Highly robust logical qubit encoding in an ensemble of V-symmetrical qutrits

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We propose using even and odd Schrödinger cat states formed from coherent states of U(3) of an ensemble of qutrits with a symmetrical V configuration (*a qubit-disguised qutrit*) to encode a logical qubit. These carefully engineered logical qubit states are parameter-independent stationary states of the effective master equation governing the evolution of the ensemble and, consequently, constitute dark states that are invulnerable to dissipation and correlated collective dephasing. In particular, the logical qubit states are immune to single-qutrit decay (the analogous of single-photon loss process for qutrits) and simultaneous decay and driving of two qutrits (the analogous of two-photon loss and driving processes for qutrits). In addition, we show how to implement the single-qubit quantum NOT gate and the Hadamard gate followed by either the phase gate or the phase and Z gates. We study analytically the case of two qutrits and conclude that the logical qubit states exhibit parity-sensitive inhomogeneous broadening and local correlated dephasing: The even logical state is completely immune to these processes, while the odd one is vulnerable. Nevertheless, in the presence of these interactions one can also define two odd states with mixed permutation symmetry (with fully distinguishable qutrits) that are immune to both inhomogeneous broadening and local correlated dephasing. We show that these results can be extended to an arbitrary number of qutrits. We also present how the logical qubit states can be defined for qutrits with an asymmetrical V configuration and that they preserve their robustness properties. The effective master equation is deduced from a physical system composed of two parametrically coupled cavities with one of them interacting dispersively with an ensemble of three-level atoms (the qutrits). A proof that the logical qubit is also part of a stationary solution of the physical system under the considered approximations is given. In principle, this physical system can be implemented by means of two coplanar waveguide resonators, a superconducting quantum interference device parametrically coupling them, and a cloud of alkali atoms close to one of the resonators.

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I. INTRODUCTION

In quantum information theory, it is common to restrict the state space of a physical system to a two-dimensional subspace in order to represent a qubit [1]. Alternatively, one can use a physical system with a higher-dimensional state space and encode the logical states of a qubit as two orthogonal pure states [2–4]. This is especially attractive because one can then design a physical system where the encoded qubit logical states can be robust against dissipation and dephasing [4–12]. A paradigmatic example consists of encoding the logical states of a qubit in a pair of even and odd (with respect to a suitable parity operator) Schrödinger cat states of a harmonic oscillator [2,4]. In the case of a single-mode electromagnetic field, these cat states can be protected against dephasing by means of two-photon loss processes but they are vulnerable to single-photon loss and this can limit their use in applications [4,5].

Another attractive possibility is to encode a qubit in an ensemble of atoms or spins because this type of system can have long coherence times [6–11]. Some of their major sources of noise are collective and local dephasing (that is, population-conserving scattering processes) and inhomogeneous broadening. For example, inhomogeneous broadening due to the Doppler effect and dephasing caused by collisions and laser fluctuations can obstruct the observation of coherent phenomena in atomic gases [13–15]. Some promising approaches rely on the use of Rydberg atoms [6–8], while other implementations consider a hybrid system of superconducting circuits and spin ensembles of either alkali atoms [8,10] or nitrogen-vacancy centers [10,11]. The former are based on

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the Rydberg dipole blockade mechanism and on multiphoton processes [6,7,16]. Moreover, for these systems it has been demonstrated that an encoded qubit can be controlled coherently and that it can be robust both to depletion of atoms and to noise [8].

Especially relevant to this article is the hybrid system approach, since Ref. [9] proposed a two-qubit simultaneous decay process that conserves the parity of the number of excited qubits and that can stabilize Schrödinger cat states of spin coherent states in the ensemble. In order to achieve this mechanism analogous to the two-photon loss process, Ref. [9] considered a system of two parametrically coupled single-mode cavities, one containing a driven *pump field* and the other containing a *signal field* that interacts dispersively with an ensemble of qubits. The effective master equation describing the evolution of the ensemble contains the desired two-qubit simultaneous decay process, as well as a linear dissipator which describes single-qubit decay, a process analogous to single-photon loss. Assuming that the number of excited qubits is much smaller than the total number of qubits and fine-tuning the system parameters, Ref. [9] found that the linear dissipator can be neglected and that the qubit ensemble is driven to an even (odd) cat state if the ensemble is initially in an even (odd) parity state. Quite remarkably, this leads to qubit Schrödinger cat states that can have a lifetime several orders of magnitude longer than cavity photonic cat states. By considering the same system but without driving and without the linear dissipator, Ref. [11] then showed that the system relaxes with high fidelity to states that can also be used as qubit logical states. In addition, it was also found [12] that the effects of inhomogeneous broadening depend on the parity and amplitude of the cat states, with the even cat state being significantly more robust against dephasing than the odd one for small amplitudes.

The same system presented in Ref. [9] has also been used to propose a protocol for the generation of spin squeezing with and without driving the pump field and obtained results comparable to those of the two-axis twisting model [10]. Furthermore, Ref. [10] discussed in detail the experimental feasibility of the protocol and proposed two implementations using two superconducting coplanar waveguide resonators (CPWR), a superconducting quantum interference device (SQUID), and an ensemble of either rubidium 87 atoms or nitrogen-vacancy centers in diamond. The SQUID is in charge of the parametric coupling between the cavities and the ensemble is placed above one of them and is magnetically coupled to it. In particular, Ref. [17] has already demonstrated both the dispersive and resonant magnetic coupling of an ensemble of ultracold rubidium 87 atoms to a CPWR.

In this article, we consider the same system as that introduced in Ref. [9] but with an ensemble of $N \geqslant 2$ qutrits with a V-type configuration (*a qubit-disguised qutrit*) replacing the qubits. The objective is to investigate if the added degree of freedom can be used to protect the logical qubit states from single qutrit decay, that is, from the process analogous to single-photon loss. The results are quite remarkable. If one chooses the logical qubit states as specific even and odd Schrödinger cat states of U(3) with a definite parity of the number of particles in the ground level, then it turns out that these logical states are parameter-independent stationary states of the effective master equation that describes the evolution of the ensemble of qutrits. Consequently, they are dark states and are immune to single qutrit decay and to correlated collective dephasing. In addition, analytical results for two qutrits show that they exhibit parity-sensitive inhomogeneous broadening and local correlated dephasing: The even cat state is immune to both, while the odd cat state is vulnerable. In the presence of these permutation symmetry breaking interactions, it turns out that one can consider another mixed permutation symmetry logical state, which is immune to these processes. Unfortunately, the trade-off is that the system does not always relax to the logical states, i.e., there are other parameter-dependent stationary states in the system. However, they are easy to create with a convenient driving process. In principle, the system considered in this article could be implemented experimentally by using the same physical system proposed in Ref. [10], since the CPWR used in Ref. [17] naturally couples three ground state hyperfine levels of rubidium 87 in a V configuration. The problem is that the coupling strength is weak and would have to be increased, for example, by decreasing the distance between the CPWR and the ensemble of atoms [17] or by using a state-of-the-art CPWR [18].

The article is structured as follows. In Sec. II, we introduce the system under study and establish the effective master equation that describes its evolution. In Sec. III, we present the parity symmetry of the master equation. Then, in Sec. IV we characterize the parameter-independent stationary states of the master equation and use them to define the logical qubit states. In Sec. V, we describe how to implement a quantum NOT gate and a Hadamard gate followed by either a phase gate or a phase and $\mathcal{Z}$ gates. In Sec. VI, we consider the case of two qutrits and determine the effects of local dephasing and inhomogeneous broadening. In Sec. VII, we deduce the effective master equation of Sec. II from a physical system composed of two parametrically coupled cavities and an ensemble of three-level atoms. In addition, we present the effect of a small asymmetry in the V configuration of the qutrits. Finally, the conclusions are in Sec. VIII. The Appendixes contain the detailed derivation of the logical qutrits and of the effective model, ending with a table summarizing the properties of the states.

II. THE MODEL

Consider a system consisting of $N \geqslant 2$ identical qutrits. Each qutrit is a quantum three-level system with a V configuration where $|1\rangle$ is the ground level and $|2\rangle$ and $|3\rangle$ are the excited levels (see Fig. 1). The angular transition frequency between levels $|j\rangle$ and $|1\rangle$ is denoted by ω_j, whereas direct transitions between levels $|2\rangle$ and $|3\rangle$ are forbidden. Assume that the qutrits are bosons and that $\omega_3 = \omega_2 = \omega_q > 0$, that is, the qutrits have a symmetrical V configuration. The case where the qutrits are identical but distinguishable, due to perturbations like local dephasing or inhomogeneous broadening, will be discussed in Sec. VI D.

The ensemble of qutrits is described in second quantization where $b_j^\dagger$ (b_j) creates (annihilates) a particle in the level $|j\rangle$ ($j = 1, 2, 3$). These operators satisfy the commutation

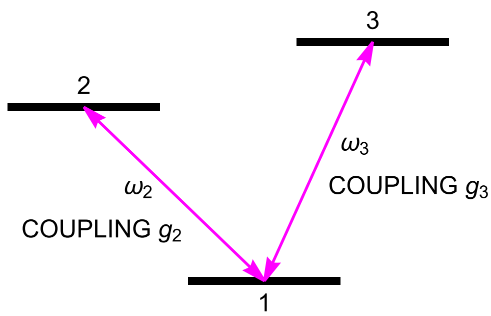

FIG. 1. Each qutrit is a quantum three-level system with a V configuration where $|1\rangle$ is the ground level and $|2\rangle$ and $|3\rangle$ are the excited levels. The angular frequency associated with the transition $|j\rangle \leftrightarrow |1\rangle$ is $\omega_j > 0$ with $j = 2, 3$. In the context of the physical system of Sec. VII, the qutrits are coupled to a single-mode cavity quantum electromagnetic field called the *signal field*. The coupling strengths are g_2 and g_3.

relations $[b_j, b_k] = 0$ and $[b_j, b_k^\dagger] = \delta_{kj}$ for $j, k = 1, 2, 3$. Here, δ_{kj} is the Kronecker delta.

An orthonormal basis β_q for the state space of the qutrits is obtained by using the occupation number states $|n_1, n_2, n_3\rangle$, where there are n_j qutrits in the level $|j\rangle$ ($j = 1, 2, 3$):

$$\beta_q = \left\{ |n_1, n_2, n_3\rangle : \ n_1, n_2, n_3 \in \mathbb{Z}^+, \ \sum_{j=1}^{3} n_j = N \right\}, \quad (1)$$

with $\mathbb{Z}^+$ the set of nonnegative integers. Then, one has

$$\begin{aligned} b_j^\dagger |n_1, n_2, n_3\rangle &= \sqrt{n_j + 1}|n_1 + \delta_{j1}, n_2 + \delta_{j2}, n_3 + \delta_{j3}\rangle, \\ b_j |n_1, n_2, n_3\rangle &= \sqrt{n_j}|n_1 - \delta_{j1}, n_2 - \delta_{j2}, n_3 - \delta_{j3}\rangle, \end{aligned} \quad (2)$$

for $j = 1, 2, 3$. For $j, k = 1, 2, 3$, define the operators

$$S_{jk} = b_j^\dagger b_k, \quad S_- = S_{12} + S_{13}. \quad (3)$$

Notice that $S_{jk}^\dagger = S_{kj}$, S_{jj} counts the number of qutrits in the level $|j\rangle$, and S_{jk} with $k \neq j$ annihilates a qutrit in the level $|k\rangle$ and creates one in the level $|j\rangle$. For an insightful interpretation of S_-, see Sec. IV A.

The effective master equation describing the evolution of the ensemble of qutrits is

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t), \quad (4)$$

where $\rho(t)$ is the density operator of the ensemble and the Liouvillian $\mathcal{L}$ is defined by

$$\mathcal{L}A = -\frac{i}{\hbar}[H_q, \ A] + \kappa_1 \mathcal{D}(S_-)A + \kappa_2 \mathcal{D}(S_-^2)A. \quad (5)$$

Here, $\kappa_1, \kappa_2 > 0$, and the superoperator $\mathcal{D}(\cdot)$ is defined by

$$\mathcal{D}(A)B = ABA^\dagger - \tfrac{1}{2}\{A^\dagger A, B\}, \quad (6)$$

where $\{\cdot, \cdot\}$ is the anticommutator. In Eqs. (5) and (6), A and B are any two linear operators. The Hamiltonian H_q is

$$\frac{1}{\hbar}H_q = \delta_1 S_{11} - \xi S_-^\dagger S_- + \delta (S_-^2)^\dagger S_-^2 + \alpha_0^* S_-^2 + \alpha_0 (S_-^2)^\dagger. \quad (7)$$

Here, ξ and δ are real constants, α_0 is a complex number, and δ_1 is a real parameter that can be tuned to zero or to a positive or negative value (see Sec. VII). This master equation is an effective model that has one possible physical origin presented in Sec. VII.

Observe that the Liouvillian $\mathcal{L}$ includes the dissipators $\kappa_1 \mathcal{D}(S_-)$ and $\kappa_2 \mathcal{D}(S_-^2)$, which, respectively, generalize to qutrits the single-qubit and two-qubit simultaneous decay processes mentioned in Sec. I. For an enlightening interpretation of the terms of the master equation, see Sec. IV A.

III. PARITY SYMMETRY OF THE MASTER EQUATION

Consider the unitary operator associated with the parity of the number of qutrits in the ground level $|1\rangle$:

$$\Pi_0 = e^{i\pi S_{11}} \ = \ (-1)^N e^{i\pi(S_{22}+S_{33})}. \quad (8)$$

In the second equality in Eq. (8), we used $S_{11} + S_{22} + S_{33} = N$. It follows that Π_0 is Hermitian and that its eigenvalues are ± 1. Whenever a ket $|\psi\rangle$ is an eigenvector of Π_0, we shall say that $|\psi\rangle$ has *the parity symmetry*. Observe that $|\psi\rangle$ can be expressed as a linear combination of occupation number states $|n_1, n_2, n_3\rangle$ where n_1 is always even (odd) if $|\psi\rangle$ is an eigenvector of Π_0 with corresponding eigenvalue $+1$ (-1).

Using how Π_0 and the operators S_- and $S_-^\dagger$ act on the kets of β_q, it is straightforward to show that

$$\begin{aligned} &[\Pi_0, S_-^2] = [\Pi_0, (S_-^\dagger)^2] = [\Pi_0, S_-^\dagger S_-] = 0, \\ &[\Pi_0, S_{11}] = [\Pi_0, (S_-^2)^\dagger S_-^2] = 0, \\ &\{\Pi_0, S_-\} = \{\Pi_0, S_-^\dagger\} = 0. \end{aligned} \quad (9)$$

As a consequence of Eq. (9), $[\Pi_0, H_q] = 0$.

Now, define the superoperator $\mathcal{P}$ by $\mathcal{P}A = \Pi_0 A \Pi_0$ for every linear operator A. Using that Π_0 is Hermitian and unitary and the properties in Eq. (9), it immediately follows that $\mathcal{P}^2 = \mathcal{I}$ with $\mathcal{I}$ the identity superoperator and that $\mathcal{PLP} = \mathcal{L}$. Then, $\mathcal{LP} = \mathcal{PL}$ and, consequently,

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) \quad \Leftrightarrow \quad \frac{d}{dt}\mathcal{P}\rho(t) = \mathcal{L}[\mathcal{P}\rho(t)]. \quad (10)$$

If $\rho(0) = \mathcal{P}\rho(0)$, then the existence and uniqueness theorem for the solution of a system of linear ordinary differential equations guarantees that $\rho(t) = \mathcal{P}\rho(t)$ for all t. In this case, $\rho(t)$ and Π_0 commute and the eigenvectors of $\rho(t)$ can be chosen to have the parity symmetry.

IV. PARAMETER-INDEPENDENT STATIONARY STATES

We are interested in determining stationary states ρ_s of the master equation in Eq. (4) that are independent of the parameters. Then, ρ_s must satisfy the conditions

$$\begin{aligned} &\mathcal{D}(S_-)\rho_s = 0, \quad [S_-^\dagger S_-, \rho_s] = 0, \\ &\mathcal{D}(S_-^2)\rho_s = 0, \quad [(S_-^2)^\dagger S_-^2, \rho_s] = 0, \\ &[S_{11}, \rho_s] = 0, \quad [\alpha_0^* S_-^2 + \alpha_0 (S_-^2)^\dagger, \rho_s] = 0. \end{aligned} \quad (11)$$

Notice that, in essence, we are *looking* for *dark states* of the Liouvillian $\mathcal{L}$ [19,20].

Consider a pure stationary state $\rho_s = |\Phi\rangle\langle\Phi|$. In Appendix A, it is shown that ρ_s satisfies (11) if and only if $S_-|\Phi\rangle = (S_-^2)^\dagger|\Phi\rangle = 0$ and $|\Phi\rangle$ is an eigenvector of S_{11}. In addition, it is also proved that $S_-|\Phi\rangle = (S_-^2)^\dagger|\Phi\rangle = 0$ and $|\Phi\rangle$ is an eigenvector of S_{11} if and only if $|\Phi\rangle$ is equal (except for a global phase factor) to one of the two following normalized states:

$$\begin{aligned} |0_L\rangle &= |z_1 = 0, z_2 = 1, z_3 = -1\rangle_N \\ &= \sum_{n_3=0}^{N} c_{n_3}|0, N - n_3, n_3\rangle, \\ |1_L\rangle &= b_1^\dagger|z_1 = 0, z_2 = 1, z_3 = -1\rangle_{N-1} \\ &= \sum_{n_3=0}^{N-1} d_{n_3}|1, N - 1 - n_3, n_3\rangle, \end{aligned} \tag{12}$$

where

$$\begin{aligned} c_{n_3} &= (-1)^{n_3}\sqrt{\frac{N!}{2^N\,(N-n_3)!\,n_3!}}, \\ d_{n_3} &= (-1)^{n_3}\sqrt{\frac{(N-1)!}{2^{N-1}\,(N-1-n_3)!\,n_3!}}, \end{aligned} \tag{13}$$

and

$$|z_1, z_2, z_3\rangle_M = \frac{1}{\sqrt{M!}}\left(\frac{z_1 b_1^\dagger + z_2 b_2^\dagger + z_3 b_3^\dagger}{\sqrt{|z_1|^2 + |z_2|^2 + |z_3|^2}}\right)^M |\mathbf{0}\rangle \tag{14}$$

represents a coherent state of U(3) for M particles [21]. Here, $|\mathbf{0}\rangle = |n_1 = 0, n_2 = 0, n_3 = 0\rangle$ is the Fock vacuum and z_i ($i = 1, 2, 3$) are complex numbers that are not all zero. In addition, note that $|0_L\rangle$ and $|1_L\rangle$ are orthogonal to each other because they are eigenvectors of S_{11} with different eigenvalues. Note that, for convenience, we use a nonconventional, redundant representation for coherent states of U(3) and that the usual one is obtained by putting $z_1 = 1$ [21]: According to Eq. (14), one has $|qz_1, qz_2, qz_3\rangle_M = (q/|q|)^M|z_1, z_2, z_3\rangle_M$ for any nonzero complex number q, so $|qz_1, qz_2, qz_3\rangle_M$ and $|z_1, z_2, z_3\rangle_M$ differ by a global phase factor.

