

Quantum Lamb model

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H. Lamb considered the classical dynamics of a vibrating particle embedded in an elastic medium before the development of quantum theory. Lamb was interested in how the back action of the elastic waves generated can damp the vibrations of the particle. We propose a quantum version of Lamb's model. We show that this model is exactly solvable by using a multimode Bogoliubov transformation. We find that the exact system ground state is a multimode squeezed-vacuum state, and we obtain the exact Bogoliubov frequencies by numerically solving a nonlinear integral equation. A closed-form expression for the damping rate of the particle is obtained, and it agrees with the result obtained by perturbation theory. The model provides a solvable example of the damped quantum harmonic oscillator.

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Introduction. Advances in the fabrication and characterization of simple mechanical systems in the nanoscopic and mesoscopic regime have facilitated experimental and theoretical investigations [1,2] into some of the foundational principles of quantum mechanics. Prominent examples of such systems include vibrating beams and mirrored surfaces that interact with laser light through its radiation pressure (optomechanics) [3], mechanical resonators coupled to electronic devices (nanoelectromechanics) [4,5], and interacting mechanical resonators (quantum acoustodynamics) [6–8]. In addition to providing a path to explore quantum science and the limits of precision measurement, such systems might be used to fashion new quantum sensors and devices for manipulating quantum information [9,10].

We consider a mechanical system whose first study predates the development of quantum mechanics. In 1900, Lamb [11] considered the dynamics of a vibrating particle embedded in an elastic medium. The back action of the elastic waves generated by the vibrations of the particle work to damp those vibrations creating a damped harmonic oscillator. In this work, we study a quantum version of Lamb's model and focus on the dynamics of the vibrational decay. Figure 1 shows a schematic consisting of a vibrating bead coupled by a spring to a long string under tension that serves as the classical basis of the model.

There have been other formulations of the damped quantum harmonic oscillator. Feshbach and Tikochinsky [12] introduced an auxiliary variable into the Lagrangian of a harmonic oscillator to get the desired effective equation of motion for the damped oscillator. They then proceeded by canonical quantization to obtain a quantum description of the damped harmonic oscillator. The auxiliary variable presumably func-

tions as a single, effective environmental degree of freedom, but the connection to the microscopic physics is not made.

Caldeira and Leggett [13] separate the system into a sum of two subsystems (oscillator and bath) plus an interaction. Using a path-integral description, the bath degrees of freedom can be integrated out to give a general quantum formulation of dissipative systems. Yurke [14] specifically considered a Lamb-type model that is a special case of the model considered here. (We will recover Yurke's results by allowing the spring that couples the bead motion to the string to be suitably stiff.) Yurke considered a string with a point mass at one end. The point mass is also coupled to a spring with a fixed end. The mass-loaded string then has a time-dependent boundary condition. As a result, the normal modes are nonorthogonal. Yurke overcame this by finding an appropriate weighting factor to use in redefining the inner product so that generalized orthogonality can be applied. He then quantized the model in the standard way.

Following Caldeira and Leggett [13], the model considered in this work expresses the Lamb Hamiltonian as a sum of two subsystems (oscillator and string) plus a coupling term. Since the coupling is bilinear in operators, the Hamiltonian is exactly diagonalizable with the use of a multimode Bogoliubov transformation. We find explicit expressions for the coefficients that diagonalize the Hamiltonian. Using the symplectic properties of the transformation, we confirm that our results satisfy the necessary identities. We then derive a nonlinear equation whose solution yields the Bogoliubov frequencies, and we use it to numerically calculate the symplectic spectrum of the model.

We show that the ground state of the quantum Lamb model is a nonclassical state—a multimode squeezed-vacuum state—and we relate this ground state to the uncoupled states (transverse phonons of the string and vibrons of the bead) of the system. Squeezed states can serve as a quantum resource for precision-sensing applications; for example, gravitational-wave detection relies on squeezing to perform displacement measurements where uncertainty in momentum is sacrificed in favor of reduced uncertainty in position. Caldeira and

