

Growth of block-diagonal operators and symmetry-resolved Krylov complexity

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This work addresses how the growth of invariant operators is influenced by their underlying symmetry structure. For this purpose, we introduce the symmetry-resolved Krylov complexity, which captures the time evolution of each block into which an operator, invariant under a given symmetry, can be decomposed. We find that, at early times, the complexity of the full operator is equal to the average of the symmetry-resolved contributions. At later times, however, the interplay among different charge sectors becomes more intricate. In general, the symmetry-resolved Krylov complexity depends on the charge sector, although in some cases this dependence disappears, leading to a form of Krylov complexity equipartition. Our analysis lays the groundwork for a broader application of symmetry structures in the study of Krylov space complexities with implications for thermalization and universality in many-body quantum systems.

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I. INTRODUCTION

Restricting the analysis of many-body quantum systems with global symmetries to a fixed-charge sector can reveal structure and correlations that remain hidden when all symmetry sectors are treated collectively [1–4]. Symmetry-resolved entanglement entropies are bipartite entanglement measures that account for a fixed value of a conserved charge restricted to a subregion. In critical systems, it has been found that they capture more universal features than the total entanglement entropy [2,3,5–17]. Furthermore, whenever we neglect microscopic details of many-body systems, symmetry-resolved entanglement entropies have been shown to be independent of the charge sector, a property known as *entanglement equipartition* [3,18–27].

These developments have motivated a broader use of symmetries—and their breaking [28–32]—as tools to probe many-body systems from a quantum information perspective (see, e.g., review [33]). In this spirit, several physical quantities have been decomposed into symmetry sectors, and their symmetry-resolved counterparts have found diverse applications [34–47]. In this work, we introduce symmetry-resolved Krylov complexity to the list.

Originally proposed as a diagnostic tool for quantum chaos [48], Krylov complexity quantifies the growth of operators undergoing time evolution. This idea was later extended to quantum states that spread through the Hilbert space during

evolution, leading to the notion of spread complexity [49]. These complexity measures have already found applications in quantum chaos [48,50–54], topological phases of matter [55,56], quantum field theory [57–63], open quantum systems, and nonunitary dynamics [64–67] (see review [68]). Moreover, operator and state complexities are also believed to capture aspects of gravity and black hole physics in anti-de Sitter spacetimes [69–72], as described by AdS/CFT [73].

In this work, we consider systems with a conserved charge and Hermitian operators that are invariant under this symmetry. These operators exhibit a block-diagonal structure across the charge sectors of the theory. We define the symmetry-resolved Krylov complexities as the Krylov complexities of these block operators. This analysis allows us to understand how different symmetry sectors contribute to the Krylov complexity of the full operator, and whether the total Krylov complexity can be meaningfully decomposed into symmetry-resolved components. We find that such a decomposition holds at early times, whereas at later times the charge sectors correlate intricately in contributing to the total complexity. In this context, it is insightful to study the dependence of symmetry-resolved Krylov complexity on the charge and investigate whether an equipartition emerges in certain regimes. We find that, in general, equipartition of Krylov complexity does not always occur; however, we present an example in which it does appear.

These results suggest that symmetry resolution can provide a fine-grained lens on operator growth, especially relevant for systems where symmetries constrain dynamics, such as those exhibiting prethermalization [74–76], deep thermalization [77–79], or quantum many-body scars [80,81]. More broadly, symmetry-resolved Krylov complexity is a promising tool for understanding the spread of information in many-body systems where symmetries and the consequent presence of charge sectors are associated with universal properties.

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II. SETUP AND TOOLS

Consider a system evolving under a time-independent Hamiltonian H . The Heisenberg evolution of a Hermitian operator $O(0)$ at a fixed time $t = 0$ reads

$$O(t) = e^{iHt} O(0) e^{-iHt} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \tilde{O}_n, \quad (1)$$

where $\tilde{O}_0 = O(0)$, $\tilde{O}_1 = [H, O(0)]$, $\tilde{O}_2 = [H, [H, O(0)]]$, and as n increases, the number of nested commutators in $\tilde{O}_n$ grows correspondingly. The goal is then to quantify this growth in time by defining the notion of operator's size, i.e., complexity. To this end, we endow the operator algebra describing our theory with a Hilbert space structure [82,83]. We define the vectors $\{|\tilde{O}_n\rangle, n = 0, 1, 2, \dots\}$ representing the operators $\tilde{O}_n$ in this Hilbert space, and we introduce an inner product. The choice of the latter is detailed in the Appendix. Given the inner product, we can orthonormalize the set $\{|\tilde{O}_n\rangle, n = 0, 1, 2, \dots\}$ through the Gram-Schmidt procedure and obtain the Krylov basis $\{|K_n\rangle, n = 0, 1, 2, \dots, \mathcal{K} - 1\}$. The number $\mathcal{K}$ of elements of the Krylov basis is bounded by the dimension of operator Hilbert space. We call $\phi_n(t)$ the amplitudes of the operator $O(t)$ on the Krylov basis. These amplitudes evolve in time satisfying a Schrödinger equation with tridiagonal Hamiltonian, whose entries are the Lanczos coefficients b_n [82,83]. The choice of representing $O(t)$ on the operator Hilbert space so that $\langle O(t) | O(t) \rangle = 1$ implies $\sum_{n=0}^{\mathcal{K}-1} |\phi_n(t)|^2 = 1$. This allows us to represent the operator evolution by a particle, described by a wavefunction $\phi_n(t)$, hopping on a one-dimensional chain (Krylov chain). As time progresses, the particle moves away from the origin, and its average position provides a useful measure of the operator's complexity as a function of time, the *Krylov complexity* [48]

$$C_K(t) = \sum_{n=0}^{\mathcal{K}-1} n |\phi_n(t)|^2. \quad (2)$$

In the Appendix, we review how to compute $C_K(t)$ in practice.

In this manuscript, we focus on systems endowed with a global symmetry generated by the charge Q , such that $[H, Q] = 0$. The presence of a conserved charge Q implies that the Hilbert space $\mathcal{H}$ organizes into superselection sectors, $\mathcal{H} = \bigoplus_q \mathcal{H}_q$, which are labeled by q , the eigenvalues of Q , corresponding to the irreducible representations of the symmetry group. We refer to the superselection sectors as *charge sectors*. Although our construction applies more generally, we focus on the case of $U(1)$ symmetries, for which the direct sum above runs over the integers $q \in \mathbb{Z}$. We now consider an invariant operator, i.e., $[O(0), Q] = [O(t), Q] = 0$, and assume that it does not commute with H , so that its time evolution is nontrivial. Due to the commutation with Q , the operator O acquires a block-diagonal structure

$$O(t) = \sum_{q \in \mathbb{Z}} O_q(t), \quad O_q(t) = \Pi_q O(t) = e^{-iHt} \Pi_q O(0) e^{iHt}. \quad (3)$$

Each of the blocks O_q is obtained by acting on O with the projector Π_q associated with the subspace $\mathcal{H}_q$ and, therefore, belongs to the charge sector labeled by the integer q . The

main question we address in this work is how the different charge sectors contribute to the Krylov complexity of an invariant operator. Thus, we first introduce a quantifier of Krylov complexity per charge sector. Considering the block operator O_q in Eq. (3), we can simply apply the same Krylov procedure as for the full operator. Namely, introduce a block operator $|O_q(t)\rangle$ and expand it on the fixed-charge Krylov basis $\{|K_n^{(q)}\rangle, n = 0, 1, 2, \dots, \mathcal{K}_q - 1\}$. The number $\mathcal{K}_q$ of elements in the Krylov basis for $O_q(t)$ satisfies $\mathcal{K}_q \leq \mathcal{K}$. Since $[H, Q] = 0$, the Hamiltonian decomposes $H = \sum_q H_q$, and the Krylov basis is obtained by the orthonormalization of nested commutators of H_q with $O_q(0)$. We call $\phi_n^{(q)}$ the amplitudes resulting from the expansion of $|O_q(t)\rangle$ on the Krylov basis. The dynamics of $\phi_n^{(q)}$ is given by a tridiagonal Hamiltonian, whose entries are the fixed-charge Lanczos coefficients $b_n^{(q)}$. This is sufficient to define the *symmetry-resolved Krylov complexity*

$$C_K^{(q)}(t) = \sum_{n=0}^{\mathcal{K}_q-1} n |\phi_n^{(q)}(t)|^2, \quad (4)$$

which quantifies the growth of the block operator associated with the charge sector q . Importantly, Eq. (4) accounts only for a single block, neglecting the existence of the others. By definition, we can compute $C_K^{(q)}(t)$ starting from the block $O_q(t)$ and extracting the amplitudes $\phi_n^{(q)}(t)$. For this purpose, we observe that $|O(t)\rangle$ and $|O_q(t)\rangle$ are related by

$$|O(t)\rangle = \sum_{q \in \mathbb{Z}} \sqrt{p_q} |O_q(t)\rangle. \quad (5)$$

The presence of the factors $\sqrt{p_q}$ is due to the assumption that both $|O(t)\rangle$ and $|O_q(t)\rangle$ are normalized to one. This also implies that $\sum_q p_q = 1$, which provides the interpretation of p_q as the probability associated with the charge sector q . The most efficient way to access $\phi_n^{(q)}(t)$ to compute Eq. (4) is described in the Appendix. The goal of the next sections is to understand if and how Eq. (4) captures the information encoded in the total Krylov complexity (2) and what can we learn from its dependence on q .

We emphasize that the symmetry resolution of entanglement measures presents a sharp difference compared to the one of Krylov complexity. In the former, the reduced density matrices are decomposed into charge sectors associated with the conserved charge *restricted to a subsystem*, while, in the latter, the charge sectors are defined with respect to the *full* conserved charge.

