

Trade-off between path entanglement and quantum sensitivity

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Entanglement often increases quantum measurement schemes' sensitivity. However, we find that in precision measurements with zero-mean Gaussian states, such as squeezed states, entanglement between different paths degrades measurement sensitivity. We prove an inverse relationship between entanglement entropy and sensitivity for measurements of single-mode phase shifts in multimode systems and for phase shifts on both modes in two-mode systems. In the two-mode case, which models devices such as interferometers, we find that entanglement strongly degrades differential phase sensitivity. Finally, we show that minimizing entanglement between paths maximizes the phase sensitivity of N -mode systems with zero-mean Gaussian state inputs.

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I. INTRODUCTION

Entanglement is a hallmark of quantum theory [1,2], critical for cutting-edge applications such as quantum computation, teleportation, and imaging [3–5]. Entanglement also plays a key role in quantum metrology, which studies how to use quantum states to achieve measurement sensitivities impossible using comparable classical resources [6–10]. It is of fundamental interest to quantitatively describe how entanglement affects measurement sensitivity.

We give such a description in the context of interferometry with zero-mean Gaussian states. Interferometers often employ Gaussian states because these states are experimentally feasible to create [11–14] and arise naturally in the description of linear quantum systems [15–17]. Zero-mean Gaussian states are of particular interest because in standard two-path interferometers, they achieve the optimal differential phase sensitivity among all initially unentangled input states with a fixed mean photon number [18].

In interferometers utilizing displaced (nonzero mean) or non-Gaussian states such as NOON states, differential phase sensitivity often improves as path entanglement increases [19–23]. We show that the opposite is true for zero-mean Gaussian states: The optimal sensitivity varies inversely with the von Neumann entanglement entropy between the two paths. Moreover, the differential phase sensitivity is most reduced by path entanglement, while the common phase sensitivity is unique in escaping this trade-off.

We generalize the trade-off to N -mode interferometry with zero-mean Gaussian states. We determine the states that optimize sensitivity, quantified through their quantum Fisher

information (QFI), and find that this sensitivity bound can be reached with zero-mean Gaussian states that possess no path entanglement.

Earlier work in Refs. [24,25] hinted that path entanglement could only provide limited gains in path-symmetric states. Meanwhile, our setting contrasts with the scenario of multi-parameter estimation, where path entanglement is sometimes beneficial for Gaussian states [26]. Finally, we distinguish our work from that of Refs. [27,28], which assessed the impact of entanglement by comparing specific, non-Gaussian states transforming under different unitaries. We isolate the role of entanglement by comparing families of Gaussian states subjected to the same phase-shift unitary, revealing a quantitative trade-off between path entanglement and optimal sensitivity.

II. GAUSSIAN STATES AND QUANTUM FISHER INFORMATION

The state inside an interferometer with N paths can be modeled as an N -mode bosonic system, which is described by N pairs of canonically conjugate operators [16,29],

$$\hat{q} = (\hat{x}_1, \hat{p}_1, \dots, \hat{x}_N, \hat{p}_N)^T := (\hat{q}_1, \dots, \hat{q}_{2N})^T. \quad (1)$$

Setting $\hbar = 2$, the nonzero canonical commutation relations are $[\hat{x}_a, \hat{p}_a] = 2i$. We call the state zero mean whenever $\langle \hat{x}_a \rangle = \langle \hat{p}_a \rangle = 0$. In this paper, we will assume that indices i, j, k range from 0 to $2N$, while indices a, b range from 0 to N .

The covariance matrix V of a zero-mean state is defined as the symmetric matrix $V_{ij} := \langle \{\hat{q}_i, \hat{q}_j\} \rangle / 2$. V_{ii} is the variance of $\hat{q}_i$, and V_{ij} is the covariance between $\hat{q}_i$ and $\hat{q}_j$. In this paper, we focus on zero-mean Gaussian states, which are completely characterized by their covariance matrix V [16,29,30]. This is equivalent to how multivariate Gaussian probability distributions are completely characterized by their means and covariances. In fact, as shown in Sec. II of the Supplemental Material [33], a pure, zero-mean Gaussian state is uniquely specified among all quantum states by its covariance matrix and the zero-mean condition.