We emphasize that

$$\begin{aligned} &S_{11}|0_L\rangle = 0, \quad S_-^\dagger|0_L\rangle = 0, \quad \Pi_0|0_L\rangle = |0_L\rangle, \\ &S_{11}|1_L\rangle = |1_L\rangle, \quad S_-^\dagger|1_L\rangle \neq 0, \quad \Pi_0|1_L\rangle = -|1_L\rangle, \\ &S_-|0_L\rangle = 0, \quad (S_-^2)^\dagger|0_L\rangle = 0, \quad \langle \mathbb{j}_L|\mathbb{k}_L\rangle = \delta_{\mathbb{j}\mathbb{k}}, \\ &S_-|1_L\rangle = 0, \quad (S_-^2)^\dagger|1_L\rangle = 0, \end{aligned} \tag{15}$$

where lowercase letters like $\mathbb{j}$ or $\mathbb{k}$ (or $\mathbb{m}$ and $\mathbb{n}$ later) represent binary digits, i.e., $\mathbb{j}, \mathbb{k} \in \{0, 1\}$. Observe that $|0_L\rangle$ is a coherent state, while $|1_L\rangle$ is a *particle-added coherent state*, to use the same nomenclature as Ref. [22] but for *photon-added coherent states*. Moreover, it is straightforward to show that

$$\begin{aligned} |\mathbb{j}_L\rangle &= \lim_{\epsilon\to 0^+}\frac{1}{2N_{\mathbb{j}}(\epsilon)}[|z_1 = \epsilon, z_2 = 1, z_3 = -1\rangle_N \\ &\qquad + (-1)^{\mathbb{j}}|z_1 = -\epsilon, z_2 = 1, z_3 = -1\rangle_N], \\ N_{\mathbb{j}}(\epsilon) &= \frac{1}{\sqrt{2}}\left[1 + (-1)^{\mathbb{j}}\left(\frac{2-\epsilon^2}{2+\epsilon^2}\right)^N\right]^{1/2}. \end{aligned} \tag{16}$$

One concludes from Eqs. (15) and (16) that $|0_L\rangle$ and $|1_L\rangle$ are highly robust even and odd Schrödinger cat states that can be used as logical states of a qubit.

The density operators $\rho_0 = |0_L\rangle\langle 0_L|$ and $\rho_1 = |1_L\rangle\langle 1_L|$ are the only parameter-independent, pure stationary states of the master equation in Eq. (4). We emphasize that they are fixed points of the Liouvillian $\mathcal{L}$ that commute with the Hamiltonian H_q and that satisfy $L\rho_{\mathbb{j}} = 0$ ($\mathbb{j} = 0, 1$) for all the Lindblad operators $L = S_-$ and S_-^2 appearing in $\mathcal{L}$. Thus, they constitute *dark states* of $\mathcal{L}$ [20] and are immune to both the linear $\kappa_1\mathcal{D}(S_-)$ and quadratic $\kappa_2\mathcal{D}(S_-^2)$ dissipators. The properties in Eq. (15) guarantee that they would also be immune to other interactions that include S_- or $(S_-^\dagger)^2$.

It is important to note that there are other parameter-dependent stationary solutions of the master equation in Eq. (4). If one uses the basis β_q to express $\mathcal{L}\rho = 0$ as a matrix equation and one takes $\delta_1 = 0$, our numerical calculations indicate that the subspace of matrices that satisfy $\mathcal{L}\rho = 0$ has dimension $N + 3$ (for $N = 2$, the result can be proved exactly). In this case, an orthonormal basis for that subspace could be constructed by starting from the matrices associated with $|\mathbb{j}_L\rangle\langle\mathbb{k}_L|$ with $\mathbb{j}, \mathbb{k} = 0, 1$ and using the Hilbert-Schmidt inner product. Notice that $\{\,|0_L\rangle,\ |1_L\rangle\}$ constitutes a *decoherence-free* subspace [23] only when $\delta_1 = 0$.

Finally, since $|0_L\rangle$ and $|1_L\rangle$ are eigenvectors of S_{11} with corresponding eigenvalues 0 and 1 [see Eq. (15)], they have a defined number of qutrits in the level $|1\rangle$ equal to 0 and 1, respectively. Therefore, the logical qubit states $|0_L\rangle$ and $|1_L\rangle$ cannot be obtained by using a Holstein-Primakoff expansion [24] where the majority of the qutrits are found in the level $|1\rangle$ (the so-called *low-excitation regime*). This is a notable difference with the case of qubits where the aforementioned approximation can be applied [9–11].

A. Coherent state creation operators

Since $S_- = S_{12} + S_{13} = b_1^\dagger(b_2 + b_3)$, one can take inspiration from the treatment of the isotropic two-dimensional harmonic oscillator in terms of creation and annihilation operators of left and right circular quanta [25] to give a simple and elegant description of both the model and the logical states $|0_L\rangle$ and $|1_L\rangle$.

Define the operators

$$b_l = \frac{1}{\sqrt{2}}(b_2 + b_3), \quad b_r = \frac{1}{\sqrt{2}}(b_2 - b_3). \tag{17}$$

Note that $[b_l, b_r] = [b_l, b_r^\dagger] = 0$ and $[b_l, b_l^\dagger] = [b_r, b_r^\dagger] = 1$. Then, $b_l^\dagger$, b_l and $b_r^\dagger$, b_r are two pairs of creation and annihilation operators of two independent *left*- and *right-handed* harmonic oscillators. Also, b_1 and $b_1^\dagger$ commute with $b_l, b_l^\dagger, b_r$, and $b_r^\dagger$, and $S_{22} + S_{33} = (b_l^\dagger b_l + b_r^\dagger b_r)$.

Using these operators, one has $S_- = \sqrt{2}b_1^\dagger b_l$, so S_- is an operator that describes a transition of a qutrit from the left mode to the ground level $|1\rangle$. Then, the Liouvillian $\mathcal{L}$ in Eq. (5) can be expressed as

$$\mathcal{L}A = -\frac{i}{\hbar}[H_q,\ A] + 2\kappa_1\mathcal{D}(b_1^\dagger b_l)A + 4\kappa_2\mathcal{D}\big[(b_1^\dagger)^2 b_l^2\big]A, \tag{18}$$

where A is any linear operator and

$$\frac{1}{\hbar}H_q = \delta_1 S_{11} - 2\xi(S_{11}+1)b_l^\dagger b_l + 4\delta(b_l^\dagger)^2 b_1^2 (b_1^\dagger)^2 b_l^2 + 2\alpha_0^*(b_1^\dagger)^2 b_l^2 + 2\alpha_0 (b_l^\dagger)^2 b_1^2. \tag{19}$$

Observe that $b_r^\dagger$ and b_r do not appear in the Liouvillian $\mathcal{L}$. This is the key to finding the parameter-independent, pure stationary states by mere inspection. In fact, $b_r^\dagger$ and b_r are associated with the *dark degrees of freedom*, while $b_l^\dagger$ and b_l are associated with *bright degrees of freedom*.

The linear dissipator $2\kappa_1\mathcal{D}(b_1^\dagger b_l)$ can be thought of as describing the decay of a single qutrit from the left (bright) mode to the ground level at a rate $2\kappa_1$, while the quadratic dissipator $4\kappa_2\mathcal{D}[(b_1^\dagger)^2 b_l^2]$ can be thought of as describing the simultaneous decay of two qutrits from the left (bright) mode to the ground level at a rate $4\kappa_2$. Also, the first two terms of the Hamiltonian H_q count the number of qutrits in the ground level and in the left (bright) mode, while the third term introduces a four-body interaction between the left (bright) mode and the ground level. Finally, H_q also includes the terms $[2\alpha_0^*(b_1^\dagger)^2 b_l^2 + 2\alpha_0(b_l^\dagger)^2 b_1^2]$, which generalize to qutrits the simultaneous two-qubit driving. This coherent two-qutrit driving describes two-qutrit simultaneous transitions from the ground level to the left (bright) mode and vice versa.

Another orthonormal basis for the state space of the N qutrits can be obtained by using the occupation number states defined by $b_1^\dagger$, $b_l^\dagger$, and $b_r^\dagger$:

$$\beta_q' = \left\{ |n_1, n_l, n_r\rangle = \frac{(b_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(b_l^\dagger)^{n_l}}{\sqrt{n_l!}} \frac{(b_r^\dagger)^{n_r}}{\sqrt{n_r!}} |\mathbf{0}\rangle : n_1, n_l, n_r \in \mathbb{Z}^+,\ n_1 + n_l + n_r = N \right\}, \tag{20}$$

with properties like those in Eq. (2) but for b_1, b_l, and b_r.

Note that, using the definition of b_l and b_r in Eq. (17) and the definition of coherent states in Eq. (14), one has

$$|n_1, n_l, n_r = 0\rangle = \frac{(b_1^\dagger)^{n_1}}{\sqrt{n_1!}} |z_1 = 0, z_2 = 1, z_3 = 1\rangle_{n_l},$$
$$|n_1, n_l = 0, n_r\rangle = \frac{(b_1^\dagger)^{n_1}}{\sqrt{n_1!}} |z_1 = 0, z_2 = 1, z_3 = -1\rangle_{n_r}, \tag{21}$$

with $n_1 + n_l + n_r = N$ and $n_1, n_l, n_r \in \mathbb{Z}^+$. Therefore, $|n_1, n_l, n_r = 0\rangle$ and $|n_1, n_l = 0, n_r\rangle$ are coherent states of n_l and n_r particles to which one adds n_1 particles in the level $|1\rangle$, respectively. Taking $n_1 = 0$, one finds that $b_l^\dagger$ and $b_r^\dagger$ are creation operators of coherent states of U(3).

From Eqs. (12) and (21), one finds that the logical qubit states can be expressed simply as

$$|0_L\rangle = |n_1 = 0, n_l = 0, n_r = N\rangle,$$
$$|1_L\rangle = |n_1 = 1, n_l = 0, n_r = N-1\rangle. \tag{22}$$

From Eqs. (18) and (19), one could have easily deduced by mere inspection that the logical states in Eq. (22) satisfy $\mathcal{L}|\mathbb{k}_L\rangle\langle\mathbb{k}_L| = 0$ with $\mathbb{k} = 0, 1$.

B. Collective dephasing

In this section, we first discuss the resilience of the logical qubit states $|0_L\rangle$ and $|1_L\rangle$ to pure collective dephasing processes, that is, the damping of coherences without changing the populations of the levels of the qutrits. The description of pure dephasing in qutrits is not as straightforward as it is for qubits [13–15,26,27]. We consider two types of pure collective dephasing described by

$$\mathcal{L}_{\text{cd}}A = \Gamma_1\mathcal{D}(S_{11})A + \Gamma_{23}\mathcal{D}(S_{22}+S_{33})A,$$
$$\mathcal{L}_{\text{ud}}A = \sum_{j=1,2,3}\Gamma_j\mathcal{D}(S_{jj})A, \tag{23}$$

where A is any linear operator. Here, $\mathcal{L}_{\text{ud}}$ describes uncorrelated collective dephasing, while the term $\mathcal{D}(S_{22}+S_{33})$ in $\mathcal{L}_{\text{cd}}$ models correlated collective dephasing.

Since the logical qubit states $|0_L\rangle$ and $|1_L\rangle$ are eigenvectors of S_{11} and $(S_{22}+S_{33}) = (N - S_{11})$ [see Eq. (15)], it is straightforward to show that $\mathcal{L}_{\text{cd}}|\mathbb{j}_L\rangle\langle\mathbb{j}_L| = 0$ for $\mathbb{j} = 0, 1$. Therefore, the logical qubit states are immune to the pure collective dephasing described by $\mathcal{L}_{\text{cd}}$.

The logical qubit states $|0_L\rangle$ and $|1_L\rangle$ are vulnerable to the pure uncorrelated, collective dephasing described by $\mathcal{L}_{\text{ud}}$ because it introduces incoherent transitions between the *bright* and *dark* modes, since

$$S_{jj} = \tfrac{1}{2}[b_l^\dagger b_l + b_r^\dagger b_r + (-1)^j(b_l^\dagger b_r + b_r^\dagger b_l)] \quad (j = 2, 3).$$

Consequently, one has $\mathcal{L}_{\text{ud}}|\mathbb{j}_L\rangle\langle\mathbb{j}_L| \neq 0$ ($\mathbb{j} = 0, 1$). The effects of $\mathcal{L}_{\text{ud}}$ along with those of local dephasing are presented for two qutrits in Sec. VI.

V. QUANTUM GATES

In this section, we show how to implement a quantum NOT gate and a Hadamard gate $\mathcal{H}$ followed by either the phase gate $\mathcal{S}$ or the phase $\mathcal{S}$ and $\mathcal{Z}$ gates [1]. All of these can be carried out simply by tuning δ_1 to zero and nonzero real values. This tuning corresponds to adjusting the frequency of a laser to be resonant or not resonant with twice the qutrits transition frequency ω_q (see Sec. VII). We assume that one can prepare the ensemble of qutrits in one of the states $|\mathbb{j}_L, \varphi\rangle$ defined below.

For any $\varphi \in (-\pi, \pi]$, consider the states

$$|\mathbb{j}_L, \varphi\rangle = \frac{1}{\sqrt{2}}[|0_L\rangle + (-1)^{\mathbb{j}}e^{i\varphi}|1_L\rangle]. \tag{24}$$

Observe that $|0_L, \varphi\rangle$ and $|1_L, \varphi\rangle$ are normalized and orthogonal to each other. From Eq. (15), it is clear that $|\mathbb{j}_L, \varphi\rangle\langle\mathbb{j}_L, \varphi|$ are stationary density operators of the master equation (4) if and only if $\delta_1 = 0$. Hence, $|0_L, \varphi\rangle$ and $|1_L, \varphi\rangle$ can also be used as robust, parameter-independent logical states of a qubit when $\delta_1 = 0$.

First assume that $\delta_1 = 0$ and that the state of the system is $\rho_q(0) = |\mathbb{j}_L, \varphi\rangle\langle\mathbb{j}_L, \varphi|$ with $\mathbb{j} = 0$ or 1. Then, the system is found in a stationary state and we want to apply either a quantum NOT gate, a $\mathcal{SH}$ transformation, or a $\mathcal{ZSH}$ operation. In order to do this, we now tune δ_1 to any nonzero value. As a consequence, $\rho_q(0)$ is no longer a stationary state and it will evolve.

Take

$$\rho_q(t) = \sum_{\mathbb{m},\mathbb{n}=0,1} A_{\mathbb{m}\mathbb{n}}(t)|\mathbb{m}_L, \varphi\rangle\langle\mathbb{n}_L, \varphi|, \tag{25}$$

where $A_{\mathbb{m}\mathbb{n}}(t)$ are complex-valued functions. Using Eq. (15), it is straightforward to show that

$$\begin{aligned}\mathcal{L}\rho_q(t) = -i\frac{\delta_1}{2}\{&[A_{01}(t) - A_{10}(t)]|0_L, \varphi\rangle\langle 0_L, \varphi| \\ &+ [A_{00}(t) - A_{11}(t)]|0_L, \varphi\rangle\langle 1_L, \varphi| \\ &+ [A_{11}(t) - A_{00}(t)]|1_L, \varphi\rangle\langle 0_L, \varphi| \\ &+ [A_{10}(t) - A_{01}(t)]|1_L, \varphi\rangle\langle 1_L, \varphi|\}.\end{aligned} \tag{26}$$

Substituting Eqs. (25) and (26) into the master equation (4), one obtains a linear system of ordinary differential equations that can be solved exactly:

$$\mathbf{A}(t) = \begin{pmatrix} \mathbb{B}_0(\delta_1 t/2) & \mathbb{B}_1(t) \\ \mathbb{B}_1(t) & \mathbb{B}_0(\delta_1 t/2) \end{pmatrix}\mathbf{A}(0), \tag{27}$$

where

$$\begin{aligned}\mathbf{A}(t) &= (A_{00}(t), A_{11}(t), A_{01}(t), A_{10}(t))^T, \\ \mathbb{B}_0(\tau) &= \begin{pmatrix} \cos^2(\tau) & \sin^2(\tau) \\ \sin^2(\tau) & \cos^2(\tau) \end{pmatrix}, \\ \mathbb{B}_1(t) &= -\frac{i}{2}\sin(\delta_1 t)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.\end{aligned} \tag{28}$$

The superscript T denotes the transpose.

Assume $A_{\mathbb{m}\mathbb{n}}(0) = \delta_{\mathbb{m}\mathbb{j}}\delta_{\mathbb{n}\mathbb{j}}$. Then, one can calculate $\rho_q(t)$ using Eqs. (25)–(28). In particular, $\rho_q(0) = |\mathbb{j}_L, \varphi\rangle\langle\mathbb{j}_L, \varphi|$ and one can produce oscillations between $|0_L, \varphi\rangle$ and $|1_L, \varphi\rangle$ by letting $\delta_1 \neq 0$.

If one tunes δ_1 to zero at $\delta_1 t_p = \pi(2p+1)$ with p an integer, then

$$\rho_q(t_p) = |(\mathbb{j} \oplus 1)_L, \varphi\rangle\langle(\mathbb{j} \oplus 1)_L, \varphi|. \tag{29}$$

Here, $\oplus$ denotes modulo 2 addition. Hence, one performs the transformation

$$|\mathbb{j}_L, \varphi\rangle\langle\mathbb{j}_L, \varphi| \to |(\mathbb{j} \oplus 1)_L, \varphi\rangle\langle(\mathbb{j} \oplus 1)_L, \varphi|,$$

and one has implemented a quantum NOT gate.

If instead one tunes δ_1 to zero at $\delta_1 t_p = \pi(2p+1)/2$ with p an integer, then

$$\rho_q(t_p) = \begin{cases} |\Psi_{+p}\rangle\langle\Psi_{+p}|, & \text{if } \mathbb{j} = 0, \\ |\Psi_{-p}\rangle\langle\Psi_{-p}|, & \text{if } \mathbb{j} = 1, \end{cases} \tag{30}$$

with

$$|\Psi_{\pm p}\rangle = \frac{1}{\sqrt{2}}[|0_L, \varphi\rangle \pm i(-1)^p|1_L, \varphi\rangle]. \tag{31}$$

Hence, one has implemented a $\mathcal{SH}$ gate if p is even or a $\mathcal{ZSH}$ gate if p is odd.

VI. THE CASE OF TWO QUTRITS

In this and only this section, we assume that $N = 2$. We first show how the logical qubit states can be prepared, we construct an orthogonal basis for the vector space of stationary solutions of the master equation in Eq. (4), and then we present the effects of uncorrelated pure collective dephasing as described by $\mathcal{L}_{\rm ud}$ in Eq. (23), as well as those of inhomogeneous broadening and local dephasing.

Assume that the kets of β_q are ordered as follows:

$$\begin{aligned}\beta_q = \{&|2, 0, 0\rangle,\ |1, 1, 0\rangle,\ |0, 2, 0\rangle,\ |1, 0, 1\rangle, \\ &|0, 1, 1\rangle,\ |0, 0, 2\rangle\}.\end{aligned} \tag{32}$$

Here, the ordered triple of numbers inside the ket symbol corresponds to the ordered triple (n_1, n_2, n_3). Notice that the state space of the qutrits has dimension 6 and that the logical states of the qubit take the form

$$\begin{aligned}|0_L\rangle &= \frac{1}{2}(|0, 2, 0\rangle - \sqrt{2}|0, 1, 1\rangle + |0, 0, 2\rangle), \\ |1_L\rangle &= \frac{1}{\sqrt{2}}(|1, 1, 0\rangle - |1, 0, 1\rangle).\end{aligned} \tag{33}$$

Also, the matrix representation of a linear operator A with respect to the basis β_q is a 6×6 complex matrix denoted by $[A]_{\beta_q}$.