III. GENERAL RESULTS

To discuss the interplay between the growths of the operator O and of its fixed-charge blocks, we study how the elements of the Krylov basis $\{|K_n\rangle\}$ can be written in terms of the fixed-charge Krylov vectors $\{|K_n^{(q)}\rangle\}$. This analysis is detailed in the Appendix; here, we report the main findings. Setting $t = 0$ in Eq. (5) and defining $|K_0\rangle = |O(0)\rangle$ and $|K_0^{(q)}\rangle = |O_q(0)\rangle$, we conclude that $|K_0\rangle$ can be written as a linear combination of Krylov vectors with the same label $n = 0$ but in different charge sectors. This is not the case for any value of n ; when $n \geq 2$, $|K_n\rangle$ cannot be written anymore

in terms of $|K_n^{(q)}\rangle$, and vectors with $m \neq n$ arise in the linear combination. This fact prevents us from writing $\phi_n(t)$ in terms of $\phi_n^{(q)}(t)$ in the different charge sectors, which implies that the total complexity $C_K(t)$ in Eq. (2) does not have a simple relation in terms of the symmetry-resolved complexities. If a relation between $C_K(t)$ and the $C_K^{(q)}(t)$ s can be found, one has to proceed case by case, as we show for the example of a general block-diagonal 4×4 operator. Nevertheless, to compare the total complexity with the cumulative information extracted from the individual charge subsectors (4), we first define an average complexity over the sectors as

$$\bar{C}(t) = \sum_{q \in \mathbb{Z}} p_q C_K^{(q)}(t), \quad (6)$$

where the Krylov complexity of each block operator is weighted with the corresponding probability p_q . We point that a different attempt to average complexity contributions in ensembles of initial states was discussed in Ref. [84]. It is then natural to ask how much information from the total complexity (2) is captured by the average (6), and more broadly, how these two quantities are related. As shown in the Appendix, by expanding $C_K(t)$ and $\bar{C}(t)$ for early times up to order t^4 , we obtain

$$C_K(t) - \bar{C}(t) = \left(\sum_{q \in \mathbb{Z}} p_q (b_1^{(q)})^4 - b_1^4 \right) \frac{t^4}{2} + O(t^6). \quad (7)$$

Expression (7) shows that, at leading order in the expansion of early times, $\bar{C}(t)$ behaves exactly as $C_K(t)$ and, at higher orders, it starts to deviate. In other words, $\bar{C}(t)$ reproduces well the early time growth of the Krylov complexity of the full operator. The coefficient of t^4 is the variance of $(b_1^{(q)})^2$ computed over the probability distribution p_q and, therefore, is positive. We conjecture that the difference in Eq. (7) is positive for any time, and we show that this holds in all the examples in the manuscript. This inequality could be justified as follows. The average Krylov complexity $\bar{C}(t)$ combines the information on the charge sectors without taking into account possible mixing during the construction of the Krylov basis $\{|K_n\rangle\}$. This mixing of the sectors can enhance the total complexity $C_K(t)$, which, therefore, is expected to be larger than $\bar{C}(t)$. Thus, $C_K(t) - \bar{C}(t)$ measures how much the charge sectors are correlated in the construction of the Krylov basis. We hope to come back to a general proof of $C_K(t) - \bar{C}(t) \geq 0$ in future work.

Finally, we provide an interpretation of the early-time behavior (7) in terms of the effective dynamics on the Krylov chain. At early times, the dynamics of the full operator can be represented by a particle for each charge sector moving on the Krylov chain, whose average positions (weighted with p_q) determine the total Krylov complexity. At later times, the fixed-charge particles do not move on the Krylov chain independently; due to the possible mixing of the charge sectors in determining the Krylov basis, the dynamics are best represented by a single degree of freedom (in which we can imagine all the fixed-charge particles coalesce), whose position on the chain gives the total Krylov complexity.

IV. EXAMPLES

In the final part of the paper, we report three examples where we compute the symmetry-resolved Krylov complexities, highlighting the properties discussed so far.

(1) Two spins: It is insightful to start from a simple model of two spins $1/2$, described by two sets of Pauli matrices X_S, Y_S, Z_S with $S \in \{L, R\}$, interacting through the Hamiltonian $H_{\text{spin}} = J(X_L X_R + Y_L Y_R + Z_L Z_R)$. This Hamiltonian commutes with the total magnetization in the Z direction $Q_{\text{spin}} = Z_L + Z_R$, which is, therefore, conserved. The operator Q_{spin} has eigenvalues $q = 0, \pm 1$, which label the three charge sectors of this system.

The left spin operator Z_L (we could choose, equivalently, Z_R) commutes with Q_{spin} , but does not with H_{spin} . Thus, it exhibits a nontrivial time evolution and a block-diagonal structure

$$Z_L(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(Jt) & i \sin(Jt) & 0 \\ 0 & -i \sin(Jt) & -\cos(Jt) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (8)$$

where the first and the last 1×1 blocks correspond to $q = 1$ and $q = -1$ respectively, while the 2×2 block to $q = 0$. Due to the small dimensionality of the Hilbert space, the Krylov basis and complexities can be obtained explicitly for the full 4×4 operator and the three blocks. Referring to the Appendix for the details, we find $C_K(t) = 1 - \cos(Jt)$, $C_K^{(q=0)}(t) = \sin^2(Jt)$, and $C_K^{(q=\pm 1)}(t) = 0$. The fact that the symmetry-resolved Krylov complexity in the sectors with $q = \pm 1$ is vanishing is consistent with the entries of the 1×1 blocks not evolving in time. Even if in a simple way, the symmetry-resolved complexities depend on the charge, not showing equipartition. We also identify the probabilities [Eq. (5), $p_{\pm 1} = \frac{1}{4}$ and $p_0 = \frac{1}{2}$, that lead, from Eq. (6), to $\bar{C}(t) = \sin^2(Jt)/2$. As expected, the early time growth coincides with the one of $C_K(t)$ above. Computing their difference, we have $C_K(t) - \bar{C}(t) = 2 \sin^4(Jt/2)$, which is positive not only at order t^4 but at any time. This is in agreement with our expectations. Interestingly, comparing C_K and $C_K^{(q=0)}$, it is straightforward to observe that, for some values of t , the symmetry-resolved component is larger than the total Krylov complexity. This makes the positivity of $C_K(t) - \bar{C}(t)$ at any time even less trivial from the mathematical point of view.

(2) General four-dimensional operator: The inequality $C_K(t) - \bar{C}(t) \geq 0$ in the previous example might be an artifact of the simplicity of the system. To improve on this, we consider the most general 4×4 Hermitian operator with two 2×2 blocks along the diagonal

$$O_4 = \begin{pmatrix} A_{11} & A_{12} + iB_{12} & 0 & 0 \\ A_{12} - iB_{12} & A_{22} & 0 & 0 \\ 0 & 0 & A_{33} & A_{34} + iB_{34} \\ 0 & 0 & A_{34} - iB_{34} & A_{44} \end{pmatrix}, \quad (9)$$

where A_{ii} ($i = 1, 2, 3, 4$), A_{12} , A_{34} , B_{12} , $B_{34} \in \mathbb{R}$. We assume that the block structure is induced by a conserved charge, which, in the diagonal basis, reads $Q_4 = \text{diag}(q_+, q_+, q_-, q_-)$.

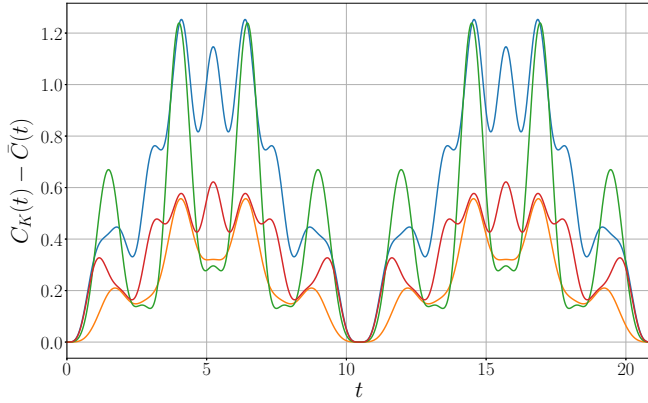


FIG. 1. Difference between the Krylov complexity of the operator (9) and the average (6) over the (two) sectors. All the curves are obtained for $E_1 = 8$, $E_2 = 5$, $E_3 = 3.4$, and $E_4 = 1$. Different colors correspond to distinct choices of the parameters A_{ii} ($i = 1, 2, 3, 4$), A_{12} , A_{34} , B_{12} , and B_{34} (see Table I in the Appendix). In all the instances shown here, $C_K(t) - \bar{C}(t) \geq 0$, supporting our surmise.

The Hamiltonian governing the (nontrivial) time evolution of O_4 is diagonal in the same basis and is given by $H_4 = \text{diag}(E_1, E_2, E_3, E_4)$.

The two blocks of O_4 evolve separately in time. The Krylov complexity of a general Hermitian operator in a two-dimensional Hilbert space has been computed in Ref. [85], and we can use those results to determine the two symmetry-resolved Krylov complexities in this example. As shown in the Appendix, we can also extract the probabilities associated with the two blocks. Combining them with the symmetry-resolved Krylov complexities as in Eq. (6), we obtain $\bar{C}(t)$. The computation of the Krylov complexity of the 4×4 operator is more involved. In the Appendix, we describe a method to obtain analytic expressions. It is insightful to plot the difference $C_K(t) - \bar{C}(t)$ as a function of time; we do it in Fig. 1 for various choices of parameters. Consistently with our expectations, all the curves are positive. In the Appendix, we provide further evidence to support this fact. As already verified for the previous example, this instance also demonstrates that the total Krylov complexity can be smaller than one of the two symmetry-resolved complexities for some time values. This is illustrated, for some parameter values of the model, in a dedicated figure in the Appendix.

