

Arbitrary-velocity Volkov wave packets

D. Ramsey¹,* J. McKeown¹, and J. P. Palastro¹*Laboratory for Laser Energetics, University of Rochester, Rochester, New York 14623-1299, USA*

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The evolution of a charged lepton in the field of an electromagnetic plane wave can be described as a superposition of Volkov states. Here, we demonstrate that imposing specific momentum correlations among Volkov states produces a spatiotemporally structured wave packet whose probability-density peak, evaluated on fixed-light-cone slices, travels at an arbitrary tailored velocity. This velocity can be chosen independently of both the field amplitude and the velocity expectation value. The imposed momentum correlations modify the expectation-value trajectory, providing a measurable signature of the arbitrary velocity within a physical observable.

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I. INTRODUCTION

The interaction of an electromagnetic field with a charged lepton induces spatiotemporal structure in the lepton's wave packet. In the case of an electromagnetic plane wave, the wave packet can be expanded in the Volkov basis [1,2], whose basis states are exact solutions of the Dirac equation in the plane wave and whose expansion coefficients are constants of motion. These states account for the field by modifying the bispinor and spatiotemporal phase of field-free solutions to the Dirac equation. The relative phases and amplitudes of the states determine the initial structure of the wave packet before it enters the field, while the Volkov modifications govern its “dressed” evolution within the field. This dressed evolution is the basis for the Furry picture of strong-field quantum electrodynamics [3], where Volkov states are used in place of field-free solutions to calculate cross sections for processes such as nonlinear Compton scattering and Breit-Wheeler pair production [4–6].

Typically, the initial wave packet is assumed to have a correlation among its momenta set only by the on-shell condition. However, the emergence of space-time structured light [7,8] and progress in wavefunction engineering [9–18] have motivated the exploration of wave packets with additional correlations. In the context of light, an optical pulse with an appropriately chosen correlation in configuration or wave number space can exhibit an intensity peak that travels at an arbitrary velocity while maintaining a near-constant profile over long distances [19–31]. The extended interaction lengths and controllable velocities enabled by these pulses benefit a wide range of applications, including particle acceleration [22,23,32–34], radiation generation [35–40], detection of strong-field quantum electrodynamical phenomena [41,42], and light detection and ranging (LIDAR) [43].

Lepton wave packets, like optical pulses, are solutions to a wave equation. As a result, the formalism underlying space-time structured light can be extended to matter waves [44]. This extension was first used to demonstrate that correlations among the momenta of a field-free lepton wave packet can produce a probability-density peak that travels at any velocity, including those exceeding the speed of light [44]. It was later applied to a specific case in an electromagnetic plane wave, where a wave packet with a negative momentum exhibited a peak traveling forward at the speed of light [45]. These realizations of space-time structured matter waves belong to the broader field of wavefunction engineering, which promises to enhance processes such as electron microscopy and electron holography [9–18].

Here, we show that the probability-density peak of a spatiotemporally structured lepton wave packet, evaluated on fixed-light-cone slices, can travel through an electromagnetic plane wave at any velocity, independent of its velocity expectation value. These wave packets are constructed by prescribing specific correlations among the momenta of the constituent Volkov states. The correlations can be built into the initial wave packet before it enters the electromagnetic field to produce a desired peak velocity within the field. The imposed correlations also modify the expectation-value trajectory, providing a measurable signature of the peak's velocity. As an extension, we show that conventional wave packets, with no prescribed momentum correlations, acquire correlations between their dressed momenta upon interaction with an electromagnetic plane wave.

The remainder of this article is organized as follows: Section II describes the momentum-dependent amplitude required to generate a lepton wave packet with a peak that travels at an arbitrary, prescribed velocity outside of a field. This description is then generalized to an arbitrary, prescribed velocity within the field. Section III presents a calculation of the peak trajectory (Sec. III A) and expectation trajectory (Sec. III B). This is followed by specific examples that compare the two trajectories, including a description of how the imposed correlations modify the expectation-value trajectory (Sec. III C). Section IV discusses the natural emergence of space-time structure during the interaction of a charged

*Contact author: dram@lle.rochester.edu

TABLE I. Summary of notation.

Symbol/expression	Description
$x = (x_0, \mathbf{x})$, $\mathbf{x} = (x_1, x_2, x_3)$	Four-position
$p = (p_0, \mathbf{p})$, $\mathbf{p} = (p_1, p_2, p_3)$	Four-momentum
$\mathbf{a}_\perp = (a_1, a_2)$	Transverse three-vector components
$k = \omega n$, $n = (1, 0, 0, 1)$	Four-wave vector of the plane wave
$A = (0, A_1(x_-), 0, 0)$	Electromagnetic four-potential
$\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$	Dirac gamma matrices
$u^\dagger, \bar{u} = u^\dagger \gamma_0$	Hermitian conjugate, Dirac adjoint
$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$	Three-vector inner product
$a \cdot b = a_0 b_0 - \mathbf{a} \cdot \mathbf{b}$	Four-vector inner product
$\bar{a}$	Field cycle-averaged quantity

lepton with an electromagnetic wave and concludes the article.

Throughout, natural units ($\hbar = c = 1$) and the mostly minus metric $(+, -, -, -)$ are used. Relativistic notation for superscripts and subscripts is not used. Four-vector products are denoted by $p \cdot x \equiv p^\mu x_\mu$, and an unindexed four-vector p appearing alone denotes the four-vector p^μ . Notational conventions are summarized in Table I.

