$$\partial_{\tau} \Phi = -\Gamma \begin{pmatrix} \eta \\ \eta^* \end{pmatrix}, \quad (16a)$$

$$\partial_{\tau} \Gamma = \Sigma^z \mathcal{H} \Sigma^z - \Gamma \mathcal{H} \Gamma, \quad (16b)$$

which converge for a sufficiently large imaginary time τ .

D. Normal squeezed modes

The normal and anomalous Green functions can be diagonalized simultaneously by a set of orthonormal functions $\{\bar{\phi}_j^{(s)}\}$ [35], i.e.,

$$G(\boldsymbol{\rho}, \boldsymbol{\rho}') = \sum_{j=1}^{\infty} N_j^{(s)} \bar{\phi}_j^{(s)}(\boldsymbol{\rho}) [\bar{\phi}_j^{(s)}(\boldsymbol{\rho}')]^{\dagger}, \quad (17a)$$

$$F(\boldsymbol{\rho}, \boldsymbol{\rho}') = \sum_{j=1}^{\infty} \sqrt{N_j^{(s)}(N_j^{(s)} + 1)} \bar{\phi}_j^{(s)}(\boldsymbol{\rho}) [\bar{\phi}_j^{(s)}(\boldsymbol{\rho}')]^T, \quad (17b)$$

where $\bar{\phi}_j^{(s)} \equiv \begin{pmatrix} \phi_{j,\uparrow}^{(s)} \\ \phi_{j,\downarrow}^{(s)} \end{pmatrix}$ is referred to as the orthonormal mode function for the j th squeezed mode which satisfy $\int d\boldsymbol{\rho} [\bar{\phi}_i^{(s)}(\boldsymbol{\rho})]^{\dagger} \bar{\phi}_j^{(s)}(\boldsymbol{\rho}) = \delta_{ij}$ and $N_j^{(s)}$ is the occupation number in mode $\bar{\phi}_j^{(s)}$.

With these squeezed modes, we may further factorize the Gaussian state wave function as [35]

$$|\Psi_{\text{GS}}\rangle = e^{\sqrt{N^{(c)}}(\hat{c}^{\dagger} - \hat{c})} \prod_{j=1}^{\infty} e^{\frac{1}{2} \xi_j (\hat{s}_j^{\dagger 2} - \hat{s}_j^2)} |0\rangle, \quad (18)$$

where $N^{(c)} = \int d\boldsymbol{\rho} n^{(c)}(\boldsymbol{\rho})$ is the atom number in the coherent state with $n^{(c)}(\boldsymbol{\rho}) = [\phi^{(c)}(\boldsymbol{\rho})]^{\dagger} \phi^{(c)}(\boldsymbol{\rho})$ being the coherent-state density and $\hat{c} = \int d\boldsymbol{\rho} [\bar{\phi}^{(c)}(\boldsymbol{\rho})]^{\dagger} \hat{\psi}(\boldsymbol{\rho})$ with

$\bar{\phi}^{(c)}(\rho) = \phi^{(c)}(\rho)/\sqrt{N^{(c)}}$ being the normalized mode function for the coherent state. In the squeezing operators, $\hat{s}_j = \int d\rho [\bar{\phi}_j^{(s)}(\rho)]^\dagger \hat{\psi}(\rho)$ is the annihilation operator for the j th squeezed mode and $\sinh \xi_j = \sqrt{N_j^{(s)}}$. Furthermore, the total number of atoms in the squeezed state is $N^{(s)} = \sum_j N_j^{(s)}$ and the total squeezed-state density is $n^{(s)}(\rho) = \sum_j N_j^{(s)} |\bar{\phi}_j^{(s)}(\rho)|^2$.

For convenience, we always assume that $N_j^{(s)}$ are sorted in descending order with respect to the index j . And numerically, it is found that in many situations N_j decreases quickly with the increase of j . Therefore only those modes that are notably populated, say $N_j^{(s)}/N^{(s)} \geq 0.1\%$, are important [35]. We may also define the quantity $\mathcal{S}$ as the number of notably populated squeezed modes, which, as shall be shown, is an important quantity for determining the quantum phases of a droplet.

E. Discussion

Before presenting our results, we would like to compare GST with EGPE. To this end, we first recall that GPE is a variational equation governing the mode function ϕ to which Bose atoms condense. The underlying many-body wave function associated with this mode function is a coherent state. As a result, the quantum fluctuation is inaccessible in GPE. To include the contribution of the quantum fluctuation induced by interactions, the Bogoliubov excitation is computed for a steady homogeneous gas, which are further used to calculate the lowest order energy correction induced by the quantum fluctuation, i.e., the LHY correction. The EGPE for an inhomogeneous gas can then be derived by incorporating the energy correction into the GPE through local-density approximation [21–24].

Apparently, EGPE is not a self-consistent theory as the perturbation to the many-body wave function is not considered. In fact, after taking into account the quantum fluctuation, the condensate wave function, i.e., the vacuum state of the Bogoliubov excitations, is a Gaussian state, i.e., Eq. (4). As shown previously, GST treats the fluctuation self-consistently by introducing G and F as new order parameters. Moreover, it was shown that the dynamical equation for ϕ in GST reduces to the EGPE if we replace G and F with those calculated using the Bogoliubov excitation of a homogeneous gas [36].

At first sight, GST may seem equivalent to HFBT as they both yield the same ground-state $\phi^{(c)}$, G , and F . However, there is a well-known issue with HFBT as it always generates gapped low energy spectra, which violates the Hugenholtz-Pines theorem [40]. GST, on the other hand, adopts a different approach to tackle the excitation of the system [36–38]. Specifically, instead of diagonalizing the single-particle mean-field Hamiltonian $\sigma^z \mathcal{H}$ originating from the variation of ϕ , GST also takes into account the variations of G and F which correspond to the two-particle excitation. As a result, GST always gives rise to the gapless excitation spectrum. In particular, we confirmed the gapless spectrum by numerically computed the excitation spectrum of a trapped condensate in Ref. [36]. It is also worthwhile to point out that although the interspecies squeezing was considered by Hu *et al.* [32], the intraspecies squeezings were completely ignored.

III. RESULTS

In this section, we systemically study the ground-state properties of the binary droplets. To this end, we numerically evolve Eqs. (15) until they converge at large imaginary time τ . This procedure yields the ground-state solution $\{\phi^{(c)}(\rho), G(\rho, \rho'), F(\rho, \rho')\}$ with $\rho = \sqrt{x^2 + y^2}$. Here to simplify the numerical calculation, we utilize the axial symmetry of the system such that the order parameters for the ground state are independent of azimuthal coordinate.

Although the mode functions for the spin- $\uparrow$ and spin- $\downarrow$ atoms are generally different, i.e., $\bar{\phi}_\uparrow^{(c)}(\rho) \neq \bar{\phi}_\downarrow^{(c)}(\rho)$ and $\bar{\phi}_{j,\uparrow}^{(s)}(\rho) \neq \bar{\phi}_{j,\downarrow}^{(s)}(\rho)$, it has been numerically verified that

$$\sqrt{a_{\uparrow\uparrow}} n_{\uparrow}^{(c,s)}(\rho) = \sqrt{a_{\downarrow\downarrow}} n_{\downarrow}^{(c,s)}(\rho) \quad (19)$$

when the atom number ratio between spin- $\uparrow$ and spin- $\downarrow$ atoms satisfies

$$\frac{N_\uparrow}{N_\downarrow} = \sqrt{\frac{a_{\downarrow\downarrow}}{a_{\uparrow\uparrow}}}, \quad (20)$$

where $n_\alpha^{(c)} = |\phi_\alpha^{(c)}|^2$ and $n_\alpha^{(s)} = \sum_j N_j^{(s)} |\phi_{j,\alpha}^{(s)}|^2$ are, respectively, the densities of the coherent and squeezed atoms for the α th spin component. As a result, for both coherent and squeezed atoms, we also have

$$\frac{N_\uparrow^{(c)}}{N_\downarrow^{(c)}} = \frac{N_\uparrow^{(s)}}{N_\downarrow^{(s)}} = \sqrt{\frac{a_{\downarrow\downarrow}}{a_{\uparrow\uparrow}}}. \quad (21)$$

Now, we may define a single coherent fraction

$$f_c \equiv \frac{N^{(c)}}{N}$$

to characterize the quantum phases of a condensate, instead of considering the coherent fraction of individual spin component. We shall refer to Eq. (20) as the optimal spin composition. As shown in Appendix A, those numerical findings can be analytically derived by minimizing the total interaction energy.

Following the experiment setup, we always assume that the optimal spin composition is satisfied in our numerical calculations. In the results presented below, we cover a wide range of parameter regime defined by the total atom number N and the effective scattering length

$$\delta a \equiv a_{\uparrow\downarrow} + \sqrt{a_{\uparrow\uparrow} a_{\downarrow\downarrow}}. \quad (22)$$

In addition, for our 2D system, there exists a virial relation (see Appendix B),

$$E_{\text{kin}} + E_{2B} + 2E_{3B} = 0, \quad (23)$$

which can be used to check the numerical results.

A. Determination of g_3

In order to carry out any numerical calculation, we must first determine the value of the unknown 3B coupling strength. Previously, g_3 of Dy atom was determined by fitting the atom number distribution of the droplet states [35]. Here we instead determine g_3 by fitting the radial size σ of the quasi-2D droplets.