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Our input state is a collection of independent squeezed states with squeezing levels $r_1, \dots, r_N$, characterized by the covariance matrix

$$\mathbf{V}_{\text{in}} = \text{diag}(e^{-2r_1}, e^{2r_1}, \dots, e^{-2r_N}, e^{2r_N}). \quad (2)$$

As discussed in Sec. II of the Supplemental Material [33], any pure, zero-mean Gaussian state can be generated by acting on a covariance matrix in the form of $\mathbf{V}_{\text{in}}$ by a passive transformation, i.e., a photon number preserving transformation that takes Gaussian states to Gaussian states. Each passive transformation is described by a symplectic orthogonal matrix $\mathbf{K}$. This transformation takes the input state $\mathbf{V}_{\text{in}}$ to a new state with covariance matrix $\mathbf{V}$ satisfying

$$\mathbf{V} = \mathbf{K}\mathbf{V}_{\text{in}}\mathbf{K}^{-1}. \quad (3)$$

We note that any passive transformation can be expressed as a sequence of beam splitters and phase shifts [31,32].

We are interested in the sensitivity of measuring small phase shifts on a zero-mean Gaussian state with covariance matrix $\mathbf{V}$. Specifically, assume each mode a receives a phase shift of magnitude $g_a\phi$, and we want to measure ϕ . For instance, $g_1 = -g_2 = 1$ represents a differential phase shift between modes 1 and 2. While we focus on phase shifts for concreteness, we note that similar results hold for any parametrized passive symplectic transformation (see Sec. IV of the Supplemental Material [33]).

The state's sensitivity to a phase shift parametrized by ϕ can be quantified through the QFI. The QFI describes how fast the state changes with respect to ϕ [34]. Specifically, fix $\phi = \phi_0$ and let the QFI be $H(\phi_0)$. Let ϕ_{est} be an estimator of ϕ satisfying $\partial_\phi \langle \phi_{\text{est}} \rangle|_{\phi=\phi_0} = 1$, and assume the estimator is based on a single positive operator-valued measurement (POVM), the most general type of measurement allowed by quantum mechanics [35]. The quantum Cramer-Rao bound states that the variance of ϕ_{est} must satisfy $\text{Var } \phi_{\text{est}} \geq 1/\sqrt{H(\phi_0)}$ [34,36–38]. In the Bayesian framework, if ϕ is tightly centered around ϕ_0 , the left-hand side can also essentially be replaced with the mean-squared error in any estimate of ϕ through the van Trees inequality [39]. The upshot is that higher QFIs yield better sensitivity.

As derived in Sec. I of the Supplemental Material [33] using properties of symplectic matrices, the QFI of a pure, zero-mean Gaussian state rotating under the phase shifts $g_a\phi$ is

$$H = \frac{\text{Tr}((\mathbf{V}\mathbf{G})^2 - \mathbf{G}^2)}{2}, \quad \text{where} \quad \mathbf{G} := \text{diag}(g_1, g_1, g_2, g_2, \dots, g_N, g_N). \quad (4)$$

Note that this is independent of ϕ_0 . For mixed states, the equality becomes an upper bound, $H \leq \text{Tr}((\mathbf{V}\mathbf{G})^2 - \mathbf{G}^2)/2$. We will argue that entanglement between modes generically decreases the QFI.

III. TRADE-OFF IN THE SINGLE PHASE SHIFT SCENARIO

A challenge in formulating the trade-off between QFI and path entanglement for multimode systems is that multipartite entanglement is difficult to define [40]. The situation