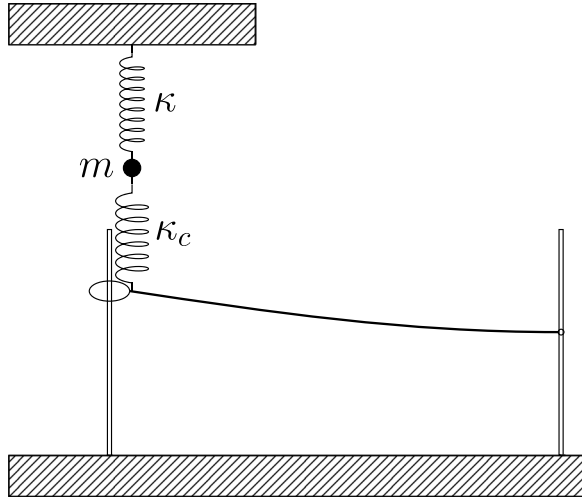


FIG. 1. Schematic of a generalization of the classical Lamb model. Bead of mass m at $x = 0$ is constrained to move in the vertical direction. The vibrating bead is coupled by a spring to a long string under tension τ . The vibrating bead creates transverse acoustic waves on the string ($\ell \gg c/\omega_0$). The bead subsequently undergoes damped harmonic motion.

Leggett [13] found, in their studies of the damped quantum oscillator, the uncertainty in position for the ground state is reduced with increasing damping, strongly so in the over-damped regime. This result is consistent with a squeezed ground state.

We then study the dynamics of the vibrational decay of the bead. Dissipation emerges in the thermodynamic limit where the number of string modes N becomes infinitely large ($N \rightarrow \infty$). In this limit, vibrational energy of bead can be radiated away. We then obtain an explicit expression for the decay rate, and we calculate the spectral distribution of single bogoliubon emission, a product of the bead decay. We show that for weak coupling strength g , the decay rate calculated agrees with both the classical result and the golden rule.

Hamiltonian. The Hamiltonian of the system in Fig. 1 is

$$H = \sum_{\alpha=0}^N \omega_{\alpha} a_{\alpha}^{\dagger} a_{\alpha} - (a_0 + a_0^{\dagger}) \sum_{n=1}^N \gamma_n (a_n + a_n^{\dagger}), \quad (1)$$

where $a_n^{\dagger}$ (a_n) creates (annihilates) a transverse acoustic phonon on the string, and $a_0^{\dagger}$ (a_0) creates (annihilates) a vibron on the bead. [We use index notation where (greek) $\alpha = 0, 1, 2, \dots$, while (roman) $n = 1, 2, \dots, N$, and work with natural units where $\hbar = 1$.] N is the number of vibrational modes for the string. We are ultimately interested in the limit $N \rightarrow \infty$ in order to obtain a description of the damped bead oscillator.

The frequency $\omega_0 = \sqrt{(\kappa + \kappa_c)/m} = \sqrt{\omega_b^2 + \omega_c^2}$ is the bead vibrational frequency with the string fixed at $x = 0$, while ω_n are the vibrational frequencies of the string (tension τ , length ℓ , mass density σ , transverse speed of sound c) subject to a spring boundary condition at $x = 0$ and a fixed condition at $x = \ell$.

The coupling parameters γ_n can be expressed in terms of physical parameters of the model [15]

$$\gamma_n = \omega_s \sqrt{\frac{\nu}{\omega_0}} \sqrt{\frac{k_n \ell}{(k_n \ell)^2 + (k_s \ell)^2}} \frac{1}{\sqrt{1 - \frac{k_s \ell}{(k_n \ell)^2 + (k_s \ell)^2}}}, \quad (2)$$

where $\omega_s = ck_s = \frac{c\kappa_c}{\tau}$ and $\nu = \frac{\tau}{2mc}$. The wave number k_n is a solution of the transcendental equation $\tan k_n \ell = -\frac{\tau}{\kappa_c} k_n$.

Bogoliubov transformation. We diagonalize the Hamiltonian in Eq. (1) with the use of a multimode Bogoliubov transformation. We look for a linear transformation (and its inverse) with the following form:

$$b_{\alpha} = \sum_{\beta} (M_{\alpha\beta} a_{\beta} + N_{\alpha\beta} a_{\beta}^{\dagger}), \quad (3)$$

$$a_{\alpha} = \sum_{\beta} (U_{\alpha\beta} b_{\beta} + V_{\alpha\beta} b_{\beta}^{\dagger}), \quad (4)$$

where $b_{\alpha}^{\dagger}$ (b_{α}) creates (destroys) a Bogoliubov excitation (bogoliubon) and M, N, U , and V are $(N+1)$ -dimensional square matrices whose elements are the coefficients of the transformation. (The string with length ℓ is a system of N discrete atoms.)

