

Redirecting countermoving swarms through collision

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(Received 11 March 2026; accepted 5 June 2026; published 26 June 2026)

Multiswarm systems, where two or more swarms of mobile agents occupy the same region of space with different parameters and goals, occur in a variety of biological, engineering, and defense applications. Composites of multiple swarms can produce hybrid spatiotemporal patterns, which compared to single swarming systems, are relatively unexplored. In this work, we develop a framework for studying the collision of countermoving swarms, each with its own preferred, stable velocity before collision. We show that redirection of such swarms after collision occurs when a stable velocity synchronized state of the multiswarm composite exists. Using a rigid-body approximation, we are able to extract how scatter-redirection transitions scale with swarm parameters in a variety of scenarios from reciprocal and nonreciprocal systems to symmetric and antagonistic parameter values. Our results compare well to simulations of both particle modeled agents and wheeled robots.

DOI: [10.1103/PhysRevResearch.8.023351](https://doi.org/10.1103/PhysRevResearch.8.023351)

I. INTRODUCTION

Swarms consist of large numbers of mobile agents that interact in space to produce collective-motion patterns [1–4]. In the most basic scenarios, swarming patterns emerge through position and velocity couplings between self-propelled agents [5–11]. More generally, a swarm’s spatial self-assembly can interact with other collective processes, such as coupled oscillations among agents, resulting in more complex and general behaviors [12–14]. Swarms can be designed, trained, and/or evolve in such a way that their collective spatiotemporal behaviors effectively solve a group-level problem [15–18]. As a consequence, swarming dynamics is found on a very wide range of space- and timescales, and across many scientific and engineering disciplines, from cellular migration, foraging, and collective assembly [19–21] to swarming insects [22,23], flocking birds [24,25], schooling fish [26,27], and swarms of mobile robots for exploration, mapping, disaster recovery, and defense missions [28–31].

An important problem for swarming research is to analyze and control a swarm’s emergent pattern formation with certain target properties, without tightly engineering all aspects of the underlying agents [32–35]. Such swarming systems are often robust to failure, heterogeneity, uncertainty, time delays, and state-space perturbations [9,25,36–41]. In the manner of physical systems, one would like to control the pattern formation of swarms in a low-dimensional or low-rank way, ideally for instance, by toggling a few control knobs or accessible parameters [42,43]. Building such swarms of autonomous mobile agents—somewhere between tightly engineered and black-box systems—can benefit from an analytical approach.

In a variety of applications, multiple swarms occupy the same space, resulting in combined pattern formation, scattering, capturing, redirection, etc. [44–48]. In the case of colliding swarms, it has been shown that the relative velocities, numbers of agents, and coupling parameters between the swarms can be tuned to produce different collision outcomes [46,49–51]. More generally, multispecies swarms are known to reshape the patterns of single swarms and produce multistability of known collective behaviors [52,53]. Yet, much remains unknown about how combined patterns form in multiswarm systems, and how pattern stability and bifurcations depend on the interactions within and between different swarms.

Motivated by these issues, we construct a modeling framework where two countermoving swarms collide, each with different preferred velocities, and focus on the case where the two swarms combine and redirect their motion after collision. We formulate and analyze redirection in terms of the appearance of a stable velocity synchronized state for the combined swarm in parameter space. Using a rigid-body approximation, we show that the velocity synchronized state corresponds to a local minimum of an effective potential function, which depends on the spatial configuration of the swarm composite and the relative velocities of the two swarms. This analysis reveals a variety of scaling patterns for different swarming systems. For example, in the case of head-on collision of swarms that differ only in their number of agents and velocities, we find that the minimum number of agents in one swarm needed to reverse the motion of a countermoving swarm is independent of the latter’s size. Moreover, for colliding swarms with antagonistic and reactive behavior, we find that redirection requires that a normalized measure of attraction between the swarms should be larger than an equivalent measure of repulsion—independent of internal forces and spatial configurations before collision. These findings are shown to be in good agreement with particle and wheeled-robot simulations.

Our work can be distinguished from earlier studies for four primary reasons. First, the collision scenario investigated

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involves swarms with different, nonhomogeneous physical and control parameters, and especially, different stable preferred velocities. Second, we identify redirection of the colliding swarms with the emergence of a combined-swarm collective behavior having synchronized velocities despite the preferred-velocity heterogeneity. Third, we analyze the bifurcation of swarm redirection using rigid-body theory—converting a high-dimensional multiagent collision problem into a tractable low-dimensional version. Fourth, we validate our results with ground-robot simulations, which include nonholonomic effects implicit in the dynamics of differential wheeled systems.

Our paper is organized as follows. In Sec. II, we introduce a model for the collision of countermoving swarms with preferred velocities. In Sec. III, we show how stable redirection of the colliding swarms emerges with a velocity synchronized state of the combined-swarm system and develop a bifurcation theory based on rigid-body approximations. In Sec. IV, we explore the consequences of our bifurcation theory in several cases and test predictions with particle and robotic swarm simulations. In Sec. V, we provide a discussion of our results and thoughts on next steps and generalizations.

II. SWARMS WITH PREFERRED VELOCITIES

The swarms of interest in this work will consist of simple mobile agents tasked with three basic objectives: to maintain group cohesion in space, to avoid contact that is too close with others, and to move collectively at a certain velocity. Instead of being tightly engineered, these objectives are encoded in parametrized interactions among the agents and controls with local rules that produce behavior with the intended properties.

Specifically, let us consider a swarm of N agents in which every agent is accelerated by several state-dependent control forces and/or physical interactions, including self-propulsion, attraction, and repulsion. We assume that the self-propulsion force tends to select a preferred velocity for every agent, analogous to a natural frequency in a limit-cycle oscillator system [54,55]. Importantly, the preferred velocities may not be identical for all swarming agents in general. In addition, attraction and repulsion interactions express the tendency for agents to cluster in space with certain agents, while avoiding contact that is too close with others. If we denote the position of the i th agent's center of mass in two spatial dimensions at time t , $\mathbf{r}_i(t) = (x_i(t), y_i(t))^T$, and its velocity and acceleration $\dot{\mathbf{r}}_i$ and $\ddot{\mathbf{r}}_i$, respectively, a simple model that captures the intended dynamics is

$$\ddot{\mathbf{r}}_i = \alpha_i[\mathbf{u}_i - \dot{\mathbf{r}}_i] + \sum_{j \neq i} \mathbf{F}_{ij}(\mathbf{r}_j - \mathbf{r}_i), \quad (1a)$$

$$\text{with } \mathbf{F}_{ij}(\mathbf{r}_j - \mathbf{r}_i) = \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|} \times \left(\frac{a_{ij}}{l_{ij}^{(a)}} e^{-|\mathbf{r}_j - \mathbf{r}_i|/l_{ij}^{(a)}} - \frac{b_{ij}}{l_{ij}^{(b)}} e^{-|\mathbf{r}_j - \mathbf{r}_i|/l_{ij}^{(b)}} \right), \quad (1b)$$

where α_i is a characteristic self-propulsion constant for agent i , $\mathbf{u}_i$ is a preferred velocity for agent i [56], a_{ij} and b_{ij} are attractive and repulsion constants, respectively, for agent i

with respect to agent j , and $l_{ij}^{(a)}$ and $l_{ij}^{(b)}$ are characteristic length scales for attraction and repulsion interactions for agent i with respect to agent j [5–11]. We note that the attraction and repulsion interactions are tangent to the displacement vector between two agents, and have magnitudes that are exponentially decaying functions of the distance between agents. This latter property encodes the tendency for local range sensing to be maximal near a given agent, but falloff rapidly over a detection zone. Other smooth short-range functions are expected to produce similar dynamics.

In a swarm robotics context, the preferred velocity may correspond to $\mathbf{u}_i = \hat{\mathbf{D}}_i/\tau_i$, for example, where $\hat{\mathbf{D}}_i$ is an estimated displacement with respect to a far-away target and τ_i is a preferred arrival time. Similar linear self-propulsion terms have been considered in engineering contexts [57–59], in contrast to nonlinear speed-selection mechanisms for undirected biophysical systems [5,6]. On the other hand, $l^{(a)}$ corresponds roughly to the sensing range of the robots, while $l^{(b)}$ is a collision avoidance length scale that is on the order of the robot's size [10,11,60,61]. We point out that if the interactions satisfy Newton's third law, then $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$. We consider this *reciprocal* limit as a starting point, but consider nonreciprocal swarms as well, $a_{ij} \neq a_{ji}$ and $b_{ij} \neq b_{ji}$. Nonreciprocal and asymmetric interactions in swarms are part of a general class for active-matter systems [47,62], which for instance, can arise if agent i tries to avoid agent j , while agent j tends to follow agent i , as in herding or predator-prey dynamics [63,64].

For single swarms with homogeneous parameters, the behaviors of systems such as Eq. (1) are well understood [60,65,66]. However, much less is known about multiswarm dynamics when two or more swarms interact in the same region with potentially different parameters and goals. In such situations, it is possible for the swarms to combine and form collective states, and for one swarm to redirect and/or capture another [46,49,50]. It is to these phenomena that we turn to in this work.

Collision of two flocking swarms

In particular, we are interested in the collision of two swarms with agents behaving according to Eq. (1). Initially, the two swarms are well separated in space. Moreover, within each swarm, the parameters in Eq. (1) are taken to be homogeneous, but in general, the control constants, preferred velocities, numbers of agents, etc., may be different between the two swarms. For convenience, let us distinguish between the two swarms with color labels “red” and “blue” or R and B, respectively. Namely, the red swarm consists of N_R agents $R = \{1, 2, \dots, N_R\}$, while the blue swarm consists of agents $B = \{N_R + 1, N_R + 2, \dots, N\}$. Agents in the red swarm have preferred velocities $\mathbf{u}_i = \mathbf{u}_R$ for $i \in R$ and $\mathbf{u}_i = \mathbf{u}_B$ for agents in the blue swarm, $i \in B$.