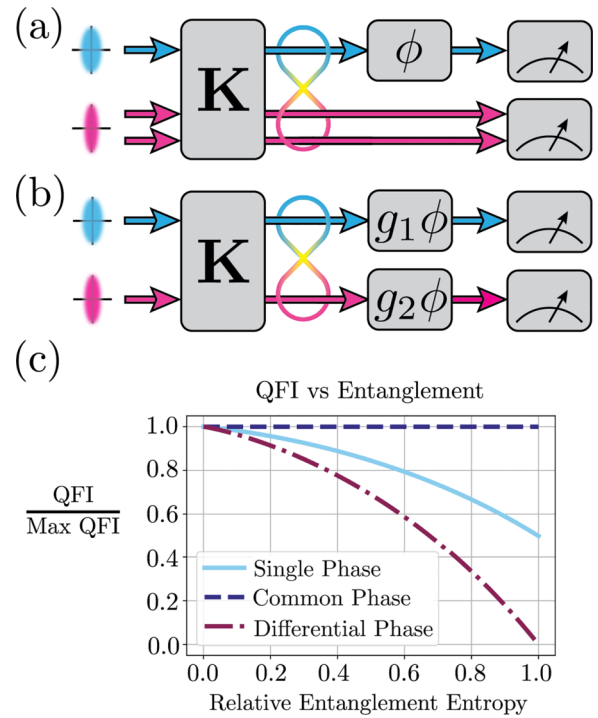


FIG. 1. The trade-off between entanglement and QFI. (a) A collection of squeezed states combine through an arbitrary multimode passive transformation $\mathbf{K}$, which generally entangles the modes. One mode then receives a phase shift ϕ . Finally, all modes are measured with some POVM. (b) A pair of modes are combined by an arbitrary multimode passive transformation $\mathbf{K}$, which generally entangles the modes. The first mode receives a phase shift $g_1\phi$ and the second receives a phase shift $g_2\phi$. (c) Plots the QFI from scenario (b), where the passive transformation corresponds to phase shifting one of the modes by θ and then mixing the two modes on a 50/50 beam splitter. θ controls the entanglement. As the entanglement increases, the common phase QFI is unaffected, while the single and differential phase QFIs decrease. This plot takes $r_1 = r_2 = 1/2$.

simplifies when there is a natural way to partition the system's modes into two sets.

We begin by considering the case when only one mode out of N receives a phase shift, as shown in Fig. 1(a). Here, the natural bipartition consists of the phase-shifted mode and the rest of the system. For convenience, assume it is the first mode that receives the phase shift, so $g_1 = 1$, $g_a = 0 \forall a > 1$. Defining $\mathbf{V}_1$ as the upper left 2×2 block of $\mathbf{V}$, Eq. (4) becomes

$$H = \frac{\|\mathbf{V}_1\|^2}{2} - 1, \quad (5)$$

where $\|\cdot\|^2$ denotes the Frobenius norm squared, the sum of the squares of the entries. In particular,

$$\|\mathbf{V}_1\|^2 = \text{Tr}(\mathbf{V}_1)^2 - 2\det(\mathbf{V}_1). \quad (6)$$

The trace and determinant have physical meaning. By definition, the phase-shifted mode has mean photon number $n_1 = \langle \hat{x}_1^2 + \hat{p}_1^2 \rangle/4 - 1/2$, so the trace of this mode's covariance matrix is given by $\text{Tr}(\mathbf{V}_1) = 4n_1 + 2$.

On the other hand, the determinant, $\det(\mathbf{V}_1)$, is related to the entanglement between the phase-shifted mode and the rest of the system. Let ρ_1 be the reduced state achieved from

tracing out all but the phase-shifted mode, and $\mu = \text{Tr}(\rho_1^2)$ its purity. The purity of this one-mode state monotonically decreases with its von Neumann entropy S , the standard measure of entanglement of a bipartite system. This follows from $S = \frac{1-\mu}{2\mu} \ln(\frac{1+\mu}{1-\mu}) - \ln(\frac{2\mu}{1+\mu})$ [30]. Furthermore, the purity is related to the determinant as $\det(V_1) = 1/\mu^2$ [30,41].

Using the formulas for trace and determinant in Eq. (6), our first main result follows from Eq. (5):

$$H = 8n_1^2 + 8n_1 - \left(\frac{1}{\mu^2} - 1\right) := H_{\text{sqz}}(n_1) - \left(\frac{1}{\mu^2} - 1\right), \quad (7)$$

where $H_{\text{sqz}}(n) := 8n^2 + 8n$ is the QFI of a single squeezed state of photon number n receiving a phase shift of ϕ [see above Eq. (16) or Refs. [38,42]]. Now, $1/\mu^2 - 1$ monotonically increases with the entanglement between the phase-shifted mode and the rest of the system, yielding a trade-off between entanglement and sensitivity.