(3) Harmonic oscillators: To include infinite-dimensional Hilbert spaces in our analysis and move toward the study of many-body quantum systems, let us consider a model of coupled complex harmonic oscillators described by N (non-Hermitian) position operators Φ_j and conjugate momenta Π_j such that $[\Phi_i, \Pi_j] = i\delta_{ij}$. If we assume that the Hamiltonian of the model can be rewritten as the sum of two quadratic Hamiltonians of independent real oscillators (see the Appendix), we can diagonalize it as [86]

$$H_{\text{HO}} = \sum_{k=1}^N \omega_k (b_k^\dagger b_k + c_k^\dagger c_k + 1), \quad (10)$$

where we have introduced two sets of creation operators $b_k^\dagger$ (particles) and $c_k^\dagger$ (antiparticles), with $k = 1, 2, \dots, N$. The

modes are labeled by the integer k and characterized by the dispersion relation ω_k , whose details depend on the specific Hamiltonian with which we start, but whose form does not affect our results. This system has a conserved charge that, in the Fock basis, reads $Q_{\text{HO}} = \sum_{k=1}^N (b_k^\dagger b_k - c_k^\dagger c_k)$ and generates a $U(1)$ symmetry. The charge Q_{HO} is interpreted as the difference between the total number of particles and antiparticles and has integer eigenvalues; thus, the charge sectors are labeled by $q \in \mathbb{Z}$. To study an operator commuting with Q_{HO} but not with H_{HO} (retaining a nontrivial dynamics), we consider

$$O_{\text{HO}} = \prod_{k=1}^N (b_k c_k + b_k^\dagger c_k^\dagger). \quad (11)$$

It is already insightful to focus on the single-mode case, i.e., $N = 1$ (we call $\omega \equiv \omega_1$). In the Appendix, we report the computation of the Krylov complexity, finding that the only nonvanishing Lanczos coefficient is $b_1 = 2\omega$ and correspondingly, $C_K(t) = \sin^2(2\omega t)$.

Since $[Q_{\text{HO}}, O_{\text{HO}}] = 0$, O_{HO} has a block-diagonal structure with infinitely many blocks labeled by all the integer numbers. In the Appendix, we find that all the fixed-charge Lanczos coefficients are the same as the one determining the Krylov chain dynamics of the full operator. Thus, also all the symmetry-resolved complexities have the same expression as the total one:

$$C_K^{(q)}(t) = C_K(t) = \sin^2(2\omega t), \quad \forall q \in \mathbb{Z}. \quad (12)$$

The independence of the charge in $C_K^{(q)}(t)$ implies that the operator in Eq. (11) exhibits the equipartition of the Krylov complexity. Despite the equipartition, the blocks of O_{HO} have a nontrivial probability distribution p_q associated with them, as shown in the Appendix. Due to Eq. (12), $C_K(t) - \bar{C}(t) = 0$ in this case. In extending the analysis to the many-body operator (11) with $N > 1$, we find that the amount of nonvanishing Lanczos coefficients grows with N and, thus, we are not able to determine a general expression for the Krylov complexity analytically. However, in the Appendix, we observe that the Lanczos coefficients associated with the evolution of O_{HO} and the fixed-charge ones are the same, and the latter are independent of q also for generic N . Thus, we conclude that the equipartition of the Krylov complexity is also present for the general many-body operator in Eq. (11). We further analytically compute the early-time growth of the Krylov complexity for generic N and its full expression when $N = 2$.

Finally, we elaborate on the occurrence of the equipartition of Krylov complexity. In general, we expect this is due to the specific choice (11). Focusing for simplicity on $N = 1$, we observe that this operator creates and destroys a particle and an antiparticle. Thus, it connects Fock states with an energy difference $\Delta E/\omega - 1 = \pm 2$. Supported by the analysis in the Appendix, we show that in this example equipartition of Krylov complexity is related to the locality in the energy basis of the considered operator. We hope to test this hypothesis in more general scenarios in future works.

V. CONCLUSIONS AND OUTLOOK

In this work, we introduced the symmetry-resolved Krylov complexity to probe the contribution of different charge

sectors to the growth of operators invariant under a given symmetry. This approach highlights the intricacy of the relationship between the total complexity $C_K(t)$ of the operator and the complexities $C_K^{(q)}(t)$ of its fixed-charge blocks. We find that, although a general relation between $C_K(t)$ and the $C_K^{(q)}(t)$'s is complicated and system-dependent, the initial growth of the total complexity is perfectly captured by the average $\bar{C}(t)$ in Eq. (6) of the symmetry-resolved complexities over the charge sectors. We interpret this as an effective early-time dynamics on the Krylov chain characterized by several charged particles hopping independently and then coalescing in a unique degree of freedom at late times. In other words, at early times, we can precisely identify the individual charge contributions, as if the fixed-charge Krylov particles have a manifest effect on the total Krylov complexity through the average $\bar{C}(t)$. On the other hand, at late times, given that $\bar{C}(t)$ deviates from $C_K(t)$, this picture of charged Krylov particles individually contributing to the total complexity breaks down. However, the charge sectors still play a collective role in determining $C_K(t)$, despite not being described by hopping particles. It would be interesting to understand how the timescale at which the behavior changes depends on the details of the system considered. Beyond the initial times, we conjecture that $C_K(t) - \bar{C}(t)$ is non-negative and captures the correlations among the charge sectors in determining the optimal basis to describe the operator growth. These correlations are not evident from other quantities, as the autocorrelation functions. Indeed, the total autocorrelation function is given by the average over the charge sectors of the resolved contributions (see the Appendix), hiding the mixing among the sectors. For this reason, $C_K - \bar{C}$, provides a helpful tool for improving our understanding of Krylov complexity as a measure of operator spread. This is supported by the examples considered in the manuscript. For a complete picture, it will be important to understand if the complexity of the full operator captures both quantum and classical correlations between its charge sectors, as it happens for the symmetry-resolved entanglement entropies [4]. Finally, we find that the $U(1)$ -invariant operator (11) involving complex harmonic oscillators exhibits the equipartition of Krylov complexity among the charge sectors.

We remark that the absence of a general relationship between Krylov complexity and symmetry-resolved Krylov complexity should not be regarded as an incompleteness. Indeed, the counterintuitive fact that, when studying the complexity of simpler blocks of a given operator, we cannot directly reconstruct the total complexity serves as evidence that Krylov complexity probes the quantum dynamics of operators at a deep level, also accounting for the aforementioned correlations between charge sectors.

This work paves the way to understand how the symmetry structure of operators in theories endowed with global symmetries determines their growth across time evolution. There are several directions to further explore in this respect. Here, we have focused on the operator growth and its complexity. However, symmetry resolution could also be used to study the spread of quantum states [87]. Indeed, when not eigenstates of a conserved charge, a state may have components in different charge sectors, i.e., distinct charge eigenspaces. The

study of state complexity can be unified with the approach presented in this work, by starting from an expansion like Eq. (5) for the evolving quantum state (not generally the associated with some operator). In addition, due to charge conservation, focusing on a charge sector means restricting to certain energy windows in the spectrum of our system and this should be related to microcanonical inner products considered in [88]. Moreover, the symmetry resolution of entanglement has shown that entanglement measures in fixed charge sectors may lead to the emergence of more universal properties of the theory compared to what is found in the unresolved analysis. Paralleling this remark, we expect the same to happen for Krylov complexities in critical systems and their continuum limit given by conformal field theories. For this purpose, investigating the Krylov chain dynamics induced by Hamiltonians symmetric under Lie groups from the perspective of symmetry resolution could be insightful [89,90]. In addition, it would be very interesting to analyze other complexity measures in quantum many-body systems and quantum field theories (see Ref. [91] for a recent review) from the perspective of symmetry resolution and understand if examples of equipartition of complexity can be found for these probes. This would be an important step toward a more detailed characterization of existing complexity measures and their role in many-body physics.

In this work, we have focused only on invariant operators, characterized by a block-diagonal structure, which naturally leads to the question of how the total Krylov complexity is affected by that of the blocks. Including charged operators in our analysis relates to recent investigations using entanglement asymmetry to detect symmetry breaking in reduced density matrices [31]. Indeed, the symmetry breaking implies a reduced density matrix that does not commute with the charge. Finding a complexity-based proxy of symmetry breaking would bring us closer to understanding how the presence or absence of symmetries influences the operator growth. Finally, the operator entanglement provides a way to characterize the complexity of an operator in terms of its approximability by matrix product operators [92–94]. It would be interesting to compare the dynamics of symmetry-resolved operator entanglement [41,44] and Krylov complexity in view of studying universal properties of many-body quantum systems.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX

In this Appendix, we provide computational details on the following aspects discussed in the main text:

- (1) Lanczos algorithm for a generic Hermitian operator.
- (2) Lanczos algorithm for the evolution of fixed-charge blocks in which invariant operators decompose.
- (3) Resolution of Krylov complexity in a two-spin system.
- (4) Krylov complexity of 4×4 block-diagonal operators and its symmetry resolution.
- (5) Equipartition of Krylov complexity in harmonic oscillator systems.

1. Review of the Lanczos algorithm

We first briefly review the iterative procedure to determine the Krylov basis associated with the dynamics of the operator O . We simply refer to this procedure as the Lanczos algorithm. To represent the operator $O(t)$ on a Hilbert space, we need to introduce an inner product. We choose the Wightman inner product

$$(A|B) = \langle e^{\beta H/2} A^\dagger e^{-\beta H/2} B \rangle_\beta = \frac{\text{Tr}(e^{-\beta H/2} A^\dagger e^{-\beta H/2} B)}{\text{Tr}(e^{-\beta H})}, \quad (\text{A1})$$

parametrized by the inverse temperature of the system $\beta = 1/T$. After introducing $|O(t)\rangle$, the Krylov basis is constructed through the recursion relations [82,83]

$$|A_n\rangle = \mathcal{L}|K_{n-1}\rangle - b_{n-1}|K_{n-2}\rangle, \quad |K_n\rangle = b_n^{-1}|A_n\rangle, \\ b_n = \sqrt{(A_n|A_n)}, \quad (\text{A2})$$

with the initial conditions

$$|K_0\rangle = |O(0)\rangle, \quad b_0 = 0. \quad (\text{A3})$$

The operator $\mathcal{L}$ in Eq. (A2) is called Liouvillian and its action is defined as

$$\mathcal{L}|O(0)\rangle = |[H, O(0)]\rangle. \quad (\text{A4})$$

Since we require $(O(t)|O(t)) = 1$, when plugged in Eq. (A1), $O(t)$ has to be appropriately normalized. The Lanczos algorithm leads to the $\mathcal{K}$ elements of the Krylov basis, on which we expand $|O(t)\rangle$ as

$$|O(t)\rangle = \sum_{n=0}^{\mathcal{K}-1} i^n \phi_n(t) |K_n\rangle. \quad (\text{A5})$$