In addition, the two swarms are assumed to collide near the origin at time $t = T$, where $T \gg 1$. In general before collision, the spatial offset between the swarms is given by a parameter δ , which is assumed to be near zero. In terms of the initial conditions, the swarms are configured into *two flocks*, such that $\mathbf{r}_i(t \ll T) = \mathbf{u}_R(t - T) + \mathbf{z}_i^* + \delta$ for $i \in R$ and $\mathbf{r}_i(t \ll T) = \mathbf{u}_B(t - T) + \mathbf{z}_i^*$ for $i \in B$, where $\mathbf{z}_i^*$ is a

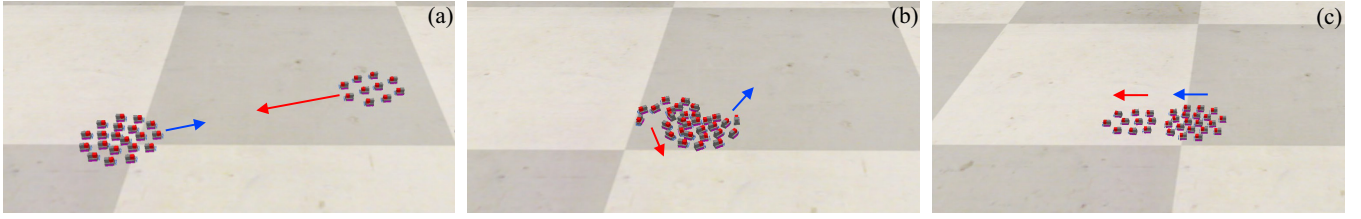


FIG. 1. Snapshots of two swarms that are redirected upon collision: (a) before, (b) during, and (c) after. In panel (a), each swarm travels as a fixed-formation flock with a velocity equal to the preferred velocity of the respective swarm. In panel (b), the swarms collide and form a transient mill. In panel (c), the swarms form a stable composite that travels at a uniform velocity, which is different than the preferred velocities of either swarm. The agents are simulated, differential-drive robots with target-velocity controls given by Eq. (1). The arrows indicate representative velocities for agents within each swarm (blue and red). Note that in this case, the collision is head on.

stable solution of a flocking configuration with vanishing self-propulsion force in Eq. (1) [46,49,60,65,67]. If we define an $N \times N$ binary matrix I whose elements are $I_{ij} = 1$ if agents i and j have the same color and zero otherwise, then $\mathbf{z}_i^*$ satisfies

$$\mathbf{0}_i = \sum_{j \neq i} I_{ij} \mathbf{F}_{ij}(\mathbf{z}_j^* - \mathbf{z}_i^*) \quad \forall i, \quad (2)$$

with $\sum_{i \in R} \mathbf{z}_i^* = \sum_{i \in B} \mathbf{z}_i^* = \mathbf{0}$.

We note that in order for collision to be feasible, $|\delta|$ must be small enough for the swarms to have appreciable interactions as they approach one another. As an example estimate, if the swarms are moving along the x axis, $\mathbf{u}_B = u_B \hat{x}$ and $\mathbf{u}_R = u_R \hat{x}$, then we can take $\delta = \delta \hat{y}$, where $\hat{x}$ and $\hat{y}$ are the unit vectors $(1, 0)^\top$ and $(0, 1)^\top$, respectively. In order for a relevant interaction to occur, the nearest distance separating agents in the blue swarm from agents in the red swarm should be less than the largest interaction length scale between the swarms, $l_{\max} \sim \max\{l_{RB}^{(a)}, l_{BR}^{(a)}\}$. Using the standard deviations of the two swarms as nominal estimates for their sizes, where the variances are $\sigma_B^2 = \sum_{i \in B} |\mathbf{z}_i^*|^2 / N_B$ and $\sigma_R^2 = \sum_{i \in R} |\mathbf{z}_i^*|^2 / N_R$, the spatial offset between the swarms should roughly satisfy $\delta - \sigma_B - \sigma_R < l_{\max}$.

Figure 1 illustrates the collision of two swarms with the setup described. In the first snapshot (a), a 10-agent red swarm approaches a 20-agent blue swarm from a configuration satisfying Eq. (2). In the second snapshot (b), the swarms collide and form a transient milling pattern. In the third snapshot (c), the swarms coalesce into a single, stable flock that reverses the direction of the blue swarm. The agents are differential-drive robots (DDR), which were simulated using COPPELIASIM software [68] with target-velocity controls given by Eq. (1) (see the Appendix for further details). Model parameters are $\alpha = 4$, $a = 0.1$, $b = 0.1$, $l^{(a)} = 2$, $l^{(b)} = 0.1$, $\mathbf{u}_R = (-0.16, 0)^\top$, and $\mathbf{u}_B = (0.05, 0)^\top$. Note that in this example, all pairwise constants are the same $\forall(i, j)$. Throughout this work, the physical units for the DDR simulations are MKS.

III. STABLE REDIRECTION AS VELOCITY SYNCHRONIZATION

Our primary goal is to examine the outcomes of two-swarm collisions described in Sec. II A and determine quantitative principles that allow us to predict and control the outcomes. For general parameters, the two swarms typically either *scatter* or *redirect* after collision. For scattering, the swarms collide but a composite state that stably contains all agents

is not accessible, and so the red and blue swarms separate into mutually uncoupled flocking configurations with altered trajectories. On the other hand, for redirection, such a stable composite is found, and the swarms merge together into a different formation that moves at a new collective velocity. Our main focus will be to determine the parameter values, such as the preferred velocities and the number of agents in the swarms, for which redirection can be expected.

In Fig. 2, we show example trajectories from DDR simulations with parameters that produce (a) scattering, (b) redirection, and (c) a transition between the two behaviors. In the examples shown, agents have similar controls and interactions within their respective swarms, but *opposite* interactions for agents in the countermoving swarm. Namely, red swarming agents are attracted to blue agents, while blue swarming agents are repelled by red agents. Parameters for Eq. (1) are $\alpha = 3$, $a_{RR} = a_{BB} = a_{RB} = b_{RR} = b_{BB} = b_{BR} = 0.1$, $a_{BR} = b_{RB} = 0$, $l_{RR}^{(a)} = l_{BB}^{(a)} = 2$, $l_{RR}^{(b)} = l_{BB}^{(b)} = 0.2$, $l_{RB}^{(a)} = l_{BR}^{(b)} = 3$, and $\mathbf{u}_B = (0.1, 0)^\top$. For each plot, we choose a different value for the preferred velocity of the red swarm $\mathbf{u}_R = (0.1, v)^\top$, where (a) $v = 0.1$, (b) $v = 0.05$, and (c) $v = 0.06$.

A. Composite flocking state

In order for all agents to be redirected after collision, a stable, composite flock containing all agents must exist for a given set of parameters. If no such state exists, then the two swarms will scatter after a transient period. Note that existence and stability of a composite are necessary but not sufficient conditions for stable redirection, since the initial conditions specified in Sec. II A may not lie in a given composite's basin of attraction [46].

To make progress in understanding redirection, we begin by describing composite flocking states. In general, such states correspond to configurations in which the displacements between *all agents* are fixed while agents travel together at a constant velocity. By substituting the ansatz $\mathbf{r}_i(t \gg T) = \mathbf{U}t + \mathbf{d}_i^*$ into Eq. (1), where $\mathbf{d}_i^*$ specifies the time-independent spatial configuration with a collective velocity $\mathbf{U}$, and taking $\ddot{\mathbf{r}}_i = \mathbf{0}_i \forall i$, we find that

$$\mathbf{U} = \frac{1}{N} \sum_i \left[\mathbf{u}_i + \frac{1}{\alpha_i} \sum_{j \neq i} \mathbf{F}_{ij}(\mathbf{d}_j^* - \mathbf{d}_i^*) \right] \quad (3a)$$

$$\text{and } \mathbf{0}_i = \alpha_i [\mathbf{u}_i - \mathbf{U}] + \sum_{j \neq i} \mathbf{F}_{ij}(\mathbf{d}_j^* - \mathbf{d}_i^*) \quad \forall i. \quad (3b)$$

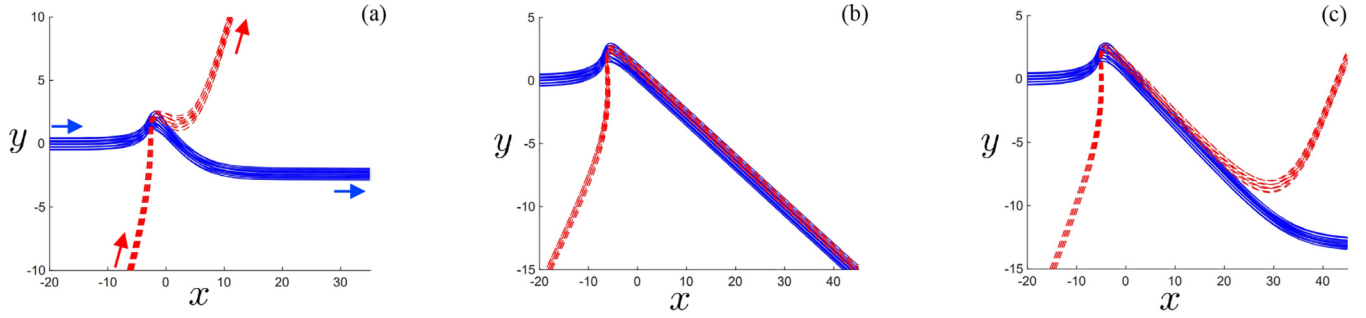


FIG. 2. Outcomes for two-swarm collisions with simulated, wheeled robots as the preferred velocity of the red swarm is varied. Shown are trajectories in the (x, y) plane for agents within each swarm: red and blue. The outcomes are: (a) scattering after collision, (b) redirection with a stable composite, and (c) weak instability of a two-swarm composite. Arrows in panel (a) show the direction of motion as time evolves.