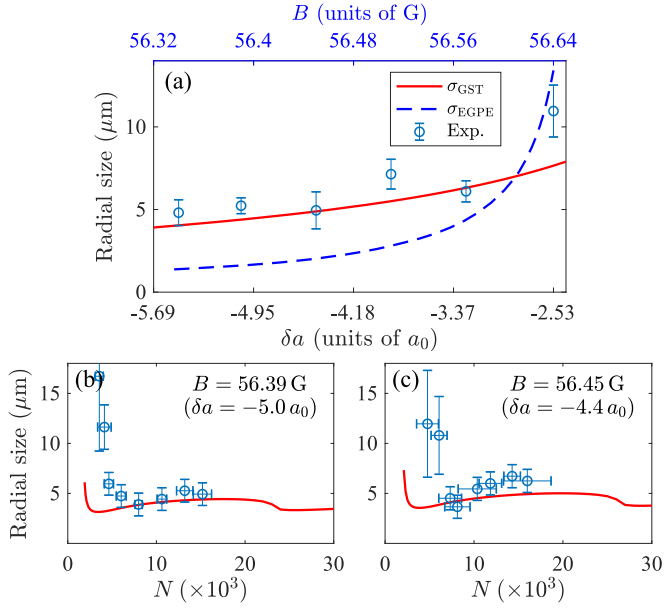


FIG. 1. (a) Radial size vs reduced scattering length for $N = 1.5 \times 10^4$. Solid line is computed using the GST with $g_3 = 6.65 \times 10^{-39} \hbar \text{ m}^6/\text{s}$. Empty circles with error bars and dashed line (both are extracted from Ref. [5]) represent the experimental data and the EGPE result, respectively. (b) and (c) show radial size as functions of atom number for various magnetic fields. Empty circles (extracted from Ref. [5]) are experimental data.

Following the experiment [5], we extract σ by fitting the total density with a two-dimensional Gaussian. Figure 1(a) shows the radial size σ_{GST} as a function of δa for $N = 1.5 \times 10^4$ with the 3B coupling strength $g_3 = 6.65 \times 10^{-39} \hbar \text{ m}^6/\text{s}$. As a comparison, we also plot the radial size, σ_{EGPE} , computed with EGPE. Here the subscript of σ is used to distinguish the theoretical approaches. As can be seen, good agreement between σ_{GST} and the experimental results is achieved. Moreover, we point out that varying the value of g_3 does not change the shape of the red line; it only shifts the red line vertically. Therefore, to achieve better agreement for the fitting, one may have to make g_3 spin dependent.

To further demonstrate the validity of the fitted g_3 , we show, in Figs. 1(b) and 1(c), the N dependence of σ_{GST} under different magnetic fields. As can be seen, although discrepancies exist at small N , good agreements are still achieved for large N . Particularly, the main feature of the experimental results is captured by the numerical simulations (see below for details). Hereafter, we shall always use the fitted g_3 for all numerical results presented below.

B. Phase diagram

Figure 2(a) shows the distribution of f_c on the $(\delta a, N)$ parameter plane. Similar to those in dipolar droplets [35], there are four distinct regions: expanding gas (EG), SVS, SCS, and CS. The EG region lies below the CAN (dash-dotted line or Δ), in which the attractive two-body interaction is insufficient for the gas to form a self-bound state. Both SVS and SCS phases contain a single squeezed mode and are macroscopic squeezed states. The difference is that SVS is

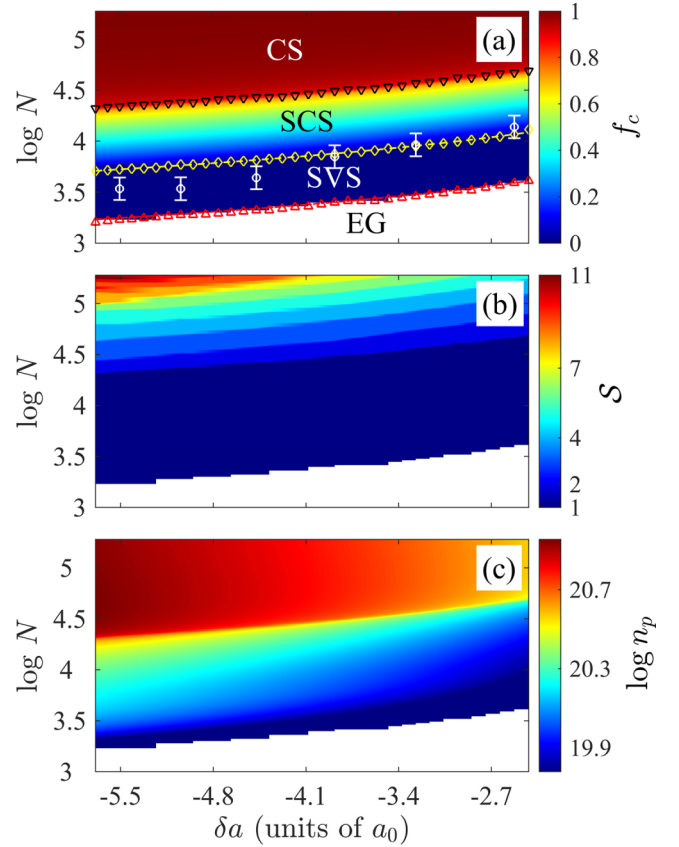


FIG. 2. Phase diagram. (a) Distribution of the coherent fraction f_c on the $(\delta a, N)$ parameter plane. Markers Δ , $\diamond$, and ∇ denote numerical results for CAN, SVS-to-SCS transition boundary, and SCS-to-CS transition boundary, respectively. The dash-dotted and dashed lines show the corresponding analytic results. Empty circles ($\circ$) with error bar are experimental data for CAN [5]. (b) and (c) demonstrate the distributions of the number of the notably-populated squeezed modes S and the peak density n_p (units of m^{-3}) on the $(\delta a, N)$ parameter plane, respectively.

a pure squeezed state while SCS also contains a coherent component. The SVS-to-SCS transition (dashed line or $\diamond$) breaks the Z_2 symmetry of the condensate and is a third-order phase transition. The CS phase is dominated by the coherent component with $f_c > 0.75$ and the SCS-to-CS transition is of first order. Interestingly, as shown by the dash-dotted and the dashed lines, the boundaries for the self-bound state and the SVS-to-SCS transition can be found nearly analytically with high accuracy (see Sec. IV C for detail). In below, we shall also determine the boundary of SCS-to-CS transition by analyzing the stability of Bogliubov excitation in the CS phase. We point out that condensates always collapse for parameters covered in Fig. 2(a) if $g_3 = 0$, which suggests that droplets are stabilized by 3B repulsion. However, one should note that this stabilization force is different from that induced by the Efimov effect when the two-body scattering length is negative and much larger than the effective two-body interaction radius [41].

The phase diagram is most conveniently reproduced by plotting the number of the notably-populated squeezed modes S . As shown in Fig. 2(b), both SVS and SCS phases are

featured by single squeezed mode; while CS phase may contains a large number of squeezed modes even if the number of squeezed atoms becomes negligibly small. Physically, squeezing in SVS and SCS phases are macroscopic quantum state originating from two-body attraction [35,36]; while squeezing in CS phase represents quantum depletion induced by 3B repulsion [35]. Similar to dipolar droplets, the presence of squeezing also significantly modifies the AND of these quantum states. Figure 2(c) shows the distribution of the peak density of the droplets n_p on the $(\delta a, N)$ parameter plane, which exhibits similar structure as f_c and S .

C. Density profiles

To gain more insight into those quantum phases, here we consider the normalized density profiles of the coherent and the squeezed atoms, i.e., $\bar{n}^{(c)}(\rho) = |\bar{\phi}^{(c)}(\rho)|^2$ and $\bar{n}^{(s)}(\rho) = n^{(s)}(\rho)/N^{(s)}$. For convenience, we introduce the width of the coherent/squeezed atoms

$$w^{(c,s)} = \left[\int dx dy \rho^2 \bar{n}^{(c,s)}(x, y) \right]^{1/2}$$

to characterize the geometry of the states. Clearly, $w^{(c,s)}$ are better defined than the radial size when $\bar{n}^{(c,s)}(\rho)$ deviate significantly from Gaussian function.

In Fig. 3, we plot $\bar{n}^{(c,s)}(\rho)$ for various N 's in SCS and CS phases. Immediately to the right of the SVS-to-SCS transition, e.g. $N = 9.5 \times 10^3$, it is found that $\bar{n}^{(c)}$ is identical to $\bar{n}^{(s)}$. This observation is understandable as the coherent atoms originates from the three-body repulsion which is simply proportional to $\bar{n}^{(s)}$ when $N^{(c)}$ is essentially zero. Mathematically, when $n^{(c)} = 0$, Eq. (27a) satisfied by $\phi^{(c)}$ reduces to Eq. (29) which governs $\phi^{(s)}$. When N is further increased, $\bar{n}^{(c)}(\rho)$ changes in according with the size of the coherent part as shown in Fig. 3(a). Particularly, in the large N limit, the top of $\bar{n}^{(c)}(\rho)$ becomes flatted and significantly deviates from a Gaussian function, which is in consistent to the liquid nature of the condensate. As to the profiles of the squeezed atoms, $\bar{n}^{(s)}(\rho = 0)$ continuously decreases with N such that $\bar{n}^{(s)}(\rho)$ is no longer a monotonic decreasing function of ρ in the CS phase, which further demonstrates the squeezing here is quantum depletion due to the three-body repulsion.