The number $\mathcal{K}$ is bounded by the dimension of the operator Hilbert space, and it is often smaller than that as the Gram-Schmidt procedure stops before the Krylov basis spans the full operator Hilbert space. The last formula in Eq. (A2) shows that, by carrying out the full Lanczos algorithm, we find the Lanczos coefficients that we can use to determine the amplitudes $\phi_n(t)$ via the Schroedinger equation

$$\partial_t \phi_n(t) = b_n \phi_{n-1}(t) - b_{n+1} \phi_{n+1}(t). \quad (\text{A6})$$

The orthonormalization procedure above can be difficult in Hilbert spaces with large dimensionality. Thus, it is helpful to extract the Lanczos coefficients differently. For that, we

introduce the autocorrelation function

$$R(t) = (O(t)|O(0)) \\ = \frac{\langle e^{\beta H/2} O(t) e^{-\beta H/2} O(0) \rangle_\beta}{\langle e^{\beta H/2} O(0) e^{-\beta H/2} O(0) \rangle_\beta} \\ = \frac{\text{Tr}(e^{-\beta H/2} O(t) e^{-\beta H/2} O(0))}{\text{Tr}(e^{-\beta H/2} O(0) e^{-\beta H/2} O(0))}. \quad (\text{A7})$$

The Lanczos coefficients are determined through the moment recursion method [82], which consists of using recursive relations between the moments of the autocorrelation function

$$\mu_n = (-i)^n \left. \frac{d^n R(t)}{dt^n} \right|_{t=0}, \quad (\text{A8})$$

and the b_n 's. If $O(t)$ is Hermitian, the autocorrelation is real and an even function of time, and, therefore, only the even moments are nonvanishing. For instance, the first two Lanczos coefficients are obtained from $\mu_2 = b_1^2$ and $\mu_4 = b_1^2(b_1^2 + b_2^2)$, and we refer the interested reader to Refs. [48,82] for more details on how to obtain the ones for larger values of n .

2. Lanczos coefficients of fixed-charge blocks

To compute the symmetry-resolved Krylov complexity, we need the fixed-charge amplitudes $\phi_n^{(q)}(t)$ obtained by expanding the block operator $|O_q(t)\rangle$ in the fixed-charge Krylov basis. Combining this expansion with Eq. (A5), we find

$$|O(t)\rangle = \sum_{q \in \mathbb{Z}} \sqrt{p_q} \sum_{n=1}^{\mathcal{K}_q-1} i^n \phi_n^{(q)}(t) |K_n^{(q)}\rangle, \quad (\text{A9})$$

which implies that, due to the lack of relation between $|K_n\rangle$ and the $|K_n^{(q)}\rangle$'s for fixed values of n (see the following section), the full complexity is not easily expressed in terms of the symmetry-resolved ones.

As discussed in the main text, the fixed-charge amplitudes are determined starting from $|O_q(t)\rangle$ and proceeding as done to compute $\phi_n(t)$. Following the discussion above, it is convenient to consider the autocorrelation function of $O_q(t)$

$$R_q(t) \equiv (O_q(t)|O_q(0)) = \frac{\langle \Pi_q e^{\beta H/2} O(t)^\dagger e^{-\beta H/2} O(0) \rangle_\beta}{\langle \Pi_q e^{\beta H/2} O(0)^\dagger e^{-\beta H/2} O(0) \rangle_\beta} \\ = \frac{\text{Tr}(\Pi_q e^{-\beta H/2} O(t) e^{-\beta H/2} O(0))}{\text{Tr}(\Pi_q e^{-\beta H/2} O(0) e^{-\beta H/2} O(0))}, \quad (\text{A10})$$

that we denote symmetry-resolved autocorrelation function. The second step in Eq. (A10) is performed by taking into account that the projectors Π_q are idempotent and commute with O and H . Also here, the normalization $(O_q(t)|O_q(t)) = 1$ requires rescaling the operators $O_q(t)$ when plugged into Eq. (A1) to compute scalar products. Once $R_q(t)$ is obtained, we can apply the moment recursion methods mentioned above to compute the fixed-charge Lanczos coefficients $b_n^{(q)}$ from the moments of $R_q(t)$, whose definition parallels Eq. (A8).

Using the inner product (A1) and the orthogonality of the projectors, i.e., $\Pi_q \Pi_{q'} = \delta_{qq'} \Pi_q$, we can also compute the

charge sector probability distribution in Eq. (3). It reads

$$p_q = |\langle O_q(0) | O(0) \rangle|^2 = \frac{\text{Tr}(\Pi_q e^{-\beta H/2} O(0) e^{-\beta H/2} O(0))}{\text{Tr}(e^{-\beta H/2} O(0) e^{-\beta H/2} O(0))}. \quad (\text{A11})$$

Using Eqs. (A7), (A10), and (A11), we can prove that

$$R(t) = \sum_q p_q R_q(t), \quad (\text{A12})$$

which provides a charge decomposition of the autocorrelation function in its symmetry-resolved components.

If the block operator O_q is known explicitly, the computation of R_q is generally straightforward. However, there are instances where accessing block operators might be difficult, as could happen for operators acting on infinite-dimensional Hilbert spaces. To bypass this problem, we take inspiration from the toolkit to study symmetry-resolved entanglement entropies [2]. Since the charge sectors are accessed via a projector $\Pi_q = \Pi_q^2$ [see Eq. (3)], we can write it in its Fourier representation:

$$\Pi_q = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{i\alpha(Q-q)}. \quad (\text{A13})$$

Plugging this relation into Eq. (A10), we find

$$R_q(t) = \frac{\int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-i\alpha q} \text{Tr}(e^{i\alpha Q} e^{-\beta H/2} O(t) e^{-\beta H/2} O(0))}{\int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-i\alpha q} \text{Tr}(e^{i\alpha Q} e^{-\beta H/2} O(0) e^{-\beta H/2} O(0))}. \quad (\text{A14})$$

Crucially, computing the right-hand side of Eq. (A14) does not require to know the block operator O_q . Instead, this is effectively replaced by inserting an Aharonv-Bohm flux in the correlation function of the full operator O and computing its Fourier transform. The formula (A14) is used to obtain the results for the last examples discussed in the main text. The Fourier representation (A13) can also be exploited to derive a useful formula for the probability distribution (A11):

$$p_q = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-i\alpha q} \frac{\text{Tr}(e^{i\alpha Q} e^{-\beta H/2} O(0) e^{-\beta H/2} O(0))}{\text{Tr}(e^{-\beta H/2} O(0) e^{-\beta H/2} O(0))}. \quad (\text{A15})$$

Expression (A15) is used in the Appendix to compute the non-trivial probability distribution of the charge sectors in which O_{HO} in Eq. (11) decomposes.

3. Lanczos algorithm and charge sectors

In this Appendix, we describe how to adapt the Lanczos algorithm to study the dynamics of fixed-charge blocks of operators invariant under a given symmetry. If the block structure of O is due to a conserved charge Q , we have that $[H, Q] = 0$, implying that the Hamiltonian H has the same block-diagonal structure as O , i.e., $H = \sum_q H_q$. This implies that $[H_q, Q_{q'}] \propto \delta_{qq'}$, and therefore $[H, O_q] = [H_q, O_q]$. As a consequence, the Liouvillian operator defined by Eq. (A4) also decomposes as

$$\begin{aligned} \mathcal{L} &= \sum_q \mathcal{L}_q, \quad \mathcal{L}_q |O_q(0)\rangle = \delta_{qq'} \mathcal{L}_q |O_q(0)\rangle \\ &= \delta_{qq'} [H_q, O_q(0)]. \end{aligned} \quad (\text{A16})$$

Since every $\mathcal{L}_q$ acts nontrivially on the block operators with the same label q , the Gram-Schmidt orthonormalization can be carried out sector by sector. This leads to the fixed-charge Krylov basis defined by the following recursive relations:

$$|A_n^{(q)}\rangle = \mathcal{L}_q |K_{n-1}^{(q)}\rangle - b_{n-1}^{(q)} |K_{n-2}^{(q)}\rangle, \quad |K_n^{(q)}\rangle = (b_n^{(q)})^{-1} |A_n^{(q)}\rangle, \quad (\text{A17})$$

with $b_n^{(q)} \equiv \sqrt{\langle A_n^{(q)} | A_n^{(q)} \rangle}$ and the initial condition $|K_0^{(q)}\rangle = |O_q(0)\rangle$.

To understand how the fixed-charge Krylov basis vectors $|K_n^{(q)}\rangle$ are related to the total ones $|K_n\rangle$, we explicitly compute $|K_0^{(q)}\rangle$, $|K_1^{(q)}\rangle$, and $|K_2^{(q)}\rangle$, and we compare them with the first Krylov vectors $|K_1\rangle$ and $|K_2\rangle$ of the full unresolved evolution. For convenience, we report these vectors here:

$$\begin{aligned} |K_1\rangle &= \frac{\mathcal{L}|K_0\rangle}{\sqrt{\langle K_0 | \mathcal{L}^2 | K_0 \rangle}}, \\ |K_2\rangle &= \frac{\mathcal{L}|K_1\rangle - \sqrt{\langle K_0 | \mathcal{L}^2 | K_0 \rangle} |K_0\rangle}{\sqrt{\langle K_1 | \mathcal{L}^2 | K_1 \rangle - \langle K_0 | \mathcal{L}^2 | K_0 \rangle}}. \end{aligned} \quad (\text{A18})$$

$n = 0$: Using the initial condition and Eq. (5) for $t = 0$, we obtain

$$|K_0\rangle = \sum_q \sqrt{p_q} |K_0^{(q)}\rangle. \quad (\text{A19})$$

$n = 1$: From Eq. (A17), we obtain

$$|K_1^{(q)}\rangle = \frac{\mathcal{L}_q |K_0^{(q)}\rangle}{\sqrt{\langle K_0^{(q)} | \mathcal{L}_q^2 | K_0^{(q)} \rangle}}, \quad (\text{A20})$$

which, from Eqs. (A18) and (5), can be used to decompose $|K_1\rangle$ in Eq. (A18) as

$$|K_1\rangle = \sum_q \sqrt{p_q} \sqrt{\frac{\langle K_0^{(q)} | \mathcal{L}_q^2 | K_0^{(q)} \rangle}{\langle K_0 | \mathcal{L}^2 | K_0 \rangle}} |K_1^{(q)}\rangle. \quad (\text{A21})$$

Notice that $|K_1\rangle$ can be decomposed in terms of the $|K_1^{(q)}\rangle$ only, without involving vectors of the fixed-charge Krylov basis with different label n .