Since $\mu \leq 1$, we also find $H \leq H_{\text{sqz}}(n_1)$. Equality holds only when the phase-shifted mode is completely unentangled with the rest of the system, since then $\mu = 1$. In this unentangled case, H reduces to the QFI of a one-mode squeezed state. This makes sense because, without entanglement, the rest of the system is entirely unaffected by the phase shift, so the situation reduces to a one-mode setup. Then, Eq. (7) expresses that the larger the entanglement, the further we get from this optimum.

Our upper bound still holds for mixed states, because Eq. (4) remains an upper bound. For pure states, we also have the following *lower* bound:

$$H = \frac{\|V_1\|^2}{2} - 1 \geq \frac{\text{Tr}(V_1)^2}{4} - 1 = 4n_1^2 + 4n_1 = \frac{H_{\text{sqz}}(n_1)}{2}. \quad (8)$$

The inequality follows from the fact that $\det(V_1) \leq \|V_1\|^2/2$, which can be shown using the Cauchy-Schwarz inequality and Eq. (6). The lower bound is saturated when V_1 is proportional to the identity, or in other words, when it represents a thermal state. Hence, this lower bound is achieved when the phase-shifted mode is maximally entangled with the rest of the system, such as for an EPR state.

Putting our two bounds together, we see that for pure states, H is constrained within a factor of 2 as

$$\frac{1}{2}H_{\text{sqz}}(n_1) \leq H \leq H_{\text{sqz}}(n_1). \quad (9)$$

IV. TRADE-OFF IN THE TWO-MODE SCENARIO

In the preceding section, the state had an arbitrary number of modes and was bipartite because only one mode was phase-shifted. Here, we instead consider two-mode states, as depicted in Fig. 1(b). Two-mode states are naturally bipartite, so we can allow a phase shift on both modes.

With two modes, Eq. (4) yields (see Sec. I of the Supplemental Material [33])

$$H = (g_1^2 - g_1g_2)H_1 + (g_2^2 - g_1g_2)H_2 + g_1g_2H_{\text{com}}, \quad (10)$$

where we have defined H_a for $a \in \{1, 2\}$ to be the QFI from a phase shift on just the a th mode, i.e., $g_a = 1$ and $g_{b \neq a} = 0$. Similarly, H_{com} is the QFI from a common phase shift, where

$g_1 = g_2 = 1$. H_1 and H_2 obey Eq. (7) and are therefore affected by the entanglement between the two modes.

Furthermore, Eq. (4) also yields $H_{\text{com}} = \text{Tr}(V^2)/2 - 2$. This is independent of entanglement: Using Eq. (3), we have $\text{Tr}(V_{\text{in}}^2) = \text{Tr}(V^2)$, so H_{com} depends only on the initial squeezed states and not the entangling transformation K . Moreover, we can compute H_{com} of the initial input states by adding their QFIs, since the inputs are independent [43]: $H_{\text{com}} = H_{\text{sqz}}(n_{\text{in},1}) + H_{\text{sqz}}(n_{\text{in},2})$. In summary, Eq. (10) decomposes the QFI into the sum of three QFIs based on simpler scenarios.

We can make the trade-off between QFI and entanglement clearer by substituting Eq. (7) into Eq. (10) and noting that tracing out either mode yields the same purity, μ , by the Schmidt decomposition [44]. We obtain

$$H = g_1g_2(H_{\text{sqz}}(n_{\text{in},1}) + H_{\text{sqz}}(n_{\text{in},2})) + (g_1^2 - g_1g_2)H_{\text{sqz}}(n_{\text{rot},1}) + (g_2^2 - g_1g_2)H_{\text{sqz}}(n_{\text{rot},2}) - (g_1 - g_2)^2 \left(\frac{1}{\mu^2} - 1\right), \quad (11)$$

where $n_{\text{in},a}$ and $n_{\text{rot},a}$ are the mean photon numbers in mode a of the initial state, V_{in} , and the entangled state, V , respectively. Recall $1/\mu^2 - 1$ is monotonic with the entanglement between the two modes. Therefore, the sensitivity of differential phase measurements, $g_1 = -g_2$, is most damaged by entanglement. The common phase measurement, $g_1 = g_2$, is unique in escaping the trade-off between entanglement and sensitivity. Indeed, because the common phase transformation commutes with the entangling transformation K , path entanglement cannot affect its QFI.