Equation (3) gives us a numerically solvable $2N$ -dimensional system for the two-swarm composite by fixing the origin $\sum_i \mathbf{d}_i^* = \mathbf{0}$. Notice that unlike the state described by Eq. (2), the self-propulsion force acting on agents does not vanish since $\mathbf{u}_i \neq \mathbf{U} \forall i$. For reference, note that if the interactions are reciprocal with forces between agents that are equal and opposite, $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, and self-propulsion constants that are also equal, then the composite velocity is simply equal to the average preferred velocity, $\mathbf{U} = \sum_i \mathbf{u}_i / N$.

The multiswarm composite of Eq. (3) describes a behavior in which swarms with different parameters nevertheless coalesce into a single collective swarm that moves coherently. In particular, agents in the combined swarm synchronize their velocities despite having heterogeneity in the preferred velocities of the agents. This behavior is similar to synchronized states of Kuramoto phase-oscillator systems with heterogeneity in the preferred velocities playing the role of quenched disorder in natural frequency [54]. For sufficiently large coupling, Kuramoto networks form a state of global frequency synchronization, in which all of the oscillators rotate at a single collective frequency [69]. Drawing on this analogy, we say that in the colliding swarms case, stable redirection corresponds to a state of *velocity synchronization*.

The next step in understanding redirection is to study the stability of velocity-synchronized states (VSSs) satisfying Eq. (3). In particular, we can determine the fixed-point bifurcations of solutions to Eq. (3) using a linear approximation. Namely, we substitute $\mathbf{r}_i(t) = \mathbf{U}t + \mathbf{d}_i^* + \epsilon_i(t)$ into Eq. (1), and collect terms to $\mathcal{O}(\epsilon)$:

$$\ddot{\epsilon}_i = -\alpha_i \dot{\epsilon}_i + \sum_j J_{ij} \epsilon_j, \quad \text{with} \quad J_{ij} = \begin{cases} \frac{\partial \mathbf{F}_{ij}}{\partial (\mathbf{d}_j^* - \mathbf{d}_i^*)}, & \text{if } j \neq i, \\ -\sum_{k \neq i} \frac{\partial \mathbf{F}_{ik}}{\partial (\mathbf{d}_k^* - \mathbf{d}_i^*)}, & \text{otherwise.} \end{cases} \quad (4)$$

The spectrum $\epsilon_i(t) = e^{\lambda t} \mathbf{v}_i$, where $\mathbf{v}_i$ is an eigenmode of Eqs. (4), gives us information about the growth, decay, and oscillations of perturbations to a VSS. The spectrum depends principally on the $2N \times 2N$ matrix J .

Here, we point out a few important properties of Eq. (4). First, because of translational invariance, the matrix J has at least two zero eigenvalues for all parameter values. Second, if

the interactions are reciprocal, then J is symmetric and has real eigenvalues only. In this case, the appearance of redirection corresponds to a saddle-node bifurcation [55,70,71]. In particular, if we denote the ordered eigenvalues of J : $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_{2N}$, where $\sigma_1 = \sigma_2 = 0$, then as we approach the saddle-node bifurcation for reciprocal swarms, $\sigma_3 \rightarrow 0$. More generally, fluctuations around a VSS in reciprocal swarms, $\mathbf{r}_i(t) = \mathbf{U}t + \mathbf{d}_i(t)$, are effectively governed by a gradient system, $\dot{\mathbf{d}}_i = -\alpha_i \dot{\mathbf{d}}_i - \partial V^{(\text{rec})} / \partial \mathbf{d}_i$, with an effective potential function

$$V^{(\text{rec})} = \frac{1}{2} \sum_{i,j \neq i} (b_{ij} e^{-|d_j - d_i|/l_{ij}^{(b)}} - a_{ij} e^{-|d_j - d_i|/l_{ij}^{(a)}}) - \sum_i \alpha_i (\mathbf{u}_i - \mathbf{U})^\top \mathbf{d}_i \quad (5)$$

similar to many phase-oscillator networks [54,72]. Hence, when interactions are reciprocal, VSSs correspond to configurations that are local minima of Eq. (5). On the other hand, for nonreciprocal interactions, the matrix J is not symmetric and has complex eigenvalues in general. Thus, for nonreciprocal colliding swarms, a VSS can change stability, not only through saddle-node bifurcations, but Hopf bifurcations as well.

In order to predict whether redirection can occur as swarm parameters are varied, therefore, one can numerically solve Eqs. (3) and (4), and track when VSS eigenvalues cross the imaginary axis [55,70]. However, this approach has downsides including high dimensionality for large swarms, the potential for multiple VSS solutions, and the possibility that any particular solution may not be accessible from the colliding-swarms initial conditions specified in Sec. II A. To make further progress, we develop a simplifying approximation that allows us to find analytical relationships and scaling patterns for redirection.

B. Rigid-body approximation for redirection

Near the transition separating scattering and redirection, we notice that the two swarms tend to settle onto a configuration (after transients) that resembles separated flocks, but with a particular spatial offset, Δ^* , that emerges after collision [see Fig. 1(c)]. Namely,

$$\mathbf{d}_i^* \approx \begin{cases} \mathbf{z}_i^* + \Delta^*, & \text{if } i \in \text{R}, \\ \mathbf{z}_i^*, & \text{if } i \in \text{B}. \end{cases} \quad (6)$$

Compared to any general solution of Eq. (3), the proposed Eq. (6) is clearly compatible with the colliding-swarms setup, and therefore we expect it to be potentially useful in capturing VSS behavior relevant for collision. We note that because the internal configurations for each swarm are taken to be approximately fixed during collision, the ansatz Eq. (6) is effectively a rigid-body approximation (RBA) for the scatter-redirection transition [65,73].

In order to fully specify the RBA, we need to determine the unknown parameter Δ^* . In general, the RBA is not a formal solution of Eq. (3). Nevertheless, we expect the RBA to apply on average, i.e., to the centers of mass for each swarm. To find Δ^* , therefore, we take the average of Eq. (3b) over red and blue agents, respectively. The result is a (vector) fixed-point equation for each swarm's center of mass:

$$\mathbf{0} = \alpha_R[\mathbf{u}_R - \mathbf{U}] + \frac{1}{N_R} \sum_{i \in R} \sum_{j \neq i} \mathbf{F}_{ij}(\mathbf{d}_j^* - \mathbf{d}_i^*)$$

and,

$$\mathbf{0} = \alpha_B[\mathbf{u}_B - \mathbf{U}] + \frac{1}{N_B} \sum_{i \in B} \sum_{j \neq i} \mathbf{F}_{ij}(\mathbf{d}_j^* - \mathbf{d}_i^*). \quad (7a)$$

Next, we point out that if the spatial configurations satisfy Eqs. (2) and (6), then there is a cancellation of all interactions between agents within the same swarm. This allows us to further simplify Eq. (7). Namely, let us subtract the blue swarm center-of-mass equation—divided by the self-propulsion constant—from the same equation for the red swarm. Doing so, we find that Δ^* is the solution of a simple two-dimensional equation:

$$\mathbf{0} = \mathbf{u}_R - \mathbf{u}_B + \frac{1}{\alpha_R} \langle \mathbf{F}_{B \rightarrow R} \rangle(\Delta^*) - \frac{1}{\alpha_B} \langle \mathbf{F}_{R \rightarrow B} \rangle(\Delta^*),$$

where

$$\langle \mathbf{F}_{B \rightarrow R} \rangle(\Delta) = \frac{1}{N_R} \sum_{i \in R} \sum_{j \in B} \mathbf{F}_{ij}(\mathbf{z}_j^* - \mathbf{z}_i^* - \Delta)$$

and,

$$\langle \mathbf{F}_{R \rightarrow B} \rangle(\Delta) = \frac{1}{N_B} \sum_{i \in B} \sum_{j \in R} \mathbf{F}_{ij}(\mathbf{z}_j^* + \Delta - \mathbf{z}_i^*). \quad (8)$$

Our final step for analyzing redirection in the general RBA is to determine the appropriate bifurcations. Redirection is expected when there is a stable solution for Δ^* in Eq. (8). Similar to Sec. III A, we can make progress by analyzing the Jacobian of Eq. (8) with respect to variation in Δ^* . Compared to the general system, the 2×2 Jacobian matrix of Eq. (8) has only real eigenvalues since $\mathbf{F}_{ij}(\mathbf{z}_j^* - \mathbf{z}_i^* + \Delta)$ can be written as a gradient,

$$\mathbf{F}_{ij}(\mathbf{z}_j^* - \mathbf{z}_i^* + \Delta) = \frac{\partial}{\partial \Delta} \left(-a_{ij} e^{-|\mathbf{z}_j^* - \mathbf{z}_i^* + \Delta|/l_{ij}^{(a)}} + b_{ij} e^{-|\mathbf{z}_j^* - \mathbf{z}_i^* + \Delta|/l_{ij}^{(b)}} \right). \quad (9)$$

The gradient property implies that the Jacobian is a Hessian matrix of second derivatives with respect to Δ^* [74]. Thus, the off-diagonal elements are equal, since the order of partial derivatives does not matter for smooth functions. As a consequence, redirection emerges in a saddle-node bifurcation

within the RBA when

$$\det \left(\frac{1}{\alpha_R} \frac{\partial \langle \mathbf{F}_{B \rightarrow R} \rangle}{\partial \Delta} - \frac{1}{\alpha_B} \frac{\partial \langle \mathbf{F}_{R \rightarrow B} \rangle}{\partial \Delta} \right) \Big|_{\Delta = \Delta^{*,s}} = 0, \quad (10)$$

where $\Delta^{*,s}$ is the collision parameter at bifurcation [75].