The considered interaction is between a charged, positive-energy lepton propagating along x_3 and an electromagnetic plane wave propagating in the positive x_3 direction. The plane wave is linearly polarized in the x_1 direction. A coordinate transformation $(x_0, x_3) \rightarrow (x_-, x_3)$ is employed, where $x_- \equiv x_0 - x_3$ is the light-cone coordinate; any subsequent use of x_0 is understood to be implicitly defined through x_- and x_3 . Since expectation values evolve in x_- rather than in time, x_0 , the bispinor inner product is constructed using $\gamma_- \equiv \gamma_0 - \gamma_3$ [46]. In what follows, the probability-density peak is understood to refer to the peak probability density evaluated on fixed- x_- slices using the γ_- inner product. A discussion of the probability-density-peak dynamics, evaluated on fixed-time slices with the inner product constructed using γ_0 , is provided in Appendix A.

II. WAVE-PACKET CONSTRUCTION

The total wave packet $\psi(x)$ of a charged lepton can be constructed from a superposition of “partial” wave packets $\Phi(x, \eta)$, labeled by a foliation parameter η :

$$\psi(x) = \int \mathcal{N}(\eta) \Phi(x, \eta) d\eta, \quad (1)$$

where the scalar weighting function $\mathcal{N}(\eta)$ is normalized such that $\int |\mathcal{N}(\eta)|^2 d\eta = 1$ and $\int \bar{\psi}(x) \gamma_- \psi(x) d^3 \mathbf{x} = 1$. Depending on the application, η may represent, for example, a temporal delay, a transverse width, or a surface in momentum space, as in the present work. In the absence of electromagnetic fields,

$$\Phi(x, \eta) = \frac{1}{(2\pi)^{3/2}} \int \frac{u(p)}{\sqrt{2p_-}} \delta(p_0 - E_{\mathbf{p}}) f(p, \eta) e^{-ip \cdot x} d^4 p, \quad (2)$$

where $p_- \equiv p_0 - p_3$, $E_{\mathbf{p}} \equiv (m^2 + |\mathbf{p}|^2)^{1/2}$, and the bispinor u is normalized according to $\bar{u} \gamma_- u = 2p_-$. The complex, scalar distribution $f(p, \eta)$ determines the η -dependent

amplitude and phase of each momentum state composing the wave packet.

The presence of an electromagnetic plane wave modifies the bispinor and phase of the partial wave packets:

$$\Phi(x, \eta) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathcal{V}}{\sqrt{2p_-}} \delta(p_0 - E_{\mathbf{p}}) f(p, \eta) e^{iS} d^4 p, \quad (3)$$

where

$$\mathcal{V}(p, x_-) \equiv \left[\mathbb{I} - \frac{\gamma_- \gamma_1 e A_1(x_-)}{2p_-} \right] u(p) \quad (4)$$

is the dressed bispinor, $\mathbb{I}$ denotes the identity matrix, and

$$S(p, x) = -p \cdot x + \frac{1}{p_-} \int_0^{x_-} \left[e p_1 A_1(x'_-) - \frac{e^2}{2} A_1^2(x'_-) \right] dx'_- \quad (5)$$

is the classical action. The resulting total wave packet is a superposition of Volkov states. While A_1 may be any function of x_- , the examples presented here adopt the specific form

$$A_1(x_-) = \xi(x_-) \cos(\omega x_- + \phi), \quad (6)$$

where the profile $\xi(x_-)$ varies slowly compared to the period $2\pi/\omega$ and ϕ is an arbitrary phase. The following subsections develop the framework for arbitrary-velocity Volkov wave packets by first reviewing the construction of conventional and field-free arbitrary-velocity wave packets.

A. Field-free conventional wave packets

Outside of an electromagnetic field, conventional wave packets are synthesized as superpositions of monoenergetic partial wave packets with $\eta = p_0$. That is, $\Phi(x, \eta)$ is chosen to be time independent up to a global phase advance, such that

$$\Phi(x, \eta) - \Phi(x_0 + T, \mathbf{x}_\perp, x_3, \eta) e^{i\eta T} = 0, \quad (7)$$

where T is an arbitrary time interval. Taking the Fourier transform of Eq. (7) and multiplying by $\bar{u} \gamma_-$ yields the constraint

$$f(p, \eta) [1 - e^{-i(p_0 - \eta)T}] = 0, \quad (8)$$

which must hold under integration over p . This can only be satisfied if $f(p, \eta) \propto \delta(\eta - p_0)$, or equivalently, if $\eta = p_0$. After imposing the on-shell condition to replace p_0 with $E_{\mathbf{p}}$ in Eq. (2), the longitudinal momentum p_3 is expressed in terms of η and $\mathbf{p}_\perp$ to enforce $\eta = E_{\mathbf{p}}$:

$$p_3(\eta, \mathbf{p}_\perp) = \pm \sqrt{\eta^2 - (m^2 + \mathbf{p}_\perp^2)}, \quad (9)$$

where $\mathbf{p}_\perp \equiv |\mathbf{p}_\perp|$. Here, the negative sign is selected so that all plane waves composing the wave packet travel in the negative x_3 direction.