We require that the transformation preserve the boson commutation rules $[b_{\alpha}, b_{\beta}^{\dagger}] = \delta_{\alpha\beta}$. As a result [16,17], the coefficients can be grouped to form a $2(N+1)$ -dimensional symplectic matrix $T \in \text{Sp}(2(N+1), \mathbb{R})$:

$$T \equiv \begin{pmatrix} M & N \\ N & M \end{pmatrix}. \quad (5)$$

T then satisfies the symplectic condition $TJT^T = J$ where the symplectic form J can be represented as

$$J \equiv \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \quad (6)$$

A number of useful coefficient identities follow [15] from the symplectic structure on this Fock space; for example, the inverse of the transformation matrix T can be obtained simply from the symplectic condition (together with $J^2 = -\mathbb{I}$):

$$T^{-1} = -JT^T J \quad (7)$$

$$= \begin{pmatrix} M^T & -N^T \\ -N^T & M^T \end{pmatrix}. \quad (8)$$

Hence, we conclude that the coefficients of the inverse transformation satisfy $U = M^T$ and $V = -N^T$. We summarize the explicit form for the transformation in Table I. The detailed calculation of the coefficients is outlined in the Supplemental Material [15].

The following Hamiltonian results

$$H = \sum_{\alpha} \Omega_{\alpha} b_{\alpha}^{\dagger} b_{\alpha}, \quad (9)$$

where the Bogoliubov frequencies $\{\Omega_{\alpha}\}$ satisfy the following summation equation

$$\Omega_{\alpha}^2 = \omega_0^2 + 4\omega_0 \sum_q \frac{\gamma_q^2 \omega_q}{\Omega_{\alpha}^2 - \omega_q^2}. \quad (10)$$

TABLE I. Coefficients for the Bogoliubov transformation $M_{\alpha\beta}$ and $N_{\alpha\beta}$ ($\alpha, \beta = 0, 1, \dots, N$ and $k, q = 1, 2, \dots, N$). Coefficients for the inverse transformation can be obtained from the transposed relations $U_{\alpha\beta} = M_{\beta\alpha}$ and $V_{\alpha\beta} = -N_{\beta\alpha}$ (see Supplemental Material [15]).

	$(\alpha 0)$	(αk)
M	$\frac{\Omega_\alpha + \omega_0}{\sqrt{4\omega_0\Omega_\alpha}} \frac{1}{\sqrt{1+4\omega_0 \sum_q \frac{\gamma_q^2 \omega_q}{(\Omega_\alpha^2 - \omega_q^2)^2}}}$	$-\frac{2\omega_0 \gamma_k}{(\Omega_\alpha - \omega_k)} \frac{1}{\sqrt{4\omega_0\Omega_\alpha}} \frac{1}{\sqrt{1+4\omega_0 \sum_q \frac{\gamma_q^2 \omega_q}{(\Omega_\alpha^2 - \omega_q^2)^2}}}$
N	$\frac{\Omega_\alpha - \omega_0}{\sqrt{4\omega_0\Omega_\alpha}} \frac{1}{\sqrt{1+4\omega_0 \sum_q \frac{\gamma_q^2 \omega_q}{(\Omega_\alpha^2 - \omega_q^2)^2}}}$	$-\frac{2\omega_0 \gamma_k}{(\Omega_\alpha + \omega_k)} \frac{1}{\sqrt{4\omega_0\Omega_\alpha}} \frac{1}{\sqrt{1+4\omega_0 \sum_q \frac{\gamma_q^2 \omega_q}{(\Omega_\alpha^2 - \omega_q^2)^2}}}$

By using the pole-expansion form (Mittag-Leffler) for cotangent, it is straightforward to show that in the thermodynamic limit ($\ell, N \rightarrow \infty$ and $\frac{N}{\ell} \rightarrow \frac{\omega_d}{\pi c}$), Eq. (10) gives Yurke's transcendental equation [14] for the special coupling case of $\omega_c = \omega_c^* \equiv \sqrt{\frac{4}{\pi}} \nu \omega_d$:

$$\Omega_\alpha^2 = \omega_b^2 + 2\nu\Omega_\alpha \cot \frac{\Omega_\alpha \ell}{c}. \quad (11)$$

The frequency ω_d is recognized as the Debye frequency of the string (the high-frequency cutoff for the string).