In general, we point out that the RBA is expected to break down when the spatial offset distance at bifurcation, $|\Delta^{*,s}|$, is small compared to the sizes of the two swarms—in which case the swarms overlap significantly in space and adjust their densities accordingly upon collision. Using the spatial standard deviations of the swarms before collision as estimates for their sizes, σ_B and σ_R , we expect the RBA to apply roughly when $|\Delta^{*,s}| > \sigma_B + \sigma_R$.

IV. EXAMPLE CASES AND COMPARISONS TO SIMULATIONS

Equations (8) and (10) give us a simple low-dimensional framework to test, predict, and control redirection in a variety of collision scenarios for two swarms, which we turn to in the following sections.

A. Reversal in collisions with symmetric interactions

The first scenario that we explore in detail involves the collision of *symmetric* swarms, i.e., swarms with different numbers of agents and preferred velocities, but otherwise equal parameters. In addition, we first consider swarms moving in opposite directions along the same axis with head-on collision. In this case, redirection corresponds to one of the swarms effectively *reversing* direction after collision, as in the example Fig. 1.

Without loss of generality, we orient the velocities for each swarm along the x axis, i.e., $\mathbf{u}_R = (u_R, 0)^T$ and $\mathbf{u}_B = (u_B, 0)^T$. Similarly, the collision parameter aligns along the same axis, $\Delta^* = (\Delta^*, 0)^T$. To start, we compare the solutions of the RBA [Eq. (8)] to DDR simulations. In Fig. 3(a), we plot the time-averaged value of the distance between the centers of mass for the two swarms after they collide versus the preferred velocity of the red swarm. Each set of blue plot markers corresponds to simulations with different numbers of red agents/robots: $N_R = 13$ (stars), $N_R = 7$ (circles), and $N_R = 3$ (squares). For all examples, $N_B = 20$. Other model parameters are the same as those specified for Fig. 1. In addition, the solution of Eq. (8) is plotted with a curve for each example, along with the saddle-node bifurcation point plotted in magenta from solving Eq. (10). In general, the agreement between theory and simulations is quite good, despite both the simplifications implicit in the RBA and unmodeled dynamics of the robots that are not included in the target-velocity controller Eq. (1) [76].

Qualitatively, we observe that the faster the red swarm moves in the opposite direction compared to the blue swarm, the larger the offset between their centers of mass in the VSS. Roughly speaking, as the separation between the two swarms approaches an inflection point of the positional interactions, such interactions are no longer sufficient to compensate for the different swarm velocities, and the saddle-node bifurcation occurs. The critical value for the separation between the swarms, $\Delta^* \rightarrow \Delta^{*,s}$, is reached, for example, when the

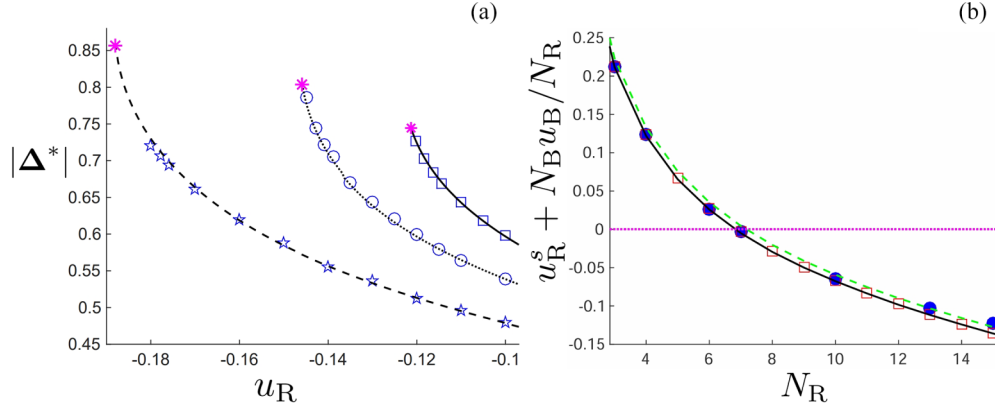


FIG. 3. Bifurcation of symmetric swarms with head-on collision. (a) Time-averaged distance between the centers of mass for two swarms after collision vs the red swarm's preferred velocity. Blue markers represent robot simulations (See the Appendix) with different numbers of red agents: $N_R = 13$ (stars), $N_R = 7$ (circles), and $N_R = 3$ (squares); $N_B = 20$ for each. Curves and magenta stars are predictions from Eqs. (8) and (10), respectively. (b) Bifurcation value of the red-swarm preferred velocity relative to the value required for reversal vs the number of red agents. Robot and particle simulation results are shown with blue circles and red squares, respectively. Predictions from solving Eqs. (8) and (10) are plotted with a solid black curve. Equation (12) predictions are plotted with a dashed-green curve. Other parameters are specified in Sec. IV A.

velocity of the red swarm approaches a limit, $u_R \rightarrow u_R^s$, for which Eqs. (8) and (10) are satisfied simultaneously. We test these predictions in Fig. 3(b) by comparing u_R^s to the largest, negative value of a red swarm's preferred velocity for which a VSS is observed in simulations. DDR and Eq. (1) simulation results are shown with blue circles and red squares, respectively. Both are in good agreement with the RBA.

Scaling with swarm parameters

We note that in Fig. 3(b), u_R^s is plotted relative to the value that produces a VSS with zero velocity. Namely, $U = 0$ implies $u_R = -N_B u_B / N_R$. The $U = 0$ line is plotted in magenta in Fig. 3(b) for reference. The general pattern is that as the number of agents in the red swarm, N_R , increases, the VSS reverses direction at a point where the black and magenta curves intersect. Note that when the VSS reverses direction, so does the blue swarm, which has a positive preferred velocity. As a general implication, if our goal is to parametrize a red swarm that reverses blue, for fixed attraction and repulsion parameters, there is some minimum number required $N_R^{(\min)}$, which in the example Fig. 3(b) corresponds to $N_R^{(\min)} = 7$.

An interesting question concerns how $N_R^{(\min)}$ scales with swarm parameters. We can estimate the answer by considering the limit of large numbers $N_R, N_B \gg 1$. In this case, the continuum approximation becomes applicable, where the configuration of each swarm is described by the same density functional form in the RBA, $\rho(\mathbf{z})$ for a flock centered at $\mathbf{z} = \mathbf{0}$ (where we drop the subscript and $*$ notation for convenience). Note that for general models, the density functions for flocking states with vanishing self-propulsion are analyzable using functional methods [65,67,77]. By replacing the sum over particles in Eq. (7b) by integrals over density, taking $U = 0$, and solving for N_R , we find the expected minimum number of red agents needed to reverse the blue

swarm:

$$N_R^{(\min)} = \frac{\alpha u_B}{S(\Delta^{*,s})} \quad \text{with,}$$

$$S(\Delta) = - \iint F^{(x)}(\mathbf{z}' - \mathbf{z} + \Delta \hat{\mathbf{x}}) \rho(\mathbf{z}) \rho(\mathbf{z}') d\mathbf{z}^2 d\mathbf{z}'^2, \quad (11)$$

where $\mathbf{z}$ and $\mathbf{z}'$ are displacement vectors from the blue and red swarm centers, respectively, $\Delta^{*,s} = \Delta^{*,s} \hat{\mathbf{x}}$ is the solution of Eq. (10), $F^{(x)}(\mathbf{z}' - \mathbf{z} + \Delta \hat{\mathbf{x}})$ is the x component of the force on a blue agent located at $\mathbf{z}$ from a red agent located at $\mathbf{z}' + \Delta \hat{\mathbf{x}}$, and $d\mathbf{z}^2$ and $d\mathbf{z}'^2$ are the differential area elements for the blue and red swarms, respectively. If we substitute the parameter values from Fig. 3(b) into Eq. (11), then we estimate the critical number to be $N_R^{(\min)} \approx 7.1$ [78], which is very near the noted observational value.

More important than the exact value for the denominator of Eq. (11) (which can be computed numerically) is the fact that the integral is a function only of the positional force parameters, $a, b, l^{(a)}$, and $l^{(b)}$. Specifically, $S(\Delta^{*,s})$ does not depend on the number of agents in the swarms or the preferred velocities. Hence, we learn that if the blue swarm doubles its velocity, $N_R^{(\min)}$ also doubles. On the other hand, if the blue swarm doubles its number of agents, this has no effect on $N_R^{(\min)}$. In fact, the red swarm does not need more agents for reversal; it needs agents that are twice as fast, according to $u_R = -N_B u_B / N_R^{(\min)}$.

