

Optimal squeezing for quantum target detection

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It is not clear if the performance of a quantum lidar or radar, without an idler and only using Gaussian resources, could exceed the performance of a semiclassical setup based on coherent states and homodyne detection. Here we prove this is indeed the case by showing that an idler-free squeezed-based setup can beat this semiclassical benchmark. More generally, we show that probes whose displacement and squeezing are jointly optimized can strictly outperform coherent states with the same mean number of input photons for both the problems of quantum illumination and reading.

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Introduction. Quantum hypothesis testing [1–4] is one of the most important theoretical areas at the basis of quantum information science [5]. In the bosonic setting [6], some of the basic protocols are those of quantum illumination [7–19], aimed at better detecting the presence of a remote target in conditions of bright thermal noise, and quantum reading [20], aimed at boosting data retrieval from an optical digital memory. These protocols can be modeled as problems of quantum channel discrimination where quantum resources are able to outperform classical strategies in detecting different amounts of channel loss.

One of the basic benchmarks which is typically considered in assessing the quality of quantum illumination is the use of coherent states and homodyne detection. This is considered the best known (semi)classical strategy and is often adopted to assess the advantage of quantum resources (e.g., entanglement) [12,17] for lidar/radar applications [21–23]. This classical strategy is clearly based on Gaussian resources (i.e., Gaussian states and measurement) and does not involve any idler system. An open question is to determine if there is another idler-free strategy based on Gaussian resources which strictly outperforms the classical one.

In this work we answer this question positively, showing the advantage of using displaced-squeezed states with a suitably optimized amount of squeezing. Such optimal probes are able to outperform coherent states for the same number of mean signal photons per mode irradiated over the unknown target. While this can be shown for quantum illumination, i.e., quantum lidar applications, the advantage becomes more evident and useful in a different regime of parameters, as is typical for quantum reading.

Optimized probes for target detection. Consider the detection of a target in terms of a binary test: The null hypothesis

H_0 corresponds to target absent, while the alternative hypothesis H_1 corresponds to target present. These hypotheses correspond to the following quantum channels acting on a single-mode input state probing the target:

H_0 : A completely thermalizing channel, i.e., a channel replacing the input state with a thermal environment state with $\bar{n}_B$ mean photons.

H_1 : A thermal-loss channel with loss $1 - \eta$, so that only a fraction η of the signal photons survives, while $\bar{n}_B$ mean thermal photons are added to the state.

Both channels can be represented by a beam splitter with transmissivity η and input thermal noise $\bar{n}'_B := \bar{n}_B/(1 - \eta)$. We have $\eta = 0$ for H_0 and some $\eta > 0$ for H_1 . In terms of quadratures $\hat{x} = (\hat{q}, \hat{p})^T$ the action of the beam splitter is $\hat{x} \rightarrow \sqrt{\eta}\hat{x} + \sqrt{1 - \eta}\hat{x}_B$, where $\hat{x}_B$ is a background mode with $\bar{n}'_B$ mean photons.

As long as there is a different amount of loss between the two channels above, it is possible to perfectly discriminate between the two hypotheses if we are allowed to use input states with arbitrary energy. However, if we assume that the input states must have a mean number of photons equal to $\bar{n}_S$, then there is an error associated with the discrimination problem.

Consider a displaced squeezed state at the input of the unknown channel. Assume that this state has $\bar{n}_A$ photons associated with its amplitude α , namely, $\bar{n}_A = |\alpha|^2$. Without losing generality, assume that $\alpha \in \mathbb{R}$, so the mean value of the state is equal to $\bar{x} = (\sqrt{2\bar{n}_A}, 0)^T$ (see [24] for details on notation). The state has covariance matrix (CM) $\mathbf{V} = (1/2)\text{diag}(r, r^{-1})$ for position squeezing $r \leq 1$ ($= 1$ corresponding to a coherent state). It is easy to compute that the mean number of photons generated by the squeezing is equal to $f_r = (r + r^{-1} - 2)/4$. Thus, the mean total number of photons associated with the state is $\bar{n}_S = \bar{n}_A + f_r$. Note that, for fixed value of $\bar{n}_S$, the amount of squeezing is bounded within the range $r_- \leq r \leq 1$, where [25]