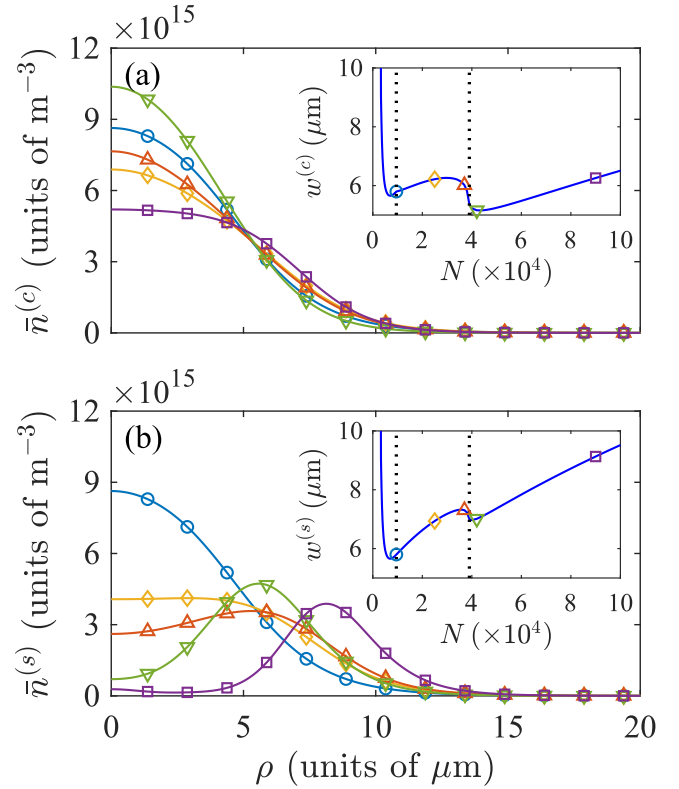


FIG. 3. Normalized density profiles of the coherent (a) and the squeezed atoms (b) with $\delta a = -3.062a_0$ and $N = 9.5 \times 10^3$ ($\circ$), 2.5×10^4 ($\diamond$), 3.7×10^4 ($\triangle$), 4.2×10^4 (∇), and 9×10^4 ($\square$). Insets in (a) and (b) show the N dependence of $w^{(c)}$ and $w^{(s)}$, respectively. Vertical dotted lines mark the locations of two phase transitions. Symbols on $w^{(c,s)}$ mark the N values used to plot the corresponding density profiles.

D. Effective equations for the coherent and squeezed modes

Apparently, solving Eq. (15) is numerically demanding compared to solving an EGPE. Here we derive a set of effective equations for the squeezed and coherent modes in the SVS and SCS phases. The computational complexity of these effective equations is equivalent to that of an EGPE.

To this end, we make use of the facts that SCS phase only contains a single squeezed mode and the number of squeezed atoms in the CS phase is negligible. The total energy of all three quantum phases can be expressed in a unified form,

$$E = \int d\rho \left[\sum_{i=c,s} \phi^{(i)*} \hat{h} \phi^{(i)} + \frac{\delta g'}{2} (n^{(c)2} + 6n^{(c)}n^{(s)} + 3n^{(s)2}) + \frac{g'_3}{3!} (n^{(c)3} + 15n^{(c)2}n^{(s)} + 45n^{(c)}n^{(s)2} + 15n^{(s)3}) \right], \quad (24)$$

where $\hat{h}(\rho) = -\hbar^2 \nabla_\rho^2 / (2M)$ and

$$\delta g' = \frac{8\pi \hbar^2 \delta a}{M} \frac{\sqrt{a_{\uparrow\uparrow} a_{\downarrow\downarrow}}}{\sqrt{2\pi} a_z (\sqrt{a_{\uparrow\uparrow}} + \sqrt{a_{\downarrow\downarrow}})^2} \quad (25)$$

is a dimensionless parameter related to the reduced scattering length. A set of time-dependent equations for $\phi^{(c,s)}(\rho, t)$ can

then be derived through the variational principle

$$i\hbar \frac{\partial \phi^{(i)}}{\partial t} = \frac{\delta E}{\delta \phi^{(i)*}}. \quad (26)$$

Moreover, the equations for stationary solutions can be obtained by substituting $\phi^{(i)}(\rho, t) = \phi^{(i)}(\rho) e^{-i\mu t}$ into the

time-dependent equations, which eventually leads to

$$\left[\hat{h} + \delta g' n^{(c)} + 3n^{(s)} + \frac{g'_3}{2} (n^{(c)2} + 10n^{(c)}n^{(s)} + 15n^{(s)2}) \right] \phi^{(c)} = \mu \phi^{(c)}, \quad (27a)$$

$$\left[\hat{h} + 3\delta g' n^{(c)} + n^{(s)} + \frac{5g'_3}{2} (n^{(c)2} + 6n^{(c)}n^{(s)} + 3n^{(s)2}) \right] \phi^{(s)} = \mu \phi^{(s)}, \quad (27b)$$

where μ is the chemical potential that ensures the normalization condition $\int d\rho[|\phi^{(c)}|^2 + |\phi^{(s)}|^2] = N$.

Compared to the GST Eq. (15), these effective equations greatly simplify the computational complexity. More importantly, we have numerically verified the validity of these effective equations. In fact, as long as we are away from the SCS-to-CS transition, they give rise to results in good agreement with those given by Eq. (15).

It is instructive to examine these equations in the limiting cases. Specifically, in the CS phase, we may assume that $\phi^{(s)}$ vanishes such that Eq. (27a) reduces to

$$\left[\hat{h} + \delta g' n^{(c)} + \frac{g'_3}{2} n^{(c)2} \right] \phi^{(c)} = \mu \phi^{(c)}, \quad (28)$$

which is exactly the GPE with 3B interaction being incorporated. On the other hand, $\phi^{(c)} = 0$ in the SVS phase and Eq. (27b) simplifies to

$$\left[\hat{h} + 3\delta g' n^{(s)} + \frac{15g'_3}{2} n^{(s)2} \right] \phi^{(s)} = \mu \phi^{(s)}, \quad (29)$$

which has the same form as GPE except for that the 2B and 3B interaction strengths are amplified by factors of 3 and 15, respectively.

As a comparison, we also write out EGPE [24]

$$\left[\hat{h} + \delta g' n^{(c)} + \frac{5}{2} g'_{5/2} n^{(c)3/2} \right] \phi^{(c)} = \mu \phi^{(c)}, \quad (30)$$

where

$$g'_{5/2} = \frac{256\sqrt{2}\hbar^2}{15\sqrt{5}\pi^{1/4}a_z^{3/2}M} \left(\frac{\sqrt{a_{\uparrow\uparrow}a_{\downarrow\downarrow}}}{1 + \sqrt{a_{\downarrow\downarrow}/a_{\uparrow\uparrow}}} \right)^{5/2} \times f\left(\frac{a_{\downarrow\downarrow}^2}{a_{\uparrow\uparrow}a_{\downarrow\downarrow}}, \sqrt{\frac{a_{\downarrow\downarrow}}{a_{\uparrow\uparrow}}} \right) \quad (31)$$

with $f(x, y) = \sum_{\pm} \frac{1}{4\sqrt{2}} (1 + y \pm \sqrt{(1-y)^2 + 4xy})^{5/2}$ and the superscript '(c)' on ϕ indicates that the EGPE is based on the coherent-state ansatz for the many-body wave function of the condensate. In Sec. IV C, we shall also use EGPE to analyze the radial size of the droplets.

E. Phase boundaries

Although all quantum phases in Fig. 2(a) are found numerically, we show here that the phase boundaries for the SVS-to-SCS and SCS-to-CS transitions can be determined nearly analytically. We point out that, in Sec. IV C, we analytically determine the boundary between EG and SVS phase, i.e., the CAN for a self-bound droplet.

Starting with the third-order SVS-to-SCS transition, the squeezed and the coherent modes share the same density profile in the vicinity of the SVS-to-SCS transition. As a result,

we may consider a homogeneous gas in the SVS phase of total density n . Assuming that the total density of the gas is increased by δn , if the system remains in the SVS phase, the total energy density becomes

$$e^{(\text{SVS})} = \frac{3\delta g'}{2} (n + \delta n)^2 + \frac{5g'_3}{2} (n + \delta n)^3. \quad (32)$$

On the other hand, if the transition to the SCS phase occurs such that the newly added atoms is in the coherent state, the total energy density is then

$$e^{(\text{SCS})} = \frac{\delta g'}{2} (3n^2 + 6n\delta n + \delta n^2) + \frac{g'_3}{6} (15n^3 + 45n^2\delta n + 15n\delta n^2 + \delta n^3). \quad (33)$$

Therefore the SVS-to-SCS transition only occurs when $e^{(\text{SCS})} < e^{(\text{SVS})}$, which, to the second order in δn , leads to the condition $-\delta g' < 5g'_3 n$. Correspondingly, the critical density for third-order phase transition is

$$n_{3\text{rd}}^* = -\frac{\delta g'}{5g'_3}. \quad (34)$$

It can be further shown that, at the critical density, the 2B and 3B interaction energies satisfy

$$-E_{2B} = 3E_{3B}. \quad (35)$$

For an inhomogeneous self-bound droplet, combining the condition (35) with the virial relation (23), one finds

$$E_{\text{kin}} = E_{3B} \quad (36)$$

at the transition point. To verify these predictions, we plot, in Fig. 4(a), the N dependence of E_{kin} , E_{2B} , and E_{3B} for $\delta a = -3.062a_0$, where the vertical dotted lines mark the atom numbers corresponding to two phase transitions. As can be seen, for the SVS-to-SCS transition (left vertical line), both Eqs. (35) and (36) are both satisfied.