We note that the Hamiltonian is no longer positive definite when the lowest Bogoliubov frequency vanishes. Thus, there is a constraint on the model; namely, using Eq. (10), we see that the following condition must be satisfied to prevent an instability:

$$\frac{4}{\omega_0} \sum_n \frac{\gamma_n^2}{\omega_n} < 1. \quad (12)$$

We define the coupling strength $g \equiv \frac{4}{\omega_0} \sum_n \frac{\gamma_n^2}{\omega_n}$ and from Eq. (12) conclude that there is a critical coupling strength $g_c = 1$ above which the model is ill defined. Using the form for γ_n in Eq. (2), we obtain an expression for g in the thermodynamic limit

$$g = \frac{2}{\pi} \left(\frac{\omega_c}{\omega_0} \right)^2 \tan^{-1} \frac{\pi \tau}{\kappa_c d}, \quad (13)$$

where $d = \ell/N$, the interatomic distance between atoms in the string. With increasing string tension τ , g asymptotically approaches $g = (\frac{\omega_c}{\omega_0})^2$, a quantity bounded by 1 (see Fig. 2). Thus, the model is stable over the range of physical parameters. (As a practical matter, before the tension τ becomes comparable to interatomic forces in the string, it is likely that the string would break.)

Multimode squeezed vacuum. Eigenstates of the system can be labeled by the set of Bogoliubov excitation numbers for the $N+1$ modes $\{|n_\alpha\rangle\}$ with corresponding energies $E(\{n_\alpha\}) = \sum_\alpha n_\alpha \Omega_\alpha + \frac{1}{2} \sum_\alpha (\Omega_\alpha - \omega_\alpha)$. The ground state of the coupled system $|\{0\}\rangle$ can be constructed from the uncoupled ground state $|\{0\}\rangle_0$ with the squeeze operator $S(\xi) = \exp(-\frac{1}{2} \sum_{\alpha\beta} \xi_{\alpha\beta} a_\alpha^\dagger a_\beta^\dagger)$:

$$|\{0\}\rangle = \mathcal{N} S(\xi) |\{0\}\rangle_0. \quad (14)$$

ξ is the (matrix) squeeze parameter and $\mathcal{N}$ is a normalization factor. (Using an identity due to Schwinger [18], we obtain the normalization constant $\mathcal{N} = \frac{1}{\sqrt{\det M}}$.)

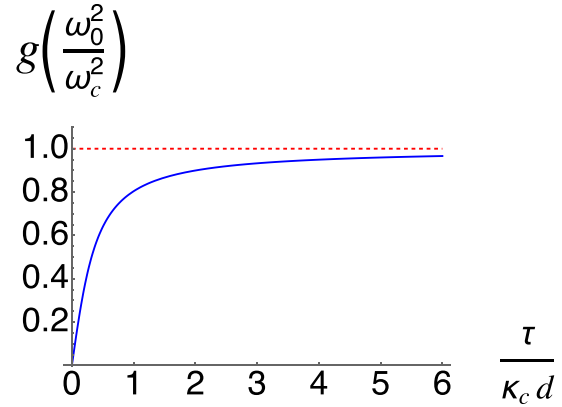


FIG. 2. Coupling strength g versus $\frac{\tau}{\kappa_c d}$ for $N, \ell \rightarrow \infty$.

We verify this by operating on Eq. (14) with b_α and using the identity $S(-\xi)a_\alpha S(\xi) = a_\alpha - \sum_\beta \xi_{\alpha\beta} a_\beta^\dagger$. We find that Eq. (14) is satisfied, provided the squeeze matrix has the value

$$\xi = M^{-1} N. \quad (15)$$

Hence, the ground state of the model is always a multimode squeezed-vacuum state [16,19] with squeeze parameter ξ determined by the Bogoliubov coefficients.

We note that the position uncertainty of the bead in the ground state can be readily evaluated with the Bogoliubov coefficients. Evaluating the bead variance gives

$$\langle \{0\} | u_0^2 | \{0\} \rangle = \frac{1}{2m\omega_0} \sum_\alpha (M_{\alpha 0} - N_{\alpha 0})^2 \quad (16)$$

$$= \frac{1}{2m} \sum_\alpha \frac{1}{\Omega_\alpha \left(1 + 4\omega_0 \sum_q \frac{\gamma_q^2 \omega_q}{(\Omega_\alpha^2 - \omega_q^2)^2} \right)}. \quad (17)$$

The sum can be evaluated analytically using complex-contour integration [15] to reveal that the position uncertainty is reduced with increasing damping ν , a result first obtained by Caldeira and Leggett [13] using the fluctuation-dissipation theorem.