Several examples demonstrating these results are shown in Fig. 4(a). For four sets of parameters, we plot $N_R^{(\min)}$ with different $(\alpha, u_B, l^{(b)})$: blue circles (1.0, 0.4, 0.1), red squares (1.0, 0.5, 0.5), magenta diamonds (1.3, 0.6, 0.1), and cyan triangles (0.5, 0.2, 0.5). To find each plotted point, we fix N_B and perform a series of collision simulations with Eq. (1) and different $N_R \in \{1, 2, \dots, 50\}$ with $u_R = -N_B u_B / N_R - 0.002$. The smallest value of N_R that produces a stable VSS is plotted. Other model parameters are $a = b = 0.1$ and $l^{(a)} = 2$. The pattern in Fig. 4(a) follows our expectations based on

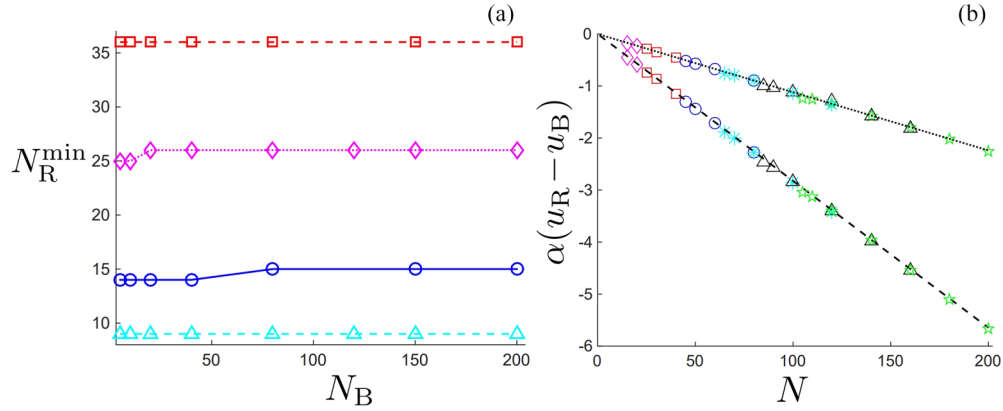


FIG. 4. Scaling for reversal when two symmetric swarms collide for a wide range of model parameters. (a) Minimum number of red agents needed to reverse a blue swarm vs the number of blue-swarm agents. Each plot label corresponds to a different set of $(\alpha, u_B, l^{(b)})$ parameters, detailed in Sec. IV A. (b) Difference between the preferred velocities of red and blue swarms at bifurcation vs the combined-swarm size. Predictions from Eq. (12) are plotted with lines for each series. The top and bottom series correspond to $l^{(b)} = 0.1$ and $l^{(b)} = 0.5$, respectively. Other parameters are specified in Sec. IV A.

Eq. (11). Namely, as N_B grows large, there is effectively no change in $N_R^{(\min)}$.

A similar analysis reveals that the critical preferred-velocity difference between the two swarms scales linearly with the total number of agents, $N = N_R + N_B \gg 1$. In particular, by repeating the continuum approximation using Eq. (8), we find that

$$\alpha(u_B - u_R^s) = NS(\Delta^{*,s}), \quad (12)$$

with the same integral appearing as in Eq. (11). Thus, in the symmetric-reversal case, we learn that larger swarms are able to accommodate more discrepancy in the velocities of the composites. These results are tested in Fig. 4(b) for two sets of repulsion length scales $l^{(b)} = 0.1$ (upper) and $l^{(b)} = 0.5$ (lower). Other force parameters are $a = 0.1$, $b = 0.1$, and $l^{(a)} = 2$. For the plotted data, α ranges from 0.2 to 2.0, u_B ranges from 0.2 to 1.0, N_B ranges from 10 to 100, and N_R ranges from 5 to 100. For each point, we fix all parameters except u_R , which is varied, and plot the most negative value for which a VSS is formed after collision in simulations of Eq. (1). Equation (12) predictions are plotted with lines along side simulation results. We observe in Fig. 3(b) that the continuum approximation correctly predicts a scaling collapse of the data onto two lines with slopes given by the particular values of the integral in Eq. (12).

B. Redirection at general angles

Next, we move on to consider collisions with more general angles, where the preferred swarm velocities are not on the same axis. Once again, our aim is to find appropriate red swarm parameters that redirect the blue swarm. Further scaling and analytical progress is possible when considering the limit in which the two colliding swarms have *isotropic mass distributions* in their flocking states before collision. Given the rotational symmetry of the attractive and repulsive forces in Eq. (1), we observe isotropy when the swarm numbers are large $N_R, N_B \gg 1$ [65]. In general, we note that the solution of the RBA Eq. (8) can be written as a local minimum of a

potential function, $-\partial V^{(\text{RBA})}/\partial \Delta|_{\Delta=\Delta^*} = \mathbf{0}$, where

$$\begin{aligned} V^{(\text{RBA})}(\Delta) &= -(\mathbf{u}_R - \mathbf{u}_B) \cdot \Delta \\ &+ \frac{1}{\alpha_R N_R} \sum_{i \in R, j \in B} (b_{ij} e^{-|\mathbf{z}_j^* - \mathbf{z}_i^* - \Delta|/l_{ij}^{(b)}} - a_{ij} e^{-|\mathbf{z}_j^* - \mathbf{z}_i^* - \Delta|/l_{ij}^{(a)}}) \\ &+ \frac{1}{\alpha_B N_B} \sum_{i \in B, j \in R} (b_{ij} e^{-|\mathbf{z}_j^* - \mathbf{z}_i^* + \Delta|/l_{ij}^{(b)}} - a_{ij} e^{-|\mathbf{z}_j^* - \mathbf{z}_i^* + \Delta|/l_{ij}^{(a)}}). \end{aligned} \quad (13)$$

If the isotropic assumption is valid, then the summations in Eq. (13) do not depend on the direction of Δ . As such, the $(\mathbf{u}_R - \mathbf{u}_B) \cdot \Delta$ term is the only one that selects a spatial direction, and therefore we expect the offset between the two swarms to align with their velocity difference, i.e.,

$$\Delta = \frac{\mathbf{u}_R - \mathbf{u}_B}{|\mathbf{u}_R - \mathbf{u}_B|} \Delta \equiv \hat{s} \Delta. \quad (14)$$

Given Eqs. (13) and (14), the RBA bifurcation point Eq. (10) in the isotropic approximation becomes the one-dimensional system:

$$0 = \frac{\partial^2 V^{(\text{RBA})}}{\partial^2 \Delta} \Big|_{\Delta=\Delta^{*,s}}. \quad (15)$$

Note that Eq. (15) is satisfied for a particular scalar $\Delta^{*,s}$, which is independent of the velocities of the swarms. Other patterns can be found by analyzing Eqs. (13) and (15) in specific scenarios.

1. Redirection angles for symmetric swarms

As in Sec. IV A, we begin with symmetric swarms. In particular, for symmetric and isotropic swarms with $N_R, N_B \gg 1$, the bifurcation condition Eq. (15) is

$$0 = \frac{\partial S}{\partial \Delta} \Big|_{\Delta=\Delta^{*,s}}. \quad (16)$$

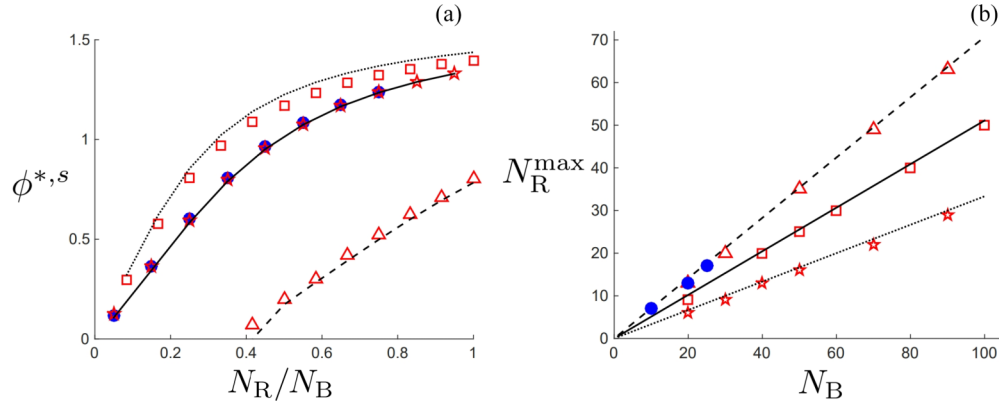


FIG. 5. Redirection at general angles. Robot and Eq. (1) simulations are plotted in blue and red, respectively. (a) Maximum redirection angle (radians) for symmetric swarms with orthogonal collision vs the size of the red swarm relative to the blue. Each series corresponds to different αu_B : 0.45 (squares), 0.25 (circles and stars), and 2.4 (triangles), as specified in Sec. IV B 1. Curves represent Eq. (17) predictions. (b) Maximum number of red agents that can be embedded in a blue swarm for the antagonistic scenario. Each series corresponds to different $(\alpha_R, \alpha_B, a_{RB}, l_{BB}^{(b)})$: red triangles and blue circles (4, 3.3, 0.06, 0.1), red squares (4.5, 3.5, 0.046, 0.2), and red stars (4.5, 3, 0.035, 0.5). Lines are predictions from Eq. (20). Other parameters are specified in Sec. IV B 2. Note that for both panels (a) and (b), $\mathbf{u}_B = u_B \hat{\mathbf{x}}$.