Choosing partial wave packets that have the time-translation property described by Eq. (7) does not imply that the total wave packet $\psi(x)$ will have the same property. However, under the right conditions, $\psi(x)$ will approximately hold this property for a finite duration. The first condition is that $\mathcal{N}(\eta)$ is narrowly peaked about some characteristic value of η (e.g., the average energy in the case of $\eta = p_0$). The second condition is that there are no correlations between

$\mathbf{p}_\perp$ and η :

$$\langle \eta \mathbf{p}_\perp \rangle - \langle \eta \rangle \langle \mathbf{p}_\perp \rangle = \mathbf{0}, \quad (10)$$

where the angled brackets denote an expectation value. In order to satisfy the second condition, $f(p, \eta)$ must take the form

$$f(p, \eta) = \left| \frac{d\eta}{dp_3} \right|^{1/2} \mathcal{T}(\mathbf{p}_\perp) \delta(\eta - p_0). \quad (11)$$

The factor $|d\eta/dp_3|^{1/2}$ ensures the probability density is separable in $\mathbf{p}_\perp$ and η . The transverse momentum distribution $\mathcal{T}(\mathbf{p}_\perp)$ is normalized so that $\int |\mathcal{T}(\mathbf{p}_\perp)|^2 d^2\mathbf{p}_\perp = 1$, and is sufficiently localized in $\mathbf{p}_\perp$ to exclude evanescent waves ($p_\perp^2 \ll \eta^2 - m^2$), making it independent of η . For computational efficiency, all numerical examples employ a reduced transverse dimensionality, with $|\mathcal{T}(\mathbf{p}_\perp)|^2 = (w/\sqrt{2\pi}) \exp(-w^2 p_1^2/2) \delta(p_2)$, where w is the minimum transverse width of the wave packet with dimensions of space (m^{-1}).

Figure 1(a) illustrates the time-independent amplitude of a conventional partial wave packet. Due to the transverse momentum spread ($\propto w^{-1}$), the partial wave packet has a minimum transverse width w at a constant location in x_3 for all x_0 . This represents a stationary focus, with the wave packet focusing along x_3 to a minimum width w before diffracting. Figure 1(b) provides a geometric interpretation of the corresponding momentum distribution [Eq. (11)], with only the negative branch of Eq. (9) selected (green curve). The momentum domain lies at the intersection of the mass shell E_p (blue) and the horizontal plane $\eta = p_0$ (translucent).

B. Field-free arbitrary-velocity wave packets

Arbitrary-velocity lepton wave packets are superpositions of propagation-invariant partial wave packets with $\eta = p_0 - v_a p_3$. The partial wave packets satisfy

$$\Phi(x, \eta) - \Phi(x_0 + T, \mathbf{x}_\perp, x_3 + v_a T, \eta) e^{i\eta T} = 0 \quad (12)$$

for an arbitrary velocity v_a and time interval T [30,44]. This property implies that the amplitude of the partial wave packets remains invariant in time along characteristics $x_3 - v_a x_0 = \text{const}$. Note that setting $v_a = 0$ recovers the same translational property expressed by Eq. (7). Thus, a conventional partial wave packet can be recovered by setting $v_a = 0$.

Taking the Fourier transform of Eq. (12) and multiplying by $\bar{u}\gamma_-$ yields the constraint

$$f(p, \eta) [1 - e^{-i(p_0 - v_a p_3 - \eta)T}] = 0, \quad (13)$$

which must hold under integration over p . This can only be satisfied if $f(p, \eta) \propto \delta(\eta - p_0 + v_a p_3)$ or $\eta = p_0 - v_a p_3$. After imposing the on-shell condition to replace p_0 with E_p , p_3 is expressed in terms of η and $\mathbf{p}_\perp$ to enforce $\eta = E_p - v_a p_3$:

$$p_3(\eta, \mathbf{p}_\perp) = \eta v_a \gamma_a^2 \pm \sqrt{(\eta \gamma_a^2)^2 - \gamma_a^2(m^2 + \mathbf{p}_\perp^2)}, \quad (14)$$

where $\gamma_a \equiv (1 - v_a^2)^{-1/2}$. Equation (14) shows that even though there are no correlations between η and $\mathbf{p}_\perp$ [Eq. (10)], the on-shell condition and constraint $\eta = E_p - v_a p_3$ impose a correlation between p_3 and $\mathbf{p}_\perp$. The sign is chosen based