A. A possible preparation of the logical states

One can prepare the system of two qutrits in the logical states $|0_L\rangle$ and $|1_L\rangle$ by means of a coherent external driving. In the context of the physical system of Sec. VII, this would be performed before coupling the qutrits to the rest of the subsystems.

If the two qutrits evolve as a closed system under the Hamiltonian

$$\begin{aligned}H_{D1}(t) &= \hbar g_d(S_{13} + S_{31}) - \hbar g_d(S_{12} + S_{21}), \\ &= -\sqrt{2}\hbar g_d(b_1^\dagger b_r + b_r^\dagger b_1),\end{aligned} \tag{34}$$

with $g_d > 0$ and the initial state is the Fock state $|2, 0, 0\rangle$ (that is, the two qutrits are initially in their respective ground levels), then with probability 1, one finds the system of two qutrits in the state $|0_L\rangle$ at times $(2n+1)\pi/(2\sqrt{2}g_d)$ with n an integer.

If instead the qutrits evolve as a closed system under the effective Hamiltonian

$$H_{D2}(t) = i\hbar g_d(S_{23} - S_{32}) = i\hbar g_d(b_r^\dagger b_l - b_l^\dagger b_r), \tag{35}$$

with $g_d > 0$ and the initial state is the Fock state $|1, 1, 0\rangle$ or the Fock state $|1, 0, 1\rangle$ (that is, one qutrit is in the ground level while the other is in an excited level), then with probability 1 one finds the system of two qutrits in the state $|1_L\rangle$ at the respective times $(4n+1)\pi/(4g_d)$ and $(4n-1)\pi/(4g_d)$ with n an integer.

B. A basis for the subspace of stationary states

Using the basis β_q, one can express $\mathcal{L}\rho = 0$ as a matrix equation. The set of all 6×6 complex matrices that satisfy this equation is a vector space S over the complex numbers that can be equipped with the Hilbert-Schmidt inner product: $(\mathbb{A}, \mathbb{B}) = \mathrm{Tr}(\mathbb{A}^\dagger\mathbb{B})$ for any two matrices $\mathbb{A}, \mathbb{B} \in \mathsf{S}$. Whenever we say that a matrix is normalized or that two matrices are orthogonal, it is to be understood that it is with respect to the Hilbert-Schmidt inner product and the associated norm. In particular, a pure state density matrix has norm 1, while a mixed state density matrix has norm <1.

It can be proved that S has dimension 3 when $\delta_1 \neq 0$. In this case, two linearly independent matrices of S are $[|0_L\rangle\langle 0_L|]_{\beta_q}$ and $[|1_L\rangle\langle 1_L|]_{\beta_q}$ but a third linearly independent matrix has a very complicated expression. Since the case $\delta_1 = 0$ has already proved to be important for the realization of quantum gates, in the rest of Sec. VI B we assume that $\delta_1 = 0$. Then, it can be proved that S has dimension 5 and we already know from Sec. IV that $\gamma_0 = \{\, [\ |\mathbb{j}_L\rangle\langle \mathbb{k}_L|\ ]_{\beta_q} : \ \mathbb{j}, \mathbb{k} = 0, 1\,\}$ is an orthonormal set contained in S. It can be extended to an orthogonal basis for S given by $\beta_{\mathsf{S}} = \gamma_0 \cup \{[\rho_{00}]_{\beta_q}\}$ where the density operator ρ_{00} still has a very complicated analytic expression. In the rest of Sec. VI B, we assume that α_0 is pure imaginary. Then, ρ_{00} has a tractable expression given by

$$[\rho_{00}]_{\beta_q} = p_3(\rho_R + i\rho_I), \tag{36}$$

where

$$\rho_R = 2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\sqrt{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 2 & -\sqrt{2} \\ 0 & 0 & 1 & 0 & -\sqrt{2} & 1 \end{pmatrix} - 2\sqrt{2}\begin{pmatrix} p_1 & 0 & -F & 0 & -\sqrt{2}F & -F \\ 0 & 0 & 0 & -\sqrt{2} & 0 & 0 \\ -F & 0 & 0 & 0 & -1 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ -\sqrt{2}F & 0 & -1 & 0 & 0 & -1 \\ -F & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\rho_I = p_2\begin{pmatrix} 0 & 0 & 1 & 0 & \sqrt{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{37}$$

and

$$F = \frac{4\kappa_2 + \kappa_1}{2\sqrt{2}\,\mathrm{Im}(\alpha_0)},$$
$$p_1 = -\sqrt{2}\left(1 + 2F^2 + \frac{p_2^2}{4}\right), \quad p_2 = \frac{2(4\delta - \xi)}{\mathrm{Im}(\alpha_0)},$$
$$p_3 = \frac{1}{12 + p_2^2 + 8F^2}, \quad 0 < p_3 \leqslant \frac{1}{12}. \tag{38}$$

Here, $\mathrm{Im}(\alpha_0)$ denotes the imaginary part of α_0.

The set β_{S} contains three density matrices, namely, those associated with the pure states $|0_L\rangle\langle 0_L|$ and $|1_L\rangle\langle 1_L|$ and with the mixed state ρ_{00}. Notice that ρ_{00} is not normalized because it is a mixed state:

$$\frac{1}{3} \leqslant \mathrm{Tr}(\rho_{00}^2) = 1 - 8p_3 < 1. \tag{39}$$

Observe that ρ_{00} tends to a pure state in a strongly dissipative system with $(4\kappa_2 + \kappa_1) \gg 2\sqrt{2}|\alpha_0|$ or if $\alpha_0 \to 0$ (the latter corresponds to no driving in the physical system of Sec. VII).

Any stationary state of the master equation in Eq. (4) can be expressed as a linear combination of the operators associated with the matrices in β_{S}. In our numerical calculations, we found that $|2, 0, 0\rangle$ relaxes to ρ_{00}, that is, $e^{\mathcal{L}t}|2, 0, 0\rangle\langle 2, 0, 0| \to \rho_{00}$ as $t \to +\infty$. Actually, one has

$$([|2, 0, 0\rangle\langle 2, 0, 0|]_{\beta_q}, [\rho_{00}]_{\beta_q}) = \mathrm{Tr}\big(\rho_{00}^2\big), \tag{40}$$

so the Cauchy-Schwarz inequality indicates that $\rho_{00} \to |2, 0, 0\rangle\langle 2, 0, 0|$ if $\mathrm{Tr}(\rho_{00}^2) \to 1$.

C. The effect of uncorrelated collective dephasing

In this and only this section, we consider the master equation (4) with $\mathcal{L}$ replaced by $\mathcal{L}_T = \mathcal{L} + \mathcal{L}_{\mathrm{ud}}$ and still denote its solution by $\rho(t)$. In general, the addition of uncorrelated pure collective dephasing described by $\mathcal{L}_{\mathrm{ud}}$ has a detrimental effect on the logical states $|0_L\rangle$ and $|1_L\rangle$. Even though we are considering two qutrits, the number of parameters in the master equation is large enough so that one does not obtain a manageable analytic expression for the stationary state. Numerically, we found that all density operators tend to a unique stationary state ρ_T that depends on the parameters. It is noteworthy to point out that ρ_T can have a large ($\geqslant$0.99) fidelity with ρ_{00} defined in Eq. (36) and also a non-negligible fidelity with both $|0_L\rangle\langle 0_L|$ and $|1_L\rangle\langle 1_L|$. The values of the fidelities depend on the parameters. It is important to note that ρ_{00} is not a stationary solution of the master equation and that it can have a large fidelity with ρ_T even when α_0 is not purely imaginary. Figure 2 illustrates the evolution of the fidelity $F_{\mathbb{j}}(t) = \mathrm{Tr}\sqrt{\sigma_{\mathbb{j}}^{1/2}\rho(t)\sigma_{\mathbb{j}}^{1/2}}$ of $\rho(t)$ with $\sigma_{\mathbb{j}} = |0_L\rangle\langle 0_L|$ (red line), $|1_L\rangle\langle 1_L|$ (blue-dashed line), and ρ_{00} (black-dotdashed line) corresponding, respectively, to $\mathbb{j} = 0, 1, 00$. The initial conditions $\rho(0) = |0_L\rangle\langle 0_L|$, $|1_L\rangle\langle 1_L|$, and ρ_{00} correspond, respectively, to Figs. 2(a)–2(c). The lifetime of the logical states is $\sim\frac{4}{\Gamma_2+\Gamma_3}$ (see Appendix E for an explanation).

D. The effect of local dephasing and inhomogeneous broadening

We now consider local dephasing and inhomogeneous broadening. For example, the latter can be due to Doppler shifts or the spatial inhomogeneity of the electromagnetic field, which interacts with the ensemble of qutrits (see Sec. VII), while the former can result from collisions within the ensemble. These physical processes distinguish the qutrits and, thus, break the permutation symmetry of the system and inhibit the collective behavior of the ensemble. In order to be able to describe these processes (we will introduce the tensor product space for N qutrits, but the results will be given for $N = 2$), one now has to work in the tensor product space of the N qutrits, which has dimension 3^N, which is isomorphic to $(\mathbb{C}^3)^{\otimes N} = \mathbb{C}^3 \otimes \cdots \otimes \mathbb{C}^3$, and has the or-

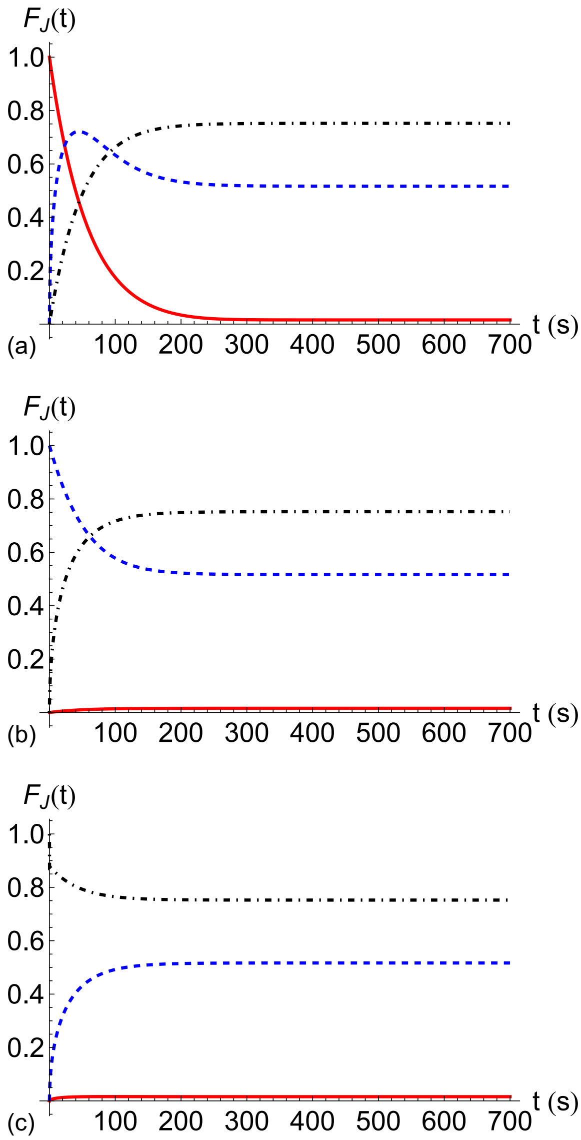

FIG. 2. The figures show the fidelity $F_{\mathbb{J}}(t)$ of $\rho(t)$ with $|0_L\rangle\langle 0_L|$ (red line), $|1_L\rangle\langle 1_L|$ (blue-dashed line), and ρ_{00} (black-dotdashed line) as a function of time t when $\rho(0)=|0_L\rangle\langle 0_L|$ [Fig. 2(a)], $|1_L\rangle\langle 1_L|$ [Fig. 2(b)], and ρ_{00} [Fig. 2(c)]. In all figures, the steady-state values of the fidelity are $F_0^\infty=0.016$, $F_1^\infty=0.52$, and $F_{00}^\infty=0.75$, $\rho(t)$ evolves under $\mathcal{L}_T=\mathcal{L}+\mathcal{L}_{\rm ud}$, and the values of the parameters (generated randomly) are $\delta_1=0$, $\xi=0.81$, $\alpha_0=6.29+i0.37$, $\delta=-3.15$, $\kappa_1=7.99$, $\kappa_2=0.65$, $\Gamma_1=0.024$, $\Gamma_2=0.032$, and $\Gamma_3=0.038$ (see Appendix E for an explanation).

thonormal basis $\beta_{tp}=\{|j_1,j_2,\ldots,j_N\rangle_{tp}:\ j_k=1,2,3;\ k=1,2,\ldots,N\}$. Here, $|j_1,j_2,\ldots,j_N\rangle_{tp}=|j_1\rangle\otimes|j_2\rangle\otimes\cdots\otimes|j_N\rangle$ is the tensor product of the kets $|j_1\rangle,|j_2\rangle,\ldots,|j_N\rangle$, where the kth qutrit is in the level $|j_k\rangle$. In addition, all operators must be expressed in terms of one-qutrit operators.

We introduce the identity operator $\mathbb{I}$ in the state space of one-qutrit, the one-qutrit transition operators

$$\sigma_{ij}=|i\rangle\langle j|\quad (i,j=1,2,3), \tag{41}$$

and their extension to the tensor product space of N qutrits

$$\sigma_{ij}^{(k)}=\overbrace{\mathbb{I}\otimes\cdots\otimes\mathbb{I}}^{k-1}\otimes\sigma_{ij}\otimes\overbrace{\mathbb{I}\otimes\cdots\otimes\mathbb{I}}^{N-k}. \tag{42}$$

If the Liouvillian of the system is invariant under permutation of the qutrits, i.e., it is written in terms of collective operators S_{ij}, then the dynamics will preserve each permutation invariant subspace in which the tensor product space $(\mathbb{C}^3)^{\otimes N}$ decomposes [28]. If the initial state ρ_0 is in the symmetric (boson) subspace, then $\rho(t)$ will be in the symmetric subspace for all t, which is why in Sec. II, we started directly with the symmetric subspace. When some interactions break the permutation symmetry, even if we start with ρ_0 in the symmetric subspace, the dynamics will produce a leakage to other permutation invariant subspaces.

It is common to call the symmetric (bosonic) subspace the *superradiant subspace* and to refer to the orthogonal subspace as the *subradiant subspace*. Thus, we divide the tensor product space of N qutrits into two parts: the superradiant subspace of dimension $D=\frac{(N+1)(N+2)}{2}$ and the subradiant subspace of dimension 3^N-D. For instance, for $N=2$, the superradiant subspace has dimension 6, whereas the subradiant subspace has dimension 3. In this case, the subradiant subspace coincides with the antisymmetric subspace. For $N=3$, the superradiant subspace has dimension 10, whereas the subradiant subspace has dimension 17. The subradiant subspace in this case contains three permutation invariant subspaces, namely, the antisymmetric subspace, of dimension 1, and two copies of the octet, each of dimension 8. For larger values of N, the subradiant subspace contains many permutation invariant subspaces. Since we are only interested in describing the leakage from the superradiant subspace and not in the details of how this leakage distributes into the different permutation invariant subspaces, it is convenient to consider only the superradiant and subradiant subspaces.

We have already shown that the logical qubit states are immune to global correlated dephasing as described by $\mathcal{L}_{\rm cd}$ in Eq. (23) but that they are vulnerable to uncorrelated collective dephasing described by $\mathcal{L}_{\rm ud}$ in Eq. (23). It turns out that they are also vulnerable to local uncorrelated dephasing. Therefore, we now consider correlated local dephasing and inhomogeneous broadening as described by the Liouvillian

$$\mathcal{L}_{\rm loc}A=-\frac{i}{\hbar}[H_{\rm B},A]+\Gamma_{\rm D}\sum_{n=1}^{2}\mathcal{D}\big[\sigma_{22}^{(n)}+\sigma_{33}^{(n)}\big]A, \tag{43}$$

where A is any operator, $\Gamma_D>0$ is the dephasing rate, and $H_{\rm B}$ is the Hamiltonian describing inhomogeneous broadening

$$H_{\rm B}=\sum_{n=1}^{2}\hbar\delta\omega_n\big[\sigma_{22}^{(n)}+\sigma_{33}^{(n)}\big], \tag{44}$$

with $\delta\omega_n$ a real number. Notice that $H_{\rm B}$ describes a type of correlated inhomogeneous broadening such as that arising from Doppler shifts because the excited levels are subject to the same shift. Also, $\Gamma_{\rm D}\mathcal{D}[\sigma_{22}^{(n)}+\sigma_{33}^{(n)}]$ represents a correlated local dephasing where the excited levels of each qutrit are affected in the same way and both qutrits are subject to the same dephasing rate.

It happens that the logical states $|0_L\rangle$ and $|1_L\rangle$ are parity sensitive to both inhomogeneous broadening and to local correlated dephasing. It is straightforward to show that $\mathcal{L}_{\rm loc}(|0_L\rangle\langle 0_L|) = 0$ independent of the parameters, so the logical qubit state $|0_L\rangle$ is immune to both inhomogeneous broadening and correlated local dephasing described by $\mathcal{L}_{\rm loc}$. On the other hand, the logical qubit state $|1_L\rangle$ is vulnerable to $\mathcal{L}_{\rm loc}$, since $\mathcal{L}_{\rm loc}(|1_L\rangle\langle 1_L|) \neq 0$. Nevertheless, for $|1_L\rangle$ it is found that the inhomogeneous broadening term produces an *oscillatory flux* between the symmetrical (boson) subspace and the antisymmetrical (fermion) subspace (see Appendix E). It is important to note that the discrepancy in the resilience to inhomogeneous broadening between the even and odd cat states $|0_L\rangle$ and $|1_L\rangle$ encoding the qubit is much more marked in the case of an ensemble of qutrits than in the case of an ensemble of qubits, since [12] found in the case of the ensemble of qubits that such discrepancy is significant only for small amplitude cat states.