Here, we use the fact that the interaction sums in Eq. (13) can be computed along any axis for Δ , i.e., $\hat{\mathbf{s}} = \hat{\mathbf{x}}$. Large- N results allow us to calculate the red-swarm parameters that deflect a blue swarm to a desired angle. To make parametric control with the red swarm one-dimensional, we assume that the red and blue swarms have orthogonal velocities, i.e., $\mathbf{u}_B = u_B \hat{\mathbf{x}}$ and $\mathbf{u}_R = u_R \hat{\mathbf{y}}$. In this case, the angle (ϕ^*) at which the blue swarm is redirected after collision is $\phi^* = \arctan(\mathbf{U} \cdot \hat{\mathbf{y}} / \mathbf{U} \cdot \hat{\mathbf{x}})$.

In order for redirection at a given ϕ^* to be stable, $\Delta^* < \Delta^{*,s}$. For fixed blue-swarm parameters, $\Delta^* = \Delta^{*,s}$ corresponds to a maximum redirection angle $\phi^{*,s}$ that is achievable by the red swarm. The maximum angle can be found by combining the RBA solution in the isotropic and symmetric limits $-\partial V^{(RBA)} / \partial \Delta|_{\Delta=\Delta^*} = 0 = |\mathbf{u}^R - \mathbf{u}^B| - NS(\Delta^*)/\alpha = 0$, with the bifurcation condition Eq. (16). Solving for $\phi^{*,s}$ gives

$$\phi^{*,s} = \arctan \left(\frac{N_R}{N_B} \sqrt{\left(\frac{NS(\Delta^{*,s})}{\alpha u_B} \right)^2 - 1} \right). \quad (17)$$

If our goal is to parametrize a red swarm that redirects a blue swarm from its preferred trajectory, then Eq. (17) gives us an upper bound for the parametric control outcome. Figure 5(a) shows tests of Eq. (17), compared to DDR and Eq. (1) simulations. Each series represents different values for (α, u_B) pairs, from top to bottom, respectively: (2.25, 0.2), (5, 0.05), and (8, 0.3). Attractive and repulsive parameters are the same as in Fig. 1. For each series, we find the maximum value of u_R that produces a VSS in simulations and plot the corresponding angle at which the blue swarm is redirected. Note that the large- N predictions from Eq. (17) are nearly identical to the RBA solutions of Eqs. (8) and (10) for finite N , and so, we simply plot the former.

Interestingly, the lower simulation dataset in Fig. 5(a) plotted with red triangles demonstrates that if blue's velocity controller, αu_B , is too large, then there are certain composite swarms that are too small to stabilize redirection by a red

swarm at any angle. In the noted example, $N_R/N_B \gtrsim 0.4$. In fact, Eq. (17) tells us that

$$N_R + N_B > \frac{\alpha u_B}{S(\Delta^{*,s})}, \quad (18)$$

giving a number bound for the controllable red swarm. Intuitively, the larger the red swarm in the symmetric scenario, the more effective the redirection of a blue swarm.

2. Antagonistic swarm collisions

The final case that we consider involves colliding swarms where red agents are attracted to blue agents, while blue agents are repelled by red agents. Because the two swarms have opposing tendencies with respect to one another, we call this scenario *antagonistic*. In particular, we assume that red agents are attracted to blue agents with a coupling constant a_{RB} and blue agents are repelled by red agents with a coupling constant b_{BR} , as in the example Fig. 2. Other interswarm couplings are taken to be zero $a_{BR} = b_{RB} = 0$. In addition, to simplify the analysis, we make the further assumption that the sensing length scales and/or interaction zones for red and blue agents with respect to one another are similar in the antagonistic scenario, i.e., $l_{RB}^{(a)} = l_{BR}^{(b)} \equiv l_{RB}$. Opposing interaction constants for the swarms with similar magnitudes can arise, for instance, if agents in one swarm effectively sense and react to changes in the motion of agents in the opposing swarm.

Despite the opposing interactions, stable redirection can be observed as in the example Fig. 2(b). For antagonistic redirection, the blue swarm is continuously repelled and deflected by the red swarm, which chases after it. Similar to the symmetric examples, we would like to parametrize the red swarm in order to produce such redirection. Here again, large- N limits are useful for extracting scalings. Note that unlike symmetric scenarios, the two swarms are not assumed to have the same internal parameters, and therefore the density functions will not be the same in general. Let us denote the densities $\rho_R(\mathbf{z}_R)$ for the red swarm and $\rho_B(\mathbf{z}_B)$ for the blue swarm, given $N_R, N_B \gg 1$. By applying Eqs. (13) and (14),

and solving for Δ^* from $-\partial V^{(\text{RBA})}/\partial \Delta|_{\Delta=\Delta^*} = 0$, we find that the collision parameter Δ^* for antagonistic swarms is the solution of

$$\frac{|\mathbf{u}_R - \mathbf{u}_B|}{\frac{N_B a_{RB}}{\alpha_R} - \frac{N_R b_{BR}}{\alpha_B}} = A(\Delta^*) \quad \text{where,}$$

$$A(\Delta) = -\frac{\partial}{\partial \Delta} \int \int e^{-|\mathbf{z}_R - \mathbf{z}_B + \Delta \hat{s}|/l_{RB}} \rho_B(\mathbf{z}_B) \times \rho_R(\mathbf{z}_R) d\mathbf{z}_B^2 d\mathbf{z}_R^2. \quad (19)$$

An important observation from Eq. (19) is that redirection cannot occur as the denominator goes to zero, since $A(\Delta)$ is finite. In this limit, only swarms with identical velocities $|\mathbf{u}_R - \mathbf{u}_B| = 0$ can be stabilized. In fact, stability of a stable composite in the antagonistic scenario generally requires

$$\frac{N_B a_{RB}}{\alpha_R} > \frac{N_R b_{BR}}{\alpha_B}. \quad (20)$$

When Eq. (20) is not satisfied, the RBA Jacobian has at least one positive eigenvalue, making redirection unstable. Effectively, Eq. (20) expresses the intuitive requirement for redirection that the attractive forces acting on a typical red agent must be larger than the opposing repulsive forces acting on a typical blue agent (normalized by self-propulsion strength). Note that Eq. (20) holds generally and is independent of the internal parameters and preferred velocities of the red and blue swarms.

In effect, Eq. (20) gives a red swarm number range for parametric control. Unlike the symmetric case, however, for antagonistic swarms, we find the opposite tendency: The red swarm cannot be too many in number in order to redirect the blue—smaller is better. What is more, the maximum number of red swarming agents that can be embedded in the blue, $N_R^{(\text{max})} = (a_{RB}/b_{BR})(\alpha_B/\alpha_R)N_B$, only depends on self-propulsion constants, interswarm coupling parameters, and the total number of blue agents. This result is tested in Fig. 5(b) against simulations of Eq. (1) (plotted in red) and DDR simulations (plotted in blue). Each series represents a different combination of $(\alpha_R, \alpha_B, a_{RB}, l_{BB}^{(b)})$: red triangles and blue circles (4, 3.3, 0.06, 0.1), red squares (4.5, 3.5, 0.046, 0.2), and red stars (4.5, 3, 0.035, 0.5). For each point, we increase N_R , while keeping all other parameters fixed, until a VSS is not observed after collision. Other model parameters are $a_{RR} = a_{BB} = b_{RR} = b_{BB} = 0.1$, $b_{BR} = 0.07$, $l_{RR}^{(a)} = l_{BB}^{(a)} = l_{RB} = 2$, $l_{RR}^{(b)} = 0.1$, $\mathbf{u}_B = (0.1, 0)^\top$, and $\mathbf{u}_R = (0.1, 0.002)^\top$.

Altogether, we see that predicting the redirection of two colliding swarms with different preferred velocities is well captured by stability analysis of a velocity synchronized composite—even when the interactions between the swarms are heterogeneous and/or antagonistic.

V. DISCUSSION

When two swarms interact in the same region of space, it is possible for one swarm to scatter, capture, redirect, or otherwise alter the motion of the other. In this work, we focused on the scenario in which two countermoving and flocking swarms collide, each with their own preferred velocity as they approach the collision. For such swarms,

we demonstrated how the existence of a stable velocity-synchronized composite of the combined swarms determines whether redirection can be expected after collision. By developing a rigid-body approximation, we were able to predict parameter values and scalings for the relative sizes, velocities, and coupling constants that separate scattering behavior from stable redirection in a variety of parameter regimes, including symmetric and antagonistic cases. For example, we were able to calculate scattering angles between symmetric colliding swarms using continuum approximations and the fact that velocity synchronized states within the rigid-body approximation can be described as minima of an effective potential function. Similarly, we found that there is an upper bound for the number of agents that can be embedded into an antagonist swarm, which depends only on interswarm couplings, self-propulsion constants, and the number of agents in the antagonist swarm. Our results were found to be in good agreement with both interacting-particle and wheeled-robot simulations.

Our work represents progress on the general problem of predicting and controlling collision outcomes in multiswarm systems. It is likely that the presented approach can shed light on more general swarming behaviors as well, for example, involving generalized synchronization in heterogeneous and networked swarms with quenched disorder. Beyond theory, our results aid in the development of robotic swarms whose emergent pattern formation can be controlled by a small number of physical and/or control parameters.