$n = 2$: Using Eq. (A17), we straightforwardly obtain

$$|K_2^{(q)}\rangle = \frac{\mathcal{L}_q |K_1^{(q)}\rangle - \sqrt{\langle K_0^{(q)} | \mathcal{L}_q^2 | K_0^{(q)} \rangle} |K_0^{(q)}\rangle}{\sqrt{\langle K_1^{(q)} | \mathcal{L}_q^2 | K_1^{(q)} \rangle - \langle K_0^{(q)} | \mathcal{L}_q^2 | K_0^{(q)} \rangle}}. \quad (\text{A22})$$

The vector $|K_2\rangle$ in Eq. (A18) can be decomposed as

$$\begin{aligned} |K_2\rangle &= \sum_q \sqrt{p_q} \sqrt{\frac{\langle K_0^{(q)} | \mathcal{L}_q^2 | K_0^{(q)} \rangle}{\langle K_0 | \mathcal{L}^2 | K_0 \rangle}} \\ &\quad \times \frac{\mathcal{L}_q |K_1^{(q)}\rangle - \frac{\langle K_0 | \mathcal{L}^2 | K_0 \rangle}{\sqrt{\langle K_0^{(q)} | \mathcal{L}_q^2 | K_0^{(q)} \rangle}} |K_0^{(q)}\rangle}{\sqrt{\langle K_1 | \mathcal{L}^2 | K_1 \rangle - \langle K_0 | \mathcal{L}^2 | K_0 \rangle}}. \end{aligned} \quad (\text{A23})$$

However, since

$$\mathcal{L} |K_1^{(q)}\rangle - \frac{\langle K_0 | \mathcal{L}^2 | K_0 \rangle}{\sqrt{\langle K_0^{(q)} | \mathcal{L}_q^2 | K_0^{(q)} \rangle}} |K_0^{(q)}\rangle \propto |K_2^{(q)}\rangle, \quad (\text{A24})$$

we cannot decompose $|K_2\rangle$ only in terms of the fixed-charge Krylov vectors with the same label $n = 2$. This property also holds for $n > 2$ and implies that the total complexity $C_K(t)$ in Eq. (2) does not have a simple relation in terms of the symmetry-resolved complexities. If a relation between $C_K(t)$ and the $C_K^{(q)}(t)$'s can be found, it has to be worked case by case, as we show later in this Appendix for the example of a general block-diagonal 4×4 operator.

The explicit construction (A17) allows us to prove that Krylov vectors associated with different charge sectors are orthogonal. Indeed, using the inner product in Eq. (A1) and the fact that the projectors commute with $O(0)$ and the Hamiltonian, we find

$$(K_0^{(q)}|K_0^{(q')}) \propto \langle e^{\beta H/2} \Pi_q O(0) e^{-\beta H/2} \Pi_{q'} O(0) \rangle_\beta \propto \delta_{qq'}, \quad (\text{A25})$$

for vectors with $n = 0$. This relation is extended to $(K_n^{(q)}|K_m^{(q')}) \propto \delta_{qq'}$ by noticing that Π_q commutes also with the nested commutators of $O(0)$ with the Hamiltonian.

To compute the fixed-charge Lanczos coefficients, we have to carry out the full Lanczos algorithm in Eq. (A17) or, alternatively, compute the moments of the symmetry-resolved autocorrelation function (A10). Expanding in series (A12), we find a tower of relations between the moments of $R_q(t)$ and those of $R(t)$:

$$\mu_n = \sum_{q \in \mathbb{Z}} p_q \mu_n^{(q)}. \quad (\text{A26})$$

Due to these formulas, we can write the first Lanczos coefficient of the full operator dynamics in terms of the first fixed-charge Lanczos coefficients, namely, $b_1^2 = \sum_q p_q (b_1^{(q)})^2$. The nonlinearity of the relations between $b_n^{(q)}$ and $\mu_n^{(q)}$ does not allow to find further simple decompositions of the Lanczos coefficients b_n in terms of their fixed-charge components. Such relations can be worked out case by case, but they are not linear in the probability distribution p_q and they become difficult to write in a closed form. We can use these decompositions to derive the formula (7). The early time growth of the full Krylov complexity reads [95]

$$C_K(t) = \mu_2 t^2 + \left(\frac{\mu_4}{6} - \frac{\mu_2^2}{2} \right) t^4 + O(t^6), \quad (\text{A27})$$

and the same holds for $C_K^{(q)}(t)$ by replacing μ_n with $\mu_n^{(q)}$. From Eq. (A26) and the first two relations between $b_n^{(q)}$ and $\mu_n^{(q)}$, we find Eq. (7).

4. Details on the two-spin example

The first example considered in the main text involves two spins $1/2$ evolving through the Hamiltonian

$$H_{\text{spin}} = J(X_L X_R + Y_L Y_R + Z_L Z_R), \quad (\text{A28})$$

where the spin variables are written in terms of the Pauli matrices X_S, Y_S, Z_S with $S \in \{L, R\}$. Here, J is the spin coupling. This Hamiltonian commutes with the total magnetization of the two spins along the Z direction, i.e.,

$$Q_{\text{spin}} = Z_L + Z_R. \quad (\text{A29})$$

We focus on a single spin operator Z_L , which commutes with Q_{spin} but not with H_{spin} , having, therefore, a nontrivial time evolution. The Heisenberg evolution of Z_L is given in Eq. (8) and exhibits the expected block-diagonal structure.

Using the inner product in Eq. (A1) with $\beta = 0$, we compute the autocorrelation function of $Z_L(t)$ and, from its moments in Eq. (A8), we find that there are only two nonvanishing Lanczos coefficients determining dynamics of $Z_L(t)$ on the Krylov chain:

$$b_1 = b_2 = J/\sqrt{2}. \quad (\text{A30})$$

This implies that the Krylov space is generated by three independent vectors. When we expand the operator on this basis, we obtain the amplitudes

$$\begin{aligned} \phi_0(t) &= \cos^2(Jt/2), & \phi_1(t) &= \frac{\sin(Jt)}{\sqrt{2}}, \\ \phi_2(t) &= \sin^2(Jt/2). \end{aligned} \quad (\text{A31})$$

From Eq. (A31), we can compute the Krylov complexity $C_K(t) = 1 - \cos(Jt)$, as reported in the main text.

Let us move to the Krylov complexities of the three blocks, i.e., the symmetry-resolved complexities. Since the two blocks associated with $q = \pm 1$ do not evolve in time, their Krylov complexity is zero. On the other hand, the 2×2 block has nontrivial time evolution and nontrivial complexity. As explained above, we can extract the Lanczos coefficients from the moments of the autocorrelation function $R_{q=0}(t)$, finding only one of them being nonvanishing, i.e., $b_1^{(q=0)} = J$. This corresponds to a two-dimensional Krylov space and fixed-charge amplitudes for the block operator expanded on the Krylov basis given by

$$\phi_0^{(q=0)}(t) = \cos(Jt), \quad \phi_1^{(q=0)}(t) = \sin(Jt). \quad (\text{A32})$$

Using these amplitudes and Eq. (4), the symmetry-resolved Krylov complexity with $q = 0$ reads $C_K^{(q=0)}(t) = \sin^2(Jt)$. From Eq. (A11), we can compute the probabilities associated with the blocks of $Z_L(t)$, finding $p_{\pm 1} = \frac{1}{4}$ and $p_0 = \frac{1}{2}$. Combining these with the symmetry-resolved Krylov complexities as in Eq. (6), we find $\bar{C}(t) = \sin^2(Jt)/2$, and we compute the difference $C_K(t) - \bar{C}(t)$ reported in the main text. It is insightful to notice that, in this example, the total Krylov complexity can be expressed in terms of the only nonvanishing symmetry-resolved component or of the average $\bar{C}(t)$. It reads $C_K(t) = 1 - \sqrt{1 - C_K^{(q=0)}(t)} = 1 - \sqrt{1 - 2\bar{C}(t)}$, which is a simple, yet nonlinear relation.

5. Krylov complexity symmetry-resolution of a block-diagonal 4×4 operator

In this Appendix, we provide the details for computing the Krylov complexity (and its symmetry resolution) of a 4×4 operator with two 2×2 blocks along its diagonal.

a. Krylov complexity of the 2×2 blocks

Let us first consider an arbitrary system defined on a two-dimensional Hilbert space and described by a Hamiltonian

that, in the energy basis, is represented as

$$H_2 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}. \quad (\text{A33})$$

The initial state operator at time $t = 0$ is taken to be the most general 2×2 Hermitian matrix O_2 :

$$O_2 = \begin{pmatrix} A_{11} & A_{12} + iB_{12} \\ A_{12} - iB_{12} & A_{22} \end{pmatrix}, \quad (\text{A34})$$

where $A_{11}, A_{22}, A_{12}, B_{12} \in \mathbb{R}$. For convenience, we define the squared sums of the diagonal and off-diagonal components of O_2 as

$$M = A_{12}^2 + B_{12}^2, \quad T = A_{11}^2 + A_{22}^2. \quad (\text{A35})$$

The Krylov complexity of the time evolution of the operator O_2 with respect to the Hamiltonian H_2 [see Eq. (1)] has been computed in Ref. [85] and reads

$$C_K^{(2 \times 2)}(t) = \frac{2M}{T + 2M} \left\{ \sin^2(\Delta E t) + \frac{8T}{T + 2M} \sin^4\left(\frac{\Delta E}{2}t\right) \right\}, \quad (\text{A36})$$

where T and M are given in Eq. (A35) and $\Delta E = E_1 - E_2$.