$$r_- := 2\bar{n}_S + 1 - 2\sqrt{\bar{n}_S(\bar{n}_S + 1)}. \quad (1)$$

Assume that the state is homodyned in the $\hat{q}$ quadrature (position). The outcome q will be distributed according to a

Gaussian distribution with mean value

$$\bar{q} = \sqrt{2(\bar{n}_S - f_r)} \geq 0 \quad (2)$$

and variance $\sigma^2 = r/2$. If homodyne is performed after the unknown beam-splitter channel, then we need to consider the transformations

$$\bar{q} \rightarrow \sqrt{\eta}\bar{q}, \sigma^2 \rightarrow \lambda_\eta^2 := \frac{2\bar{n}_B + 1 - \eta(1-r)}{2}. \quad (3)$$

By measuring the $\hat{q}$ quadrature for M times and adding the outcomes, the total variable z will be distributed according to a Gaussian distribution $P_\eta(z)$ with mean value $\bar{z} := M\sqrt{\eta}\bar{q} = M\sqrt{2\eta(\bar{n}_S - f_r)}$ and variance $\sigma_z^2 := M\lambda_\eta^2$. Note that, for H_0 , we have a Gaussian $P_0(z)$ centered in $\bar{z} = 0$ with variance $\sigma_z^2 = M\lambda_0^2 = (M/2)(2\bar{n}_B + 1)$. For H_1 , we have instead $P_1(z) = P_\eta(z)$ with $\eta > 0$.

Let us adopt a maximum likelihood test with some threshold value $t > 0$ (implicitly optimized), where we select H_1 if $z > t$ (otherwise we select the null hypothesis H_0). The false-alarm probability p_{FA} and the misdetection probability p_{MD} are therefore given by [26]

$$p_{FA} := \text{prob}(H_1|H_0) = \int_t^{+\infty} P_0(z)dz = \frac{1}{2} \left\{ 1 - \text{erf} \left[\frac{t}{\sqrt{M(2\bar{n}_B + 1)}} \right] \right\}, \quad (4)$$

$$p_{MD} := \text{prob}(H_0|H_1) = \int_{-\infty}^t P_1(z)dz = \frac{1}{2} \left\{ 1 + \text{erf} \left[\frac{t - M\sqrt{2\eta(\bar{n}_S - f_r)}}{\sqrt{M[2\bar{n}_B + 1 - \eta(1-r)]}} \right] \right\}. \quad (5)$$

For equal priors $\text{prob}(H_0) = \text{prob}(H_1) = 1/2$, the mean error probability is given by $p_{\text{err}} = (p_{FA} + p_{MD})/2$.

It is clear that the performance of the displaced squeezed states is at least as good as that of the coherent states, because the optimization over the squeezing parameter r (within the constraint imposed by $\bar{n}_S$) includes the point $r = 1$. The goal is therefore to show that some amount of squeezing can be useful to strictly outperform the coherent-state probes. For this purpose, the first step is to correctly quantify the amount of thermal noise $\bar{n}_B$ that is seen by a free-space lidar receiver.