Of course, real swarming systems exhibit effects due to noise, agent heterogeneity, sensing and communication latencies, and other constraints; however, we believe our results will be generally robust to moderate levels of such complications. First, we expect the analysis to be generally accurate for small amounts of noise on timescales that are shorter than the characteristic timescales for observing large fluctuations to unstable swarm configurations. For example, if the swarm interactions are reciprocal, the mean observation time for large fluctuations is exponential in an action function, which for the case of swarm redirection with independent and identically distributed additive Gaussian noise in the agent accelerations is proportional to the difference between the potential function Eq. (5) evaluated at a swarm saddle point and the velocity-synchronized state; more complex noise produces more complicated actions, but with similar results [72,79,80]. In terms of parameter heterogeneity, we expect our analysis to be qualitatively robust as well. For instance, for reciprocal swarms, we expect the emergence of a velocity synchronized state through a saddle-node bifurcation to carry over to more general parameter distributions, beyond the bimodal case effectively studied in this work, because the swarm state can still be described as the minimum of an effective potential function—thus preserving the bifurcation topology, as in the analytically similar Kuramoto model [69]. Finally, in terms of latency, previous studies have demonstrated that if latency produces time delays that are on the same order as the typical relaxation timescales for a swarming pattern, then one can expect spatial rotational modes to be excited [39,61]. For context, in the wheeled-robot simulations used in this work, the relaxation timescales were on the order of a second. Yet, central control of the robots occurred with fixed

latency, 0.05 s. If the two timescales had been more similar, additional behaviors in the colliding robots would have been observed.

Extending our results to include more detailed analysis of real-swarm complications and more general collision scenarios are important avenues for future work.

ACKNOWLEDGMENTS

J.H., I.B.S., C.B.P., and L.M. were supported by the U.S. Naval Research Laboratory funding (No. N0001424WX00021). J.H. and I.B.S. were also supported by the Office of Naval Research funding (No. N0001425GI01182 and No. N0001425GI01158).

DATA AVAILABILITY

The data that support the findings of this article are not publicly available because they are owned by a third party and the

terms of use prevent public distribution. The data are available from the authors upon reasonable request. Simulation code associated with this work can be found at Ref. [76].

APPENDIX: COPPELIASIM ROBOT SIMULATIONS

COPPELIASIM is a medium-fidelity simulation environment with built-in object collision handling, ideal for multiagent applications [68]. Individual agents consist of geometric 3D models and associated behavior scripts. The robots modeled in this work are differential-drive robots with a maximum angular velocity of 1 rad/s based on the Anki Vector mobile robotic platform. $N \times M$ agents can be initialized at a desired Euclidean spacing and initial heading. Velocity commands are generated every 50 ms by each agent using Eq. (1). Each agent's state information is shared globally, enabling synchronized flocking behavior. As such, interagent communication is simplified in this study, but specific communication architectures can be implemented in subsequent studies.

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- [1] T. Vicsek and A. Zafeiris, Collective motion, *Phys. Rep.* **517**, 71 (2012).
 - [2] L. Psimeni, *Active Matter Within and Around Us: From Self-Propelled Particles to Flocks and Living Forms*, 1st ed. (Springer Nature, Cham, 2021).
 - [3] F. Cucker and S. Smale, Emergent behavior in flocks, *IEEE Trans. Autom. Control* **52**, 852 (2007).
 - [4] A. Cavagna, A. Cimorelli, I. Giardina, G. Parisi, R. Santagati, F. Stefanini, and M. Viale, Scale-free correlations in starling flocks, *Proc. Natl. Acad. Sci. USA* **107**, 11865 (2010).
 - [5] H. Levine, W.-J. Rappel, and I. Cohen, Self-organization in systems of self-propelled particles, *Phys. Rev. E* **63**, 017101 (2000).
 - [6] M. R. D'Orsogna, Y. L. Chuang, A. L. Bertozzi, and L. S. Chayes, Self-propelled particles with soft-core interactions: Patterns, stability, and collapse, *Phys. Rev. Lett.* **96**, 104302 (2006).
 - [7] J. Hindes, K. Daley, G. Stantchev, and I. B. Schwartz, Swarming network inference with importance clustering of relative interactions, *J. Phys.: Complexity* **5**, 045009 (2024).
 - [8] R. Olfati-Saber, Flocking for multi-agent dynamic systems: Algorithms and theory, *IEEE Trans. Autom. Control* **51**, 401 (2006).
 - [9] J. Hindes, V. Edwards, K. S. Kasraie, G. Stantchev, and I. B. Schwartz, Swarm shedding in networks of self-propelled agents, *Sci. Rep.* **11**, 13544 (2021).
 - [10] M. Melchiorre, L. Salamina, L. S. Scimmi, S. Mauro, and S. Pastorelli, Experiments on the artificial potential field with local attractors for mobile robot navigation, *Robotics* **12**, 81 (2023).
 - [11] M. Melchiorre, L. S. Scimmi, L. Salamina, S. Mauro, and S. P. Pastorelli, Robot collision avoidance based on artificial potential field with local attractors, in *Proceedings of the 19th International Conference on Informatics in Control, Automation and Robotics* (SciTePress, Lisbon, Portugal, 2022), pp. 340–350.
 - [12] K. P. O'Keeffe, H. Hong, and S. H. Strogatz, Oscillators that sync and swarm, *Nat. Commun.* **8**, 1504 (2017).
 - [13] G. K. Sar, K. O'Keeffe, J. U. Lizárraga, M. A. de Aguiar, C. Bettstetter, and D. Ghosh, Interplay of sync and swarm: Theory and application of swarmalators, *Phys. Rep.* **1167**, 1 (2026).
 - [14] S. Acharya, G. K. Sar, and D. Ghosh, A two-dimensional swarmalator system with Winfree-type phase coupling, *Proc. R. Soc. A* **481**, 20250315 (2025).
 - [15] E. Bonabeau, M. Dorigo, and G. Theraulaz, *Swarm Intelligence: From Natural to Artificial Systems* (Oxford University Press, New York, NY, 1999).
 - [16] X.-S. Yang, S. Deb, Y.-X. Zhao, S. Fong, and X. He, Swarm intelligence: Past, present, and future, *Soft Computing* **22**, 5923 (2018).
 - [17] M. N. Ab Wahab, S. Nefti-Meziani, and A. Atyabi, A comprehensive review of swarm optimization algorithms, *PLoS One* **10**, e0122827 (2015).
 - [18] X. Huang, W. Luo, L. Li, H. Zhang, H. Jiang, and X. Gao, Competitive swarm reinforcement learning improves stability and performance of deep reinforcement learning, *Sci. Rep.* **15**, 43637 (2025).
 - [19] M. F. Copeland and D. B. Weibel, Bacterial swarming: A model system for studying dynamic self-assembly, *Soft Matter* **5**, 1174 (2009).
 - [20] A. Be'er and G. Ariel, A statistical physics view of swarming bacteria, *Mov. Ecol.* **7**, 9 (2019).
 - [21] B. L. Rocha-Gregg and A. Huttenlocher, Swarming motility in host defense, *Science* **372**, 1262 (2021).
 - [22] I. Giardina, Collective behavior in animal groups: Theoretical models and empirical studies, *HFSP J.* **2**, 205 (2008).
 - [23] M. Sinhuber, K. van der Vaart, Y. Feng, A. M. Reynolds, and N. T. Ouellette, An equation of state for insect swarms, *Sci. Rep.* **11**, 3773 (2021).
 - [24] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale, and V. Zdravkovic, Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study, *Proc. Natl. Acad. Sci. USA* **105**, 1232 (2008).