The formula simplifies when the initial squeezed states are equally squeezed and thus have the same mean photon number. After the entangling transformation K , the mean photon numbers of the two modes will remain the same (see Sec. III B of the Supplemental Material [33]). Therefore, we can write $n_{\text{in},1} = n_{\text{rot},1} = n_{\text{in},2} = n_{\text{rot},2} := n$, simplifying Eq. (11) to

$$H = (g_1^2 + g_2^2)H_{\text{sqz}}(n) - (g_1 - g_2)^2 \left(\frac{1}{\mu^2} - 1\right). \quad (12)$$

Additionally, for the maximally path entangled EPR state, the saturation of the lower bound from Eq. (9) yields $1/\mu^2 - 1 = H_{\text{sqz}}(n)/2$, so Eq. (12) simplifies further to

$$H = \frac{(g_1 + g_2)^2}{2}H_{\text{sqz}}(n). \quad (13)$$

Thus, EPR states are sensitive to common phase shifts, where $g_1 = g_2$, but completely insensitive to differential phase shifts, where $g_1 = -g_2$.

Figure 1(c) plots the sensitivity of a pair of equally squeezed states to single, common, and differential phase shifts as the path entanglement is varied. As expected, the common phase sensitivity is independent of entanglement. Meanwhile, when the state is maximally path entangled, the single-mode phase sensitivity is halved, while the differential phase sensitivity vanishes.

V. DECOUPLED SQUEEZED STATES OPTIMIZE THE QFI

In the N -mode case with general g_a values, many relevant notions of path entanglement exist due to the lack of a natural

bipartition, making it challenging to give a formula for the QFI as a function of path entanglement. However, there is always a well-defined notion of *no* entanglement: the state decomposes as a tensor product. We show that the QFI attains its maximum when path entanglement is absent.

To formalize this, we say V is *decoupled* when the covariances vanish between all pairs of modes. In our setup, this condition is equivalent to the absence of path entanglement:

Proposition 1. $V = KV_{\text{in}}K^{-1}$ is decoupled if and only if the state is a tensor product of squeezed states with V_{in} 's input squeezing levels, $r_1 \cdots r_N$, in some order and orientation.

For a proof, see Sec. III A of the Supplemental Material [33].

We say a state is *properly ordered* if for any modes a and b , if $g_a^2 > g_b^2$, then their mean photon numbers satisfy $n_a \geq n_b$. For convenience, label the modes such that $g_1^2 \geq \cdots \geq g_N^2$. Furthermore, by absorbing permutations of modes into K , we may assume the input squeezing levels satisfy $r_1 \geq \cdots \geq r_N$. We have (proof in Sec. III of the Supplemental Material [33]):

Theorem 1. Optimal QFI states are exactly those that can be reached from a decoupled, properly ordered state by applying passive transformations within modes having the same g_a . These optimal QFI states have

$$H = 2 \sum_{a=1}^N g_a^2 \sinh^2(2r_a). \quad (14)$$

For instance, since V_{in} is already decoupled and properly ordered, it is an optimal QFI state; we need not perform a passive transformation at all to reach the optimal QFI. We also note that if the g_a are distinct, the passive transformations in Theorem 1 can only be phase shifts. Thus, every optimal QFI state in this case is decoupled.