To better understand the ground state, the average number of uncoupled excitations (phonons and vibrons) in the mode α contained in the coupled ground state $|\{0\}\rangle$ can also be expressed in terms of Bogoliubov coefficients, with

$$n_\alpha = \sum_\beta N_{\beta\alpha}^2. \quad (18)$$

An example is given in Fig. 3 for a coupling strength of $g = 0.7$. There is a small fraction of a vibron contributed by the bead ($\alpha = 0$), with an equal total amount of phonons on the string approximately uniformly distributed across the modes at this coupling strength.

Vibron decay. We now consider the dynamics of the vibrational decay of the bead. We start in the ground state $|\{0\}\rangle$, displace the bead by δ to create the initial state $|\Psi(0)\rangle = \exp(-ip_0\delta)|\{0\}\rangle$, and compute the expectation of the bead's position at time t :

$$\langle u_0(t) \rangle = \langle \Psi(t) | u_0 | \Psi(t) \rangle. \quad (19)$$

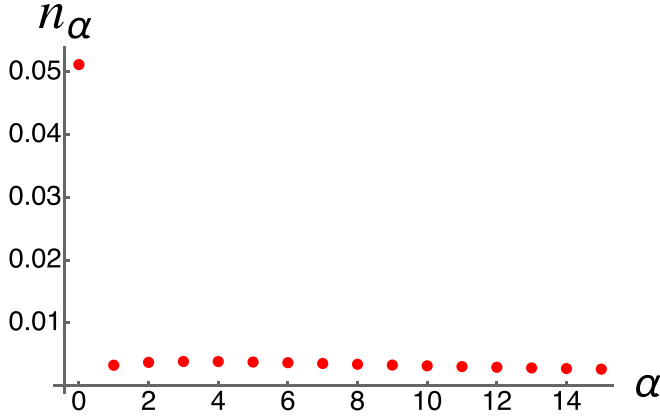


FIG. 3. Distribution of the average number of uncoupled excitations (phonons and vibrons) in the coupled ground state $n_\alpha = \langle \{0\} | a_\alpha^\dagger a_\alpha | \{0\} \rangle$ for parameter values $N = 15$, $g = 0.7$.

The expectation can be expressed in terms of Bogoliubov coefficients [15]

$$\langle u_0(t) \rangle = \delta \operatorname{Re} \sum_{\alpha} (U_{0\alpha}^2 - V_{0\alpha}^2) \exp(-i\Omega_{\alpha} t) \quad (20)$$

$$= \delta \operatorname{Re} \sum_{\alpha} \frac{\exp(-i\Omega_{\alpha} t)}{1 + 4\omega_0 \sum_n \frac{\gamma_n^2 \omega_n}{(\Omega_{\alpha}^2 - \omega_n^2)^2}}. \quad (21)$$

We identify the factor $(U_{0\alpha}^2 - V_{0\alpha}^2)$ in the summand of Eq. (20) as the spectral density of the decay:

$$\rho(\Omega_{\alpha}) = U_{0\alpha}^2 - V_{0\alpha}^2. \quad (22)$$

This spectral density satisfies a sum rule [15] $\sum_{\alpha} \rho(\Omega_{\alpha}) = 1$, and the width of this spectral density is the decay rate of the bead displacement [20] (see Fig. 4).

Using contour integration in the complex plane, the sum can be evaluated [15] and the decay rate Γ can be obtained:

$$\Gamma = \frac{\omega_r}{\sqrt{2}} \left(\sqrt{1 + \frac{\Gamma_r^4}{\omega_r^4}} - 1 \right)^{\frac{1}{2}}, \quad (23)$$

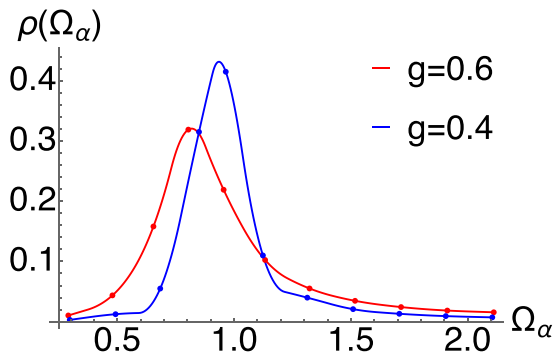


FIG. 4. Spectral distribution function $\rho(\Omega_{\alpha})$ versus Ω_{α}/ω_0 for $g = 0.4$ and 0.6 . It satisfies the sum rule $\sum_{\alpha} \rho(\Omega_{\alpha}) = 1$ and its width gives the decay rate Γ of $\langle u_0(t) \rangle$.