One can take advantage of the oscillatory flux (see Appendix E) to construct a superposition of symmetric and antisymmetric states that is immune to inhomogeneous broadening and, as a bonus, that is also invulnerable to correlated local dephasing. Define the one-qutrit *right state* $|r\rangle = \frac{|2\rangle - |3\rangle}{\sqrt{2}}$ and the antisymmetrical odd state

$$\left|1_L^A\right\rangle = \frac{1}{\sqrt{2}}[|1\rangle \otimes |r\rangle - |r\rangle \otimes |1\rangle], \tag{45}$$

which is a stationary state (dark state) of the antisymmetrical version of the Liouvillian $\mathcal{L}$ in Eq. (5), and the mixed permutation symmetry odd states

$$|\tilde{1}_L\rangle_\pm = \frac{1}{\sqrt{2}}\left(|1_L\rangle \pm \left|1_L^A\right\rangle\right) = \begin{cases} |1\rangle \otimes |r\rangle, & \text{if } +, \\ |r\rangle \otimes |1\rangle, & \text{if } -. \end{cases} \tag{46}$$

Notice that $|\tilde{1}_L\rangle_\pm$ are neither symmetrical nor antisymmetrical. In fact, in the two states $|\tilde{1}_L\rangle_\pm$ the two qutrits are fully distinguishable and they are related to each other by permuting the two particles. One can show that $\mathcal{L}(|\tilde{1}_L\rangle_{\pm\pm}\langle\tilde{1}_L|) = \mathcal{L}_{\rm loc}(|\tilde{1}_L\rangle_{\pm\pm}\langle\tilde{1}_L|) = \mathcal{L}_{\rm cd}(|\tilde{1}_L\rangle_{\pm\pm}\langle\tilde{1}_L|) = 0$ independently of the parameters. Recall that $\mathcal{L}$ and $\mathcal{L}_{\rm cd}$ are defined in Eqs. (5) and (23).

In conclusion, $|0_L\rangle$ and $|\tilde{1}_L\rangle_\pm$ are immune to all the processes embodied by $\mathcal{L}$, as well as to inhomogeneous broadening and correlated global and local dephasing described by $\mathcal{L}_{\rm cd}$ and $\mathcal{L}_{\rm loc}$. It is important to emphasize that this immunity is independent of the parameters appearing in all the Liouvillians (they are dark states). Therefore, $|0_L\rangle$ and $|\tilde{1}_L\rangle_\pm$ could also be used as highly robust states of a logical qubit in the presence of local dephasing interactions. Unfortunately, both states are vulnerable to global and local uncorrelated dephasing.

The main conclusions obtained in this section for two qutrits can be extended to an arbitrary number of qutrits. In particular, the construction of the antisymmetric odd state (45) and mixed symmetry odd state (46) now translates to an odd subradiant state (pertaining to the direct sum of all representations except the symmetric, superradiant representation) and a mixed symmetry odd state, and follows the same lines that in the case of two qutrits. The only difference is that more freedom appears as the number of qutrits is increased.

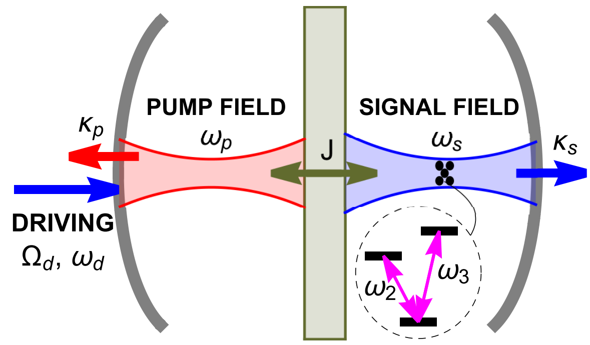

FIG. 3. The figure presents a schematic of the physical system under consideration. It is composed of a collection of $N \geqslant 2$ qutrits (represented by the black dots on the right-hand side) and two single-mode cavity electromagnetic fields called the *pump field* and the *signal field*. The pump field has frequency ω_p, is subject to dissipation at a rate κ_p, and is also driven by a classical field with frequency ω_d and driving strength Ω_d. The signal field has frequency ω_s, is subject to dissipation at a rate κ_s, and interacts with the pump field with coupling strength J. The qutrits have a V configuration with transition frequencies ω_j and interact with the signal field with coupling strengths g_j ($j = 2, 3$) (see also Fig. 1).

For instance, for $N = 3$ qutrits the mixed permutation symmetry odd states that are stationary under the Liouvillian $\mathcal{L}$ in Eq. (5) and inhomogeneous broadening and global and local correlated dephasing described by $\mathcal{L}_{\rm cd}$ and $\mathcal{L}_{\rm loc}$ are the states (with fully distinguishable qutrits): $|1\rangle \otimes |r\rangle \otimes |r\rangle$ and the ones obtained by permuting the particles.

This is easily generalized to an arbitrary number N of particles. The mixed permutation symmetry odd states that are stationary under the Liouvillian $\mathcal{L}$ in Eq. (5) and inhomogeneous broadening and global and local correlated dephasing described by $\mathcal{L}_{\rm cd}$ and $\mathcal{L}_{\rm loc}$ are the N states (with fully distinguishable qutrits): $|1\rangle \otimes |r\rangle \otimes \cdots \otimes |r\rangle$ and the ones obtained by permuting the particles.

VII. AN ORIGIN OF THE MODEL

In this section, we present one possible origin of the effective model presented in Sec. II.

Consider a tripartite quantum system composed of $N \geqslant 2$ identical three-level atoms with a V configuration and two single-mode cavity electromagnetic fields, which will be called the *pump* and *signal* fields. The atoms, which will henceforth be called *qutrits*, are bosons that only interact with the signal field, while the pump field is driven by a classical field and is parametrically coupled to the signal field. A schematic of the system is presented in Fig. 3.

The Hamiltonian of the system is

$$\begin{aligned} H(t) = {} & \sum_{l=p,s} \hbar\omega_l a_l^\dagger a_l + \sum_{j=2,3} \hbar\omega_j S_{jj} \\ & + \hbar\Omega_d\left(a_p^\dagger e^{-i\omega_d t} + a_p e^{i\omega_d t}\right) + \hbar J\left[a_p (a_s^\dagger)^2 + a_p^\dagger a_s^2\right] \\ & + \hbar \sum_{j=2,3} g_j (a_s^\dagger S_{1j} + a_s S_{1j}^\dagger). \end{aligned} \tag{47}$$

Here, $\omega_l > 0$ is the angular frequency of the pump (signal) field if $l = p$ (s) and $a_l^\dagger$ and a_l are the corresponding creation and annihilation operators. As presented in Sec. II, the qutrits are described in second quantization and we have chosen the energy scale so that the ground level of the qutrits has energy equal to zero. The terms in the first line of the right-hand side of Eq. (47) correspond to the free energies of all the component subsystems, while the terms in the second line describe the driving (in the rotating-wave approximation) of the pump field and the parametric coupling between the two fields. The driving angular frequency $\omega_d > 0$ is quasiresonant with the frequency of the pump field $\omega_d \simeq \omega_p$ and the driving strength Ω_d (a real constant) is weak $|\Omega_d| \ll \omega_d$. Meanwhile, the strength of the parametric coupling is described by the real quantity J. The third line of the right-hand side of Eq. (47) describes the interaction (in the rotating-wave approximation) between the qutrits and the signal field where g_j is a real parameter representing the single-qutrit coupling strength with the signal field (it is the same for all the qutrits). Notice that we have not yet imposed the conditions that $\omega_2 = \omega_3$ and that $g_2 = g_3$. This will be done at the end.

The set of Fock states $\{|n_l\rangle : n \in \mathbb{Z}^+\}$ is an orthonormal basis for the state space of the pump (signal) field if $l = p$ (s), while the set β_q of occupation number states in Eq. (1) is an orthonormal basis for the state space of the N qutrits. An orthonormal basis for the complete system is obtained by taking the tensor product of the previous three bases.

We assume that the pump $(l = p)$ and signal $(l = s)$ fields are coupled to independent thermal baths of harmonic oscillators at zero temperature (actually, one only requires that each thermal bath has a temperature $T_l > 0$ such that the expected number of thermal photons of frequency ω_l at that temperature is sufficiently small so that it can be neglected [29]). Then, we model the effect of these thermal baths on the pump $(l = p)$ and signal $(l = s)$ fields by means of the dissipators $\kappa_l \mathcal{D}(a_l)$ defined in Eq. (6) with $\kappa_l > 0$ the decay rates. The evolution of the complete system is then described by the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) master equation

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H(t), \rho(t)] + \sum_{l=p,s} \kappa_l \mathcal{D}(a_l)\rho(t), \tag{48}$$

with $\rho(t)$ the density operator of the complete system.

The idea of how one expects the system to behave is the following. First, $\omega_d \simeq \omega_p$ so the driving is quasiresonant with the pump field and, consequently, the pump field will approximately be driven to a coherent state where the expected value of the number of photons is >0. Second, $\kappa_s, \omega_s \gg \omega_p, \omega_2, \omega_3$ so the signal field will be approximately in the vacuum state and the signal field and the ensemble of qutrits interact dispersively. The purpose of the signal field is to act as an intermediary between the pump field and the ensemble of qutrits. Third, $\omega_p \simeq 2\omega_2 \simeq 2\omega_3$ so there is an effective interaction between the pump field and the qutrits mediated by the signal field and the processes where a pump photon is emitted or absorbed by two qutrits are favored (see Ref. [9]). In the following, it is important to keep this intuitive picture in mind so that one recognizes how the successive approximations lead precisely to this behavior. In particular, the first adiabatic approximation will confirm that the signal field is approximately in the vacuum state and that it acts as an intermediary between the pump field and the qutrits, while the averaging will make explicit the aforementioned effective interaction between the pump field and the ensemble qutrits and the second adiabatic approximation will confirm that the pump field is approximately in a coherent state.

To eliminate the time dependence of the Hamiltonian, first pass to an interaction picture (IP) using the unitary transformation

$$U_I(t) = e^{-\frac{i}{\hbar}H_0 t}, \tag{49}$$

where

$$H_0 = \hbar\omega_d a_p^\dagger a_p + \frac{\hbar\omega_d}{2} a_s^\dagger a_s - \frac{\hbar\omega_d}{2} S_{11}. \tag{50}$$

Then, the density operator in the IP is given by $\rho_I(t) = U_I^\dagger(t)\rho(t)U_I(t)$ and its evolution is described by the IP master equation

$$\begin{aligned}\frac{d}{dt}\rho_I(t) &= \mathcal{L}_I\rho_I(t),\\ &= -\frac{i}{\hbar}[H_I, \rho_I(t)] + \sum_{l=p,s}\kappa_l\mathcal{D}(a_l)\rho_I(t),\end{aligned} \tag{51}$$

where

$$\begin{aligned}\frac{1}{\hbar}H_I = {}& \delta_s a_s^\dagger a_s + \delta_p a_p^\dagger a_p + \sum_{j=2,3}\omega_j S_{jj} + \frac{\omega_d}{2}S_{11}\\ &+ \Omega_d(a_p^\dagger + a_p) + J\big[a_p(a_s^\dagger)^2 + a_p^\dagger a_s^2\big]\\ &+ \sum_{j=2,3} g_j(a_s^\dagger S_{1j} + a_s S_{1j}^\dagger),\end{aligned} \tag{52}$$

and δ_p and δ_s are defined in Appendix D. All the parameters that appear with the successive approximations are not presented below. They are listed in Appendix D where they are given a physical interpretation and they are grouped according to the approximation performed.

A. First adiabatic approximation

The complete system can be decomposed into two subsystems: the *signal field* on one hand, and the *qutrits+driven pump field* on the other.

Assume

$$\epsilon_1 = \max\left\{\left|\frac{\delta_p}{\kappa_s}\right|, \frac{\omega_j}{\kappa_s}, \left|\frac{\Omega_d}{\kappa_s}\right|, \left|\frac{J}{\kappa_s}\right|, \left|\frac{g_j}{\kappa_s}\right|, \frac{\omega_d}{2\kappa_s}, \frac{\kappa_p}{\kappa_s} : j = 2, 3\right\} \ll 1. \tag{53}$$

To preserve the order of the asymptotic expansion, also assume that each of the nonzero elements of the set in Eq. (53) is $\gg \epsilon_1^2$ and that $|\delta_s/\kappa_s| \gg \epsilon_1$. Observe that the quantities in the set (53) come from the coefficients of the terms appearing in the master equations (51) and (52) and dividing them by κ_s.

The assumption implies that the signal field evolves rapidly and is strongly dissipative, while the qutrits + pump field is a subsystem that evolves slowly. Hence, one can adiabatically eliminate the signal field. In addition, measuring time in units of $1/\kappa_s$ [that is, expressing the master equation in Eq. (51) in terms of the dimensionless time $\tau = \kappa_s t$] leads to the conclusion that, to order zero in ϵ_1, the signal field evolves according to the master equation of a damped harmonic oscillator at zero temperature:

$$\frac{d}{dt}\rho_I(t) = -\frac{i}{\hbar}[\hbar\delta_s a_s^\dagger a_s,\, \rho_I(t)] + \kappa_s\mathcal{D}(a_s)\rho_I(t). \tag{54}$$

As a consequence of Eq. (54), the signal field is a stable subsystem that converges to the stationary state $|0_s\rangle\langle 0_s|$.

Adiabatically eliminating the signal field from Eq. (51) [30,31], to second order in ϵ_1 the reduced dynamics of the *qutrits+driven pump field* subsystem is given by

$$\begin{aligned}\frac{d}{dt}(\rho_{qp})_I(t) = &-\frac{i}{\hbar}[H_{qp},\ (\rho_{qp})_I(t)] + \kappa_p'\mathcal{D}(a_p)(\rho_{qp})_I(t)\\ &+\kappa_s\mathcal{D}(S)(\rho_{qp})_I(t),\end{aligned} \tag{55}$$

where

$$\begin{aligned}\frac{1}{\hbar}H_{qp} &= \delta_p' a_p^\dagger a_p + \sum_{j=2,3}\omega_j S_{jj} + \frac{\omega_d}{2}S_{11} - \delta_s S^\dagger S\\ &\quad + \Omega_d(a_p^\dagger + a_p) + \langle 0_s|\frac{1}{\hbar}H_{\rm int}|0_s\rangle,\\ \frac{1}{\hbar}H_{\rm int} &= J\big[a_p(a_s^\dagger)^2 + a_p^\dagger a_s^2\big] + \kappa_s'(a_s^\dagger S + a_s S^\dagger).\end{aligned} \tag{56}$$

Here, we have introduced the effective qutrit operator S and the IP density operator $(\rho_{qp})_I(t)$ of the qutrits + driven pump field subsystem by the equations

$$\begin{aligned}S &= \frac{1}{\kappa_s'}(g_2S_{12} + g_3S_{13}),\\ (\rho_{qp})_I(t) &= ({\rm Tr}_s[\rho(t)])_I(t).\end{aligned} \tag{57}$$

In Eqs. (55)–(57), we have introduced the parameters κ_p', δ_p', and κ_s' defined in Appendix D.

Notice that ${\rm Tr}_s$ is the partial trace with respect to the degrees of freedom of the signal field, so

$$\begin{aligned}(\rho_{qp})_I(t) = &\, e^{-i(\omega_d t/2)S_{11}}e^{i(\omega_d t)a_p^\dagger a_p}{\rm Tr}_s[\rho(t)]\\ &\times e^{-i(\omega_d t)a_p^\dagger a_p}e^{i(\omega_d t/2)S_{11}}.\end{aligned} \tag{58}$$

Comparing Eqs. (51) and (52) with Eqs. (55) and (56), one finds that the effect of the interaction of the signal field with the *qutrits+driven pump field* consists of an increase of the pump field decay rate and in the appearance of incoherent transitions from the excited levels to the ground level in the qutrits, as described, respectively, by the increase of κ_p to κ_p' and by the dissipator $\kappa_s\mathcal{D}(S)$. It also includes a shift of the pump field frequency from δ_p to δ_p', a coherent interaction among qutrits described by $-\delta_s S^\dagger S$, and a coherent interaction between the pump field and the qutrits embodied by $\langle 0_s|(1/\hbar)H_{\rm int}|0_s\rangle$. In particular, observe that expanding $-\delta_s S^\dagger S$ using the definition of S in (57) leads to

$$-\delta_s S^\dagger S = -\frac{\delta_s}{(\kappa_s')^2}b_1b_1^\dagger\left[\sum_{j=2,3}g_j^2S_{jj} + g_2g_3(S_{23}+S_{32})\right],$$

so the interaction of the qutrits with the signal field also leads to shifts of the transition frequencies ω_j $(j = 2, 3)$ and to an effective coupling between the excited levels of the qutrits, all of which depend on the number of particles in the ground level.

Observe that $\langle 0_s|H_{\rm int}|0_s\rangle = 0$, so the pump field and the qutrits would evolve independently of one another according to Eqs. (55) and (56). One would have to preserve terms up to order 3 in ϵ_1 to include the effective interaction between the qutrits and the pump field induced by the signal field. This is rather difficult to calculate [30,31], so we will take advantage of the different timescales involved in the evolution of the system to average $H_{\rm int}$ and obtain an approximate Hamiltonian whose expected value in the state $|0_s\rangle$ is not zero. This averaging has to be done to third order to be able to describe the effective qutrits-pump field interaction induced by the signal field.

B. Averaging

Assume that

$$|J|,\ |g_2|,\ |g_3| \ll \omega_s \tag{59}$$

and

$$\omega_s \gg \omega_p > \omega_j \quad (j = 2, 3). \tag{60}$$

Then, one can use *James' averaging method* [32,33] to third order (see Appendix B) to obtain the approximation

$$\begin{aligned}&\langle 0_s|\frac{1}{\hbar}H_{\rm int}|0_s\rangle\\ &\quad\simeq -(S_{11}+1)\left[\frac{g_2^2}{\Delta_2}S_{22} + \frac{g_3^2}{\Delta_3}S_{33} + g_{23}(S_{32}+S_{32}^\dagger)\right]\\ &\qquad - 2\left(\frac{J^2}{\Delta_s}\right)a_p^\dagger a_p + \sum_{j=2,3}\chi_j\big[S_{1j}^2a_p^\dagger + (S_{1j}^\dagger)^2a_p\big]\\ &\qquad + \chi_{23}[S_{12}S_{13}a_p^\dagger + S_{13}^\dagger S_{12}^\dagger a_p].\end{aligned} \tag{61}$$

Here, we have introduced the parameters Δ_j, g_{23}, Δ_s, χ_j, and χ_{23} $(j = 2, 3)$ defined in Appendix D. Then, $\langle 0_s|\frac{1}{\hbar}H_{\rm int}|0_s\rangle$ is to be replaced by the right-hand side of Eq. (61) in the effective master equation in Eqs. (55) and (56). In the right-hand side of Eq. (61), the terms multiplied by χ_2, χ_3, and χ_{23} are of order 3, while the rest of the terms are of order 2. Notice that the terms of order 3 describe the effective interactions between the qutrits and the pump field where a pump photon is absorbed or emitted by two qutrits. Meanwhile, all the terms of order 2 can be combined with terms already present in the master equation and simply lead to a modification of the values of δ_p' and $\delta_s/(\kappa_s')^2$.

C. Second adiabatic approximation

To obtain an effective master equation describing the dynamics of the N qutrits, we adiabatically eliminate the driven pump field.

Assume

$$\epsilon_2 = \max\left\{\frac{\omega_j}{\kappa_p'},\ \frac{\omega_d}{2\kappa_p'},\ \frac{|\chi_j|}{\kappa_p'},\ \frac{|\chi_{23}|}{\kappa_p'},\ \frac{1}{\kappa_p'}\left|g_{23} + \frac{\delta_s g_2g_3}{(\kappa_s')^2}\right|,\ \frac{1}{\kappa_p'}\left|\frac{\delta_s g_j^2}{(\kappa_s')^2} + \frac{g_j^2}{\Delta_j}\right|,\ \frac{\kappa_s|g_jg_k|}{\kappa_p'(\kappa_s')^2}\ :\ j,k = 2,3\right\} \ll 1. \tag{62}$$

In order to preserve the order of the asymptotic expansion, each nonzero element of the set in the definition of ϵ_2 must be $\gg \epsilon_2^2$ and one must also have $\epsilon_2 \ll |\Omega_d|/\kappa_p'$ and $\epsilon_2 \ll |\delta_p' - 2J^2/\Delta_s|/\kappa_p'$ (the last one if $|\delta_p' - 2J^2/\Delta_s| \neq 0$).