Now, we turn to determine the boundary between the SCS and CS phases by analyzing the stability of the Bogoliubov excitation in the CS phase. Again, we consider a homogeneous gas of total density n in the CS phase which is described by the GPEs

$$i\partial_t \phi_\alpha = \left(-\frac{\hbar^2 \nabla^2}{2M} - \mu_\alpha + \sum_\beta g'_{\alpha\beta} |\phi_\beta|^2 + \frac{g'_3}{2} \sum_{\beta,\gamma} |\phi_\beta|^2 |\phi_\gamma|^2 \right) \phi_\alpha. \quad (37)$$

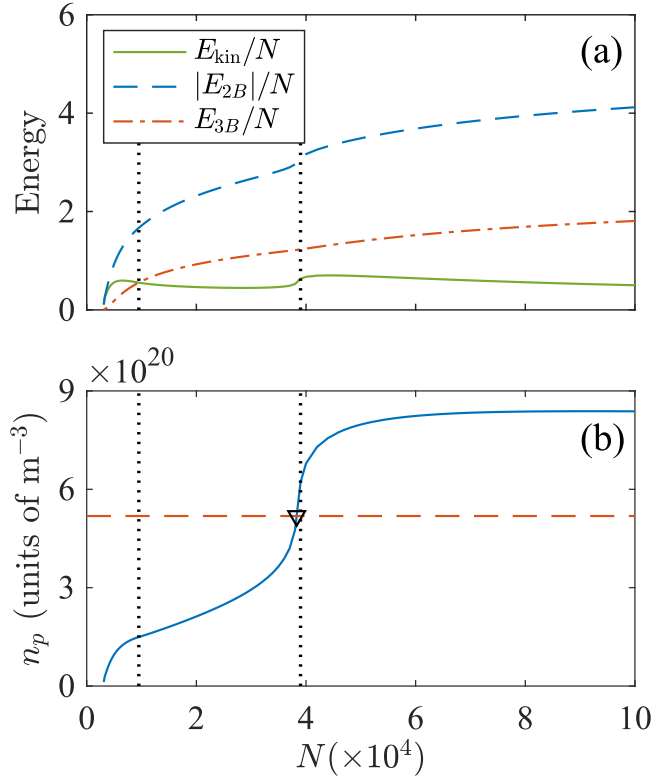


FIG. 4. (a) E_{kin}/N , E_{2B}/N , and E_{3B}/N as functions of N . Energy is in units of $\hbar^2/(ML^2)$ with $L = 6 \mu\text{m}$. (b) n_p as a function of N . The effective scattering length is $\delta a = -3.062a_0$. The left and right vertical dotted lines denote the critical atom numbers for the SVS-to-SCS and SCS-to-CS transitions, respectively. In (b), the horizontal dashed line marks the critical condensate density n_{1st}^* and ∇ represents the intersection of the solid and dashed lines.

For a ground state, the chemical potentials can be determined as

$$\mu_\alpha = \sum_\beta g'_{\alpha\beta} n_\beta + \frac{g'_3}{2} \sum_{\beta,\gamma} n_\beta n_\gamma. \quad (38)$$

Then, around the ground state, the Hamiltonian can be expanded in the momentum space as

$$H \approx E + \sum_{k \neq 0} \frac{1}{2} \delta \hat{\Psi}_k^\dagger \mathcal{H}_k \delta \hat{\Psi}_k, \quad (39)$$

where $E = \mathcal{V} \sum_\alpha \mu_\alpha n_\alpha$ is the ground state energy with $\mathcal{V}$ being the volume of the system, $\delta \hat{\Psi}_k$ is the Fourier transform of $\delta \hat{\Psi}(\mathbf{r})$, and $\mathcal{H}_k = \begin{pmatrix} \mathcal{E}_k & \Delta_k \\ \Delta_k^\dagger & \mathcal{E}_k^T \end{pmatrix}$. Furthermore,

$$\mathcal{E}_k = \begin{pmatrix} \epsilon_k^0 + (g'_{\uparrow\uparrow} + g'_3 n) n_\uparrow & (g'_{\uparrow\downarrow} + g'_3 n) \sqrt{n_\uparrow n_\downarrow} \\ (g'_{\downarrow\uparrow} + g'_3 n) \sqrt{n_\uparrow n_\downarrow} & \epsilon_k^0 + (g'_{\downarrow\downarrow} + g'_3 n) n_\downarrow \end{pmatrix}$$

and

$$\Delta_k = \begin{pmatrix} (g'_{\uparrow\uparrow} + g'_3 n) n_\uparrow & (g'_{\uparrow\downarrow} + g'_3 n) \sqrt{n_\uparrow n_\downarrow} \\ (g'_{\downarrow\uparrow} + g'_3 n) \sqrt{n_\uparrow n_\downarrow} & (g'_{\downarrow\downarrow} + g'_3 n) n_\downarrow \end{pmatrix},$$

where $\epsilon_k^0 = \hbar^2 k^2 / (2M)$ is the free-particle dispersion. After diagonalizing $\mathcal{H}_k$, we obtain the spectrum for the Bogoliubov excitation as

$$\zeta_{\pm,k}^2 = \frac{1}{2} \left[\epsilon_{\uparrow,k}^2 + \epsilon_{\downarrow,k}^2 \pm \sqrt{(\epsilon_{\uparrow,k}^2 - \epsilon_{\downarrow,k}^2)^2 + 16(g'_{\uparrow\downarrow} + g'_3 n)^2 n_\uparrow n_\downarrow (\epsilon_k^0)^2} \right],$$

where $\epsilon_{\alpha,k} = \sqrt{\epsilon_k^0 [\epsilon_k^0 + 2(g'_{\alpha\alpha} + g'_3 n) n_\alpha]}$ is the familiar quasi-particle energies for the α th spin component. Apparently, the CS phase becomes unstable when the lower-branch quasi-particle energy is imaginary, i.e.,

$$(\epsilon_{\uparrow,k}^2 + \epsilon_{\downarrow,k}^2)^2 < (\epsilon_{\uparrow,k}^2 - \epsilon_{\downarrow,k}^2)^2 + 16(g'_{\uparrow\downarrow} + g'_3 n)^2 n_\uparrow n_\downarrow (\epsilon_k^0)^2. \quad (40)$$

After some straightforward calculations, we find that the first-order transition to the SCS phase occurs when

$$n < n_{1st}^* \equiv \frac{g'_{\uparrow\downarrow} - g'_{\uparrow\uparrow} g'_{\downarrow\downarrow}}{g'_3 (g'_{\uparrow\uparrow} + g'_{\downarrow\downarrow} - 2g'_{\uparrow\downarrow})}, \quad (41)$$

As an example, the solid line, in Fig. 4(b), shows the numerically computed peak condensate density n_p as a function of N for $\delta a = -3.062a_0$. The horizontal dashed line marks the critical density n_{1st}^* predicted by Eq. (41). The intersection of the solid and dashed lines (∇) yields the critical atom number for the transition. As can be seen, this critical atom number is in very good agreement with the one (vertical dotted line) obtained with full numerical calculations.

IV. EXPERIMENTAL DETECTION

We now turn to explore the experimental detection of the predicted quantum states in binary droplet. As shall be shown, all squeezed-state related phases distinguish themselves from the CS phase by their statistical properties, such as the atom number distribution (AND) and the second-order correlation function. In addition, we show that a more readily accessible detection is the measurement of droplet's radial size which, other than the quantum states, also contains the complete information of the stabilization mechanism.

A. Atom number distribution

The particle number statistic is an important feature of a many-body quantum state. Therefore it can be used to distinguish different quantum phases. Following the approach used in dipolar droplets [35], here we derive the particle-number distribution of a Gaussian state describing a binary condensate. To this end, we expand the coherent mode $\hat{c}$ in terms of the squeezed modes as

$$\hat{c} = \sum_{i=1}^{\infty} \alpha_i \hat{s}_i + \alpha_\perp \hat{c}_\perp, \quad (42)$$

where $\alpha_i = \int d\boldsymbol{\rho} [\bar{\phi}_i^{(s)}(\boldsymbol{\rho})]^\dagger \bar{\phi}^{(c)}(\boldsymbol{\rho})$, $\alpha_\perp^2 = 1 - \sum_i |\alpha_i|^2$, and $\hat{c}_\perp$ represents the mode that is perpendicular to all $\hat{s}_i$. The

Gaussian state wave function can now be expressed as

$$|\Psi_{GS}\rangle = e^{\sqrt{N^{(c)}}\alpha_{\perp}(\hat{c}_{\perp}^{\dagger} - \hat{c}_{\perp})} \times \prod_{i=1}^{\infty} e^{\sqrt{N^{(c)}}(\alpha_i \hat{s}_i^{\dagger} - \alpha_i^* \hat{s}_i)} e^{\frac{1}{2}\xi_i(\hat{s}_i^{\dagger 2} - \hat{s}_i^2)} |0\rangle. \quad (43)$$

The AND for mode $\hat{c}_{\perp}$ is

$$p_{\perp}(\ell) = e^{-N^{(c)}|\alpha_{\perp}|^2} (N^{(c)}|\alpha_{\perp}|^2)^{\ell} / \ell!$$

and that for mode $\hat{s}_i$ is (see, e.g., Ref. [42])

$$p_i(\ell) = \frac{(\frac{1}{2} \tanh \xi_i)^{\ell}}{\ell! \cosh \xi_i} |H_{\ell}[\gamma_i (\sinh 2\xi_i)^{-1/2}]|^2 \times \exp \left[-N^{(c)}|\alpha_i|^2 - \frac{N^{(c)}}{2}(\alpha_i^{*2} + \alpha_i^2) \tanh \xi_i \right], \quad (44)$$

where $\gamma_i = \sqrt{N^{(c)}}\alpha_i \cosh \xi_i + \sqrt{N^{(c)}}\alpha_i^* \sinh \xi_i$ and $H_{\ell}(x)$ are Hermite polynomials. The AND of $|\Psi_{GS}\rangle$ can then be expressed into a recursive form as follows. Let $P_i(n)$ denote the AND of the state containing first i squeezed modes, the AND containing first $i+1$ squeezed modes can then be expressed as

$$P_{i+1}(n) = \sum_{\ell=0}^n P_i(n-\ell) p_{i+1}(\ell), \quad (45)$$

where $P_0(n) \equiv p_{\perp}(n)$. Applying this equation successively will eventually leads to the total AND $P(n)$.

Figure 5 shows the typical ANDs corresponding to states in SVS, SCS, and CS phases. Due to the squeezing, $P(n)$ is asymmetric with a long tail at the large n side, which was observed in dipolar droplet experiments [2,4]. However, as f_c increases, $P(n)$ becomes less asymmetric. Other properties of the particle number distribution includes that $P(n)$ has a peak at $n \approx N^{(c)}$ and the mean value of atom number is $\sum_n nP(n) = N^{(c)} + N^{(s)} = N$.