Consider a receiver with aperture radius a_R , angular field of view Ω_{fov} (in steradians), detector bandwidth W , and spectral filter $\Delta\lambda$ (the latter can be very small thanks to the interferometric effects occurring at the homodyne detector). Compactly, we may define the photon collection parameter $\Gamma_R := \Delta\lambda W^{-1} \Omega_{\text{fov}} a_R^2$ (see Ref. [27] for more details). Considering that sky brightness at $\lambda = 800$ nm is $B_\lambda^{\text{sky}} \simeq 1.5 \times 10^{-1} \text{ W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$ [28,29] (in cloudy conditions), the mean number of thermal photons per mode hitting the receiver is

$$\bar{n}_B = \frac{\pi\lambda}{hc} B_\lambda^{\text{sky}} \Gamma_R. \quad (6)$$

Assuming $a_R = 10$ cm, $\Omega_{\text{fov}} \simeq 3 \times 10^{-6}$ sr ($\Omega_{\text{fov}}^{1/2} = 0.1^\circ$), $W = 100$ MHz, and $\Delta\lambda = 10^{-4}$ nm, we get $\bar{n}_B \simeq 5.8 \times 10^{-2}$ mean thermal photons per mode.

Let us take $\bar{n}_S = 0.1$ signal photons per mode and assume $\eta = 0.2$ for the reflectivity of the target (the latter quantity

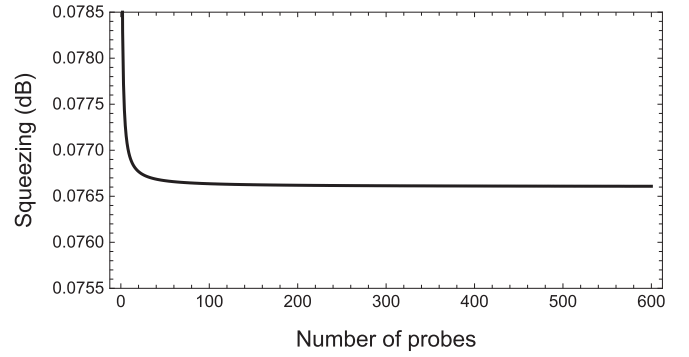


FIG. 1. Optimal squeezing $-10 \log_{10} r$ versus number of probes/modes M for the problem of target discrimination. Parameters are $\eta = 0.2$ for target present (otherwise $\eta = 0$), $\bar{n}_S = 0.1$ mean photons per signal mode, and $\bar{n}_B \simeq 5.8 \times 10^{-2}$ mean thermal photons per background mode. The threshold value t is implicitly optimized for each point.

implies either a proximity of the target or very good reflectivity properties, i.e., very limited diffraction at the target). For realistic values of $M \lesssim 10^3$ [21–23], we can see that the optimal probes are not coherent states but rather states that are both displaced and squeezed. For the regime of parameters considered, the difference is small but still very significant from a conceptual point of view. As we can see in Fig. 1, the amount of squeezing is small, i.e., less than 0.08 dB. (See [30] for the *Mathematica* files associated with this Letter.)

The significance of the result relies on the fact that the use of coherent states and homodyne detection might be considered to be the optimal Gaussian strategy for quantum illumination in the absence of idlers. This is not exactly true. One can find regimes of parameters where the presence of squeezing can strictly outperform coherent states, even if the advantage can be very small. As we discuss below, the difference becomes more appreciable in problems of quantum reading [20] or short-range quantum scanning [31], where the transmissivities associated with the hypotheses are relatively high.

Optimized probes for quantum reading or scanning. Note that the probabilities p_{FA} and p_{MD} discussed above can be extended to the general case where $P_0(z) = P_{\eta_0}(z)$ and $P_1(z) = P_{\eta_1}(z)$ for arbitrary $0 \leq \eta_0 \leq \eta_1 \leq 1$. In such a case, we just write $p_{FA} = \frac{1}{2}(1 - \Omega_0)$ and $p_{MD} = \frac{1}{2}(1 + \Omega_1)$, where we define (for $u = 0, 1$)

$$\Omega_u = \text{erf} \left(\frac{t - M\sqrt{2\eta_u(\bar{n}_S - f_r)}}{\sqrt{M[2\bar{n}_B + 1 - \eta_u(1-r)]}} \right). \quad (7)$$

This scenario can refer to the readout of an optical cell with two different reflectivities [20], or to the scan of a biological sample to distinguish between a blank from a contaminated sample [31].