In the next section, we use Eq. (A36) as symmetry-resolved Krylov complexity of the 4×4 operator O_4 (9). For notational convenience, we label the two 2×2 blocks along its diagonal as O_+ and O_- , i.e.,

$$O_4 = \begin{pmatrix} O_+ & 0 \\ 0 & O_- \end{pmatrix}. \quad (\text{A37})$$

The blocks $O_\pm$ have Krylov complexities $C_K^{(\pm)}(t)$ written as in Eq. (A36) in terms of the energy differences $\Delta E_+ = E_1 - E_2$ and $\Delta E_- = E_3 - E_4$ and the quantities defined in Eq. (A35) specified for the two sets of entries of the blocks. We denote these latter auxiliary quantities for the upper (lower) block as M_+ and T_+ (M_- and T_-). As $O_\pm$ are interpreted as labeling the charge sectors induced by a conserved charge, we think of $C_K^{(\pm)}(t)$ as symmetry-resolved Krylov complexities of O_4 .

b. Krylov complexity of a 4×4 operator

We move now to discuss the total Krylov complexity of O_4 evolving via the Hamiltonian $H_4 = \text{diag}(E_1, E_2, E_3, E_4)$. When represented as a vector in a Hilbert space, the evolution of O_4 is realized by acting with the Liouvillian in Eq. (A4) associated with H_4 . We denote it as $\mathcal{L}_4$. The Hilbert space considered here is equipped with the inner product (A1) with $\beta = 0$. Since $\mathcal{L}_4$ is Hermitian, we can diagonalize it as

$$\mathcal{L}_4 = V D V^{-1}, \quad (\text{A38})$$

where D is a diagonal matrix whose entries are the real eigenvalues λ_j of $\mathcal{L}_4$, and the columns of V are the corresponding eigenvectors. When we represent the evolving operator in a Hilbert space, the autocorrelation function can be seen as a return amplitude. This, in turn, is equal to the complex conjugate of the amplitude of $|O_4(t)\rangle$ on the first Krylov vector. We have [48]

$$R(t) = \phi_0^*(t) = \sum_k |V_{0k}|^2 e^{i\lambda_k t}, \quad V_{0k} \equiv (O_4(0)|V|k), \quad (\text{A39})$$

where $|k\rangle$ is the eigenvector of $\mathcal{L}_4$ associated with λ_k . It is convenient to introduce the time-dependent moments as

$$\mu_n(t) = (-i)^n \frac{\partial_t^n \phi_0(t)}{\phi_0(t)} = \frac{\sum_k |V_{0k}|^2 \lambda_k^n e^{i\lambda_k t}}{\sum_k |V_{0k}|^2 e^{i\lambda_k t}}, \quad (\text{A40})$$

which, when $t = 0$, reduce to the moments (A8). By applying the moments recursion method on the autocorrelation function (A39), we find that there are only four nonvanishing Lanczos coefficients. Now, we can use Eq. (A6) to express the other four nonvanishing amplitudes $\phi_n(t)$, with $n = 1, 2, 3, 4$, in terms of the Lanczos coefficients and $\phi_0(t)$ and its derivatives. We obtain

$$\phi_1(t) = -\frac{1}{b_1} \partial_t \phi_0(t), \quad \phi_2(t) = \frac{b_1^2 + (\partial_t)^2}{b_1 b_2} \phi_0(t), \quad (\text{A41})$$

$$\phi_3(t) = -\frac{(b_1^2 + b_2^2)(\partial_t) + (\partial_t)^3}{b_1 b_2 b_3} \phi_0(t),$$

$$\phi_4(t) = \frac{b_1^2 b_3^2 + (b_1^2 + b_2^2 + b_3^2)(\partial_t)^2 + (\partial_t)^4}{b_1 b_2 b_3 b_4} \phi_0(t). \quad (\text{A42})$$

From Eq. (A40), we re-express these amplitudes in terms of time-dependent moments as

$$\phi_1(t) = -\frac{1}{b_1} i \mu_1(t) \phi_0(t), \quad \phi_2(t) = \frac{b_1^2 - \mu_2(t)}{b_1 b_2} \phi_0(t), \quad (\text{A43})$$

$$\phi_3(t) = -\frac{(b_1^2 + b_2^2) i \mu_1(t) - i \mu_3(t)}{b_1 b_2 b_3} \phi_0(t),$$

$$\phi_4(t) = \frac{b_1^2 b_3^2 - (b_1^2 + b_2^2 + b_3^2) \mu_2(t) + \mu_4(t)}{b_1 b_2 b_3 b_4} \phi_0(t). \quad (\text{A44})$$

At this point, we have all the ingredients to explicitly compute the amplitudes $\phi_n(t)$, given that we know the time-dependent moments from Eq. (A40) and the Lanczos coefficients from the initial moments $\mu_n(0)$. This allows us to compute the Krylov complexity of $O_4(t)$, which reads

$$C_K(t) = \sum_{n=0}^4 n |\phi_n(t)|^2 = g_4(t) |\phi_0(t)|^2, \quad (\text{A45})$$

where we have defined

$$g_4(t) \equiv \left| \frac{\mu_1(t)}{b_1} \right|^2 + 2 \left| \frac{b_1^2 - \mu_2(t)}{b_1 b_2} \right|^2$$

$$+ 3 \left| \frac{(b_1^2 + b_2^2) \mu_1(t) - \mu_3(t)}{b_1 b_2 b_3} \right|^2$$

$$+ 4 \left| \frac{b_1^2 b_3^2 - (b_1^2 + b_2^2 + b_3^2) \mu_2(t) + \mu_4(t)}{b_1 b_2 b_3 b_4} \right|^2. \quad (\text{A46})$$

Although we already know the Krylov complexity of the two 2×2 time-dependent matrices [see Eq. (A36)], it is convenient to adapt the approach shown here to the 2×2 blocks $O_\pm$. Starting from the knowledge of the two amplitudes $\phi_0^{(\pm)}(t)$ [or, equivalently, the two autocorrelation functions $R_\pm(t)$], we can extract the only nonvanishing Lanczos coefficients $b_1^{(\pm)}$ and $b_2^{(\pm)}$. Following what we did before, we

compute the other two nonvanishing amplitudes for each block operator, finding

$$\begin{aligned}\phi_1^{(\pm)}(t) &= -\frac{1}{b_1^{(\pm)}} \partial_t \phi_0^{(\pm)}(t), \\ \phi_2^{(\pm)}(t) &= \frac{(b_1^{(\pm)})^2 + \partial_t^2}{b_1^{(\pm)} b_2^{(\pm)}} \phi_0^{(\pm)}(t).\end{aligned}\quad (\text{A47})$$

Adapting the definition (A40) to $\mu_n^{(\pm)}(t)$ defined in terms of $\phi_0^{(\pm)}(t)$, we can write the Krylov complexity of $O_{\pm}$ as

$$\begin{aligned}C_K^{(\pm)}(t) &= \sum_{n=0}^2 n |\phi_n^{(\pm)}(t)|^2 = f_2^{(\pm)}(t) |\phi_0^{(\pm)}(t)|^2, \\ f_2^{(\pm)}(t) &\equiv \left| \frac{\mu_1^{(\pm)}(t)}{b_1^{(\pm)}} \right|^2 + 2 \left| \frac{(b_1^{(\pm)})^2 - \mu_2^{(\pm)}(t)}{b_1^{(\pm)} b_2^{(\pm)}} \right|^2.\end{aligned}\quad (\text{A48})$$

As expected, these expressions become Eq. (A36) when we write the Lanczos coefficients and the time-dependent moments in terms of the entries of $O_{\pm}$.

When we regard the operators $O_{\pm}$ as the two blocks along the diagonal of O_4 [see Eq. (A37)], we can decompose the amplitude ϕ_0 in terms of $\phi_0^{(\pm)}$, namely,

$$\phi_0(t) = p_+ \phi_0^{(+)}(t) + p_- \phi_0^{(-)}(t). \quad (\text{A49})$$

This fact comes from taking the complex conjugate of decomposition (A12). The probabilities $p_{\pm}$ can be written in terms of the parameters of the entries of $O_{\pm}$ as

$$p_{\pm} \equiv \frac{T_{\pm} + 2M_{\pm}}{T_+ + 2M_+ + T_- + 2M_-}, \quad (\text{A50})$$

where $T_{\pm}$ and $M_{\pm}$ are defined below (A37). Since the autocorrelation function in Eq. (A39) is real for Hermitian operators, $\phi_0(t)$, $\phi_0^{(\pm)}(t)$ are real. Using the fact that these amplitudes are real, from Eq. (A49), we can write the square modulus of $\phi_0(t)$ as

$$\begin{aligned}|\phi_0(t)|^2 &= p_+^2 (\phi_0^{(+)}(t))^2 + p_-^2 (\phi_0^{(-)}(t))^2 \\ &\quad + 2p_+ p_- \phi_0^{(+)}(t) \phi_0^{(-)}(t).\end{aligned}\quad (\text{A51})$$

Plugging this expression into Eq. (A45) and using Eq. (A48), we obtain

$C_K(t)$

$$= g_4(t) \left(p_+^2 \frac{C_K^{(+)}(t)}{f_2^{(+)}(t)} + p_-^2 \frac{C_K^{(-)}(t)}{f_2^{(-)}(t)} + 2p_+ p_- \phi_0^{(+)}(t) \phi_0^{(-)}(t) \right) \quad (\text{A52})$$

$$= g_4(t) \left(p_+ \sqrt{\frac{C_K^{(+)}(t)}{f_2^{(+)}(t)}} + s_{\pm}(t) p_- \sqrt{\frac{C_K^{(-)}(t)}{f_2^{(-)}(t)}} \right)^2, \quad (\text{A53})$$

where $g_4(t)$ is given in Eq. (A46), $f_2^{(\pm)}(t)$ in Eq. (A48), and $p_{\pm}$ in Eq. (A50), and we have defined $s_{\pm}(t) \equiv \text{sgn}[\phi_0^{(+)}(t) \phi_0^{(-)}(t)]$. Let us comment on this result. This expression highlights the intricacy of reconstructing full complexity from the contributions of the blocks. The weights from the different charge sectors occur through the squares of the

TABLE I. The table shows the values of the parameters of the operator in Eq. (9) chosen to draw the curves in Figs. 1 and 2.