The assumption implies that the pump field evolves rapidly and is strongly dissipative, while the qutrits evolve slowly. Hence, one can adiabatically eliminate the pump field. In addition, measuring time in units of $1/\kappa_p'$ [that is, expressing the master equation in Eqs. (55), (56), and (61) in terms of the dimensionless time $\tau = \kappa_p' t$] leads to the conclusion that, to order zero in ϵ_2, the pump field evolves according to the master equation of a driven and damped harmonic oscillator at zero temperature:

$$\frac{d}{dt}(\rho_{qp})_I(t) = -\frac{i}{\hbar}[\hbar\delta_p'' a_p^\dagger a_p + \hbar\Omega_d(a_p^\dagger + a_p),\ (\rho_{qp})_I(t)] + \kappa_p'\mathcal{D}(a_p)(\rho_{qp})_I(t). \tag{63}$$

The harmonic oscillator angular frequency δ_p'' and the coherent state parameter α_p mentioned in the following are defined in Appendix D. As a consequence of Eq. (63), the pump field is a stable subsystem that converges to the stationary state $|\alpha_p\rangle\langle\alpha_p|$ where $|\alpha_p\rangle$ is a coherent state.

Adiabatically eliminating the pump field from the master equation in Eqs. (55), (56), and (61) [30,31], to second order in ϵ_2 the reduced dynamics of the ensemble of N qutrits is given by

$$\begin{aligned}\frac{d}{dt}(\rho_q)_I(t) &= \mathcal{L}_G(\rho_q)_I(t),\\ &= -\frac{i}{\hbar}[H_{\text{eff}},\ (\rho_q)_I(t)] + \kappa_s\mathcal{D}(S)(\rho_q)_I(t)\\ &\quad + \kappa_p'\mathcal{D}(S_q)(\rho_q)_I(t),\end{aligned} \tag{64}$$

where the effective Hamiltonian H_{eff} of the qutrits is

$$\begin{aligned}\frac{1}{\hbar}H_{\text{eff}} &= \frac{\omega_d}{2}S_{11} + \sum_{j=2,3}[\xi_j + (\xi_j - \omega_j)S_{11}]S_{jj}\\ &\quad - \xi_{23}(S_{11}+1)(S_{32} + S_{32}^\dagger)\\ &\quad - \delta_p'' S_q^\dagger S_q + (\alpha_q^* S_q + \alpha_q S_q^\dagger).\end{aligned} \tag{65}$$

Here, we have introduced the qutrit operator

$$S_q = \frac{1}{\kappa_p''}\big(\chi_2 S_{12}^2 + \chi_3 S_{13}^2 + \chi_{23}S_{12}S_{13}\big), \tag{66}$$

and the parameters ξ_2, ξ_3, ξ_{23}, α_q, and κ_p'' are defined in Appendix D. Also, $(\rho_q)_I(t)$ is the density operator of the qutrits in the IP. Explicitly,

$$(\rho_q)_I(t) = e^{-i(\omega_d t/2)S_{11}}\text{Tr}_{p+s}[\rho(t)]e^{i(\omega_d t/2)S_{11}}, \tag{67}$$

where Tr_{s+p} denotes the partial trace with respect to the degrees of freedom of the pump and signal fields.

Comparing Eqs. (55) and (56) with Eqs. (64) and (65), one finds that the interaction between the qutrits and the pump field induces a quadratic dissipator $\kappa_p'\mathcal{D}(S_q)$ for the qutrits, as well as shifts for ω_2, ω_3, and the strength of the excited levels coupling in the second line of Eq. (65). Also notice the appearance of the coherent interactions in the last line of Eq. (65): The term $(\alpha_q^* S_q + \alpha_q S_q^\dagger)$ is similar to a squeezing operator for b_j and $b_j^\dagger$ $(j = 2, 3)$, while $-\delta_p'' S_q^\dagger S_q$ has fourth-order terms involving b_2, b_3, and their adjoints. Finally, observe that the linear dissipator $\kappa_s\mathcal{D}(S)$ is not affected.

D. The final model

Now, assume that

$$g_2 = g_3 = g > 0, \quad \omega_2 = \omega_3 = \omega_q, \quad \chi > 0, \tag{68}$$

and consider the operator S_- defined in Eq. (3).

If one uses Eq. (68) and $(S_{11} + S_{22} + S_{33}) = N$ because there are N qutrits, then the master equation in Eqs. (64) and (65) takes the form of the master equation in Eqs. (4)–(7). The parameters κ_1, κ_2, Δ, δ_1, δ, ξ, α_0, and χ and the relationships with the previous quantities are presented in Appendix D.

Note that the density operator $\rho(t)$ appearing in Eq. (4) is actually $(\rho_q)_I(t)$, which is defined in Eq. (67). In particular, the detuning δ_1 can be tuned to zero or to a positive or negative value because one can adjust the frequency ω_d of the classical field driving the pump field.

E. The case of qubits

We now discuss what happens if one has qubits instead of qutrits. Moreover, it allows one to compare the master equation in Eqs. (4)–(7) with that deduced in Ref. [9].

To obtain the case of qubits, one simply has to take

$$\omega_3 = g_3 = 0, \quad \omega_2 = \omega_q, \quad g_2 = g > 0, \quad \chi_2 = \chi > 0, \tag{69}$$

in the original Hamiltonian in Eq. (47). Then, one performs the same approximations as before to obtain the effective N qubit master equation

$$\begin{aligned}\frac{d}{dt}(\rho_q)_I(t) &= -\frac{i}{\hbar}[H_{\text{eff}},\ (\rho_q)_I(t)] + \kappa_1\mathcal{D}(S_{12})(\rho_q)_I(t)\\ &\quad + \kappa_2\mathcal{D}\big(S_{12}^2\big)(\rho_q)_I(t),\end{aligned} \tag{70}$$

with the quantities defined in Eq. (69), Appendix D, and

$$\begin{aligned}\frac{1}{\hbar}H_{\text{eff}} &= \frac{\omega_d}{2}S_{11} + (\omega_q - \xi - \xi S_{11})S_{22}\\ &\quad + \delta\big(S_{12}^2\big)^\dagger S_{12}^2 + \alpha_0^* S_{12}^2 + \alpha_0\big(S_{12}^2\big)^\dagger.\end{aligned} \tag{71}$$

This master equation was obtained from Eqs. (64)–(66) by applying Eq. (69). In particular, notice that Eq. (69) leads to

$$\xi_3 = \chi_3 = 0, \quad S = \frac{g}{\kappa_s'}S_{12},$$

$$\xi_{23} = \chi_{23} = g_{23} = 0, \quad S_q = \frac{\chi}{\kappa_p''}S_{12}^2.$$

We now reexpress H_{eff}. Using the commutation relations of b_j and their adjoints and the fact that there are N qubits (so $S_{11} + S_{22} = N$), one has

$$\frac{\omega_d}{2}S_{11} + (\omega_q - \xi - \xi S_{11})S_{22} = \omega_q N + \delta_1 S_{11} - \xi S_{12}^\dagger S_{12}.$$

Substituting this result into Eq. (71) leads to

$$\frac{1}{\hbar}H_{\text{eff}} = \delta_1 S_{11} - \xi S_{12}^\dagger S_{12} + \delta\big(S_{12}^2\big)^\dagger S_{12}^2 + \alpha_0^* S_{12}^2 + \alpha_0\big(S_{12}^2\big)^\dagger, \tag{72}$$

where we have discarded the term $\omega_q N$ because it is a scalar that disappears when one substitutes H_{eff} in the commutator of the master equation in Eq. (70).

Observe that the N qubit master equation in Eqs. (70) and (72) has exactly the same form as the N qutrit master equation in Eqs. (4)–(7) because $S_- = S_{12}$ for qubits (there is no third level).

We now compare the N qubit master equation in Eqs. (70) and (72) with the one deduced in Ref. [9]. The N qubit master equation in Ref. [9] has exactly the same dissipators and the same driving in the Hamiltonian but with κ_{1at}/N, κ_{2at}/N^2, and $i\chi_{2at}/N$ replacing our κ_1, κ_2, and α_0^*, respectively. Reference [9] considers $\kappa_p' \simeq \kappa_p$; $|\delta_s| \gg (\kappa_s/2)$; $\kappa_p'/2 \gg |\delta_p''|$; $\kappa_p'/2 \gg |2J^2/\Delta_s - \delta_p'|$; and $\omega_d = \omega_p = 2\omega_q$. Using these conditions, one finds that $\kappa_2 = \kappa_{2at}/N^2$, $\kappa_1 = \kappa_{1at}/N$, $\alpha_0^* = i\chi_{2at}/N$, and $\delta_1 = 0$, so the two master equations are identical under these conditions except that [9] neglects the second and third terms of the right-hand side of Eq. (72). A deeper comparison is in Appendix E.

F. Approximate stationary solutions of the original master equation

Section IV presented two parameter-independent stationary states, $|0_L\rangle\langle 0_L|$ and $|1_L\rangle\langle 1_L|$, of the effective master equation in Eqs. (4) and (5) that can be used as logical states of a qubit. In this section, we show that they are part of an approximate stationary solution of the original master equation in Eqs. (51) and (52) when Eq. (68) and the assumptions of the first (53) and second (62) adiabatic approximations hold [see also the paragraphs below Eqs. (53) and (62)].

Using Eq. (68) and $N = S_{11} + S_{22} + S_{33}$, one can write the Hamiltonian in Eq. (52) as

$$\begin{aligned}\frac{1}{\hbar}H_I &= \delta_s a_s^\dagger a_s + \delta_p a_p^\dagger a_p + \omega_q(N - S_{11}) + \frac{\omega_d}{2}S_{11}\\ &\quad + \Omega_d(a_p^\dagger + a_p) + J\big[a_p(a_s^\dagger)^2 + a_p^\dagger a_s^2\big]\\ &\quad + g(a_s^\dagger S_- + a_s S_-^\dagger).\end{aligned} \tag{73}$$

Now, consider the density operators ($\mathfrak{j} = 0, 1$):

$$\rho_{\mathfrak{j}} = |0_s\rangle\langle 0_s| \otimes |\alpha_{po}\rangle\langle\alpha_{po}| \otimes |\mathfrak{j}_L\rangle\langle\mathfrak{j}_L|, \tag{74}$$

where

$$\alpha_{po} = \frac{\Omega_d}{-\delta_p + i\left(\frac{\kappa_p}{2}\right)}. \tag{75}$$

It is important to note that $|0_s\rangle\langle 0_s|$ is the steady-state solution of the damped harmonic oscillator master equation at zero temperature in Eq. (54), while $|\alpha_{po}\rangle\langle\alpha_{po}|$ is the steady-state solution of the driven and damped harmonic oscillator master equation at zero temperature in Eq. (63) with δ_p and κ_p replacing δ_p'' and κ_p', respectively. Notice that we use α_{po} instead of α_p because it is δ_p and κ_p that appear in the original master equations (51) and (73) instead of δ_p'' and κ_p'. Using these observations and the assumptions of the first and second adiabatic approximations, one can show (see Appendix C) that for $n = 1, 2, \ldots$ and $\mathfrak{j} = 0, 1$ one has

$$\begin{aligned}\mathcal{L}_I^n\rho_{\mathfrak{j}} \simeq -i\sqrt{2}J[&\alpha_{po}(-k_s - i2\delta_s)^{n-1}|2_s\rangle\langle 0_s|\\ &- \alpha_{po}^*(-\kappa_s + i2\delta_s)^{n-1}|0_s\rangle\langle 2_s|]\\ &\otimes |\alpha_{po}\rangle\langle\alpha_{po}| \otimes |\mathfrak{j}_L\rangle\langle\mathfrak{j}_L|.\end{aligned} \tag{76}$$

Since the interaction and the Schrödinger pictures coincide at time $t = 0$ [see Eq. (49)], using Eq. (76), one finds that, according to the original master equation, $\rho_{\mathfrak{j}}$ evolves as

$$\begin{aligned}\rho_{\mathfrak{j}}^I(t) &= e^{\mathcal{L}_I t}\rho_{\mathfrak{j}} = \rho_{\mathfrak{j}} + \sum_{n=1}^{\infty}\frac{t^n}{n!}\mathcal{L}_I^n\rho_{\mathfrak{j}},\\ &\simeq \rho_{\mathfrak{j}} + R_s(t) \otimes |\alpha_{po}\rangle\langle\alpha_{po}| \otimes |\mathfrak{j}_L\rangle\langle\mathfrak{j}_L|,\end{aligned} \tag{77}$$

where

$$R_s(t) = i\frac{\sqrt{2}J\alpha_{po}}{\kappa_s + i2\delta_s}[e^{-(\kappa_s + i2\delta_s)t} - 1]|2_s\rangle\langle 0_s| + \text{H.c.} \tag{78}$$

Here, H.c. stands for *Hermitian conjugate*.

Observe that

$$\lim_{t\to+\infty} R_s(t) = -i\frac{\sqrt{2}J\alpha_{po}}{\kappa_s + i2\delta_s}|2_s\rangle\langle 0_s| + \text{H.c.}, \tag{79}$$

and, using the definition of α_{po} in Eq. (75), that

$$\left|-i\frac{\sqrt{2}J\alpha_{po}}{\kappa_s + i2\delta_s}\right| \leqslant 2\sqrt{2}\epsilon_1\frac{|\Omega_d|}{\kappa_p}. \tag{80}$$

By the first and second adiabatic approximations, the right-hand side of Eq. (80) is $\ll 1$, so one concludes from Eqs. (77)–(80) that $\rho_{\mathfrak{j}}^I(t)$ approximately tends to $\rho_{\mathfrak{j}}$ as $t \to +\infty$.

G. The effect of $g_2 \neq g_3$ or $\omega_2 \neq \omega_3$

The assumption of a perfectly symmetric V configuration (i.e., $\omega_2 = \omega_3$ and $g_2 = g_3$) for the qutrits has been central to the analytical results presented in Secs. II–VI. In this section, we briefly discuss what happens when this condition does not hold. The details and extensions of the results will be presented in future work.

Assume that $\omega_2 \neq \omega_3$ or $g_2 \neq g_3$. After the first adiabatic, averaging, and second adiabatic approximations, the master equation describing the evolution of the qutrits is given in Eqs. (64)–(67). In particular, the Liouvillian of the master equation is $\mathcal{L}_G$.

Introduce the *left (bright)* and *right (dark)* operators

$$b_l = \frac{g_2 b_2 + g_3 b_3}{\sqrt{g_2^2 + g_3^2}}, \quad b_r = \frac{g_3 b_2 - g_2 b_3}{\sqrt{g_2^2 + g_3^2}}. \tag{81}$$

Notice that Eq. (81) reduces to the original definition in Eq. (17) when $g_2 = g_3 = g > 0$. Using Eq. (81), define the logical qubit states $|0_L\rangle$ and $|1_L\rangle$ as before [see Eqs. (20) and (22)].

If $g_2 \neq g_3$ and $\omega_2 = \omega_3$, then one can show that $|0_L\rangle$ and $|1_L\rangle$ continue to be dark states of $\mathcal{L}_G$.

In the following, assume that $\omega_2 < \omega_3$ and let

$$\omega_{\text{avg}} = \frac{\omega_3 + \omega_2}{2}, \quad \Delta_q = \frac{\omega_3 - \omega_2}{2} > 0 \tag{82}$$

be the average of the qutrit transition frequencies and the detuning between them, respectively.

Observe that the first adiabatic, averaging, and second adiabatic approximations require

$$\left|\frac{\Delta_q}{\omega_s-\omega_{\rm avg}}\right|,\quad \left|\frac{\Delta_q}{\omega_s-(\omega_p-\omega_{\rm avg})}\right| \ll 1, \tag{83}$$

because $\omega_s \gg \omega_p \simeq 2\omega_2 \simeq 2\omega_3$ in order to have an effective interaction between the pump field and the qutrits where the processes where a pump photon is emitted or absorbed by two qutrits are favored (see the discussion of the model in the first part of Sec. VII).

Now, in $\mathcal{L}_G$ make a linear approximation in the quantities appearing in Eq. (83) and denote the resulting superoperator by $\mathcal{L}_{GL}$. Then, one can show that

$$\mathcal{L}_{GL}|\mathbb{j}_L\rangle\langle\mathbb{j}_L| = -\frac{i}{\hbar}[H_{\rm pert},\ |\mathbb{j}_L\rangle\langle\mathbb{j}_L|] \quad (\mathbb{j}=0,1), \tag{84}$$

where

$$\frac{1}{\hbar}H_{\rm pert} = -\Delta_q f(S_{11})(b_l^\dagger b_r + b_r^\dagger b_l), \tag{85}$$

with

$$f(x) = \frac{g_3g_2}{g_3^2+g_2^2}\left[2-\frac{(x+1)}{(\omega_s-\omega_{\rm avg})^2}\left(g_3^2+g_2^2\right)\right]. \tag{86}$$

The appearance of $(b_l^\dagger b_r + b_r^\dagger b_l)$ fundamentally alters the dynamics of the ensemble of qutrits acting as a direct exchange interaction between the dark and bright modes. This means that the logical states $|0_L\rangle$ and $|1_L\rangle$ will continuously rotate out of the protected b_r mode and leak into the unprotected b_l mode.

A good approach aimed at restoring the robust encoding of our logical qubit states is to use an elegant experimental technique, which has to do with microwave dressed states (*Autler-Townes effect*) [26], quantum Zeno dynamics [34], and continuous dynamical decoupling [35]. Since $H_{\rm pert}$ forces qutrits to jump from the safe dark state into the bright state, introduce the qutrit driving

$$\begin{aligned}H_\Omega &= \hbar\Omega(S_{32}+S_{23}),\\ &= \hbar\Omega\left(\frac{2g_3g_2}{g_3^2+g_2^2}\right)(b_l^\dagger b_l - b_r^\dagger b_r)\\ &\quad + \hbar\Omega\left(\frac{g_3^2-g_2^2}{g_3^2+g_2^2}\right)(b_l^\dagger b_r + b_r^\dagger b_l),\end{aligned} \tag{87}$$

which drives transitions between the two excited levels $|2\rangle$ and $|3\rangle$. The term $(1/\hbar)H_\Omega$ would be added to the right-hand side of Eq. (52) and, consequently, the Liouvillian of the master equation would be $\mathcal{L}_G - (i/\hbar)[H_\Omega,\cdot]$. The strength of the driving Ω would have to be consistent with the first adiabatic, averaging, and second adiabatic approximations. Notice that the term in the last line of Eq. (87) can be used to cancel $H_{\rm pert}$ when $g_2 \neq g_3$.