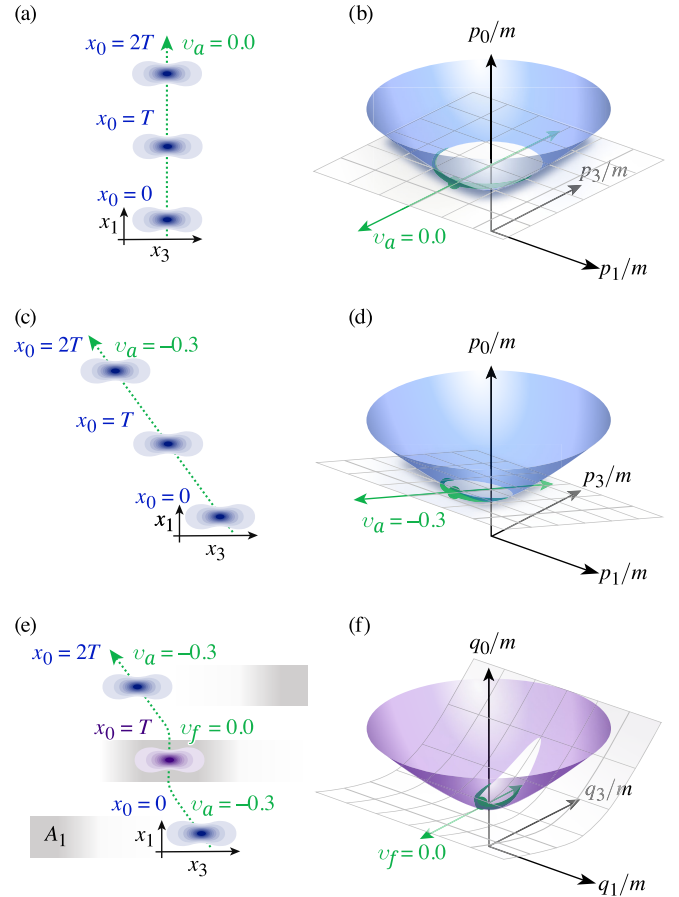


FIG. 1. Spatiotemporal evolution and momentum representations of partial wave packets. In each case, v_a denotes the prescribed out-of-field velocity of the peak. Snapshots at three times showing the advection of the density $\bar{\Phi}\gamma_-\Phi$ for (a) a conventional wave packet with $v_a = 0.0$, (c) an arbitrary-velocity wave packet with $v_a = -0.3$, and (e) an arbitrary-velocity Volkov wave packet with $v_a = -0.3$ outside the field and $v_{f*} = 0.0$ at the maximum vector potential $|e|\xi_*/m = 3.0$. The dotted green lines track the location of the peak amplitudes. The gray shading illustrates the location of the electromagnetic plane wave. (b), (d) Corresponding momentum representations. The green curves indicate the intersection of the mass shell E_p (blue) with the plane $\eta = p_0 - v_a p_3$ (translucent, with slope marked by the green arrow). (f) Dressed momentum representation of the Volkov wave packet. Here, the green curve indicates the intersection of the dressed mass shell U_q (purple) and nonlinear surface $\lambda(\eta, q_-) = q_0 - v_{f*} q_3$ (translucent). For all panels, $p_-(\eta, 0) = 3m$, yielding $\eta = 1.67m$ for $v_a = 0.0$ and $\eta = 1.27m$ for $v_a = -0.3$.

on the value of v_a . For $|v_a| \leq 1$, the negative branch ensures that the longitudinal momenta are negative in a Lorentz frame moving with velocity v_a . For $|v_a| > 1$, the positive branch ensures that the longitudinal momenta are negative in a Lorentz frame moving with velocity $1/v_a$. These two cases correspond to wave packets with time-independent and longitudinal-coordinate-independent “foci” in the Lorentz frame, respectively (see Ref. [44] for details). Limiting cases of Eq. (14) are discussed in Appendix B.

As before, the total wave packet $\psi(x)$ will approximately hold the property described by Eq. (12) if $\mathcal{N}(\eta)$ is narrowly peaked about a characteristic value of η and

$\langle \eta \mathbf{p}_\perp \rangle - \langle \eta \rangle \langle \mathbf{p}_\perp \rangle = \mathbf{0}$. This latter condition requires that $f(p, \eta)$ take the form

$$f(p, \eta) = \left| \frac{d\eta}{dp_3} \right|^{1/2} \mathcal{T}(\mathbf{p}_\perp) \delta(\eta - p_0 + v_a p_3), \quad (15)$$

which builds the correlation into the wave packet. The transverse momentum distribution $\mathcal{T}(\mathbf{p}_\perp)$ is again normalized so that $\int |\mathcal{T}(\mathbf{p}_\perp)|^2 d^2 \mathbf{p}_\perp = 1$ and excludes evanescent waves ($p_\perp^2 \ll \gamma_a^2 \eta^2 - m^2$).

Figure 1(c) illustrates the time-dependent density of an arbitrary-velocity partial wave packet $\tilde{\Phi} \gamma_- \Phi$ with a peak that moves at $v_a = -0.3$. Like the conventional case, the space-time wave packet has a minimum width w determined by its transverse momentum spread. However, now the focusing and diffraction occur with respect to the coordinate $x_3 - v_a x_0$. Figure 1(d) provides the geometric interpretation of the momentum distribution. The domain lies along the negative branch of Eq. (14) (green curve), corresponding to the intersection of the mass shell $E_{\mathbf{p}}$ (blue) and the plane $\eta = p_0 - v_a p_3$ (translucent). Along the intersection,

$$\left(\frac{\partial E_{\mathbf{p}}}{\partial p_3} \right)_\eta = v_a, \quad (16)$$

where $(\)_\eta$ denotes that the derivative is taken with η held fixed. This relation indicates that $f(p, \eta)$ features a unique correlation among the momenta, i.e., $\partial E_{\mathbf{p}} / \partial p_3|_\eta \neq 0$. Nevertheless, when superposing partial wave packets, η varies on the mass shell, such that $\partial E_{\mathbf{p}} / \partial p_3 = p_3 / E_{\mathbf{p}}$. Thus, while v_a can take any value, the expectation value of the longitudinal velocity, $\langle p_3 / E_{\mathbf{p}} \rangle$, is always subluminal.