For our numerical investigation, we consider high transmissivities $\eta_0 = 0.9$ and $\eta_1 = 0.98$, and relatively-high signal energy $\bar{n}_S = 1$. The other parameters are the same as above for target detection. Thus, we study the performance for equal-prior symmetric hypothesis testing, plotting the mean error probability p_{err} as a function of the number of probes M . As we can see from Fig. 2, the optimized displaced-squeezed

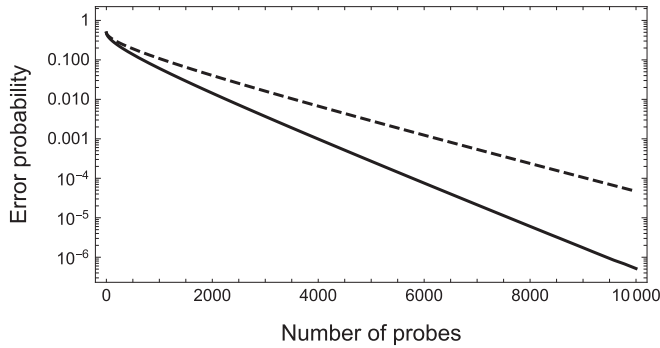


FIG. 2. Optimal displaced-squeezed probes for quantum reading and scanning. We plot the mean error probability achievable with the optimal displaced-squeezed probes (solid) with respect to just-displaced probes, i.e., coherent states (dashed). Parameters are $\eta_0 = 0.9$, $\eta_1 = 0.98$, $\bar{n}_S = 1$ mean photons per signal mode, and $\bar{n}_B \simeq 5.8 \times 10^{-2}$ mean thermal photons per background mode. The squeezing parameter r and the threshold value t are implicitly optimized for each point.

probes (here corresponding to about 4 dB of squeezing) clearly outperform coherent states with orders of magnitude advantage for increasing M .

We also consider asymmetric hypothesis testing [12,32,33] plotting the receiver operating characteristic (ROC), expressed by the misdetection probability versus the false-alarm probability for some fixed number of probes. As we can see from Fig. 3, for the case of $M = 500$, we have a clear advantage of the optimized probes with respect to coherent states. This behavior is generic and holds for other values of M .

Conclusions. In this work we have investigated the use of displaced-squeezed probes for problems of bosonic loss discrimination, i.e., quantum illumination and quantum reading. We have compared the performance of these probes with respect to that of purely displaced ones, i.e., coherent states,

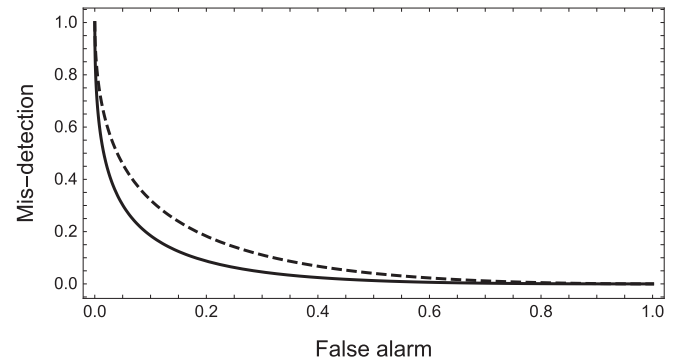


FIG. 3. Receiver operating characteristic (ROC) p_{MD} as a function of p_{FA} . We compare the performance of the optimal displaced-squeezed probes (solid) with respect to just-displaced probes, i.e., coherent states (dashed). Parameters are $M = 500$, $\eta_0 = 0.9$, $\eta_1 = 0.98$, $\bar{n}_S = 1$ mean photons per signal mode, and $\bar{n}_B \simeq 5.8 \times 10^{-2}$ mean thermal photons per background mode. The squeezing parameter r is implicitly optimized for each point.

showing that a strict advantage can be obtained by optimizing over the amount of squeezing while keeping the input mean number of photons as a constant. For the specific case of target detection, our results show that there exists an idler-free Gaussian-based detection strategy outperforming the typical (semi)classical benchmark considered in the literature, which is based on coherent states and homodyne detection. Due to the intrinsic Gaussian nature of the process, the dependence of the quantum advantage versus the various parameters is continuous and expected to be maintained in the presence of small experimental imperfections of the devices.