If $g_2 \neq g_3$, then one can adjust Ω to take the value

$$\Omega = \left(\frac{g_3^2+g_2^2}{g_3^2-g_2^2}\right)\Delta_q f(\mathbb{j}) \quad (\mathbb{j}=0,1), \tag{88}$$

so that

$$\mathcal{L}_{GL}|\mathbb{j}_L\rangle\langle\mathbb{j}_L| - \frac{i}{\hbar}[H_\Omega, |\mathbb{j}_L\rangle\langle\mathbb{j}_L|] = 0. \tag{89}$$

Hence, the logical qubit state $|\mathbb{j}_L\rangle$ would be a dark state of the master equation.

Finally, consider that $g_2 = g_3 = g$. Let us restrict ourselves to the subspace spanned by the two states

$$\begin{aligned}|\mathtt{1}\rangle &= |\mathbb{j}_L\rangle = |n_1=\mathbb{j}, n_l=0, n_r=N-\mathbb{j}\rangle,\\ |\mathtt{2}\rangle &= |n_1=\mathbb{j}, n_l=1, n_r=N-\mathbb{j}-1\rangle,\end{aligned} \tag{90}$$

where $|\mathtt{1}\rangle$ is the logical qubit state with $\mathbb{j}=0$ or $\mathbb{j}=1$, and $|\mathtt{2}\rangle$ results from transferring a dark quantum to a bright one. If the state of the ensemble of qutrits at time $t=0$ is $|\mathtt{1}\rangle$ and evolves under the restriction of the Hamiltonian $(H_{\rm pert}+H_\Omega)$ to the subspace spanned by $|\mathtt{1}\rangle$ and $|\mathtt{2}\rangle$, then the probability to find the ensemble of qutrits in the state $|\mathtt{2}\rangle$ at any time $t \geqslant 0$ satisfies

$$\mathcal{P}(t) \leqslant \frac{[\Delta_q f(\mathbb{j})]^2(N-\mathbb{j})}{\Omega^2+[\Delta_q f(\mathbb{j})]^2(N-\mathbb{j})}. \tag{91}$$

Choosing $|\Omega| \gg |\Delta_q f(\mathbb{j})\sqrt{N-\mathbb{j}}|$ (which is acceptable as long as Ω is consistent with the approximations), this probability can be made small enough. It follows that H_Ω creates an energetic wall between right and left modes, which decouples the *dark sector* from the *bright sector*, effectively reducing the detuning Δ_q to zero. This is one way to successfully keep the robust Schrödinger cat encoding we proposed, even with asymmetric V-configuration qutrits.

VIII. CONCLUSIONS

We have presented a special case of a cat code where a logical qubit is encoded using engineered even and odd Schrödinger cat states of an ensemble of identical qutrits with a symmetrical V configuration (*a qubit-disguised qutrit*). These logical states have either zero or one qutrits in the ground level and constitute dark states. In particular, they are immune to single-qutrit decay, two-qutrit decay and driving processes, and collective correlated dephasing. It is important to emphasize that these logical states do not depend on the parameters of the effective master equation describing the evolution of the ensemble of qutrits. In the case of two qutrits, the logical state encoded in the even cat state is immune to both inhomogeneous broadening and to local correlated dephasing, while the logical state encoded in the odd cat state is vulnerable to both processes. Nevertheless, in the presence of these permutation symmetry breaking interactions one can replace the fragile logical odd state with one with mixed permutation symmetry (with fully distinguishable qutrits, in fact) that is immune to both inhomogeneous broadening and local correlated dephasing and also to all the interactions to which the fragile logical state is invulnerable. This robust mixed permutation symmetry state was extended to an arbitrary number of qutrits.

The effective master equation describing the evolution of the ensemble of qutrits can be deduced from the master equation describing a physical system composed of two parametrically coupled cavity fields where one is driven by a classical field and the other interacts dispersively with an ensemble of three-level atoms that play the role of the qutrits. Both cavity fields are subject to decay by the single-photon loss process and are assumed to be strongly dissipative so that they can be adiabatically eliminated. It is shown analytically that the logical qubit is part of an approximate stationary state of the physical system under the assumptions made to derive the effective model.

In principle, this physical system can be implemented experimentally using an extension of the setup originally proposed in Ref. [10]: One considers two state-of-the-art superconducting CPWR, an SQUID, and an ensemble of alkali atoms placed close to one of the CPWR. The SQUID is in charge of parametric coupling between the fields in each CPWR and, in particular, [8] already realized the dispersive coupling between an ensemble of rubidium 87 atoms and a CPWR. The CPWR in Ref. [8] naturally couples three hyperfine ground state levels of rubidium 87 in a V configuration: $|F=2, M_F=-2\rangle$ and $|F=2, M_F=0\rangle$ play the role of the excited levels, while $|F=1, M_F=-1\rangle$ plays the role of the ground level. Although the aforementioned excited levels have different frequencies in the setup of Ref. [8], we showed that the logical qubit states can also be defined for an asymmetric V configuration where the excited levels of the qutrits have different frequencies and coupling strengths. Quite remarkably, they still preserve their robustness properties.

In future work, we will investigate in greater depth the effect of a detuning between the excited state frequencies (i.e., asymmetrical V configuration) on the stationary states of the effective qutrit master equation. In particular, we shall study the robustness of such states under dephasing processes, as well as the interaction of two ensembles of qutrits (each one corresponding to a logical qubit) in order to define two-qubit logical gates. We also aim at investigating the advantages and disadvantages offered by ensembles of qutrits with a Λ or Σ configuration. The inclusion of a thermal bath with $T>0$ for the signal and pump fields and how this will affect the robustness of the logical qubit here introduced will also be addressed.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: PARAMETER-INDEPENDENT STATIONARY SOLUTIONS

In this Appendix, we first prove three propositions that are then used to calculate the parameter-independent stationary states of the master equation in Eq. (4).

We first prove the following proposition:

$$\lambda \text{ is an eigenvalue of } S_- \quad \Rightarrow \quad \lambda = 0. \tag{A1}$$

Let the complex number λ be an eigenvalue of S_-. Assume $\lambda \neq 0$. Then, there is a nonzero ket $|\Phi\rangle$ such that $S_-|\Phi\rangle = \lambda|\Phi\rangle$.

Since β_q is an orthonormal basis for the state space of the N qutrits, one has the closure relation

$$\sum_{n_3=0}^{N}\sum_{n_2=0}^{N-n_3} |N-n_2-n_3, n_2, n_3\rangle\langle N-n_2-n_3, n_2, n_3| = \mathbb{1}_q, \tag{A2}$$

where $\mathbb{1}_q$ is the identity operator in the state space of the N qutrits. Introducing this closure relation between S_- and $|\Phi\rangle$, it is straightforward to show that

$$\begin{aligned} S_-|\Phi\rangle = \sum_{n_3=0}^{N-1}\sum_{n_2=0}^{N-n_3-1} &\sqrt{N-n_2-n_3}\big[c_{n_2+1,n_3}\sqrt{n_2+1} \\ &+ c_{n_2,n_3+1}\sqrt{n_3+1}\big]|N-n_2-n_3, n_2, n_3\rangle, \end{aligned} \tag{A3}$$

where

$$c_{n_2,n_3} = \langle N-n_2-n_3, n_2, n_3|\Phi\rangle. \tag{A4}$$

Using the expansions of $S_-|\Phi\rangle$ and $|\Phi\rangle$ with respect to the basis β_q, it follows that

$$\begin{aligned} &S_-|\Phi\rangle = \lambda|\Phi\rangle \\ &\Leftrightarrow \begin{cases} c_{0,N} = 0, \\ c_{N-m_3,m_3} = 0 \ \ (m_3 = 0, 1, \ldots, N-1), \\ \lambda c_{n_2,n_3} = \sqrt{N-n_2-n_3}\big[c_{n_2+1,n_3}\sqrt{n_2+1} \\ \quad +c_{n_2,n_3+1}\sqrt{n_3+1}\big] \\ (n_2 = 0, 1, \ldots, N-n_3-1; \\ n_3 = 0, 1, \ldots, N-1). \end{cases} \end{aligned} \tag{A5}$$

It is clear that all the conditions on the right-hand side of Eq. (A5) are satisfied if $c_{n_2,n_3} = 0$ for all $n_2 = 0, \ldots, N-n_3$ and $n_3 = 0, \ldots, N$. We now prove that these conditions imply that $c_{n_2,n_3} = 0$ for all $n_2 = 0, \ldots, N-n_3$ and $n_3 = 0, \ldots, N$.

Assume that all the conditions on the right-hand side of Eq. (A5) hold. We claim that $c_{n_2,n_3} = 0$ for all $n_2 = 0, 1, \ldots, N-n_3-1$ and $n_3 = 0, 1, \ldots, N-1$.

We prove the claim by induction over μ_3 with $n_3 = N-1-\mu_3$ for all $\mu_3 = 0, 1, \ldots, N-1$. The induction parameter is μ_3 instead of n_3, so one can start with $\mu_3 = 0$ and $n_3 = N-1$ instead of $n_3 = 0$.

First assume that $\mu_3 = 0$. Then, $n_3 = N-1$ and $n_2 = 0$. From the third condition in Eq. (A5), one has

$$\lambda c_{0,N-1} = c_{1,N-1} + c_{0,N}\sqrt{N}. \tag{A6}$$

Using the first and the second conditions in Eq. (A5), one finds that the right-hand side of Eq. (A6) is zero. Since $\lambda \neq 0$, it follows that $c_{0,N-1} = 0$ and the claim holds for $\mu_3 = 0$.

Now, assume that the claim holds for some $\mu_3 \in \{0, \ldots, N-2\}$. We prove that it holds for $\mu_3 + 1$. In this case, $n_3 = N-2-\mu_3$ and $n_2 = 0, 1, \ldots, \mu_3+1$.

From the third condition in Eq. (A5) and for $n_2 = \mu_3 + 1$,

$$\begin{aligned} \lambda c_{\mu_3+1,N-2-\mu_3} = \ &c_{\mu_3+2,N-2-\mu_3}\sqrt{\mu_3+2} \\ &+ c_{\mu_3+1,N-1-\mu_3}\sqrt{N-1-\mu_3}. \end{aligned} \tag{A7}$$

Using the first and the second conditions in Eq. (A5), one finds that the right-hand side of Eq. (A7) is zero, so

$$c_{\mu_3+1,N-2-\mu_3} = 0. \tag{A8}$$

Applying the induction hypothesis to the third condition in Eq. (A5) for $n_2 = 0, 1, \ldots, \mu_3$, one obtains

$$\begin{aligned}\lambda c_{\mu_3,N-2-\mu_3} &= c_{\mu_3+1,N-2-\mu_3}\sqrt{2(\mu_3+1)},\\ \lambda c_{\mu_3-1,N-2-\mu_3} &= c_{\mu_3,N-2-\mu_3}\sqrt{3\mu_3},\\ &\vdots\\ \lambda c_{0,N-2-\mu_3} &= c_{1,N-2-\mu_3}\sqrt{(\mu_3+2)}.\end{aligned} \tag{A9}$$

Substituting Eq. (A8) in the right-hand side of the first equation in Eq. (A9), one finds that $c_{\mu_3,N-2-\mu_3} = 0$. Substituting this result in the second equation of Eq. (A9), one obtains that $c_{\mu_3-1,N-2-\mu_3} = 0$. Continuing in this manner, one concludes that $c_{\mu_3-2,N-2-\mu_3} = 0, \ldots, c_{0,N-2-\mu_3} = 0$. Therefore, the claim holds for $\mu_3 + 1$. Consequently, the claim holds for all $\mu_3 = 0, 1, \ldots, N-1$.

Using the claim, it follows that the conditions in Eq. (A5) are equivalent to $c_{n_2,n_3} = 0$ for all $n_2 = 0, 1, \ldots, N - n_3$ and $n_3 = 0, 1, \ldots, N$, which in turn is equivalent to $|\Phi\rangle = 0$. However, $|\Phi\rangle \neq 0$ because it is an eigenvector. Hence, we arrive at a contradiction that stems from the assumption that $\lambda \neq 0$. Therefore, we conclude that $\lambda = 0$, so the proposition in Eq. (A1) is true.

Consider a pure state $|\Phi\rangle\langle\Phi|$. We now prove that

$$\mathcal{D}(S_-)|\Phi\rangle\langle\Phi| = 0 \Leftrightarrow S_-|\Phi\rangle = 0. \tag{A10}$$

Inspecting the form of $\mathcal{D}(S_-)|\Phi\rangle\langle\Phi|$, it is clear that $\mathcal{D}(S_-)|\Phi\rangle\langle\Phi| = 0$ if $S_-|\Phi\rangle = 0$.

Now, assume that $\mathcal{D}(S_-)|\Phi\rangle\langle\Phi| = 0$. Since $|\Phi\rangle$ is normalized, it follows that

$$\begin{aligned}0 &= \langle\Phi|[\mathcal{D}(S_-)|\Phi\rangle\langle\Phi|]|\Phi\rangle\\ &= \langle\Phi|\big[S_-|\Phi\rangle\langle\Phi|S_-^\dagger - \tfrac{1}{2}\{S_-^\dagger S_-, |\Phi\rangle\langle\Phi|\}\big]|\Phi\rangle\\ &= |\langle\Phi|S_-|\Phi\rangle|^2 - \langle\Phi|S_-^\dagger S_-|\Phi\rangle\\ &= |\langle\Phi|S_-\Phi\rangle|^2 - \langle S_-\Phi|S_-\Phi\rangle,\end{aligned} \tag{A11}$$

so

$$|\langle\Phi|S_-\Phi\rangle|^2 = \langle S_-\Phi|S_-\Phi\rangle. \tag{A12}$$

From the Cauchy-Schwarz inequality, one has

$$|\langle\Phi|S_-\Phi\rangle|^2 \leqslant \langle S_-\Phi|S_-\Phi\rangle. \tag{A13}$$

From Eq. (A12), it follows that the equality in Eq. (A13) is realized, so the Cauchy-Schwarz inequality tells us that there is a complex number λ such that $S_-|\Phi\rangle = \lambda|\Phi\rangle$, so $|\Phi\rangle$ is an eigenvector of S_- with eigenvalue λ. From Eq. (A1), it follows $\lambda = 0$. Therefore, $S_-|\Phi\rangle = 0$ and this completes the proof of Eq. (A10).

Consider a pure state $\rho_s = |\Phi\rangle\langle\Phi|$. We now prove that

$$\rho_s \text{ satisfies(11)} \Leftrightarrow \begin{cases} S_-|\Phi\rangle = (S_-^2)^\dagger|\Phi\rangle = 0,\\ |\Phi\rangle \text{ is an eigenvector of } S_{11}.\end{cases} \tag{A14}$$

Inspecting Eq. (11), it is clear that Eq. (11) holds if $S_-|\Phi\rangle = (S_-^2)^\dagger|\Phi\rangle = 0$ and $|\Phi\rangle$ is an eigenvector of S_{11}.

Now, assume that ρ_s satisfies Eq. (11). Since $\mathcal{D}(S_-)(\rho_s) = 0$, it follows from Eq. (A10) that $S_-|\Phi\rangle = 0$. Using $S_-|\Phi\rangle = 0$, the last equation in Eq. (11) takes the form

$$\alpha_0(S_-^2)^\dagger\rho_s - \alpha_0^*\rho_s S_-^2 = 0. \tag{A15}$$

Applying the operator on the left-hand side of Eq. (A15) to $|\Phi\rangle$ and using once more that $S_-|\Phi\rangle = 0$, one obtains that

$$\alpha_0(S_-^2)^\dagger|\Phi\rangle = 0. \tag{A16}$$

Since this equation must hold for any complex number α_0, one concludes that $(S_-^2)^\dagger|\Phi\rangle = 0$.

Finally, from the first equation in the last line of Eq. (11), it follows that

$$0 = S_{11}|\Phi\rangle - |\Phi\rangle\langle\Phi|S_{11}|\Phi\rangle, \tag{A17}$$

so $|\Phi\rangle$ is an eigenvector of S_{11}. This completes the proof of Eq. (A14).

At last, we calculate the normalized kets $|\Phi\rangle$ that satisfy the conditions that $S_-|\Phi\rangle = (S_-^2)^\dagger|\Phi\rangle = 0$ and that $|\Phi\rangle$ is an eigenvector of S_{11}.

Assume that

$$\begin{aligned}&S_-|\Phi\rangle = (S_-^2)^\dagger|\Phi\rangle = 0, \quad \langle\Phi|\Phi\rangle = 1,\\ &S_{11}|\Phi\rangle = s_{11}|\Phi\rangle, \quad \text{for some real number } s_{11}.\end{aligned} \tag{A18}$$

First, observe that one can choose $|\Phi\rangle$ to have the parity symmetry associated with Π_0 defined in Eq. (8). Since Π_0 anticommutes with S_- and commutes with both $(S_-^2)^\dagger$ and S_{11} [see Eq. (9)], from Eq. (A18) one has

$$A[\Pi_0|\Phi\rangle] = 0, \quad S_{11}[\Pi_0|\Phi\rangle] = s_{11}\Pi_0|\Phi\rangle, \tag{A19}$$

for $A = S_-$ and $(S_-^2)^\dagger$. Then, $|\pm\rangle = |\Phi\rangle \pm \Pi_0|\Phi\rangle$ satisfy

$$\begin{aligned}&S_-|\pm\rangle = (S_-^2)^\dagger|\pm\rangle = 0, \quad S_{11}|\pm\rangle = s_{11}|\pm\rangle,\\ &\Pi_0|\pm\rangle = \pm|\pm\rangle\end{aligned} \tag{A20}$$

and at least one of $|+\rangle$ and $|-\rangle$ is not zero. Therefore, one can look for $|\Phi\rangle$ among the kets that are linear combinations of occupation number states $|n_1, n_2, n_3\rangle$, where n_1 is always even or always odd or, equivalently, where $(n_2 + n_3)$ is always even or always odd.

Assume N is an even, positive integer. Let

$$
\begin{aligned}
|\Phi_+\rangle &= \sum_{n_3=0}^{N}\sum_{m=\lceil\frac{n_3}{2}\rceil}^{N/2} c(n_3,m)|N-2m,2m-n_3,n_3\rangle,\\
|\Phi_-\rangle &= \sum_{n_3=0}^{N-1}\sum_{m=\lceil\frac{n_3-1}{2}\rceil}^{\lfloor(N-1)/2\rfloor} d(n_3,m)|N-2m-1,2m+1-n_3,n_3\rangle,
\end{aligned}
\tag{A21}
$$

where $\lceil x\rceil$ and $\lfloor x\rfloor$ are the *ceil* and *floor* functions, that is, $\lceil x\rceil$ is the smallest integer $\geqslant x$ and $\lfloor x\rfloor$ is the largest integer $\leqslant x$. Notice that $|\Phi_\pm\rangle$ have the parity symmetry.

One finds that $S_-|\Phi_+\rangle = 0$ if and only if

$$
c(n_3,m) = c(0,m)(-1)^{n_3}\sqrt{\frac{(2m)!}{(2m-n_3)!\,n_3!}}, \tag{A22}
$$

for $m = \lceil n_3/2\rceil,\ \dots,\ N/2$ and $n_3 = 0, 1, \dots, N$. Similarly, $S_-|\Phi_-\rangle = 0$ if and only if

$$
d(n_3,m) = d(0,m)(-1)^{n_3}\sqrt{\frac{(2m+1)!}{(2m+1-n_3)!\,n_3!}}, \tag{A23}
$$

for $m = \lceil(n_3-1)/2\rceil,\ \dots,\ \lfloor(N-1)/2\rfloor$ and $n_3 = 0,\ \dots,\ N-1$.