C. Arbitrary-velocity Volkov wave packets

In the presence of an electromagnetic plane wave, field-dependent terms in the phase S [Eq. (5)] modify the constraint required for propagation invariance. To determine this constraint, consider a charged lepton interacting with a four potential that has a constant strength $\xi = \xi_*$, such that $A_1(x_-) = \xi_* \cos(\omega x_- + \phi)$. The partial wave packets are chosen to be propagation-invariant up to a global phase advance when averaged over a cycle of the potential:

$$\int_0^{\frac{2\pi}{\omega}} [\Phi(x, \eta) - \Phi(x_0 + T, \mathbf{x}_\perp, x_3 + v_f T, \eta) e^{i\eta T}] dx_- = 0, \quad (17)$$

where T is an arbitrary time interval and v_f is the arbitrary velocity in the field. Taking the Fourier transform of Eq. (17), multiplying by $\bar{u} \gamma_-$, and applying the Jacobi-Anger expansion to simplify the cycle average yields the constraint

$$f(p, \eta) [1 - e^{-i(q_0 - v_f q_3 - \eta)T}] = 0. \quad (18)$$

Here, the field-dressed four momenta q are defined as

$$q \equiv p + \frac{e^2 \xi_*^2}{4p_-} n, \quad (19)$$

subject to the dressed-shell condition $q_0 = U_{\mathbf{q}} \equiv (m^2 + e^2 \xi_*^2 / 2 + |\mathbf{q}|^2)^{1/2}$ [2]. The constraint expressed by Eq. (18) can only be satisfied if $f(p, \eta) \propto \delta(\eta - q_0 + v_f q_3)$, which

shows that propagation invariance in an electromagnetic plane wave requires correlations among the dressed momenta.

At this stage, the wave packet construction could proceed along the same path as in the previous subsections, with the momenta replaced by the dressed momenta. However, this approach has two practical limitations. First, a physically realizable wave packet would typically be prepared outside the field, where imposing correlations among dressed momenta—whose dependence on the momenta is nonlinear—would be challenging. Second, the strength of a physical four-potential is never constant, i.e., ξ depends on x_- , such that $A_1(x_-) = \xi(x_-) \cos(\omega x_- + \phi)$. As a result, the propagation-invariant property described by Eq. (17) can only occur at a target potential strength $\xi = \xi_*$.

Notably, these limitations do not apply in the special case of $v_f = 1$, where $\eta = p_-$ and the probability-density peak travels at the constant velocity $v_a = v_f = 1$ regardless of the local value of $\xi(x_-)$ [45]. This occurs because $p_- = q_-$ is a conserved quantity, ensuring that Eq. (17) is satisfied across all potential strengths.

Here, the goal is to prepare a wave packet with a linear momentum correlation outside the field, so its probability-density peak travels at a desired velocity v_{f*} at a target strength ξ_* inside the field. This is achieved by appropriately setting the out-of-field velocity v_a . To determine this velocity, the out-of-field plane $\eta = p_0 - v_a p_3$ is expressed in terms of the dressed momenta at $\xi = \xi_*$, yielding the surface

$$\lambda(\eta, q_-) - q_0 + v_{f*} q_3 = 0, \quad (20)$$

where

$$\lambda \equiv \left(\frac{1 - v_{f*}}{1 - v_a} \right) \eta - \left(\frac{v_a - v_{f*}}{1 - v_a} \right) q_- + (1 - v_{f*}) \frac{e^2 \xi_*^2}{4q_-}. \quad (21)$$

The surface is locally planar with slope v_{f*} relative to q_3 if

$$\frac{\partial \lambda}{\partial q_3} = \left(\frac{v_a - v_{f*}}{1 - v_a} \right) + (1 - v_{f*}) \frac{e^2 \xi_*^2}{4q_-^2} = 0. \quad (22)$$

When evaluated on-shell, $q_- = p_- = E_{\mathbf{p}} - p_3(\eta, \mathbf{p}_\perp)$ [see Eq. (14)], this condition depends on $\mathbf{p}_\perp$ and η . To first approximation, this dependence can be removed by evaluating q_- at the expectation values $\langle \mathbf{p}_\perp \rangle$ and $\langle \eta \rangle$, providing the out-of-field velocity

$$v_a \approx \frac{v_{f*} - (1 - v_{f*}) \frac{e^2 \xi_*^2}{4\mathcal{P}_-^2}}{1 - (1 - v_{f*}) \frac{e^2 \xi_*^2}{4\mathcal{P}_-^2}}, \quad (23)$$

where $\mathcal{P}_- \equiv p_-(\eta = \langle \eta \rangle, \mathbf{p}_\perp = \langle \mathbf{p}_\perp \rangle)$. A validity condition for this approximation can be derived by Taylor expanding Eq. (22) about $\mathbf{p}_\perp = \langle \mathbf{p}_\perp \rangle$. The first-order term vanishes when $\langle \mathbf{p}_\perp \rangle = 0$. The second-order term is negligible compared to the leading-order term when

$$w^2 \gg |(1 + v_a)^2 \gamma_a^2 \mathcal{P}_-^2 - m^2|^{-1}. \quad (24)$$

When this condition is not met, an exact construction based on partial wave packets satisfying Eq. (18) must be used instead. The value of v_a for which the right-hand side of Eq. (24) diverges is discussed in Appendix B.