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- [1] C. W. Helstrom, *Quantum Detection and Estimation Theory*, Mathematics in Science and Engineering, Vol. 123 (Academic, New York, 1976).
- [2] A. Chefles, Quantum state discrimination, *Contemp. Phys.* **41**, 401 (2000).
- [3] U. Herzog and J. A. Bergou, Optimum unambiguous discrimination of two mixed quantum states, *Phys. Rev. A* **71**, 050301(R) (2005).
- [4] S. M. Barnett and S. Croke, Quantum state discrimination, *Adv. Opt. Photonics* **1**, 238 (2009).
- [5] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).
- [6] S. Pirandola, B. Roy Bardhan, T. Gehring, C. Weedbrook, and S. Lloyd, Advances in photonic quantum sensing, *Nat. Photonics* **12**, 724 (2018).
- [7] S. Lloyd, Enhanced sensitivity of photodetection via quantum illumination, *Science* **321**, 1463 (2008).
- [8] S. H. Tan, B. I. Erkmen, V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, S. Pirandola, and J. H. Shapiro, Quantum Illumination with Gaussian States, *Phys. Rev. Lett.* **101**, 253601 (2008).
- [9] E. D. Lopaeva, I. Ruo Berchera, I. P. Degiovanni, S. Olivares, G. Brida, and M. Genovese, Experimental Realization of Quantum Illumination, *Phys. Rev. Lett.* **110**, 153603 (2013).
- [10] Z. Zhang, M. Tengner, T. Zhong, F. N. C. Wong, and J. H. Shapiro, Entanglement's Benefit Survives an Entanglement-Breaking Channel, *Phys. Rev. Lett.* **111**, 010501 (2013).
- [11] S. Barzanjeh, S. Guha, C. Weedbrook, D. Vitali, J. H. Shapiro, and S. Pirandola, Microwave Quantum Illumination, *Phys. Rev. Lett.* **114**, 080503 (2015).
- [12] Q. Zhuang, Z. Zhang, and J. H. Shapiro, Entanglement enhanced Neyman-Pearson target detection using quantum illumination, *J. Opt. Soc. Am. B* **34**, 1567 (2017).
- [13] G. De Palma and J. Borregaard, Minimum error probability of quantum illumination, *Phys. Rev. A* **98**, 012101 (2018).