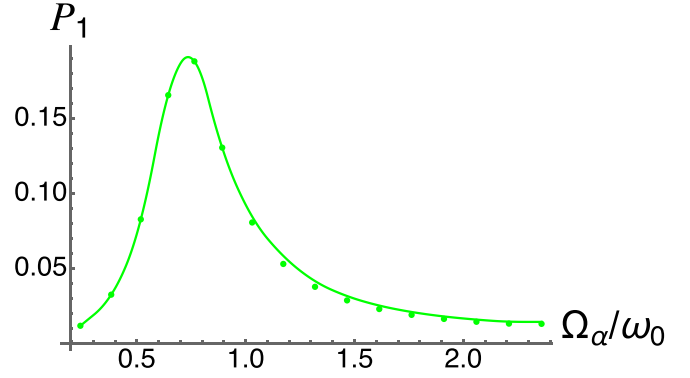


FIG. 5. Spectral probability distribution for single bogoliubon emission $P_1(\Omega_{\alpha})$ from the decay of a vibron $|1; \{0\}\rangle$. Parameter values are $g = 0.7$ and $N = 15$. The spectrum of single bogoliubons comprise 90.7% of the total emission.

where $\omega_r \approx \omega_0$ and $\Gamma_r \approx \sqrt{\nu\omega_0}$. For the case of light damping where $\Gamma_r \ll \omega_r$, Eq. (23) gives $\Gamma \approx \nu$, in agreement with the classical result.

Using Fermi's Golden Rule, we obtain for weak coupling strength the decay rate Γ_{GR} for a transition $|1; \{0\}\rangle \rightarrow |0; \{1_n\}\rangle$ with the bead losing $\Delta E = \omega_0$ to the string

$$\begin{aligned} \Gamma_{GR} &= 2\pi \sum_n |\langle 1; \{0\} | H_i | 0; \{1_n\} \rangle|^2 \delta(\omega_n - \omega_0) \\ &= 2\pi \int d\omega \mathcal{D}(\omega) \gamma^2(\omega) \delta(\omega - \omega_0) \\ &= 2\pi \frac{\nu\omega_0 c}{\ell\omega_0} \frac{\ell}{\pi c} = 2\nu. \end{aligned} \quad (24)$$

That Γ_{GR} is twice the decay rate for the bead displacement is expected, since Γ_{GR} is the energy decay rate, while Γ is the displacement decay rate. As energy of the bead varies as the square of the oscillation amplitude, $\Gamma_{GR} = 2\Gamma$.

We now turn to the radiation spectrum from the vibrating bead. The decay of the vibrating bead is accompanied by the emission of bogoliubons. The probability of the emission of a single bogoliubon of frequency Ω_{α} can be expressed in terms of Bogoliubov coefficients

$$P_1(\Omega_{\alpha}) = |\langle \{0\} | b_{\alpha} a_0^{\dagger} | \rangle_0|^2 = M_{0\alpha}^{-2} (\det M)^{-1}. \quad (25)$$

A plot of the spectral probability distribution for single bogoliubon emission is given in Fig. 5.

Summary. We analyzed the dynamics of a vibrating particle coupled to an environment by extending a generalization of the Lamb model to the quantum regime. The model provides an exactly solvable example of a damped quantum harmonic oscillator. These results may apply to a variety of related quantum systems, e.g., a local vibrational mode in a magnetic insulator (vibron-magnon) or coupled to an electromagnetic cavity (vibron-photon).

Our solution explicitly calculates the coefficients of the multimode Bogoliubov transformation that diagonalize the Hamiltonian, and we use these coefficients to describe the properties of the system. We found that the true ground state

of the system is always a multimode squeezed vacuum state where the displacement uncertainty is reduced with increasing damping rate ν . We then obtained an explicit expression for the vibrational decay rate of the bead and found that it recovered the classical damping rate in the light damping regime.

We examined the acoustic radiation spectrum emitted by the vibrating particle. We obtained an expression for the probability of single bogoliubon emission in terms of the Bogoliubov coefficients, and we observed that the spectral

emission has a nearly symmetric lineshape about a slightly red-shifted peak frequency.

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Data availability. No data were created or analyzed in this study.

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