B. Second-order correlation function

Quantum phases can also be distinguished by measuring the second-order correlation function. To this end, let us first consider the second-order correlation function between the spin components α and β :

$$g_{\alpha\beta}^{(2)}(\rho, \rho) = \frac{\langle \hat{\psi}_{\alpha}^{\dagger}(\rho) \hat{\psi}_{\beta}^{\dagger}(\rho) \hat{\psi}_{\beta}(\rho) \hat{\psi}_{\alpha}(\rho) \rangle}{\langle \hat{\psi}_{\alpha}^{\dagger}(\rho) \hat{\psi}_{\alpha}(\rho) \rangle \langle \hat{\psi}_{\beta}^{\dagger}(\rho) \hat{\psi}_{\beta}(\rho) \rangle}. \quad (46)$$

Making use of the relations (19), it can be shown that

$$g_{\alpha\beta}^{(2)}(\rho, \rho) = \frac{2[2n^{(c)} + n^{(s)}]n^{(s)} + [n^{(c)} + \chi_{\alpha\beta}n^{(s)}]^2}{n^2}, \quad (47)$$

where $\chi_{\alpha\beta} = \sqrt{1 + \delta_{\alpha\beta}/[N^{(s)}]^{-2}}$. Particularly, $\chi_{\alpha\beta} \approx 1$ if $N^{(s)} \gg 1$. As a result, $g_{\alpha\beta}^{(2)}$ is spin independent and the total second-order correlation function is

$$g^{(2)}(\rho, \rho) = \frac{\sum_{\alpha\beta} \langle \hat{\psi}_{\alpha}^{\dagger}(\rho) \hat{\psi}_{\beta}^{\dagger}(\rho) \hat{\psi}_{\beta}(\rho) \hat{\psi}_{\alpha}(\rho) \rangle}{\sum_{\alpha} \langle \hat{\psi}_{\alpha}^{\dagger}(\rho) \hat{\psi}_{\alpha}(\rho) \rangle \sum_{\beta} \langle \hat{\psi}_{\beta}^{\dagger}(\rho) \hat{\psi}_{\beta}(\rho) \rangle} \approx \frac{1}{n^2} [n^{(c)2} + 6n^{(c)}n^{(s)} + 3n^{(s)2}], \quad (48)$$

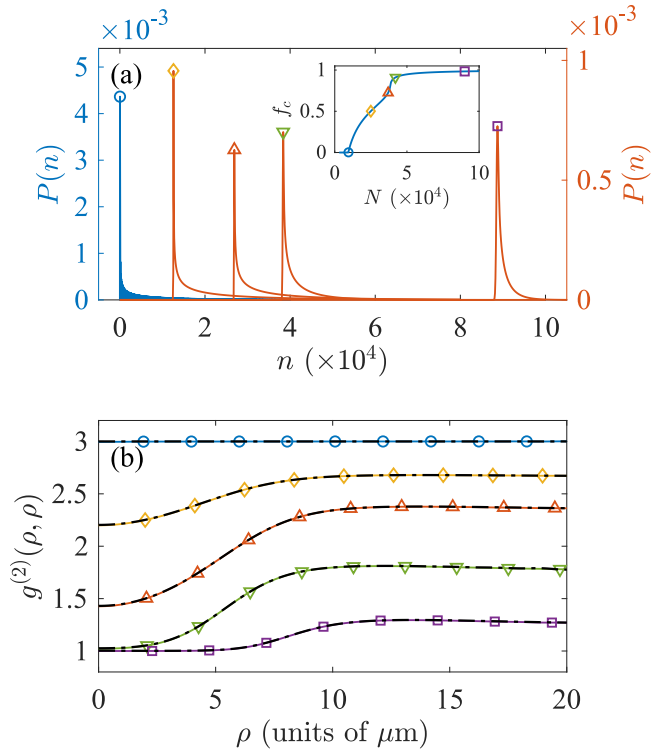


FIG. 5. (a) Atom number distributions $P(n)$ for the states with mean atom numbers (from left to right) $N = 9.5 \times 10^3$ ($\circ$), 2.5×10^4 ($\diamond$), 3.7×10^4 ($\triangle$), 4.2×10^4 (∇), and 9×10^4 ($\square$). The y axis of the blue line (red lines) is on the left (right). The inset shows the coherent fraction as a function of mean atom number where the markers denote the mean atom numbers used to plot $P(n)$'s. The reduced scattering length is $\delta a = -3.062a_0$. (b) Second-order correlation functions $g^{(2)}(\rho, \rho)$ for the same set of parameters as (a). The solid and dash-dotted lines are plotted using Eqs. (47) and (48), respectively.

where the last line is obtained by assuming $\chi_{\alpha\beta} = 1$. It can be easily shown that $g^{(2)}$ is bound between 1 and 3. In particular, $g^{(2)} = 1$ and 3 for a pure coherent and squeezed states, respectively.

Figure 5(b) shows $g^{(2)}(\rho, \rho)$ calculated using both Eqs. (47) and (48) for $\delta a = -3.062a_0$ with various mean atom numbers N . Equation (48) is clearly a very good approximation as two results are visually indistinguishable. Moreover, the behavior of $g^{(2)}(\rho, \rho)$ is in agreement with the densities shown in Fig. 3. For a pure squeezed state, we find $g^{(2)}(\rho, \rho) = 3$. When the coherent fraction grows, $g^{(2)}(\rho, \rho)$ starts to decrease along the whole radial direction. In particular, $g^{(2)}(\rho, \rho)$ reaches its minimum at $\rho = 0$ where the coherent density has the largest value. Eventually, $g^{(2)}(\rho, \rho)$ approaches unit around the center of the droplet; while away from this high density region, $g^{(2)}(\rho, \rho)$ is still significantly deviated from unit as there still is large fraction of atoms in the squeeze state.

C. Radial size

Here we show that the predicted quantum phases and even the stabilization mechanism can be identified by measuring the readily accessible radial size of a droplet. To this end,

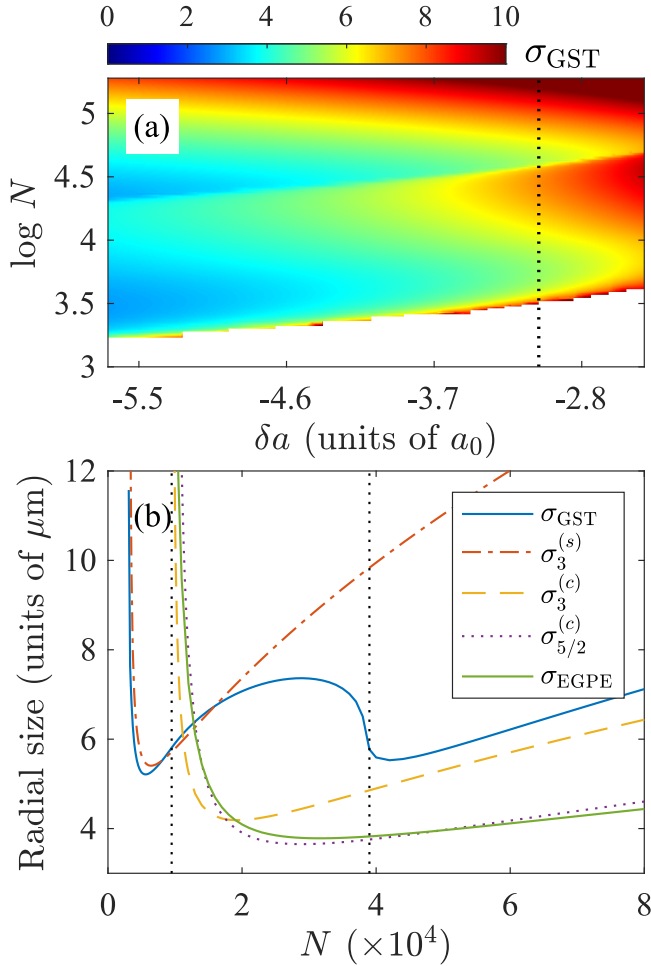


FIG. 6. (a) Distribution of the radial size σ_{GST} (units of μm) on the $(\delta a, N)$ parameter plane. (b) Radial size versus mean atom number for $\delta a = -3.062a_0$. Vertical dotted lines mark the locations of two phase transitions.

we shall analyze how a droplet's radial size depends on the quantum phases and the stabilization mechanisms which lead to distinct expressions for energy. More specifically, we shall focus on SVS and CS phases, for which the coefficients for both 2B and 3B interaction energies are distinct.

In Fig. 6(a), we map out the radial size numerically computed with the GST, i.e., σ_{GST} , on the δa - N parameter plane. Compared to f_c , $\mathcal{S}$, and n_p shown, respectively, in Figs. 2(a)–2(c), σ_{GST} exhibits complicate structure. To gain more details, we plot the N dependence of σ_{GST} for $\delta a = -3.062a_0$ in Fig. 6. As a comparison, the radial size numerically computed via EGPE, σ_{EGPE} , is also plotted. Surprisingly, σ_{GST} is W-shaped function of N , in striking contrast to the V-shaped σ_{EGPE} .