Curve	A_{11}	A_{22}	A_{33}	A_{44}	A_{12}	B_{12}	A_{34}	B_{34}
	0.7	0.2	2.1	1.9	0.4	0.9	1.5	0.7
	2.2	1.5	4.7	0.4	0.7	-0.8	0.9	-1.3
	0.3	1.9	1.1	1.8	-0.6	0.2	2.4	0.6
	0.5	0.3	4.9	2.2	1.0	0.3	-1.7	0.8

probabilities, and an interference term $\phi_0^{(+)}(t) \phi_0^{(-)}(t)$ mixing the distinct sectors also emerges. We can use the analytical expressions (A53), (A48), and (A50) to compute $\bar{C}(t)$ in Eq. (6) and study the behavior of $C_K(t) - \bar{C}(t)$. This is reported in Figs. 1 and 2. Curves in different panels are drawn for different sets of energies E_i (specified in the captions), while curves in the same panel differ by the choice of the entries of the initial operator in Eq. (9) (see Table I). We observe that $C_K(t) - \bar{C}(t)$ is positive in all the cases considered. Despite not being a proof, these curves support our expectation on the sign of $C_K(t) - \bar{C}(t)$. The fact that difference $C_K(t) - \bar{C}(t)$ is positive for any time is mathematically even less trivial if we notice that one of the symmetry-resolved Krylov complexity in Eq. (A48), i.e., $C_K^{(+)}(t)$, can be either larger or smaller than $C_K(t)$ in Eq. (A53), depending on t . This is shown in Fig. 3, where we report these three quantities as a function of time for certain values of parameters of the model. The plot shows that there is no fixed order among these complexities, as it changes with t .

We hope that the form (A53) of the Krylov complexity may offer a valuable framework for studying more general block-diagonal operators defined in higher-dimensional Hilbert spaces. In particular, it could help verify the expectation that $C_K(t) - \bar{C}(t)$ is non-negative while also shedding light on how correlations across different charge sectors contribute to this behavior.

6. Krylov complexity of complex harmonic oscillators

In this final Appendix, we detail the computations for the Krylov complexity and its resolution for $U(1)$ -invariant operators defined in systems of coupled complex harmonic oscillators.

a. Example of model of coupled oscillators

As discussed in the main text, any Hamiltonian of non-Hermitian canonically commuting operators Φ_j and Π_j that can be rewritten as the sum of two independent real oscillator Hamiltonians is diagonalized in the form (10). Here, for concreteness, we provide an example of a Hamiltonian with these properties. Consider the one-dimensional chain

H_{HO}

$$= \sum_{j=1}^N (\Pi_j^\dagger \Pi_j + \omega^2 \Phi_j^\dagger \Phi_j + (\Phi_{j+1}^\dagger - \Phi_j^\dagger)(\Phi_{j+1} - \Phi_j)), \quad (\text{A54})$$

where periodic boundary conditions $\Phi_{j+N} = \Phi_j$ are imposed. We introduce two sets of Hermitian canonically commuting

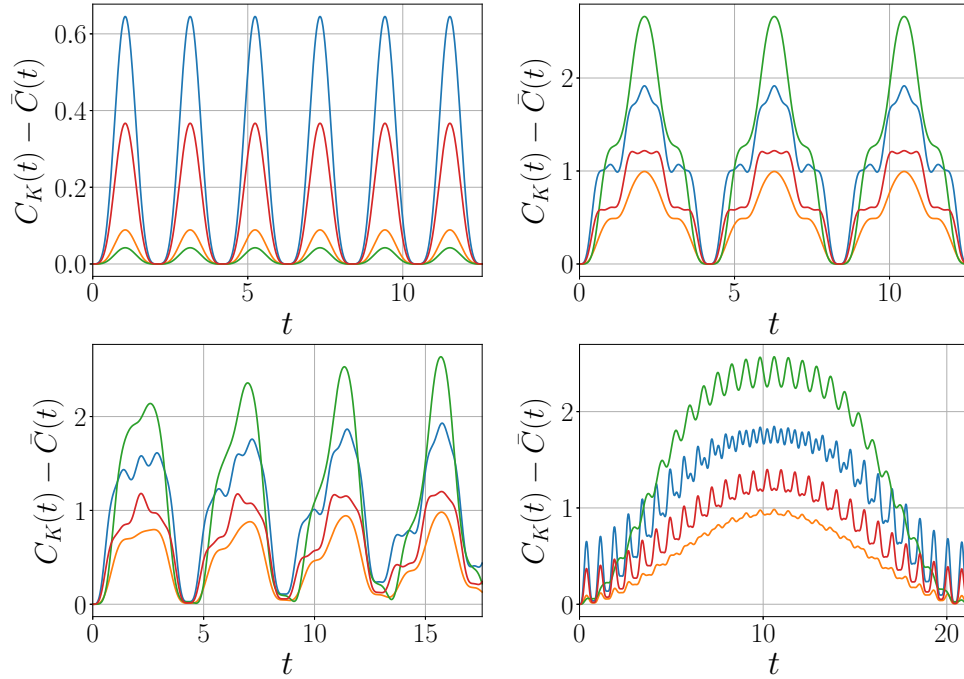


FIG. 2. Difference between the Krylov complexity of the operator (9) and the average (6) over the (two) sectors. Different colors correspond to distinct choices of the parameters A_{ii} ($i = 1, 2, 3, 4$), A_{12} , A_{34} , B_{12} , B_{34} (see Table I). In all the instances shown here, $C_K(t) - \bar{C}(t) \geq 0$, supporting our surmise. Curves in different panels are obtained for different sets of energies: top left $E_1 = 8.00$, $E_2 = 5.00$, $E_3 = 1.00$, $E_4 = 1.00$; top right $E_1 = 8.00$, $E_2 = 2.00$, $E_3 = 2.50$, $E_4 = 1.00$; bottom left $E_1 = 7.10$, $E_2 = 2.72$, $E_3 = 3.14$, $E_4 = 1.73$; bottom right $E_1 = 10.50$, $E_2 = 2.20$, $E_3 = 3.00$, $E_4 = 3.30$.

operators $\Phi_j = (\varphi_j + i\tilde{\varphi}_j)/\sqrt{2}$, $\Pi_j = (p_j + i\tilde{p}_j)/\sqrt{2}$ so that $[\varphi_j, p_l] = [\tilde{\varphi}_j, \tilde{p}_l] = i\delta_{jl}$ and all the other commutators are vanishing. In these variables, Eq. (A54) becomes

$$H_{\text{HO}} = \sum_{j=1}^N \left(\frac{1}{2} p_j^2 + \frac{1}{2} \omega^2 \varphi_j^2 + \frac{1}{2} (\varphi_{j+1} - \varphi_j)^2 \right) + \sum_{j=1}^N \left(\frac{1}{2} \tilde{p}_j^2 + \frac{1}{2} \omega^2 \tilde{\varphi}_j^2 + \frac{1}{2} (\tilde{\varphi}_{j+1} - \tilde{\varphi}_j)^2 \right). \quad (\text{A55})$$

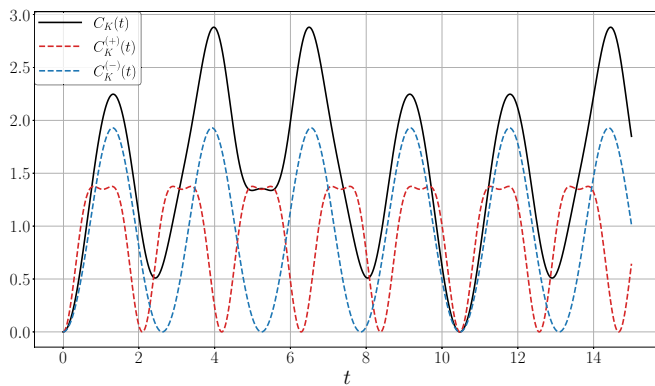


FIG. 3. Krylov complexity (A53) and symmetry-resolved complexities (A48) for the two charge sectors plotted as functions of time. The curves have been obtained for $E_1 = 8.0$, $E_2 = 5.0$, $E_3 = 3.4$, $E_4 = 1.0$, and A_{ij} and B_{ij} in the first line of Table I. We observe that, for some values of time, the symmetry-resolved Krylov complexity $C_K^{(+)}(t)$ is larger than the Krylov complexity of the total 4×4 operator.

We have written H_{HO} as a sum of two real oscillator Hamiltonians with the same frequency. Thus, we can recast it in the diagonal form (10) after having introduced two sets of creation and annihilation operators. In this example, we know the explicit expression of the dispersion relation, which reads [86]

$$\omega_k = \sqrt{\omega^2 + 4 \sin^2 \left(\frac{\pi k}{N} \right)}, \quad k = 1, \dots, N. \quad (\text{A56})$$

However, this expression does not affect our analysis, and we can maintain it general in what follows.

b. Krylov complexity of the operator (11) with $N = 1$ and its symmetry resolution

Here, we compute the Krylov complexity of the full operator $O_{\text{HO}}(t)$ in Eq. (11) and its components resolved with respect to the $U(1)$ symmetry generated by Q_{HO} given in the main text. For this purpose, we need to compute the autocorrelation functions $R(t)$ and $R_q(t)$. For evaluating $R(t)$, we observe that

$$\text{Tr}(e^{-H\beta/2} O_{\text{HO}}(t) e^{-H\beta/2} O_{\text{HO}}(0)) = \frac{\cos(2\omega t)}{8 \sinh^4(\beta\omega/2)}. \quad (\text{A57})$$

Notice that here we are considering general value of β in the inner product (A1). The return amplitude is obtained by dividing Eq. (A57) by its value at initial time $t = 0$. We obtain $R(t) = \cos(2\omega t)$. Applying the moment recursion method to this autocorrelation function, we find, as described in the main text, $b_1 = 2\omega$ and all the other Lanczos coefficients equal to zero. This leads to $C_K(t) = \sin^2(2\omega t)$.