- [14] C. W. Sandbo Chang, A. M. Vadiraj, J. Bourassa, B. Balaji, and C. M. Wilson, Quantum-enhanced noise radar, *Appl. Phys. Lett.* **114**, 112601 (2019).
- [15] S. Barzanjeh, S. Pirandola, D. Vitali, and J. M. Fink, Experimental microwave quantum illumination, *Sci. Adv.* **6**, eabb0451 (2020).
- [16] R. Nair and M. Gu, Fundamental limits of quantum illumination, *Optica* **7**, 771 (2020).
- [17] A. Karsa, G. Spedalieri, Q. Zhuang, and S. Pirandola, Quantum Illumination with a generic Gaussian source, *Phys. Rev. Research* **2**, 023414 (2020).
- [18] S.-Y. Lee, Y. S. Ihn, and Z. Kim, Quantum illumination via quantum-enhanced sensing, *Phys. Rev. A* **103**, 012411 (2021).
- [19] H. Yang, W. Roga, J. D. Pritchard, and J. Jeffers, Gaussian state-based quantum illumination with simple photodetection, *Opt. Express* **29**, 8199 (2021).
- [20] S. Pirandola, Quantum Reading of a Classical Digital Memory, *Phys. Rev. Lett.* **106**, 090504 (2011).
- [21] I. S. Merrill, *Introduction to Radar Systems* (McGraw-Hill, New York, 1981).
- [22] J. I. Marcum, *Statistical Theory of Target Detection by Pulsed Radar: Mathematical Appendix*, RAND Corporation, Santa Monica, CA, Research Memorandum RM-753, July 1, 1948. Reprinted in IRE Trans. Info. Theory IT-6, 59 (1960).
- [23] W. Albersheim, A closed-form approximation to Robertson's detection characteristics, *Proc. IEEE* **69**, 839 (1981).
- [24] We work in the standard quantum optics notation, so the annihilation operator reads $\hat{a} = (\hat{q} + i\hat{p})/\sqrt{2}$ in terms of the quadratures $\hat{q}$ (position) and $\hat{p}$ (momentum), with $[\hat{q}, \hat{p}] = i$. This means that the amplitude reads $\alpha = (q + ip)/\sqrt{2}$, so the mean value is given by $\bar{x} = (q, p)^T = \sqrt{2}(\text{Re } \alpha, \text{Im } \alpha)^T$. Then, the number operator takes the form $\hat{n} := \hat{a}^\dagger \hat{a} = (\hat{q}^2 + \hat{p}^2 - 1)/2$, implying that a displaced and q - (or p -)squeezed state has mean number of photons $\bar{n} = (\langle \hat{q}^2 \rangle + \langle \hat{p}^2 \rangle - 1)/2 = |\alpha|^2 + [\text{var}(\hat{q}) + \text{var}(\hat{p}) - 1]/2$. Finally, note that the quadrature variance of a thermal state corresponds to $\bar{n} + 1/2$, where $\bar{n}$ is the mean number of photons and $1/2$ is the vacuum shot noise.
- [25] Note that, in general, we write $r_- \leq r \leq r_+$, where $r_+ := 2\bar{n}_S + 1 + 2\sqrt{\bar{n}_S(\bar{n}_S + 1)} \geq 1$. However, since we consider position squeezing in our analysis, we implicitly have $r \leq 1$, so the upper bound r_+ is just replaced by 1.
- [26] Recall that, for a Gaussian $g(z) = (\sigma\sqrt{2\pi})^{-1} \exp[-(z - \bar{z})^2/(2\sigma^2)]$ we have $\int_t^{+\infty} g(z)dz = (1 - \Omega)/2$ and $\int_{-\infty}^t g(z)dz = (1 + \Omega)/2$ where $\Omega := \text{erf}[(t - \bar{z})/(\sigma\sqrt{2})]$.
- [27] S. Pirandola, Limits and security of free-space quantum communications, *Phys. Rev. Research* **3**, 013279 (2021).
- [28] E.-L. Miao, Z.-F. Han, S.-S. Gong, T. Zhang, D.-S. Diao, and G.-C. Guo, Background noise of satellite-to-ground quantum key distribution, *New J. Phys.* **7**, 215 (2005).
- [29] C. Liorni, H. Kampermann, and D. Brūß, Satellite-based links for quantum key distribution: beam effects and weather dependence, *New J. Phys.* **21**, 093055 (2019).
- [30] See https://github.com/softqanta/squeezed_radar for the *Mathematica* file.
- [31] G. Spedalieri, L. Piersimoni, O. Laurino, S. L. Braunstein, and S. Pirandola, Detecting and tracking bacteria with quantum light, *Phys. Rev. Research* **2**, 043260 (2020).
- [32] G. Spedalieri and S. L. Braunstein, Asymmetric quantum hypothesis testing with Gaussian states, *Phys. Rev. A* **90**, 052307 (2014).
- [33] K. Li, Second-order asymptotics for quantum hypothesis testing, *Ann. Stat.* **42**, 171 (2014).