With v_a determined, the out-of-field distribution $f(p, \eta)$ is given by Eq. (15). The resulting in-field distribution produces

the desired linear correlation between the dressed momenta at $\xi = \xi_*$: $f(p, \eta) \propto \delta(\Lambda - q_0 + v_{f*} q_3)$ in the vicinity of $q_- = \mathcal{P}_-$ and $\mathbf{p}_\perp = \langle \mathbf{p}_\perp \rangle$, where $\Lambda \equiv \lambda(\langle \eta \rangle, \mathcal{P}_-)$. Together with the on-shell condition, $q_0 = U_{\mathbf{q}}$, the constraint imposed by the delta function reveals a correlation between q_3 and $\mathbf{p}_\perp$:

$$q_3(\Lambda, \mathbf{p}_\perp) \approx \Lambda v_{f*} \gamma_{f*}^2 \pm \sqrt{(\Lambda \gamma_{f*}^2)^2 - \gamma_{f*}^2 \left(m^2 + \frac{e^2}{2} \xi_*^2 + \mathbf{p}_\perp^2 \right)}, \quad (25)$$

where $\gamma_{f*} \equiv (1 - v_{f*}^2)^{-1/2}$, which is a dressed version of Eq. (14).

Figure 1(e) illustrates the dynamic density peak of an arbitrary-velocity-Volkov partial wave packet $\Phi_{\mathcal{P}_-} \Phi$. The wave packet starts outside the field, where its peak travels at $v_a = -0.3$. Upon entering the field and encountering the target potential strength $|e|\xi_*/m = 3$, the peak travels at the designed velocity $v_{f*} = 0.0$. After leaving the field, the peak again travels at $v_a = -0.3$. Figures 1(d) and 1(f) provide the geometric interpretation. Outside the field, the momenta lie along the intersection of the mass shell and plane $\eta = p_0 - v_a p_3$ [Fig. 1(d)]. Inside the field, the dressed momenta lie along the intersection of the dressed-shell and nonlinear surface $\lambda(\eta, q_-) = q_0 - v_{f*} q_3$ [Fig. 1(f)]. The out-of-field velocity v_a is chosen so that the nonlinear surface has the desired slope v_{f*} in the vicinity of the momentum expectation values. That is, along the intersection and in the vicinity of $\mathbf{p}_\perp = \langle \mathbf{p}_\perp \rangle$,

$$\left(\frac{\partial U_{\mathbf{q}}(\xi_*)}{\partial q_3(\xi_*)} \right)_{\eta=\langle \eta \rangle} \approx v_{f*} \quad (26)$$

[see green arrow in Fig. 1(f)], which indicates that $f(p, \eta)$ locally produces the desired linear correlation among the dressed momenta.

III. TRAJECTORIES

This section examines the distinct dynamics of the expectation and peak-probability trajectories by presenting their equations of motion, comparing their evolution, and analyzing the influence of momentum correlations and v_a on the expectation trajectory. The probability densities and expectation-value trajectories depicted in this section are numerical evaluations of the exact Volkov superposition [i.e., Eqs. (1), (3), (5), and (15)]. The superposition is constructed using a value of v_a obtained from the approximate velocity-design relation [Eq. (23)]. With v_a selected, the wave packet is propagated through the field and its probability density evaluated exactly, without approximation. The details and parameters of the calculations are presented in Appendix C.

Because v_a is selected using the approximate expression [Eq. (23)], the exact Volkov superposition realizes v_{f*} as long as the condition in Eq. (24) is satisfied. In each example, this condition is well satisfied, and the peak-probability density travels at v_{f*} where $\xi = \xi_*$. The analytic expressions derived in this section rely on the approximations in Table II and are used to guide interpretation. While these expressions describe both the non-cycle-averaged and cycle-averaged dynamics, the numerical examples lie in a regime where the

TABLE II. Summary of approximations used in deriving the approximate peak-probability and expectation-value trajectories, and their regimes of validity.

Approximation	Regime of validity
Narrowband $\mathcal{N}(\eta)$	$\Delta\eta \ll \langle \eta \rangle$
Paraxial expansion	$\mathbf{p}_\perp^2 \sim w^{-2} \ll \langle \eta \rangle^2 \gamma_a^2 - m^2 $
Transverse distortion	$\mathbf{p}_\perp^2 \sim w^{-2} \gg \left \frac{\omega \mathcal{P}_-^2 (\mathcal{P}_3 - v_a \mathcal{E})}{e \xi_* (1 - v_a)} \right ^{2/3}$
Eq. (23) validity	$\mathbf{p}_\perp^2 \sim w^{-2} \gg (1 + v_a)^2 \gamma_a^2 \mathcal{P}_-^2 - m^2 $
Slowly varying envelope	$ \xi^{-1} d\xi/dx_- \ll \omega$
Cycle average	$\Delta x_- \gg 1/\omega$

cycle-averaged dynamics meaningfully describe the wave packet's evolution. This requires the duration or lifetime over which $\psi(x)$ retains the propagation-invariance property (defined in Appendix D), as well as the duration of the envelope ξ , to be long compared to $1/\omega$.