Quite generally, for a 2D droplet of radial size σ , the total energy per atom, $\varepsilon \equiv E/N$, can be expressed as

$$\varepsilon(\sigma) = \kappa \left[\frac{1}{\sigma^2} - \frac{\tilde{g}_2 N}{\sigma^2} + \frac{\tilde{g}_v N^{v-1}}{\sigma^{2(v-1)}} \right] \quad (\text{for } v > 2), \quad (49)$$

where κ is a coefficient associated with the kinetic energy, the first term in the square bracket is the kinetic energy, the second term is contributed by the 2B interaction, and the

last term is the energy associated with the stabilization force, i.e., $v = 3$ for 3B repulsion and $5/2$ for quantum fluctuation. Moreover, $\tilde{g}_2 (> 0)$ and $\tilde{g}_v (> 0)$ are reduced interaction parameters (RIPs). From Eq. (49), the equilibrium size can be immediately found to be

$$\sigma = \sqrt{N} \left[\frac{(\nu - 1)\tilde{g}_v N}{\tilde{g}_2 N - 1} \right]^{1/[2(\nu-2)]} \quad (\text{for } N > 1/\tilde{g}_2), \quad (50)$$

which is a V-shaped function of N . More specifically, σ diverges as N approaches the CAN

$$N_{\text{cri}} \equiv \frac{1}{\tilde{g}_2}, \quad (51)$$

grows asymptotically as $\sigma \approx \sqrt{N}[(\nu - 1)\tilde{g}_v/\tilde{g}_2]^{1/[2(\nu-2)]}$ for large N , and reaches its minimal value at the DAN

$$N_{\text{dip}} \equiv \frac{\nu - 1}{(\nu - 2)\tilde{g}_2}. \quad (52)$$

In contrast, we note that the peak density, $n_p \propto N/\sigma^2 = \{(\tilde{g}_2 N - 1)/[(\nu - 1)\tilde{g}_v N]\}^{1/(\nu-2)}$, is always a monotonically increasing function of N and converges to a constant proportional to $\{\tilde{g}_2/[(\nu - 1)\tilde{g}_v]\}^{1/(\nu-2)}$ for large N . It is worthwhile to point out that $\sigma(N)$ depends not only on the stabilization mechanism through ν but also on the many-body wave function via the RIPs ($\tilde{g}_2, \tilde{g}_v$).

To carry out quantitative comparisons, we assume that all density profiles are of Gaussian form, i.e.,

$$\bar{\phi}(\mathbf{r}) = \frac{1}{\sqrt{\pi}\sigma} e^{-(x^2+y^2)/(2\sigma^2)}, \quad (53)$$

which yields the kinetic energy coefficient $\kappa = \hbar^2/(2M)$. More importantly, for pure coherent states, we obtain two sets of RIPs ($\tilde{g}_2^{(c)}, \tilde{g}_3^{(c)}$) and ($\tilde{g}_2^{(c)}, \tilde{g}_{5/2}^{(c)}$) corresponding to different stabilization mechanisms, where $\tilde{g}_2^{(c)} = M\delta g'/(\pi\hbar^2)$, $\tilde{g}_3^{(c)} = Mg'_3/(9\pi^2\hbar^2)$, and $\tilde{g}_{5/2}^{(c)} = 4Mg'_{5/2}/(5\pi^{3/2}\hbar^2)$. Furthermore, for the pure squeezed states with 3B interaction, the corresponding RIPs are ($\tilde{g}_2^{(s)}, \tilde{g}_3^{(s)}$), where $\tilde{g}_2^{(s)} = 3\tilde{g}_2^{(c)}$ and $\tilde{g}_3^{(s)} = 15\tilde{g}_3^{(c)}$.

Armed with this information, we plot droplet's radial size, $\sigma_v^{(c,s)}(N)$, using Eq. (50) in Fig. 6, where the subscript denotes the stabilization mechanism and the superscript denotes the quantum states. As can be seen, $\sigma_3^{(s)}$ agrees very well with σ_{GST} around the left dip where the condensate is a SVS. This agreement also indicates that the density profile of a SVS is well approximated by a Gaussian function. Around the right dip, the quantum state of the condensate is more complicated than a pure coherent state. Therefore $\sigma_3^{(c)}$ only exhibits rough agreement with σ_{GST} for large N where the quantum state is dominated by coherent component. Still, the remaining discrepancy can be explained as the density profile of the coherent state is significantly deviated from a Gaussian function. As to $\sigma_{5/2}^{(c)}$, it agrees well with σ_{EGPE} , which confirms the variational calculation. One should note that, although $\sigma_{5/2}^{(c)}$ and $\sigma_3^{(c)}$ have the same CANs, they define different DANs.

It is now clear that the W-shaped σ_{GST} stems from the multiple quantum phases in the self-bound droplets. Remarkably, this feature can be vaguely seen in the experimental data [5]. In fact, as shown in Figs. 1(b) and 1(c), after the completion

of the left dip, the experimental data of the radial size starts to decrease again as one further increases N . Since radial size will eventually grow as $\sqrt{N}$ in the liquid phase, the decrease of σ_{GST} after the first dip then signals the existence of the second dip and, consequently, the W shape. As the coherent-state-based EGPE theory only predicts a V-shaped radial size curve, a clear observation of the double dips may qualitatively exclude the EGPE explanation of the droplet states.

More interestingly, variational analyses can even lead to quantitative criteria for quantum phases and stabilization mechanisms. To see this, let us examine N_{cri} , the CAN extracted analytically from Eq. (50). At first sight, it may seem surprise that N_{cri} is independent of $\tilde{g}_v$. This can be understood as follows. Because ε_{kin} and ε_{2B} have the same power dependence on σ , the 2B attraction can be canceled out exactly by the kinetic energy when $N = N_{\text{cri}}$. Then, in the absence of stabilization force, a self-bound state can be of arbitrary radial size. Once the stabilization force is turned on, in order to minimize total energy, the radial size of the droplet has to be infinite such that the repulsive interaction energy vanishes. In this case, we have a self-bound droplet of infinite radial size. This universality of N_{cri} is highly nontrivial as it allows our theory to be examined by experimental data regardless of the accuracy of the fitted g_3 . As shall be shown, the above analysis is not applicable in 3D geometry where a self-bound droplet always has a finite size.

One should note that CAN still depends on the quantum state of the condensate through the RIP $\tilde{g}_2$. Then corresponding to the pure squeezed and coherent states, we have distinct CANs $N_{\text{cri}}^{(s)}$ and $N_{\text{cri}}^{(c)}$ which satisfy $N_{\text{cri}}^{(c)} = 3N_{\text{cri}}^{(s)}$. The precise measurements of CAN can therefore be used to distinguish the quantum state of the condensates. In Fig. 2(a), we show the experimentally measured CAN versus the reduced scattering length. From the rightmost three data points, it seems that the quantum state at the self-bound boundary is coherent state; while the quantum state for the other three data point is unclear. However, after carefully examining these data, we find that high precision measurements are still needed to accurately determine the quantum phase. In fact, to precisely determine CAN, the experimental data should be fitted using proper curves corresponding to appropriate stabilization mechanisms. In particular, the experimental data should be of V shape close to the self-bound boundary.

More remarkably, we note that even the stabilization mechanism can be identified using DAN N_{dip} , a quantity that depends on v . In fact, corresponding to three $\sigma_v^{(c,s)}(N)$, Eq. (52) indicates that there exist three distinct DANs satisfying $N_{\text{dip},3}^{(c)} = 3N_{\text{dip},3}^{(s)}$ and $N_{\text{dip},5/2}^{(c)} = (9/2)N_{\text{dip},3}^{(s)}$, where $N_{\text{dip},3}^{(s)} = 2/\tilde{g}_2^{(s)}$. Therefore precise measurement of DAN allows us to distinguish not only the quantum phase but also the stabilization mechanism. Moreover, N_{dip} is also independent of $\tilde{g}_v$, which makes the determination of the stabilization mechanism independent of the fitted g_3 .

Finally, we point out that some features of the radial size maintain even in the 3D self-bound binary droplets. As shown in Fig. 7(a), the GST result for the σ - N curve is also of W shape, which again is the manifestation of the SVS, SCS, and CS phases containing in the droplet state. And, similar to the quasi-2D cases, the SVS-to-SCS and SCS-to-CS transitions marked by the vertical dotted lines in Fig. 7(a) are of third

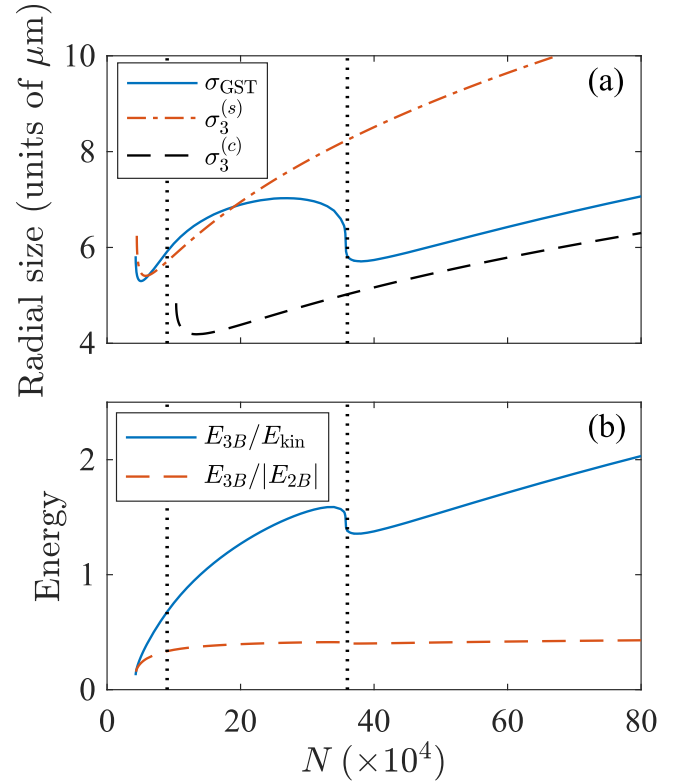


FIG. 7. (a) GST result of the N dependence of the radial size σ_{GST} for a self-bound 3D binary droplet with $\delta a = -3.062a_0$. Vertical dotted lines mark the phase boundaries for three quantum phases, i.e., SVS, SCS, and CS phases from left to right. The dash-dotted and dashed lines are the radial sizes for the SVS and CS phases obtained via variational analysis, respectively. (b) E_{3B}/E_{kin} and $E_{3B}/|E_{2B}|$ versus N for $\delta a = -3.062a_0$.

and first orders, respectively. Using a Gaussian density profile, we also perform the variational analysis for the radial size, which, combined with 3B repulsion, gives rise to two V-shaped curves for the SVS and CS phases in Fig. 7(a). As can be seen, $\sigma_3^{(s)}$ is in good agreement with σ_{GST} within the SVS phase. We would also like to point out that the CAN now depends on the stabilization force as a 3D self-bound droplet is always of finite size. For example, using the Gaussian variational analysis, the CAN of the SVS phase with 3B repulsion can be expressed as

$$N_{\text{cri}}^{(s)} = \frac{2\pi(\sqrt{a_{\uparrow\uparrow}} + \sqrt{a_{\downarrow\downarrow}})^4}{81a_{\uparrow\uparrow}a_{\downarrow\downarrow}\delta a^2} \sqrt{\frac{10Mg_3}{\sqrt{3}\pi^3\hbar^2}}. \quad (54)$$

However, as shown in Fig. 7(b), the relative contribution from the 3B interaction becomes less important when the atom number approaches CAN, which again indicates that the CAN is insensitive to the stabilization mechanism even for 3D droplets.