Moving to the symmetry resolution, computing $R_q(t)$ is not straightforward, as we do not have direct access to the fixed-charge blocks. Indeed, the full Hilbert space of the system and the fixed-charge Hilbert spaces are infinite-dimensional. To overcome this difficulty, we resort to the formula (A14). Thus, we first compute the trace above with the insertion of a flux operator $e^{i\alpha Q_{\text{HO}}}$, obtaining

$$\begin{aligned} & \text{Tr}(e^{i\alpha Q_{\text{HO}}} e^{-H\beta/2} O_{\text{HO}}(t) e^{-H\beta/2} O_{\text{HO}}(0)) \\ &= \frac{\cos(2\omega t)}{2[\cos(\alpha) - \cosh(\beta\omega)]^2}. \end{aligned} \quad (\text{A58})$$

We see that the dependence on time and the one on α factorize, which implies that, after the Fourier transform in α , we find $\cos(2\omega t)$ multiplied by a nontrivial function of q . Dividing this outcome by itself evaluated at $t = 0$, the dependence on q drops out and we find $R_q(t) = \cos(2\omega t)$ for any integer q . This implies that the fixed-charge Lanczos coefficients are the same as the ones for the dynamics of the full operator, and as a consequence, the symmetry-resolved complexity is $C_K^{(q)}(t) = C_K(t) = \sin^2(2\omega t)$ in any charge sector. The fact that $C_K^{(q)}(t)$ is independent of q realizes the equipartition of Krylov complexity. Before commenting on the possible reason behind the equipartition, we observe that, using Eq. (A11), the probability associated with each sector reads

$$p_q = e^{-|q|\omega\beta} (|q| + \coth(\beta\omega)) \tanh^2(\beta\omega/2), \quad (\text{A59})$$

whose normalization implies that $C_K(t) - \bar{C}(t) = 0$ in this case. The distribution (A59) shows that the various charge sectors of the operator O_{HO} have different properties. This makes the occurrence of equipartition of Krylov complexity even less trivial.

c. Equipartition and its breaking

To investigate possible reasons for the occurrence of the equipartition of Krylov complexity, we consider a generic operator invariant under the $U(1)$ symmetry generated by Q_{HO} . We preliminarily characterize the fixed-charge Hilbert spaces in which the total Hilbert space decomposes. The Hilbert space of the q th sector (with q integer) is generated by the basis $\{|n, q\rangle, n - |q|/2 \geq 0\}$, where the integer or half-integer n is related to the energy eigenvalues E of H_{HO} as $n = \frac{E}{2\omega} - \frac{1}{2}$. Using these bases, a generic operator commuting with Q_{HO} (but not necessarily with H_{HO}) can be written as

$$O = \sum_{q \in \mathbb{Z}} O_q, \quad O_q = \sum_{n, n' = |q|/2}^{\infty} O_{n, n'}^{(q)} |n, q\rangle \langle n', q|, \quad (\text{A60})$$

which automatically provides an expression for the fixed-charge block O_q .

The representation in Eq. (A60) allows us to provide an alternative formula for the symmetry-resolved return amplitude $R_q(t)$ associated with each block operator. Plugging it into the definition of the inner product $(O_q(t)|O_q(0))$, using the orthonormality of the bases $\{|n, q\rangle\}$ and the orthogonality among different charge sectors, we find

$$R_q(t) = \frac{\sum_{n, n' = |q|/2}^{\infty} |O_{n, n'}^{(q)}|^2 e^{-\beta\omega(n+n'+1)} e^{-2i\omega t(n'-n)}}{\sum_{n, n' = |q|/2}^{\infty} |O_{n, n'}^{(q)}|^2 e^{-\beta\omega(n+n'+1)}}. \quad (\text{A61})$$

Moreover, using again the inner product (A1), we could also conveniently write the probability $p_q = |(O_q(0)|O(0))|^2$ associated with the corresponding charge sector as

$$p_q = \frac{\sum_{n, n' = |q|/2}^{\infty} |O_{n, n'}^{(q)}|^2 e^{-\beta\omega(n+n'+1)}}{\sum_{q=-\infty}^{\infty} \sum_{n, n' = |q|/2}^{\infty} |O_{n, n'}^{(q)}|^2 e^{-\beta\omega(n+n'+1)}}. \quad (\text{A62})$$

We can rewrite the operator O_{HO} in the basis (A60), finding the components

$$(O_{\text{HO}}^{(q)})_{n, n'} = \sqrt{n^2 - \frac{q^2}{4}} \delta_{n, n'+1} + \sqrt{n'^2 - \frac{q^2}{4}} \delta_{n', n+1}, \quad (\text{A63})$$

and their squared norm

$$|(O_{\text{HO}}^{(q)})_{n, n'}|^2 = \left(n^2 - \frac{q^2}{4}\right) \delta_{n, n'+1} + \left(n'^2 - \frac{q^2}{4}\right) \delta_{n', n+1}. \quad (\text{A64})$$

We can use this expression in Eq. (A61) to confirm our result for the symmetry-resolved return amplitude. Moreover, we could straightforwardly generalize our findings for the operator given by $\tilde{O}_{n, n'}^{(q)} = o_n^{(q)} \delta_{n, n'+a} + o_{n'}^{(q)} \delta_{n', n+a}$ with $a \in \mathbb{Z}$ and $o_n^{(q)} \in \mathbb{R}$. Importantly, the operator with components $\tilde{O}_{n, n'}^{(q)}$ is the most general example of an operator local in the energy space, as it connects only Fock states with a fixed energy difference $\Delta E/\omega - 1 = \pm 2a$. Its symmetry-resolved autocorrelation function $R_q(t)$ reads

$$R_q(t) = \cos(2a\omega t), \quad (\text{A65})$$

and it coincides with the total autocorrelation function $R(t)$. This implies that, for a general operator local in the energy space, we always find equipartition of Krylov complexity. If we instead consider an operator $\hat{O}$ that connects Fock states in such a way that $\Delta E/\omega - 1 = 2a, 2b$ ($a \neq b \in \mathbb{Z}$), i.e., $\hat{O}_{n, n'}^{(q)} = A_n^{(q)} \delta_{n, n'+a} + A_{n'}^{(q)} \delta_{n', n+a} + B_n^{(q)} \delta_{n, n'+b} + B_{n'}^{(q)} \delta_{n', n+b}$, a similar computation shows that the corresponding symmetry-resolved autocorrelation function acquires a dependence of q , which leads to a breaking of the equipartition of Krylov complexity. This shows that, as soon as we do not consider operators local in the energy space, the equipartition of Krylov complexity breaks down. In the example of complex harmonic oscillators, locality in energy space appears crucial for equipartition. It would be interesting to investigate whether this connection manifests in other systems.

d. Equipartition of Krylov complexity in the many-body case

We finally generalize our computation to the operator (11) defined for a system of $N > 1$ coupled harmonic oscillators. Since the considered operator has the factorized form $O_{\text{HO}} = \prod_{k=1}^N O^{(k)}$ with each single-mode operator defined by $O^{(k)} = c_k b_k + c_k^\dagger b_k^\dagger$, the corresponding time-evolved state reads $|O_{\text{HO}}(t)\rangle = \bigotimes_{k=1}^N |O^{(k)}(t)\rangle$. Due to this factorization, the total autocorrelation function also factorizes into a product of the autocorrelation functions for each oscillator mode:

$$R(t) = \prod_{k=1}^N (O^{(k)}(t)|O^{(k)}(0)) = \prod_{k=1}^N \cos(2\omega_k t), \quad (\text{A66})$$

where, in the last step, we have used the result for the single oscillator derived in the previous sections.

In this many-body case, the fixed-charge blocks in which O_{HO} decomposes are even more intricate. Since the sectors are labeled by the integer values of the charge, we can still use Eq. (A14) to compute the symmetry-resolved autocorrelation functions. We obtain

$$R_q(t) = \frac{\int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-i\alpha q} \prod_{k=1}^N \text{Tr}(e^{i\alpha Q_{\text{HO}}} e^{-\beta H/2} O^{(k)}(t) e^{-\beta H/2} O^{(k)}(0))}{\int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-i\alpha q} \prod_{k=1}^N \text{Tr}(e^{i\alpha Q_{\text{HO}}} e^{-\beta H/2} O^{(k)}(0) e^{-\beta H/2} O^{(k)}(0))}, \quad (\text{A67})$$

where the traces in the integrand can be computed and read

$$\text{Tr}(e^{i\alpha Q_{\text{HO}}} e^{-\beta H/2} O^{(k)}(t) e^{-\beta H/2} O^{(k)}(0)) = \frac{\cos(2\omega_k t)}{2[\cos(\alpha) - \cosh(\beta\omega_k)]^2}. \quad (\text{A68})$$

In Eq. (A67), the mode factorization occurs before the Fourier transform and, therefore, the symmetry-resolved autocorrelation does not factorize. This hampers the direct evaluation of the integrals. However, using Eq. (A68), we find again that the dependence on time and α factorizes. Thus, the time dependence at the numerator of Eq. (A67) can be brought out from the integral, and the two complicated integrals at the

numerator and denominator cancel. As a result, $R_q(t) = R(t)$ for any $q \in \mathbb{Z}$, which, as discussed before, leads to the equipartition of Krylov complexity. This example demonstrates that equipartition is not a feature of the single-body model, but can also occur in many-body scenarios.

To compute the Krylov complexity and its resolution for a generic value of N , we have to apply the moment recursion method to Eq. (A66). Doing so, we find that the number of nonvanishing Lanczos coefficients [and corresponding amplitudes $\phi_n(t)$] grows with N and these quantities are hard to express in a closed form. This does not allow us to provide a formula for the Krylov complexity of $O_{\text{HO}}(t)$ for any N . For gaining some analytical understanding, we can compute the early-time growth of $C_K(t)$ at the first two nonvanishing orders, obtaining

$$C_K(t) = C_K^{(q)}(t) = 4t^2 \sum_{k=1}^N \omega_k^2 - \frac{16}{3} t^4 \sum_{k=1}^N \omega_k^4 + O(t^6). \quad (\text{A69})$$

Here, we notice that the oscillator modes do not couple with each other at early times. To verify whether this feature survives at higher orders, we can compute the exact expression of the Krylov complexity for the operator with $N = 2$. Here, the nonvanishing Lanczos coefficients are only three, and this allows us to find the formula

$$C_K(t) = C_K^{(q)}(t) = \frac{3}{2} - \frac{(\omega_1 + \omega_2)^2 \cos(4t(\omega_1 - \omega_2)) + (\omega_1 - \omega_2)^2 \cos(4t(\omega_1 + \omega_2)) + 4\omega_1^2 \cos(4t\omega_2) + 4\omega_2^2 \cos(4t\omega_1)}{4(\omega_1^2 + \omega_2^2)}. \quad (\text{A70})$$

Expanding this result for small times, we see that it is enough to reach the order $O(t^6)$ to find products between ω_1 and ω_2 , corresponding to a coupling between the oscillator modes.

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