A. Peak-probability trajectory

When the weighting function $\mathcal{N}(\eta)$ is narrowly peaked about the expectation value $\langle \eta \rangle$, the phase S [Eq. (5)] can be expanded about the intersection of $\langle \eta \rangle$ and the mass shell. When the paraxial approximation is valid, $\mathbf{p}_\perp^2 \ll |\langle \eta \rangle^2 \gamma_a^2 - m^2|$, the phase can also be expanded to second order about $\mathbf{p}_\perp = \langle \mathbf{p}_\perp \rangle = 0$. Together, these approximations yield

$$S \approx \mathcal{P}_3 x_3 - \mathcal{E} x_0 - \frac{1}{2\mathcal{P}_-} \mathcal{I}_2(x_-) + p_1 x_1 + \frac{p_1}{\mathcal{P}_-} \mathcal{I}_1(x_-) + p_2 x_2 + \frac{\mathbf{p}_\perp^2}{2(\mathcal{E} v_a - \mathcal{P}_3)} \left(x_3 - v_a x_0 - \frac{1 - v_a}{2\mathcal{P}_-^2} \mathcal{I}_2(x_-) \right), \quad (27)$$

where $\mathcal{P}_3 \equiv p_3(\eta = \langle \eta \rangle, \mathbf{p}_\perp = 0)$ with p_3 given by Eq. (14), $\mathcal{E} \equiv (m^2 + \mathcal{P}_3^2)^{1/2}$, and

$$\mathcal{I}_1(x_-) = \int_0^{x_-} e A_1(x'_-) dx'_-,$$

$$\mathcal{I}_2(x_-) = \int_0^{x_-} e^2 A_1^2(x'_-) dx'_-.$$

Although the final term has an apparent singularity at $v_a = \mathcal{P}_3/\mathcal{E}$, this value of v_a lies outside the regime of validity of the approximations made to derive Eq. (27) (see Appendix B).

Each term in Eq. (27) can be understood by conceptualizing the wave packet as an envelope modulated by a carrier wave. The carrier describes the motion of the phase fronts, which is governed by the terms of $\mathcal{O}(\mathbf{p}_\perp^0)$ [first line of Eq. (27)]. The envelope describes the trajectory of the probability-density peak $\mathbf{x}_f$, which is governed by the remaining terms [second and third lines of Eq. (27)]. The transverse and longitudinal locations of the peak are determined by the vanishing of the $\mathcal{O}(\mathbf{p}_\perp)$ and $\mathcal{O}(\mathbf{p}_\perp^2)$ terms, respectively, for all p . For $v_a \neq 1$, this results in the equations of

motion:

$$\frac{dx_{f1}}{dx_-} = -\frac{eA_1}{\mathcal{P}_-}, \quad (28)$$

$$\frac{dx_{f2}}{dx_-} = 0, \quad (29)$$

$$\frac{dx_{f3}}{dx_-} = \frac{v_a}{1-v_a} + \frac{e^2 A_1^2}{2\mathcal{P}_-^2}. \quad (30)$$

For $v_a = v_f = 1$, $dx_{f3}/dx_0 = 1$ and the longitudinal trajectory is nonoscillatory at any field amplitude [45]. With $A_1 = \xi(x_-) \cos(\omega x_- + \phi)$, the cycle average (denoted by a tilde) of Eq. (30) becomes

$$\frac{d\tilde{x}_{f3}}{dx_-} = \frac{v_a}{1-v_a} + \frac{e^2 \xi^2(x_-)}{4\mathcal{P}_-^2}. \quad (31)$$

This expression can be recast in terms of a local effective velocity $v_f(x_-)$,

$$\frac{d\tilde{x}_{f3}}{dx_-} = \frac{v_f(x_-)}{1-v_f(x_-)}, \quad (32)$$

where

$$v_f(x_-) = \frac{v_a + (1-v_a)\frac{e^2}{4\mathcal{P}_-^2}\xi^2(x_-)}{1 + (1-v_a)\frac{e^2}{4\mathcal{P}_-^2}\xi^2(x_-)}. \quad (33)$$

Equation (33) shows that the longitudinal velocity of the probability-density peak depends on the local value of ξ and confirms that a constant velocity v_{f*} can only be achieved at a target value $\xi = \xi_*$. Note that even when $v_a = 0$, the probability-density peak follows a trajectory determined by the local potential strength.

B. Expectation-value trajectory

The equations of motion for the position expectation values follow from

$$\frac{d\langle \mathbf{x} \rangle}{dx_-} = \frac{d}{dx_-} \int \mathbf{x} \bar{\psi}(x) \gamma_- \psi(x) d^3 \mathbf{x}.$$

Componentwise, this gives

$$\frac{d\langle x_1 \rangle}{dx_-} = - \int \left(\frac{eA_1}{p_-} \right) |\mathcal{N}(\eta)|^2 |\mathcal{T}(\mathbf{p}_\perp)|^2 d^2 \mathbf{p}_\perp d\eta, \quad (34)$$

$$\frac{d\langle x_2 \rangle}{dx_-} = 0, \quad (35)$$

$$\frac{d\langle x_3 \rangle}{dx_-} = \int \left(\frac{p_3}{p_-} + \frac{e^2 A_1^2}{2p_-^2} \right) |\mathcal{N}(\eta)|^2 |\mathcal{T}(\mathbf{p}_\perp)|^2 d^2 \mathbf{p}_\perp d\eta, \quad (36)$$

where $\mathcal{T}(\mathbf{p}_\perp)$ is assumed to be an even function of $\mathbf{p}_\perp$ and the momenta are on shell: $p_-(\eta, \mathbf{p}_\perp) = E_{\mathbf{p}} - p_3(\eta, \mathbf{p}_\perp)$ with $p_3(\eta, \mathbf{p}_\perp)$ given by Eq. (14).