V. CONCLUSION AND DISCUSSION

We have performed an in-depth study on the quantum phases of self-bound droplets of quasi-2D binary condensates by presenting a detailed phase diagram. Among all phases found, SVS and SCS represent two macroscopic squeezed

states exhibiting novel statistical properties. For experimental detection of the quantum phase and the stabilization mechanism, we have found three easily accessible criteria: (i) the observation of multiple dips on the σ - N curve may rule out the single-quantum-state based explanation of the liquid droplets; (ii) the precise measurements of N_{cri} or N_{dip} further allow us to distinguish the squeezed and the coherent states; and (iii) we may even determine the stabilization mechanism from N_{dip} . Remarkably, because both N_{cri} and N_{dip} are independent of the strength of the stabilization force, the comparison between the theoretical and the experimental results immune to any unknown parameters. Finally, by noting that most experimental data were obtained in the parameter regime close to the self-bound boundary where droplets have lowest density and the role played by the stabilization force is of least importance, it becomes understandable that EGPE theory may provide satisfactory explanations to some experiments. This statement is also true for 3D droplets.

ACKNOWLEDGMENTS

We thank enlightening discussions with H. Hu, X.-J. Liu, C. R. Cabrera, C. Navarrete-Benlloch, T. Pfau, and H. P. Buechler. This work was supported by National Key Research and Development Program of China (Grant No. 2021YFA0718304), by the NSFC (Grants No. 12135018, No. 11974363, and No. 12047503), by the Strategic Priority Research Program of CAS (Grant No. XDB28000000), and by the Open Project of Shenzhen Institute of Quantum Science and Engineering (Grant No. SIQSE202006).

APPENDIX A: OPTIMAL SPIN COMPOSITION

Armed with information about the quantum phases, we can analytically derive the optimal atom number ratio, $N_{\uparrow}/N_{\downarrow} = \sqrt{a_{\downarrow\downarrow}/a_{\uparrow\uparrow}}$, by considering a homogeneous gas.

To this end, we first note that the atom number ratio is independent of the three-body interactions as we have assumed that the three-body coupling constant g_3 is independent of the atom species. Then, for both SVS and CS, the two-body interaction energy density takes the form

$$e_{2B} \propto (n_{\uparrow} \quad n_{\downarrow}) \begin{pmatrix} a_{\uparrow\uparrow} & a_{\uparrow\downarrow} \\ a_{\downarrow\uparrow} & a_{\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} n_{\uparrow} \\ n_{\downarrow} \end{pmatrix}, \quad (\text{A1})$$

where $n_{\uparrow}$ and $n_{\downarrow}$ are the densities for the spin- $\uparrow$ and $\downarrow$ components, respectively. Similar to Ref. [24], it can be shown that, in the small δa limit ($|\delta a| \ll a_{\uparrow\uparrow}, a_{\downarrow\downarrow}$), E_{2B} is minimized when $n_{\uparrow}/n_{\downarrow} = \sqrt{a_{\downarrow\downarrow}/a_{\uparrow\uparrow}}$.

Now, for SCS, the two-body interaction energy becomes

$$\begin{aligned} e_{2B} \propto & a_{\uparrow\uparrow}(n_{\uparrow}^{(c)2} + 3n_{\uparrow}^{(s)2} + 6n_{\uparrow}^{(c)}n_{\uparrow}^{(s)}) + a_{\downarrow\downarrow}(n_{\downarrow}^{(c)2} \\ & + 3n_{\downarrow}^{(s)2} + 6n_{\downarrow}^{(c)}n_{\downarrow}^{(s)}) + 2a_{\uparrow\downarrow}(n_{\uparrow}^{(c)}n_{\downarrow}^{(c)} + 3n_{\uparrow}^{(s)}n_{\downarrow}^{(s)} \\ & + n_{\uparrow}^{(c)}n_{\downarrow}^{(s)} + n_{\downarrow}^{(c)}n_{\uparrow}^{(s)} + 4\sqrt{n_{\uparrow}^{(c)}n_{\downarrow}^{(c)}n_{\uparrow}^{(s)}n_{\downarrow}^{(s)}}), \end{aligned} \quad (\text{A2})$$

where $n_{\alpha}^{(c)}$ and $n_{\alpha}^{(s)}$ are, respectively, the number densities of the coherent and the squeezed atoms for the α th spin component. Since the total density $n = \sum_{\alpha} [n_{\alpha}^{(c)} + n_{\alpha}^{(s)}]$ is a constant, we may express, e.g., $n_{\downarrow}^{(s)}$, in terms of other densities. As a

result, E_{2B} reduces to a function of $n_{\uparrow}^{(c)}$, $n_{\downarrow}^{(c)}$, and $n_{\uparrow}^{(s)}$. To minimize E_{2B} , we require

$$\frac{\partial e_{2B}}{\partial n_{\uparrow}^{(c)}} = \frac{\partial e_{2B}}{\partial n_{\downarrow}^{(c)}} = \frac{\partial e_{2B}}{\partial n_{\uparrow}^{(s)}} = 0, \quad (\text{A3})$$

which, in the small δa limit, give rise to the relation

$$\left(\sqrt{\frac{a_{\downarrow\downarrow}}{a_{\uparrow\uparrow}}} \frac{n_{\downarrow}^{(s)}}{n_{\uparrow}^{(s)}} - 1 \right) \left(\sqrt{\frac{a_{\downarrow\downarrow}}{a_{\uparrow\uparrow}}} \frac{n_{\downarrow}^{(c)}}{n_{\uparrow}^{(c)}} - 1 \right) = 0. \quad (\text{A4})$$

Therefore the necessary condition for E_{2B} to be minimized is

$$\frac{n_{\uparrow}^{(c)}}{n_{\downarrow}^{(c)}} = \sqrt{\frac{a_{\downarrow\downarrow}}{a_{\uparrow\uparrow}}} \text{ or } \frac{n_{\uparrow}^{(s)}}{n_{\downarrow}^{(s)}} = \sqrt{\frac{a_{\downarrow\downarrow}}{a_{\uparrow\uparrow}}}. \quad (\text{A5})$$

In fact, these two conditions are always satisfied simultaneously because if we substitute either of them into Eq. (A3), the other can be derived, which is exactly what we find in numerical simulations.

APPENDIX B: VIRIAL RELATION

Here we derive a virial relation among different contributions to the total energy. To this end, we shall make use of the fact that different spin states share same normalized density profile. Hence we can use $\bar{\phi}^{(c)}(\mathbf{r})$ to represent the normalized mode function for the coherent component and for i th squeezed mode the normalized mode functions are $\bar{\phi}_i^{(s)}(\mathbf{r})$.

Since the total energy E is stationary with respect to any variation of $\bar{\phi}^{(c)}(\mathbf{r})$ and $\bar{\phi}_i^{(s)}(\mathbf{r})$ around the ground state solution, we can choose a scaling transformation of the form $\phi(x, y) \rightarrow (1 + \lambda)^{1/2} \phi[(1 + \lambda)x, y]$. After applying this transformation to E_{kin} , E_{2B} , and E_{3B} , we obtain the variations of the energy components as follows:

$$\begin{aligned} \delta E_{\text{kin}} &= [(1 + \lambda)^2 - 1] E_{\text{kin}}^x, \\ \delta E_{2B} &= \lambda E_{2B}, \\ \delta E_{3B} &= [(1 + \lambda)^2 - 1] E_{3B}, \end{aligned} \quad (\text{B1})$$

where

$$\begin{aligned} E_{\text{kin}}^x &= -\frac{1}{2M} \int d\mathbf{r} \left(N^{(c)} \bar{\phi}^{(c)*}(\mathbf{r}) \frac{\partial^2}{\partial x^2} \bar{\phi}^{(c)}(\mathbf{r}) \right. \\ &\quad \left. + \sum_i N_i^{(s)} \bar{\phi}_i^{(s)*}(\mathbf{r}) \frac{\partial^2}{\partial x^2} \bar{\phi}_i^{(s)}(\mathbf{r}) \right). \end{aligned} \quad (\text{B2})$$

After imposing the constraint $\delta E_{\text{kin}}^x + \delta E_{2B} + \delta E_{3B} = 0$ to first order of λ , we obtain

$$2E_{\text{kin}}^x + E_{2B} + 2E_{3B} = 0. \quad (\text{B3})$$

Similar expression can be obtained by applying the same scaling transform in y directions, which leads to the virial relation Eq. (23) after summing over the x and y directions.