Now, we apply $(S_-^2)^\dagger$ to $|\Phi_\pm\rangle$. Actually, one can choose $|\Phi_+\rangle$ to be an eigenvector of $S_-^\dagger$ with eigenvalue 0. Using Eq. (A21), one obtains that $S_-^\dagger|\Phi_+\rangle = 0$ if and only if

$$
\begin{aligned}
c(n_3,m) = 0,\quad &\text{for } m = \left\lceil\frac{n_3}{2}\right\rceil,\ \dots,\ \frac{N}{2}-1,\\
&n_3 = 0,\ \dots,\ N.
\end{aligned}
\tag{A24}
$$

Combining Eqs. (A22) and (A24), one concludes that $S_-|\Phi_+\rangle = S_-^\dagger|\Phi_+\rangle = 0$ if and only if

$$
|\Phi_+\rangle = c\left(0,\frac{N}{2}\right)\sum_{n_3=0}^{N}|0,N-n_3,n_3\rangle(-1)^{n_3}\sqrt{\frac{N!}{(N-n_3)!\,n_3!}}. \tag{A25}
$$

Here, $c(0,N/2)$ can be used to normalize $|\Phi_+\rangle$. Choosing $c(0,N/2) > 0$, one finds that $|\Phi_+\rangle$ is normalized if

$$
c\left(0,\frac{N}{2}\right) = 2^{-N/2}. \tag{A26}
$$

In addition, observe that $S_{11}|\Phi_+\rangle = 0$.

We now consider $|\Phi_-\rangle$. First, $|\Phi_-\rangle$ cannot be chosen to be an eigenvector of $S_-^\dagger$ with eigenvalue zero, so one must use $(S_-^2)^\dagger$. Using Eq. (A21), one obtains that $(S_-^2)^\dagger|\Phi_-\rangle = 0$ if and only if

$$
\begin{aligned}
d(n_3,m) = 0,\quad &\text{for } m = \left\lceil\frac{n_3-1}{2}\right\rceil,\ \dots,\ \frac{N}{2}-2,\\
&n_3 = 0,\ \dots,\ N-1.
\end{aligned}
\tag{A27}
$$

Combining Eqs. (A23) and (A27), one concludes that $S_-|\Phi_-\rangle = (S_-^2)^\dagger|\Phi_-\rangle = 0$ if and only if

$$
\begin{aligned}
|\Phi_-\rangle = d\left(0,\frac{N}{2}-1\right)&\sum_{n_3=0}^{N-1}|1,N-1-n_3,n_3\rangle\\
&\times(-1)^{n_3}\sqrt{\frac{(N-1)!}{(N-1-n_3)!\,n_3!}}.
\end{aligned}
\tag{A28}
$$

Choosing $d(0,N/2-1) > 0$, one finds that $|\Phi_-\rangle$ is normalized if

$$
d\left(0,\frac{N}{2}-1\right) = 2^{-(N-1)/2}. \tag{A29}
$$

In addition, observe that $S_{11}|\Phi_-\rangle = |\Phi_-\rangle$.

Using how $b_j^\dagger$ $(j = 1, 2, 3)$ act on occupation number states, from Eqs. (A25), (A26), (A28), and (A29) one has

$$
\begin{aligned}
|\Phi_+\rangle &= \frac{1}{\sqrt{2^N\,N!}}(b_2^\dagger - b_3^\dagger)^N|\mathbf{0}\rangle,\\
|\Phi_-\rangle &= \frac{1}{\sqrt{2^{N-1}\,(N-1)!}}b_1^\dagger(b_2^\dagger - b_3^\dagger)^{N-1}|\mathbf{0}\rangle,
\end{aligned}
\tag{A30}
$$

where $|\mathbf{0}\rangle = |n_1 = 0, n_2 = 0, n_3 = 0\rangle$. Using the definition of SU(3) coherent states in Eq. (14), from Eq. (A30) one concludes that

$$
\begin{aligned}
|\Phi_+\rangle &= |z_1 = 0, z_2 = 1, z_3 = -1\rangle_N,\\
|\Phi_-\rangle &= b_1^\dagger|z_1 = 0, z_2 = 1, z_3 = -1\rangle_{N-1}.
\end{aligned}
\tag{A31}
$$

Therefore, when N is an even, positive integer one finds that Eq. (A18) holds if and only if $|\Phi\rangle$ is equal (except for a global phase factor) to $|\Phi_+\rangle$ or $|\Phi_-\rangle$.

Now, assume that N is an odd, positive integer. Motivated by the case where N is an even, positive integer, propose

$$
\begin{aligned}
|\Phi_+\rangle &= b_1^\dagger|z_1 = 0, z_2 = 1, z_3 = -1\rangle_{N-1},\\
|\Phi_-\rangle &= |z_1 = 0, z_2 = 1, z_3 = -1\rangle_N.
\end{aligned}
\tag{A32}
$$

Then, it is straightforward to verify that

$$
\begin{aligned}
&S_-|\Phi_\pm\rangle = 0,\quad (S_-^2)^\dagger|\Phi_\pm\rangle = 0,\quad S_{11}|\Phi_+\rangle = |\Phi_+\rangle,\\
&S_-^\dagger|\Phi_+\rangle \neq 0,\quad S_-^\dagger|\Phi_-\rangle = 0,\quad S_{11}|\Phi_-\rangle = 0.
\end{aligned}
\tag{A33}
$$

APPENDIX B: AVERAGING

We present James' averaging method [32,33] to third order (the reference only presents it to second order). It allows one to obtain an effective Hamiltonian that accounts for the most important effects of an interaction when considering a rapidly oscillating Hamiltonian.

Consider a quantum system whose evolution is governed by a Hamiltonian $H(t)$. Note that $H(t)$ may or may not be explicitly time dependent. We denote the state of the system at time t by $|\psi(t)\rangle$.

Now, pass to an IP defined by the unitary operator

$$
U_I(t,t_0) = e^{-\frac{i}{\hbar}H_0(t-t_0)}, \tag{B1}
$$

where H_0 is a time-independent Hermitian operator and t_0 is a fixed time.

The state of the system in the IP is given by

$$
|\psi_I(t)\rangle = U_I^\dagger(t,t_0)|\psi(t)\rangle, \tag{B2}
$$

and the IP Schrödinger equation is

$$i\hbar\frac{d}{dt}|\psi_I(t)\rangle = H_I(t)|\psi_I(t)\rangle, \tag{B3}$$

with

$$H_I(t) = U_I^\dagger(t,t_0)H(t)U_I(t,t_0) - H_0. \tag{B4}$$

Then, the IP evolution operator $U_{\rm IP}(t,t_0)$ is defined by the initial value problem

$$i\hbar\frac{\partial}{\partial t}U_{\rm IP}(t,t_0) = H_I(t)U_{\rm IP}(t,t_0), \quad U_{\rm IP}(t_0,t_0) = \mathbb{I}, \tag{B5}$$

where $\mathbb{I}$ is the identity operator.

We now introduce the averaging. Given a (possibly time-dependent) linear operator $A(t)$, define the averaged linear operator $\overline{A(t)}$ by

$$\overline{A(t)} = \int_{-\infty}^{+\infty} dt' f(t-t')A(t'), \tag{B6}$$

where $f(t)$ is a real-valued function such that $\int_{-\infty}^{+\infty} dt\, f(t) = 1$. In principle, one should also demand that $f(t) = 0$ for $t < 0$ to avoid problems with causality but this condition can be omitted in the present context. The purpose of the function $f(t)$ is to eliminate high-frequency terms by averaging them to zero. We use an *ideal low pass filter*

$$f(t) = \frac{\sin(\omega_0 t)}{\pi t}, \tag{B7}$$

whose Fourier transform is

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dt\ f(t)e^{-i\omega t} = \frac{\theta(\omega+\omega_0)\theta(\omega_0-\omega)}{\sqrt{2\pi}}. \tag{B8}$$

Here, $\omega_0 > 0$ is a cutoff frequency and θ is the Heaviside step function: $\theta(x) = 1$ if $x \geqslant 0$ and $\theta(x) = 0$ if $x < 0$. To illustrate how the filter works, consider a linear operator of the form $A(t) = A_0 e^{-i\Omega(t-t_0)}$ where Ω is a real quantity. Using Eqs. (B6)–(B8), the strong Parseval formula, and the Dirac delta function $\delta(\omega-\Omega)$, one has

$$\begin{aligned}\overline{A(t)} &= A(t)\int_{-\infty}^{+\infty} d\tau\ f(\tau)e^{i\Omega\tau},\\ &= A(t)\int_{-\infty}^{+\infty} d\omega \hat{f}(\omega)\sqrt{2\pi}\,\delta(\omega-\Omega),\\ &= \begin{cases} A(t), & \text{if } \Omega \in (-\omega_0,\omega_0),\\ 0, & \text{in any other case.}\end{cases}\end{aligned} \tag{B9}$$

Hence, the averaging eliminates high-frequency terms with $|\Omega| > \omega_0$.

From Eq. (B6), it is straightforward to show that

$$\overline{A^\dagger(t)} = [\overline{A(t)}]^\dagger, \quad \frac{d}{dt}\overline{A(t)} = \overline{\frac{dA}{dt}(t)}. \tag{B10}$$

Averaging Eq. (B5) and using Eq. (B10), one obtains

$$i\hbar\frac{\partial}{\partial t}\overline{U_{\rm IP}(t,t_0)} = \overline{H_I(t)U_{\rm IP}(t,t_0)}. \tag{B11}$$

Motivated by Eq. (B11), define the operator $\mathsf{H}_{\rm avg}(t)$ by

$$i\hbar\frac{\partial}{\partial t}\overline{U_{\rm IP}(t,t_0)} = \mathsf{H}_{\rm avg}(t)\overline{U_{\rm IP}(t,t_0)}. \tag{B12}$$

From Eqs. (B11) and (B12), it follows that

$$\mathsf{H}_{\rm avg}(t) = \overline{H_I(t)U_{\rm IP}(t,t_0)}[\overline{U_{\rm IP}(t,t_0)}]^{-1}. \tag{B13}$$

Usually, $\mathsf{H}_{\rm avg}(t)$ in Eq. (B13) is not Hermitian. The averaged Hamiltonian is then defined to be

$$H_{\rm avg}(t) = \tfrac{1}{2}[\mathsf{H}_{\rm avg}(t) + \mathsf{H}_{\rm avg}^\dagger(t)]. \tag{B14}$$

Now, assume that $H_I(t)$ is a perturbation. Then, one has the expansion

$$U_{\rm IP}(t,t_0) = \mathbb{I} + \sum_{n=1}^{+\infty} U_{\rm IP}^{(n)}(t,t_0), \tag{B15}$$

where $(n = 2, 3, \ldots)$

$$\begin{aligned}U_{\rm IP}^{(1)}(t,t_0) &= -\frac{i}{\hbar}\int_{t_0}^{t} dt_1 H_I(t_1),\\ U_{\rm IP}^{(n)}(t,t_0) &= \left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\\ &\quad \times H_I(t_1)H_I(t_2)\cdots H_I(t_n).\end{aligned} \tag{B16}$$

Notice that

$$\left[U_{\rm IP}^{(1)}(t,t_0)\right]^\dagger = -U_{\rm IP}^{(1)}(t,t_0). \tag{B17}$$

Using the linearity of the average (B6), the expansion in Eqs. (B15)–(B17), and the Maclaurin series of $1/(1+x)$, it is straightforward to show that

$$\overline{H_I(t)U_{\rm IP}(t,t_0)} = \overline{H_I(t)} + \sum_{n=1}^{+\infty}\overline{H_I(t)U_{\rm IP}^{(n)}(t,t_0)}, \tag{B18}$$

and

$$\begin{aligned}[\overline{U_{\rm IP}(t,t_0)}]^{-1} &= \left[\mathbb{I} + \sum_{n=1}^{+\infty}\overline{U_{\rm IP}^{(n)}(t,t_0)}\right]^{-1}\\ &= \mathbb{I} - \overline{U_{\rm IP}^{(1)}(t,t_0)} - \overline{U_{\rm IP}^{(2)}(t,t_0)} - \overline{U_{\rm IP}^{(3)}(t,t_0)}\\ &\quad + \left[\overline{U_{\rm IP}^{(1)}(t,t_0)}\right]^2 + \left[\overline{U_{\rm IP}^{(1)}(t,t_0)}\right]\left[\overline{U_{\rm IP}^{(2)}(t,t_0)}\right]\\ &\quad + \left[\overline{U_{\rm IP}^{(2)}(t,t_0)}\right]\left[\overline{U_{\rm IP}^{(1)}(t,t_0)}\right] - \left[\overline{U_{\rm IP}^{(1)}(t,t_0)}\right]^3\\ &\quad + \cdots.\end{aligned} \tag{B19}$$

Here, and in the following, the dots $\cdots$ indicate terms of order $\geqslant 4$ in $H_I(t)$. Then, it follows from Eqs. (B13), (B18), and (B19) that

$$\begin{aligned}\mathsf{H}_{\rm avg}(t) &= \overline{H_I(t)}\Big\{\mathbb{I} - \overline{U_{\rm IP}^{(1)}(t,t_0)} - \overline{U_{\rm IP}^{(2)}(t,t_0)}\\ &\quad + \left[\overline{U_{\rm IP}^{(1)}(t,t_0)}\right]^2\Big\} + \overline{H_I(t)U_{\rm IP}^{(1)}(t,t_0)}\\ &\quad - \left[\overline{H_I(t)U_{\rm IP}^{(1)}(t,t_0)}\right]\overline{U_{\rm IP}^{(1)}(t,t_0)}\\ &\quad + \overline{H_I(t)U_{\rm IP}^{(2)}(t,t_0)} + \cdots.\end{aligned} \tag{B20}$$

Observe from Eq. (B20) that it is advantageous to choose (when possible) H_0 such that

$$\overline{H_I(t)} = 0, \tag{B21}$$

because all terms between the curly brackets disappear. In particular, all the terms of order 1 in $H_I(t)$ are eliminated. In the following, we assume that (B21) holds.

Neglecting terms of order $\geqslant 4$ in $H_I(t)$, it follows from Eqs. (B14), (B20), and (B21) that

$$H_{\rm avg}(t) \simeq H^{(2)}_{\rm avg}(t) + H^{(3)}_{\rm avg}(t), \tag{B22}$$

where $H^{(j)}_{\rm avg}(t)$ includes the terms of order j ($j = 2, 3$) in $H_I(t)$ and is given by

$$\begin{aligned} H^{(2)}_{\rm avg}(t) &= \tfrac{1}{2}\overline{[H_I(t),\ U^{(1)}_{\rm IP}(t,t_0)]},\\ H^{(3)}_{\rm avg}(t) &= \tfrac{1}{2}\Big\{\overline{H_I(t)U^{(2)}_{\rm IP}(t,t_0)} + \Big[\overline{H_I(t)U^{(2)}_{\rm IP}(t,t_0)}\Big]^\dagger\\ &\quad - \Big[\overline{H_I(t)U^{(1)}_{\rm IP}(t,t_0)}\Big]\ \overline{U^{(1)}_{\rm IP}(t,t_0)}\\ &\quad - \overline{U^{(1)}_{\rm IP}(t,t_0)}\ \Big[\overline{U^{(1)}_{\rm IP}(t,t_0)H_I(t)}\Big]\Big\}. \end{aligned} \tag{B23}$$

We now apply the averaging method with the same notation as that used above. Consider the Hamiltonian

$$H = H_0 + H_{\rm int}, \tag{B24}$$

where $H_{\rm int}$ is defined in Eq. (56) and

$$\frac{1}{\hbar}H_0 = \delta_p a_p^\dagger a_p + \delta_s a_s^\dagger a_s + \sum_{j=2,3}\omega_j S_{jj} + \frac{\omega_d}{2}S_{11}. \tag{B25}$$

Observe that H in Eq. (B24) coincides with H_I in Eq. (52) when one takes $\Omega_d = 0$. We take $\Omega_d = 0$ because we want to obtain an approximation for the *qutrits-signal* and *pump-signal* interactions described by $H_{\rm int}$.

First, pass to the IP by means of the unitary transformation in Eq. (B1) with H_0 in Eq. (B25). Then, the IP Schrödinger equation (B3) has the Hamiltonian

$$\begin{aligned} \frac{1}{\hbar}H_I(t) &= \frac{1}{\hbar}U_I^\dagger(t)H_{\rm int}U_I(t),\\ &= J\big[a_p(a_s^\dagger)^2e^{i\zeta_1 t} + a_p^\dagger a_s^2 e^{-i\zeta_1 t}\big]\\ &\quad + a_s^\dagger\sum_{j=2,3} g_j S_{1j}e^{i\zeta_j t} + a_s\sum_{j=2,3} g_j S_{1j}^\dagger e^{-i\zeta_j t},\\ &= \frac{1}{\hbar}\sum_{j=1,2,3}\big(F_j e^{-i\zeta_j t} + F_j^\dagger e^{i\zeta_j t}\big). \end{aligned} \tag{B26}$$

Here, we have introduced the operators and the detunings

$$\begin{aligned} &F_1 = \hbar J a_p^\dagger a_s^2, \qquad F_j = \hbar g_j a_s S_{1j}^\dagger,\\ &\zeta_1 = 2\omega_s - \omega_p, \quad \zeta_j = \omega_s - \omega_j \quad (j = 2, 3). \end{aligned} \tag{B27}$$

The form of $H_I(t)$ in the last line of Eq. (B26) is convenient to carry out the lengthy calculations of the method.

In the following, assume that Eqs. (59) and (60) hold. Then, $\zeta_1 \geqslant \omega_s$ and $\zeta_j \sim \omega_s$ for $j = 2, 3$.

Consider a cutoff frequency $\omega_0 > 0$ such that

$$\begin{aligned} |\zeta_j + \zeta_k - \zeta_1| = |\omega_j + \omega_k - \omega_p| < \omega_0 \quad (j, k = 2, 3),\\ |\zeta_3 - \zeta_2| = |\omega_3 - \omega_2| < \omega_0, \end{aligned} \tag{B28}$$

and

$$\begin{aligned} n\zeta_{l_1},\ \zeta_2 + \zeta_3,\ \zeta_1 \pm \zeta_j, \zeta_{l_1} + \zeta_{l_2} + \zeta_{l_3} > \omega_0,\\ \big|\zeta_{l_1} + \zeta_{l_2} - \zeta_{l_3}\big| > \omega_0, \end{aligned} \tag{B29}$$

for $l_1, l_2, l_3 = 1, 2, 3$ and $n = 1, 2$ and $j, k = 2, 3$ with $|\zeta_{l_1} + \zeta_{l_2} - \zeta_{l_3}|$ not of the form $|\zeta_j + \zeta_k - \zeta_1|$. The assumption in Eq. (60) guarantees that the quantities that appear on the left-hand side of the inequalities in Eq. (B29) are $\gtrsim \omega_s$, while the quantities that appear on the left-hand side of the inequalities in Eq. (B28) are $\ll \omega_s$.