Theorem 1 demarcates the maximal phase sensitivity achievable with any passive transformation based on the initial collection of squeezed states. Constraining the squeezing of the initial states to at most r , the bound becomes

$$H \leq \|G\|^2 \sinh^2(2r). \quad (15)$$

A different QFI bound has been derived in Ref. [45] by constraining the total mean photon number instead of the mean photon number per mode, as we do here. The total photon number constraint results in a bound with a similar form to ours, with the Schatten 2-norm of Eq. (15) replaced with a Schatten- ∞ norm ([46], p. 7). Experimental limitations tend to set a bound on the mean photon number per mode as opposed to the total mean photon number because the squeezing level per mode is constrained by loss and phase noise [47], whereas there is no similar experimental constraint on the total mean photon number in a multimode system.

We also note that Eq. (15) breaks down if we remove the zero-mean condition. This is because the maximal squeezing level present in the initial squeezed states cannot be increased by a passive transformation, whereas if the initial states were displaced, passive transformations could increase their maximal displacement. As a result, with enough modes, it can become advantageous to inject displaced states instead of squeezed vacuum states, evading the bound. Indeed, taking $g_1 = 1$ and $g_a = 0 \forall a > 1$, the right-hand side of Eq. (15) is independent of N . However, with displacement in the

initial states, K can siphon all the displacement into the phase-shifted mode, ensuring the QFI scales as N . Then with enough modes, the displacement strategy will become more effective.

As another corollary of Theorem 1, since a one-mode squeezed state is trivially decoupled and properly ordered, its QFI immediately follows as $H = 2g_1^2 \sinh^2(2r)$. Since for a one-mode squeezed state, $n = \sinh^2(r)$, by taking $g_1 = 1$, we find $H_{\text{sqz}}(n) = 8n^2 + 8n$. In general, letting $n_{\text{in},i}$ be the mean photon number in mode i , we could then rewrite the optimal QFI as

$$H = \sum_{a=1}^N g_a^2 H_{\text{sqz}}(n_{\text{in},a}). \quad (16)$$

This is the sum of the QFIs from each mode of a decoupled, properly ordered state, as expected by the additivity of the QFI when both the state and unitary have an appropriate tensor product structure [[43], Eq. (72)].

Theorem 1 continues to hold in the case of equal losses ϵ on all modes when the input states have equal squeezing levels. In this case, the QFI is modified to (see Sec. V of the Supplemental Material [33])

$$H \rightarrow \frac{(1-\epsilon)^2}{1+2\epsilon(1-\epsilon)\sinh^2(r)} H. \quad (17)$$

The theorem still holds in this case because the prefactor is independent of the passive transformation.

Finally, we can gain intuition for why path entanglement degrades phase sensitivity in zero-mean Gaussian states by comparing squeezed states with Fock (number) states. Fock states are invariant with respect to phase shifts, but path entanglement breaks this invariance, improving their phase sensitivity. In contrast, in phase space, the initial squeezed states have their major and minor axes in planes acted on by phase shifts. Gaussian path-entangling transformations act as higher-dimensional rotations, tilting the axes out of those planes. This means that phase shifts no longer rotate the most antisqueezed major axes into the most squeezed minor axes, effectively reducing the amount of squeezing and, hence, their phase sensitivity.

VI. CONCLUSION

In this paper, we elucidate a trade-off between entanglement among the modes of a zero-mean Gaussian system and the sensitivity of the system to phase shifts. In fact, the analysis applies to the broader class of parametrized passive symplectic transformations, as shown in Sec. IV of the Supplemental Material [33]. In the case of measuring phase shifts, we find that unless two paths receive the same phase shift, entanglement between these paths degrades the QFI. We quantify the trade-off using entanglement entropy for naturally bipartite systems, these being two-mode Gaussian states and N -mode Gaussian states where only one path is phase shifted. Finally, in the general N -mode case, we determine all optimal QFI states, showing they have no entanglement between paths receiving different phase shifts. The value of this optimal QFI is of independent interest. Fixing the g_a in Eq. (16) and taking the limit of large squeezing

ratios r_i , the QFI scales as the photon number squared, which represents Heisenberg scaling.

Lastly, we note that our analysis uses a frequency-independent quantum optics formalism, relevant for experiments with pulsed light or continuous-wave light without frequency-dependent elements such as optical cavities [15,16]. A full description of experiments employing continuous-wave Gaussian states requires a frequency-dependent extension of this formalism [17,48–51]. We leave this for future studies.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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