- [1] H. Kadau, M. Schmitt, M. Wenzel, C. Wink, T. Maier, I. Ferrier-Barbut, and T. Pfau, Observing the Rosensweig instability of a quantum ferrofluid, *Nature* **530**, 194 (2016).
- [2] M. Schmitt, M. Wenzel, F. Böttcher, I. Ferrier-Barbut, and T. Pfau, Self-bound droplets of a dilute magnetic quantum liquid, *Nature* **539**, 259 (2016).
- [3] L. Chomaz, S. Baier, D. Petter, M. J. Mark, F. Wächtler, L. Santos, and F. Ferlaino, Quantum-Fluctuation-Driven Crossover from a Dilute Bose-Einstein Condensate to a Macrodroplet in a Dipolar Quantum Fluid, *Phys. Rev. X* **6**, 041039 (2016).
- [4] F. Böttcher, M. Wenzel, J.-N. Schmidt, M. Guo, T. Langen, I. Ferrier-Barbut, T. Pfau, R. Bombín, J. Sánchez-Baena, J. Boronat, and F. Mazzanti, Dilute dipolar quantum droplets beyond the extended Gross-Pitaevskii equation, *Phys. Rev. Res.* **1**, 033088 (2019).
- [5] C. R. Cabrera, L. Tanzi, J. Sanz, B. Naylor, P. Thomas, P. Cheiney, and L. Tarruell, Quantum liquid droplets in a mixture of Bose-Einstein condensates, *Science* **359**, 301 (2018).
- [6] P. Cheiney, C. R. Cabrera, J. Sanz, B. Naylor, L. Tanzi, and L. Tarruell, Bright Soliton to Quantum Droplet Transition in a Mixture of Bose-Einstein Condensates, *Phys. Rev. Lett.* **120**, 135301 (2018).
- [7] G. Semeghini, G. Ferioli, L. Masi, C. Mazzinghi, L. Wolswijk, F. Minardi, M. Modugno, G. Modugno, M. Inguscio, and M. Fattori, Self-Bound Quantum Droplets of Atomic Mixtures in Free Space, *Phys. Rev. Lett.* **120**, 235301 (2018).
- [8] C. D'Errico, A. Burchianti, M. Prevedelli, L. Salasnich, F. Ancilotto, M. Modugno, F. Minardi, and C. Fort, Observation of quantum droplets in a heteronuclear bosonic mixture, *Phys. Rev. Res.* **1**, 033155 (2019).
- [9] A. Burchianti, C. D'Errico, M. Prevedelli, L. Salasnich, F. Ancilotto, M. Modugno, F. Minardi, and C. Fort, A dual-species Bose-Einstein condensate with attractive interspecies interactions, *Condens. Matter* **5**, 21 (2020).
- [10] G. Natale, R. M. W. van Bijnen, A. Patscheider, D. Petter, M. J. Mark, L. Chomaz, and F. Ferlaino, Excitation Spectrum of a Trapped Dipolar Supersolid and Its Experimental Evidence, *Phys. Rev. Lett.* **123**, 050402 (2019).
- [11] L. Tanzi, S. Roccuzzo, E. Lucioni, F. Famà, A. Fioretti, C. Gabbanini, G. Modugno, A. Recati, and S. Stringari, Supersolid symmetry breaking from compressional oscillations in a dipolar quantum gas, *Nature* **574**, 382 (2019).
- [12] G. Ferioli, G. Semeghini, L. Masi, G. Giusti, G. Modugno, M. Inguscio, A. Gallellí, A. Recati, and M. Fattori, Collisions of Self-Bound Quantum Droplets, *Phys. Rev. Lett.* **122**, 090401 (2019).
- [13] F. Böttcher, J.-N. Schmidt, M. Wenzel, J. Hertkorn, M. Guo, T. Langen, and T. Pfau, Transient Supersolid Properties in an Array of Dipolar Quantum Droplets, *Phys. Rev. X* **9**, 011051 (2019).
- [14] L. Tanzi, E. Lucioni, F. Famà, J. Catani, A. Fioretti, C. Gabbanini, R. N. Bisset, L. Santos, and G. Modugno, Observation of a Dipolar Quantum Gas with Metastable Supersolid Properties, *Phys. Rev. Lett.* **122**, 130405 (2019).
- [15] L. Chomaz, D. Petter, P. Ilzhöfer, G. Natale, A. Trautmann, C. Politi, G. Durastante, R. M. W. van Bijnen, A. Patscheider, M. Sohmen, M. J. Mark, and F. Ferlaino, Long-Lived and Transient Supersolid Behaviors in Dipolar Quantum Gases, *Phys. Rev. X* **9**, 021012 (2019).
- [16] M. Guo, F. Böttcher, J. Hertkorn, J.-N. Schmidt, M. Wenzel, H. P. Büchler, T. Langen, and T. Pfau, The low-energy goldstone mode in a trapped dipolar supersolid, *Nature* **574**, 386 (2019).
- [17] Z.-H. Luo, W. Pang, B. Liu, Y.-Y. Li, and B. A. Malomed, A new form of liquid matter: Quantum droplets, *Front. Phys.* **16**, 32201 (2021).
- [18] F. Böttcher, J.-N. Schmidt, J. Hertkorn, K. Ng, S. Graham, M. Guo, T. Langen, and T. Pfau, New states of matter with fine-tuned interactions: Quantum droplets and dipolar supersolids, *Rep. Prog. Phys.* **84**, 012403 (2021).
- [19] M. Guo and T. Pfau, A new state of matter of quantum droplets, *Front. Phys.* **16**, 32202 (2021).
- [20] T. D. Lee, K. Huang, and C. N. Yang, Eigenvalues and eigenfunctions of a bose system of hard spheres and its low-temperature properties, *Phys. Rev.* **106**, 1135 (1957).
- [21] R. Schützhold, M. Uhlmann, Y. Xu, and U. R. Fischer, Mean-field expansion in Bose-Einstein condensates with finite-range interactions, *Int. J. Mod. Phys. B* **20**, 3555 (2006).
- [22] A. R. P. Lima and A. Pelster, Quantum fluctuations in dipolar bose gases, *Phys. Rev. A* **84**, 041604(R) (2011).
- [23] A. R. P. Lima and A. Pelster, Beyond mean-field low-lying excitations of dipolar bose gases, *Phys. Rev. A* **86**, 063609 (2012).
- [24] D. S. Petrov, Quantum Mechanical Stabilization of a Collapsing Bose-Bose Mixture, *Phys. Rev. Lett.* **115**, 155302 (2015).
- [25] H. Saito, Path-integral Monte Carlo study on a droplet of a dipolar Bose-Einstein condensate stabilized by quantum fluctuation, *J. Phys. Soc. Jpn.* **85**, 053001 (2016).
- [26] A. Macia, J. Sánchez-Baena, J. Boronat, and F. Mazzanti, Droplets of Trapped Quantum Dipolar Bosons, *Phys. Rev. Lett.* **117**, 205301 (2016).
- [27] F. Cinti, A. Cappellaro, L. Salasnich, and T. Macrì, Superfluid Filaments of Dipolar Bosons in Free Space, *Phys. Rev. Lett.* **119**, 215302 (2017).
- [28] R. Bombin, J. Boronat, and F. Mazzanti, Dipolar Bose Supersolid Stripes, *Phys. Rev. Lett.* **119**, 250402 (2017).
- [29] V. Cikojević, L. V. Markić, G. E. Astrakharchik, and J. Boronat, Universality in ultradilute liquid Bose-Bose mixtures, *Phys. Rev. A* **99**, 023618 (2019).
- [30] M. Ota and G. E. Astrakharchik, Beyond Lee-Huang-Yang description of self-bound Bose mixtures, *SciPost Phys.* **9**, 020 (2020).
- [31] A. Boudjemâa and N. Guebli, Quantum correlations in dipolar droplets: Time-dependent Hartree-Fock-Bogoliubov theory, *Phys. Rev. A* **102**, 023302 (2020).
- [32] H. Hu, J. Wang, and X.-J. Liu, Microscopic pairing theory of a binary bose mixture with interspecies attractions: Bosonic BEC-BCS crossover and ultradilute low-dimensional quantum droplets, *Phys. Rev. A* **102**, 043301 (2020).
- [33] H. Hu and X.-J. Liu, Consistent Theory of Self-Bound Quantum Droplets with Bosonic Pairing, *Phys. Rev. Lett.* **125**, 195302 (2020).
- [34] Q. Gu and L. Yin, Phonon stability and sound velocity of quantum droplets in a boson mixture, *Phys. Rev. B* **102**, 220503(R) (2020).
- [35] Y. Wang, L. Guo, S. Yi, and T. Shi, Theory for self-bound states of dipolar Bose-Einstein condensates, *Phys. Rev. Res.* **2**, 043074 (2020).
- [36] T. Shi, J. Pan, and S. Yi, Trapped Bose-Einstein condensates with attractive s-wave interaction, [arXiv:1909.02432](https://arxiv.org/abs/1909.02432).

- [37] T. Guaita, L. Hackl, T. Shi, C. Hubig, E. Demler, and J. I. Cirac, Gaussian time-dependent variational principle for the Bose-Hubbard model, [Phys. Rev. B](#) **100**, 094529 (2019).
- [38] L. Hackl, T. Guaita, T. Shi, J. Haegeman, E. Demler, and J. I. Cirac, Geometry of variational methods: Dynamics of closed quantum systems, [SciPost Phys.](#) **9**, 048 (2020).
- [39] T. Shi, E. Demler, and J. I. Cirac, Variational study of fermionic and bosonic systems with non-Gaussian states: Theory and applications, [Ann. Phys.](#) **390**, 245 (2018).
- [40] N. M. Hugenholtz and D. Pines, Ground-state energy and excitation spectrum of a system of interacting bosons, [Phys. Rev.](#) **116**, 489 (1959).
- [41] A. Bulgac, Dilute Quantum Droplets, [Phys. Rev. Lett.](#) **89**, 050402 (2002).
- [42] C. Gerry and P. Knight, *Introductory Quantum Optics* (Cambridge University Press, New York, 2005).