It is important to note that Eq. (59) must hold in order to consider $H_I(t)$ a perturbation. The reason for this is that, when one expresses the IP Schrödinger equation in terms of the dimensionless time $\tau = \omega_s t$, all the parameters J, g_2, and g_3 in $H_I(t)$ are divided by ω_s and one requires the quotients J/ω_s, g_2/ω_s, and g_3/ω_s to be small so that high-frequency terms average to zero.

Applying the averaging method with $t_0 = 0$ and with Eqs. (B28) and (B29) leads to $\overline{H_I(t)} = 0$ and, consequently, to the following approximation to third order in $H_I(t)$:

$$H_I(t) \simeq H^{(2)}_{\rm avg}(t) + H^{(3)}_{\rm avg}(t), \tag{B30}$$

where $H^{(2)}_{\rm avg}(t)$ and $H^{(3)}_{\rm avg}(t)$ are given by Eq. (B23).

Returning to the original picture, one obtains from Eqs. (B26) and (B30) that

$$\begin{aligned} H_{\rm int} &= U_I(t)H_I(t)U_I^\dagger(t),\\ &\simeq U_I(t)\big[H^{(2)}_{\rm avg}(t) + H^{(3)}_{\rm avg}(t)\big]U_I^\dagger(t). \end{aligned} \tag{B31}$$

Carrying out the calculations and taking the expected value in the state $|0_s\rangle$ leads to Eq. (61).

APPENDIX C: APPROXIMATE EVOLUTION

In this Appendix, we show that Eq. (76) holds if Eq. (68) and the assumptions of the first (53) and second (62) adiabatic approximations are satisfied [see also the paragraphs below Eqs. (53) and (62)].

To have succinct expressions, we introduce the density operators of the pump+qutrits subsystem

$$\rho^{(\jmath)}_{pq} = |\alpha_{po}\rangle\langle\alpha_{po}| \otimes |\jmath_L\rangle\langle\jmath_L| \quad (\jmath = 0, 1), \tag{C1}$$

where $|0_L\rangle$ and $|1_L\rangle$ are defined in Eq. (12) and $|\alpha_{po}\rangle$ is a coherent state with α_{po} in Eq. (75).

In all that follows, it is used without mention that $|0_L\rangle$ and $|1_L\rangle$ satisfy the properties in Eq. (15), that $|0_s\rangle\langle 0_s|$ is the steady-state solution of the damped harmonic oscillator master equation at zero temperature in Eq. (54), and that $|\alpha_{po}\rangle\langle\alpha_{po}|$ is the steady-state solution of the driven and damped harmonic oscillator master equation at zero temperature in Eq. (63) with δ_p and κ_p replacing δ_p'' and κ_p', respectively.

First, it is straightforward to show from Eqs. (51), (73), and (74) that one has

$$\mathcal{L}_I\rho_\jmath = -i\sqrt{2}J[\alpha_{po}|2_s\rangle\langle 0_s| - \alpha^*_{po}|0_s\rangle\langle 2_s|] \otimes \rho^{(\jmath)}_{pq}, \tag{C2}$$

and that

$$\begin{aligned} \mathcal{L}_I|0_s\rangle\langle 2_s| \otimes \rho^{(0)}_{pq} &= \big[(-\kappa_s + i2\delta_s)|0_s\rangle\langle 2_s| + R^{(0)}_s\big] \otimes \rho^{(0)}_{pq}\\ &\quad + i\sqrt{2}J|0_s\rangle\langle 0_s| \otimes \rho^{(0)}_{pq}a_p,\\ \mathcal{L}_I|0_s\rangle\langle 2_s| \otimes \rho^{(1)}_{pq} &= \big[(-\kappa_s + i2\delta_s)|0_s\rangle\langle 2_s| + R^{(0)}_s\big] \otimes \rho^{(1)}_{pq}\\ &\quad + i\sqrt{2}J|0_s\rangle\langle 0_s| \otimes \rho^{(1)}_{pq}a_p\\ &\quad + i\sqrt{2}g|0_s\rangle\langle 1_s| \otimes \rho^{(1)}_{pq}S_-, \end{aligned} \tag{C3}$$

where

$$R^{(0)}_s = -i\sqrt{2}J\alpha_{po}|2_s\rangle\langle 2_s| + i\sqrt{12}J\alpha^*_{po}|0_s\rangle\langle 4_s|. \tag{C4}$$

Now, let us compare the magnitude of the coefficients of the terms appearing in Eqs. (C3) and (C4):

$$\left|\frac{i\sqrt{2}J}{-\kappa_s+i2\delta_s}\right| \leqslant \sqrt{2}\epsilon_1\ , \quad \left|\frac{i\sqrt{2}g}{-\kappa_s+i2\delta_s}\right| \leqslant \sqrt{2}\epsilon_1,$$
$$\left|\frac{-i\sqrt{2}J\alpha_{po}}{-\kappa_s+i2\delta_s}\right| \lesssim 2\sqrt{2}\epsilon_1, \quad \left|\frac{i\sqrt{12}J\alpha^*_{po}}{-\kappa_s+i2\delta_s}\right| \lesssim 2\sqrt{12}\epsilon_1, \quad \text{(C5)}$$

where ϵ_1 is defined in Eq. (53) and where $\lesssim$ appears instead of $\leqslant$ because the second adiabatic approximation requires $|\Omega_d/\kappa_p| \sim 1$ [see the paragraph below Eq. (62)]. Since the first adiabatic approximation requires $\epsilon_1 \ll 1$, to good approximation one can neglect all the terms appearing on the right-hand side of Eq. (C3) except for the first one:

$$\mathcal{L}_I|0_s\rangle\langle 2_s|\otimes\rho^{(\mathfrak{j})}_{pq} \simeq (-\kappa_s+i2\delta_s)|0_s\rangle\langle 2_s|\otimes\rho^{(\mathfrak{j})}_{pq}. \quad \text{(C6)}$$

Since

$$\mathcal{L}_I|2_s\rangle\langle 0_s|\otimes\rho^{(\mathfrak{j})}_{pq} = \left[\mathcal{L}_I|0_s\rangle\langle 2_s|\otimes\rho^{(\mathfrak{j})}_{pq}\right]^\dagger, \quad \text{(C7)}$$

from Eqs. (C2) and (C6), one can obtain Eq. (76).

APPENDIX D: PARAMETERS OF THE MODEL

In this Appendix, we list the effective parameters appearing in the model, group them according to the approximation performed, and give a physical interpretation.

(1) Original model pump field-driving detuning and signal field-driving detuning

$$\delta_p = \omega_p - \omega_d, \quad \delta_s = \omega_s - \frac{\omega_d}{2}.$$

(2) First adiabatic approximation

(a) Rate appearing in S

$$\kappa'_s = \sqrt{\left(\frac{\kappa_s}{2}\right)^2 + \delta_s^2}.$$

(b) Effective pump field damping rate

$$\kappa'_p = \kappa_p + \kappa_s\left(\frac{J}{\kappa'_s}\right)^2.$$

(c) Shifted pump field-driving detuning

$$\delta'_p = \delta_p - \delta_s\left(\frac{J}{\kappa'_s}\right)^2.$$

(3) Averaging.

(a) Signal field-pump field detuning

$$\Delta_s = 2\omega_s - \omega_p.$$

(b) Detuning between the signal field frequency and the atomic transition frequency between levels $|j\rangle$ and $|1\rangle$ ($j = 2, 3$)

$$\Delta_j = \omega_s - \omega_j.$$

(c) Effective pump field-atoms couplings for the simultaneous transition of two atoms from level $|j\rangle$ to $|1\rangle$ and vice versa ($j = 2, 3$)

$$\chi_j = g_j^2 J\left[\frac{1}{2\Delta_j^2} + \frac{1}{\Delta_s(\Delta_s-\Delta_j)}\right],$$

and from levels $|2\rangle$ and $|3\rangle$ to $|1\rangle$ and vice versa

$$\chi_{23} = g_2g_3J\sum_{j=2,3}\left[\frac{1}{\Delta_j(\Delta_2+\Delta_3)} + \frac{1}{\Delta_s(\Delta_s-\Delta_j)}\right].$$

(d) Effective coupling for the atomic transition between levels $|2\rangle$ and $|3\rangle$

$$g_{23} = \frac{g_2g_3}{2}\left(\frac{1}{\Delta_2} + \frac{1}{\Delta_3}\right).$$

(4) Second adiabatic approximation

(a) Effective harmonic oscillator frequency

$$\delta''_p = \delta'_p - 2\frac{J^2}{\Delta_s}.$$

(b) Coherent state parameter

$$\alpha_p = \frac{\Omega_d}{-\delta''_p + i\left(\frac{\kappa'_p}{2}\right)}.$$

(c) Shifted atomic transition frequency between levels $|j\rangle$ and $|1\rangle$ ($j = 2, 3$)

$$\xi_j = \omega_j - g_j^2\left[\frac{1}{\Delta_j} + \frac{\delta_s}{(\kappa'_s)^2}\right].$$

(d) Effective coupling for the atomic transition between levels $|2\rangle$ and $|3\rangle$

$$\xi_{23} = g_2g_3\left[\frac{1}{2}\left(\frac{1}{\Delta_2} + \frac{1}{\Delta_3}\right) + \frac{\delta_s}{(\kappa'_s)^2}\right].$$

(e) Rate appearing in S_q

$$\kappa''_p = \sqrt{\left(\frac{\kappa'_p}{2}\right)^2 + (\delta''_p)^2}.$$

(f) Two-atom driving strength

$$\alpha_q = \alpha_p\kappa''_p.$$

(5) The final model ($g_2 = g_3 = g > 0,\ \omega_2 = \omega_3 = \omega_q$)

(a) Signal field-qutrit detuning

$$\Delta = \omega_s - \omega_q.$$

One has

$$\Delta_2 = \Delta_3 = \Delta.$$

(b) Driving-qutrit detuning

$$\delta_1 = \frac{\omega_d}{2} - \omega_q.$$

(c) Effective pump field-atoms coupling

$$\chi = g^2J\left[\frac{1}{2\Delta^2} + \frac{1}{\Delta_s(\Delta_s-\Delta)}\right].$$

One has

$$\chi_2 = \chi_3 = \tfrac{1}{2}\chi_{23} = \chi.$$

(d) Effective single-atom and two-atom decay rates

$$\kappa_1 = \frac{\kappa_s g^2}{(\kappa'_s)^2}, \quad \kappa_2 = \frac{\kappa'_p\chi^2}{(\kappa''_p)^2}.$$

TABLE I. Table indicating, for each one of the stationary states constructed in this paper and in Ref. [9], if they remain stationary (✓) or not (×) under each one of the considered perturbations. The superscripts refer to the numbered explanations in Appendix E. For the details on states $|\mathcal{C}_\pm\rangle$, see Ref. [9].

State (Eq.)	$\mathcal{L}$ (5)	$\mathcal{L}+\mathcal{L}_{\rm cd}$ (23)	$\mathcal{L}+\mathcal{L}_{\rm ud}$ (23)	$\mathcal{L}+\mathcal{L}_{\rm loc}$ (43)
$\lvert 0_L\rangle$ (12)	✓	✓	$\times^1$	✓
$\lvert 1_L\rangle$ (12)	✓	✓	$\times^2$	$\times^5$
$\lvert 1_L^A\rangle$ (45)	✓	✓	$\times^3$	$\times^6$
$\lvert \tilde{1}_L\rangle$ (46)	✓	✓	$\times^4$	✓
$\lvert \mathcal{C}_+\rangle$ [9]	✓	–	×	×
$\lvert \mathcal{C}_-\rangle$ [9]	✓	–	×	×

(e) Effective two-atom driving strength

$$\alpha_0 = \alpha_p|\chi|.$$

One has

$$\alpha_q = \alpha_0\sqrt{\frac{\kappa_p'}{\kappa_2}}.$$

(f) Coupling for the atomic quadratic term

$$\xi = g^2\left[\frac{1}{\Delta} + \frac{\delta_s}{(\kappa_s')^2}\right].$$

One has

$$\xi_{23} = \xi, \quad \xi_2 = \xi_3 = \omega_q - \xi.$$

(g) Coupling for the atomic quartic term

$$\delta = -\delta_p''\left(\frac{\kappa_2}{\kappa_p'}\right).$$

(h) Effective operators

$$S = \sqrt{\frac{\kappa_1}{\kappa_s}}S_-, \quad S_q = \sqrt{\frac{\kappa_2}{\kappa_p'}}S_-^2.$$

APPENDIX E: SUMMARY TABLE OF STATIONARY STATES

In this Appendix, we collect in Table I all stationary states obtained in this paper and all possible perturbation Liouvillians indicating if they are stationary under them. We also discuss, in the case of failure, their evolution for small times.

Consider the Hilbert space $\mathcal{H}$ that can be decomposed into the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, with P_1 and P_2 the corresponding orthogonal projectors verifying $P_1 + P_2 = I_{\mathcal{H}}$. Then, the space of bounded operators $B(\mathcal{H})$ decomposes into four orthogonal subspaces.

Define the map $\mathbb{P}_{1,2} : B(\mathcal{H}) \to \mathcal{M}_{2\times 2}$ given by

$$\mathbb{P}_{1,2}(A) = \begin{pmatrix} ||P_1AP_1|| & ||P_1AP_2|| \\ ||P_2AP_1|| & ||P_2AP_2|| \end{pmatrix}, \tag{E1}$$

with $||\cdot||$ the Frobenius norm. The matrix $\mathbb{P}_{1,2}(A)$ visualizes the size of the different projections of A into each one of the four subspaces into which $B(\mathcal{H})$ divides.

Consider a Liouvillian $\mathcal{L}_T = \mathcal{L} + \mathcal{L}_p$ on $\mathcal{H}$. Then, for a given initial density matrix $\rho(0)$ stationary under $\mathcal{L}$, its evolution for small times is given by

$$\begin{aligned}\rho(t) &= e^{t\mathcal{L}_T}\rho(0), \\ &= \rho(0) + t\mathcal{L}_p[\rho(0)] + \tfrac{1}{2}t^2\mathcal{L}_T[\mathcal{L}_p[\rho(0)]] + \cdots.\end{aligned} \tag{E2}$$

Denote $\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{(1)}_{1,2} = \mathbb{P}_{1,2}[\mathcal{L}_p[\rho(0)]]$ and $\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{(2)}_{1,2} = \mathbb{P}_{1,2}\mathcal{L}_T[\mathcal{L}_p[\rho(0)]]$, which provide information about the size of the projections onto each one of the four subspaces for the first- and second-order terms in the expansion (E2). The entries a, b, c, d will be either 0 or $*$, the last case indicating a nonzero value. For $m \geqslant 0$, $*^m$ indicates that the nonzero value is of order m in the coupling constants of the associated Liouvillians.

When we consider the Liouvillian $\mathcal{L}$ given by Eq. (5) and the perturbations $\mathcal{L}_{\rm cd}$ and $\mathcal{L}_{\rm ud}$ given by Eq. (23), the Hilbert space will be that of the symmetric representation of N qutrits, as discussed in Sec. II. In this case, the Hilbert space is decomposed into even and odd subspaces under the parity operator given in Eq. (8), with associated projection operators $P_{\rm even}$ and $P_{\rm odd}$.

When we consider the Liouvillian $\mathcal{L}$ and the perturbation $\mathcal{L}_{\rm loc}$ given in Eq. (43) (Sec. VI D), the Hilbert space is the tensor product of N copies of the one-qutrit Hilbert space. We can consider two kinds of decompositions: the extension of the even and odd subspaces to the tensor product space with the corresponding projection operators $P^{\rm TP}_{\rm even}$ and $P^{\rm TP}_{\rm odd}$, and the superradiant (symmetric) and subradiant (direct sum of nonsymmetric) subspaces, with the corresponding projection operators $P^{\rm TP}_{\rm sup}$ and $P^{\rm TP}_{\rm sub}$.

When we consider the stationary states $|1^A_L\rangle$ and $|\tilde{1}_L\rangle$, the Hilbert space will be always the tensor product Hilbert space.

In the following, $\Gamma = |\Gamma_2 + \Gamma_3|$ and $\delta\omega = |\delta\omega_2 - \delta\omega_1|$.

(1) $\Gamma\begin{pmatrix} *^0 & 0 \\ 0 & 0\end{pmatrix}^{(1)}_{\rm even,odd}$, $\Gamma\begin{pmatrix} *^1 & 0 \\ 0 & *^1\end{pmatrix}^{(2)}_{\rm even,odd}$, there is dephasing only in the even subspace at first order, but in the even and in the odd subspaces at second order in time. This explains why F_0^∞ is very small, while F_1^∞ is relatively large in Fig. 2(a).

(2) $\Gamma\begin{pmatrix} 0 & 0 \\ 0 & *^0\end{pmatrix}^{(1)}_{\rm even,odd}$, $\Gamma\begin{pmatrix} 0 & 0 \\ 0 & *^1\end{pmatrix}^{(2)}_{\rm even,odd}$, there is dephasing only in the odd subspace at first and second order in time. This explains why F_0^∞ is very small, while F_1^∞ is relatively large in Fig. 2(b).

(3) $\Gamma\begin{pmatrix} 0 & 0 \\ 0 & *^0\end{pmatrix}^{(1)}_{\rm even,odd}$, $\Gamma\begin{pmatrix} 0 & 0 \\ 0 & *^1\end{pmatrix}^{(2)}_{\rm even,odd}$, there is dephasing only in the odd subspace at first and second order in time. With respect to superradiant and subradiant subspaces, it remains always in the subradiant subspace.

(4) $\Gamma\begin{pmatrix} 0 & 0 \\ 0 & *^0\end{pmatrix}^{(1)}_{\rm even,odd}$, $\Gamma\begin{pmatrix} 0 & 0 \\ 0 & *^1\end{pmatrix}^{(2)}_{\rm even,odd}$, there is dephasing only in the odd subspace at first and second order in time. With respect to superradiant and subradiant subspaces, $\Gamma\begin{pmatrix} *^0 & *^0 \\ *^0 & *^0\end{pmatrix}^{(1)}_{\rm sup,sub}$, $\Gamma\begin{pmatrix} *^1 & *^1 \\ *^1 & *^1\end{pmatrix}^{(2)}_{\rm sup,sub}$, there is dephasing into the four subspaces at first and second order in time.

(5) For inhomogeneous broadening: $\delta\omega\begin{pmatrix} 0 & *^0 \\ *^0 & 0\end{pmatrix}^{(1)}_{\rm sup,sub}$, $\delta\omega\begin{pmatrix} *^1 & 0 \\ 0 & *^1\end{pmatrix}^{(2)}_{\rm sup,sub}$, there is a periodic flux between the superradiant and subradiant subspaces at first order and dephasing in the superradiant and in the subradiant subspaces at second order in time. For local dephasing: $\Gamma_D\begin{pmatrix} *^0 & 0 \\ 0 & *^0\end{pmatrix}^{(1)}_{\rm sup,sub}$,

$\Gamma_D \begin{pmatrix} *^1 & 0 \\ 0 & *^1 \end{pmatrix}^{(2)}_{\mathrm{sup,sup}}$, there is dephasing in the superradiant and in the subradiant subspaces at first and second order in time.

(6) For both inhomogeneous broadening and local dephasing, the effect is the same as that of $|1_L\rangle$, but with opposite signs in such a way that they cancel in $|\tilde{1}_L\rangle$.

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