- [25] G. F. Young, L. Scardovi, A. Cavagna, I. Giardina, and N. E. Leonard, Starling flock networks manage uncertainty in consensus at low cost, *PLoS Comput. Biol.* **9**, e1002894 (2013).
- [26] K. Tunström, Y. Katz, C. C. Ioannou, C. Huepe, M. J. Lutz, and I. D. Couzin, Collective states, multistability and transitional behavior in schooling fish, *PLoS Comput. Biol.* **9**, e1002915 (2013).
- [27] D. S. Calovi, U. Lopez, S. Ngo, C. Sire, H. Chaté, and G. Theraulaz, Swarming, schooling, milling: Phase diagram of a data-driven fish school model, *New J. Phys.* **16**, 015026 (2014).
- [28] X. Huang, F. Arvin, C. West, S. Watson, and B. Lennox, Exploration in extreme environments with swarm robotic system, in *Proceedings of the 2019 IEEE International Conference on Mechatronics (ICM)* (IEEE, Ilmenau, Germany, 2019), Vol. 1, pp. 193–198.
- [29] R. J. Amala Arokia Nathan, I. Kurmi, and O. Bimber, Drone swarm strategy for the detection and tracking of occluded targets in complex environments, *Commun. Eng.* **2**, 55 (2023).
- [30] F. Berlinger, M. Gauci, and R. Nagpal, Implicit coordination for 3D underwater collective behaviors in a fish-inspired robot swarm, *Sci. Rob.* **6**, eabd8668 (2021).
- [31] L. Gladence, V. M. Anu, A. Anderson, I. Stanley, J. A. Fernando J, and S. Revathy, Swarm intelligence in disaster recovery, in *Proceedings of the 2021 5th International Conference on Intelligent Computing and Control Systems (ICICCS)* (IEEE, Madurai, India, 2021), pp. 1–8.
- [32] V.-L. Heuthe, P. Iyer, G. Gompper, and C. Bechinger, Tunable colloidal swarmalators with hydrodynamic coupling, *Nat. Commun.* **16**, 10984 (2025).
- [33] J. Hinds, G. Stantchev, K. S. Kasraie, and I. B. Schwartz, Target-density formation in swarms with stochastic sensing and dynamics, *Phys. Rev. Res.* **7**, 043167 (2025).
- [34] E. M. Z. Sánchez, S. Billès, P.-H. Glinel, N. Bredeche, and R. Candelier, Maze-solving with density-driven swarms, *arXiv:2509.18359*.
- [35] S. Ceron, W. Xiao, and D. Rus, Reciprocal and non-reciprocal swarmalators with programmable locomotion and formations for robot swarms, in *Proceedings of the 2024 IEEE International Conference on Robotics and Automation (ICRA)* (IEEE, Yokohama, Japan, 2024), pp. 12233–12239.
- [36] K. Szwaykowska, L. M.-y.-T. Romero, and I. B. Schwartz, Collective motions of heterogeneous swarms, *IEEE Trans. Autom. Sci. Eng.* **12**, 810 (2015).
- [37] K. Szwaykowska, L. Mier-y Teran-Romero, and I. B. Schwartz, Patterned dynamics of delay-coupled swarms with random communication graphs, in *Proceedings of the 2015 54th IEEE Conference on Decision and Control (CDC)* (IEEE, Osaka, Japan, 2015), pp. 6496–6501.
- [38] M. S. Islam and I. A. Faruque, Insect visuomotor delay adjustments in group flight support swarm cohesion, *Sci. Rep.* **13**, 6407 (2023).
- [39] J. Hinds, V. Edwards, S. Kamimoto, I. Triandaf, and I. B. Schwartz, Unstable modes and bistability in delay-coupled swarms, *Phys. Rev. E* **101**, 042202 (2020).
- [40] A. Tiwari, S. Devasia, and J. J. Riley, Low-distortion information propagation with noise suppression in swarm networks, *Proc. Natl. Acad. Sci. USA* **120**, e2219948120 (2023).
- [41] M. Moussa and G. Beltrame, On the robustness of consensus-based behaviors for robot swarms, *Swarm Intell* **14**, 205 (2020).
- [42] A. Barciś, M. Barciś, and C. Bettstetter, Robots that sync and swarm: A proof of concept in ROS 2, in *Proceedings of the 2019 International Symposium on Multi-Robot and Multi-Agent Systems (MRS)* (IEEE, New Brunswick, NJ, 2019), pp. 98–104.
- [43] E. S. Medeiros and U. Feudel, Local control for the collective dynamics of self-propelled particles, *Phys. Rev. E* **109**, 014312 (2024).
- [44] C. Walton, I. Kaminer, Q. Gong, A. H. Clark, and T. Tsatsanifos, Defense against adversarial swarms with parameter uncertainty, *Sensors* **22**, 4773 (2022).
- [45] T. Tsatsanifos, A. H. Clark, C. Walton, I. Kaminer, and Q. Gong, Modeling large-scale adversarial swarm engagements using optimal control, in *Proceedings of the 2021 60th IEEE Conference on Decision and Control (CDC)* (IEEE, Austin, TX, 2021), pp. 1244–1249.
- [46] J. Hinds, V. Edwards, M. A. Hsieh, and I. B. Schwartz, Critical transition for colliding swarms, *Phys. Rev. E* **103**, 062602 (2021).
- [47] A. Lama, M. di Bernardo, and S. H. L. Klapp, Nonreciprocal field theory for decision-making in multi-agent control systems, *Nat. Commun.* **16**, 8450 (2025).
- [48] S. Shahid, Z. Zhen, U. Javaid, and L. Wen, Offense-defense distributed decision making for swarm vs. swarm confrontation while attacking the aircraft carriers, *Drones* **6**, 271 (2022).
- [49] D. Armbruster, S. Martin, and A. Thatcher, Elastic and inelastic collisions of swarms, *Physica D* **344**, 45 (2017).
- [50] S. Kamimoto, J. Hinds, and I. B. Schwartz, The chaotic milling behaviors of interacting swarms after collision, *Chaos* **33**, 081106 (2023).
- [51] C. Kolon and I. Schwartz, *The Dynamics of Interacting Swarms*, NRL Memo Report No. NRL/MR/6790-18-9782, US Naval Research Laboratory, Washington, DC, 2018.
- [52] S. Chatterjee, M. Mangeat, C.-U. Woo, H. Rieger, and J. D. Noh, Flocking of two unfriendly species: The two-species Vicsek model, *Phys. Rev. E* **107**, 024607 (2023).
- [53] A. K. Dutta, M. Mangeat, H. Rieger, R. Paul, and S. Chatterjee, Stability of flocking in the reciprocal two-species Vicsek model: Effects of relative population, motility, and noise, *Phys. Rev. E* **112**, 024137 (2025).
- [54] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort, and R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, *Rev. Mod. Phys.* **77**, 137 (2005).
- [55] Y. Kuramoto, Chemical oscillations, waves, and turbulence, in *Springer Series in Synergetics* (Dover Publications, New York, 1984).
- [56] Note that each agent will approach its preferred velocity asymptotically when the attraction and repulsion interactions vanish.
- [57] W. Ren, Consensus based formation control strategies for multi-vehicle systems, in *Proceedings of the 2006 American Control Conference* (IEEE, Minneapolis, MN, 2006), p. 6.
- [58] W. Ren, Formation keeping and attitude alignment for multiple spacecraft through local interactions, *J. Guid. Control. Dyn.* **30**, 633 (2007).
- [59] A. N. Montanari, A. E. D. Barioni, C. Duan, and A. E. Motter, Optimal flock formation induced by agent heterogeneity, *Nat. Commun.* **16**, 9626 (2025).

- [60] Y.-L. Chuang, Y. R. Huang, M. R. D’Orsogna, and A. L. Bertozzi, Multi-vehicle flocking: Scalability of cooperative control algorithms using pairwise potentials, in *Proceedings of the 2007 IEEE International Conference on Robotics and Automation* (IEEE, Rome, Italy, 2007), pp. 2292–2299.
- [61] V. Edwards, P. deZonia, M. A. Hsieh, J. Hindes, I. Triandaf, and I. B. Schwartz, Delay induced swarm pattern bifurcations in mixed reality experiments, *Chaos* **30**, 073126 (2020).
- [62] Z. You, A. Baskaran, and M. C. Marchetti, Nonreciprocity as a generic route to traveling states, *Proc. Natl. Acad. Sci. USA* **117**, 19767 (2020).
- [63] M. A. Tsyganov, J. Brindley, A. V. Holden, and V. N. Biktashev, Quasisoliton interaction of pursuit-evasion waves in a predator-prey system, *Phys. Rev. Lett.* **91**, 218102 (2003).
- [64] A. Lama and M. di Bernardo, Shepherding and herdability in complex multiagent systems, *Phys. Rev. Res.* **6**, L032012 (2024).
- [65] Yao-li Chuang, M. R. D’Orsogna, D. Marthaler, A. L. Bertozzi, and L. S. Chayes, State transitions and the continuum limit for a 2D interacting, self-propelled particle system, *Physica D* **232**, 33 (2007).
- [66] G. Albi, D. Balagué, J. A. Carrillo, and J. von Brecht, Stability analysis of flock and mill rings for second order models in swarming, *SIAM J. Appl. Math.* **74**, 794 (2014).
- [67] A. J. Bernoff and C. M. Topaz, A primer of swarm equilibria, *SIAM J. Appl. Dyn. Syst.* **10**, 212 (2011).
- [68] E. Rohmer, S. P. N. Singh, and M. Freese, V-REP: A versatile and scalable robot simulation framework, in *Proceedings of the 2013 IEEE/RSJ International Conference on Intelligent Robots and Systems* (IEEE, Tokyo, Japan, 2013), pp. 1321–1326.
- [69] Florian Dörfler and F. Bullo, Synchronization in complex networks of phase oscillators: A survey, *Automatica* **50**, 1539 (2014).
- [70] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory* (Springer-Verlag, Berlin, 1998), 2nd ed.
- [71] D. Manik, D. Witthaut, B. SchÄpf, M. Matthiae, A. Sorge, M. Rohden, E. Katifori, and M. Timme, Supply networks: Instabilities without overload, *Eur. Phys. J.: Spec. Top.* **223**, 2527 (2014).
- [72] J. Hindes and I. B. Schwartz, Rare slips in fluctuating synchronized oscillator networks, *Chaos* **28**, 071106 (2018).
- [73] J. R. Taylor, *Classical Mechanics* (University Science Books, Sausalito, CA, 2005).
- [74] K. Binmore and J. Davies, *Calculus: Concepts and Methods* (Cambridge University Press, New York, NY, 2002).
- [75] One can solve the nonlinear system [Eqs. (8) and (10)] numerically with, e.g., matlab’s fsolve function.
- [76] C. Prasad, J. Hindes, R. Suitor, and L. McGuire, Anki-swarm, 2025, <https://github.com/chinthanprasad/Anki-Swarm>.
- [77] A. J. Leverentz, C. M. Topaz, and A. J. Bernoff, Asymptotic dynamics of attractive-repulsive swarms, *SIAM J. Appl. Dyn. Syst.* **8**, 880 (2009).
- [78] The integrals over density in this work were numerically approximated by simulating two separate flocking swarms, each with 1000 agents, from random initial conditions with Eq. (1) until $t = 10^4$. Using each swarm’s steady-state configuration, $\mathbf{z}_B$ and $\mathbf{z}_R$ for the blue and red swarms, respectively, the integral operator $\int \int \rho_B(\mathbf{z}_B) \rho_R(\mathbf{z}_R) d\mathbf{z}_B^2 d\mathbf{z}_R^2$, can be estimated by $\frac{1}{N_B N_R} \sum_{i \in B, j \in R}$, similar to a Monte Carlo integration approximation.
- [79] J. Hindes, I. B. Schwartz, and M. Tyloo, Stability of Kuramoto networks subject to large and small fluctuations from heterogeneous and spatially correlated noise, *Chaos* **33**, 113129 (2023).
- [80] J. Hindes, P. Jacquod, and I. B. Schwartz, Network desynchronization by non-Gaussian fluctuations, *Phys. Rev. E* **100**, 052314 (2019).