An approximate expression for the cycle-averaged longitudinal velocity is obtained by expanding the terms in parentheses about $\eta = \langle \eta \rangle$ and $\mathbf{p}_\perp = 0$ to second order, leading to

$$\begin{aligned} \frac{d\langle \tilde{x}_3 \rangle}{dx_-} &\approx \frac{v_{1D}}{1-v_{1D}} \\ &+ \frac{1-v_f v_{1D}}{2\mathcal{P}_-^2(v_f - v_{1D})} \int p_\perp^2 |\mathcal{T}(\mathbf{p}_\perp)|^2 d^2 \mathbf{p}_\perp, \end{aligned} \quad (37)$$

where

$$v_{1D} = \frac{\mathcal{P}_3/\mathcal{E} + (1-\mathcal{P}_3/\mathcal{E})\frac{e^2}{4\mathcal{P}_-^2}\xi^2(x_-)}{1 + (1-\mathcal{P}_3/\mathcal{E})\frac{e^2}{4\mathcal{P}_-^2}\xi^2(x_-)} \quad (38)$$

is the one-dimensional drift velocity of a planar wave packet. The second term in Eq. (37) accounts for the effect of transverse momentum and depends explicitly on v_f (and thus on v_a). This dependence is a consequence of enforcing separability between η and $\mathbf{p}_\perp$, which necessarily introduces correlations among other momentum combinations. For example, a conventional wave packet ($v_a = 0$) exhibits no correlation between p_0 and $\mathbf{p}_\perp$, i.e., $\langle p_0 \mathbf{p}_\perp \rangle - \langle p_0 \rangle \langle \mathbf{p}_\perp \rangle = \mathbf{0}$, whereas an arbitrary-velocity wave packet ($v_a \neq 0$) does: $\langle p_0 \mathbf{p}_\perp \rangle - \langle p_0 \rangle \langle \mathbf{p}_\perp \rangle \neq \mathbf{0}$. Note that the apparent singularity at $v_{1D} = v_f$ is equivalent to the singularity observed in Eq. (27) [cf. Eqs. (37) and (33) and see Appendix B].

C. Trajectory comparison

Figure 2 displays the total probability density $\bar{\psi} \gamma_- \psi$ generated from Eq. (1) for three peak trajectories. The densities are evaluated at $x_1 = x_{f1}(x_-)$, which tracks the oscillating transverse location of the peak. This makes the longitudinal motion more visually apparent. In panels (a)–(c), the out-of-field velocities $v_a = -0.3, 0$, and 19.5 are chosen to yield the in-field velocities $v_{f*} = 0, 0.2$, and -4.1 at $\xi = \xi_*$, respectively. The dashed green line shows $\tilde{x}_{f3}$ and the solid pink line $\langle \tilde{x}_3 \rangle$. To emphasize that $\tilde{x}_{f3}$ is independent of $\langle \tilde{x}_3 \rangle$, the $\langle \tilde{x}_3 \rangle$ trajectories are chosen to be nearly identical in all three cases.

In each case, $\mathcal{P}_- = 3m$, $w = 170m^{-1}$, and $\mathcal{N}(\eta)$ is defined so that its Fourier transform exhibits a region of near-constant amplitude in time, modeled here as a tenth-order super-Gaussian (see Appendix C for details). An illustrative, short-duration vector potential with $\omega = 0.01m$ is used for simultaneous visualization of the in-field and out-of-field dynamics within a single frame. The dotted black line shows the profile of the potential, $\xi(x_-) = \xi_* \exp(-x_-^8/\tau^8)$, where $|e|\xi_*/m = 3$ and $\tau = 2.5 \times 10^4 m^{-1}$, corresponding to a full width at half maximum (FWHM) of $4.8 \times 10^4 m^{-1}$.

Figure 2(a) shows the dynamics illustrated in Fig. 1(e), where a desired in-field velocity $v_{f*} = 0.0$ is achieved by setting $v_a = -0.3$. Figure 2(b) demonstrates that a conventional wave packet ($v_a = 0$) acquires a nonzero velocity ($v_f \neq 0$) upon entering the field. This behavior confirms the prediction of Eq. (33) and establishes that wave packets initialized without prescribed momentum correlations acquire correlations between their dressed momenta upon interaction with a field.

In each panel of Fig. 2, the peak trajectory is only distinguishable within an envelope that follows the expectation-value trajectory. The finite lifetime of the peak is due to interference among partial wave packets, which restricts the translational property described by Eqs. (12) and (17) to a finite duration. In all cases, the envelopes have equal lengths in x_3 , and the peaks are apparent over the interval required to traverse this length. This is most clearly illustrated in Fig. 2(c), where the large disparity between the expectation velocity ($\langle \tilde{v}_3 \rangle = -0.8$ outside the field) and the peak-probability velocity ($v_{f*} = -4.1$) results in a shorter lifetime compared to Figs. 2(a) and 2(b). When the lifetime of the peak-probability

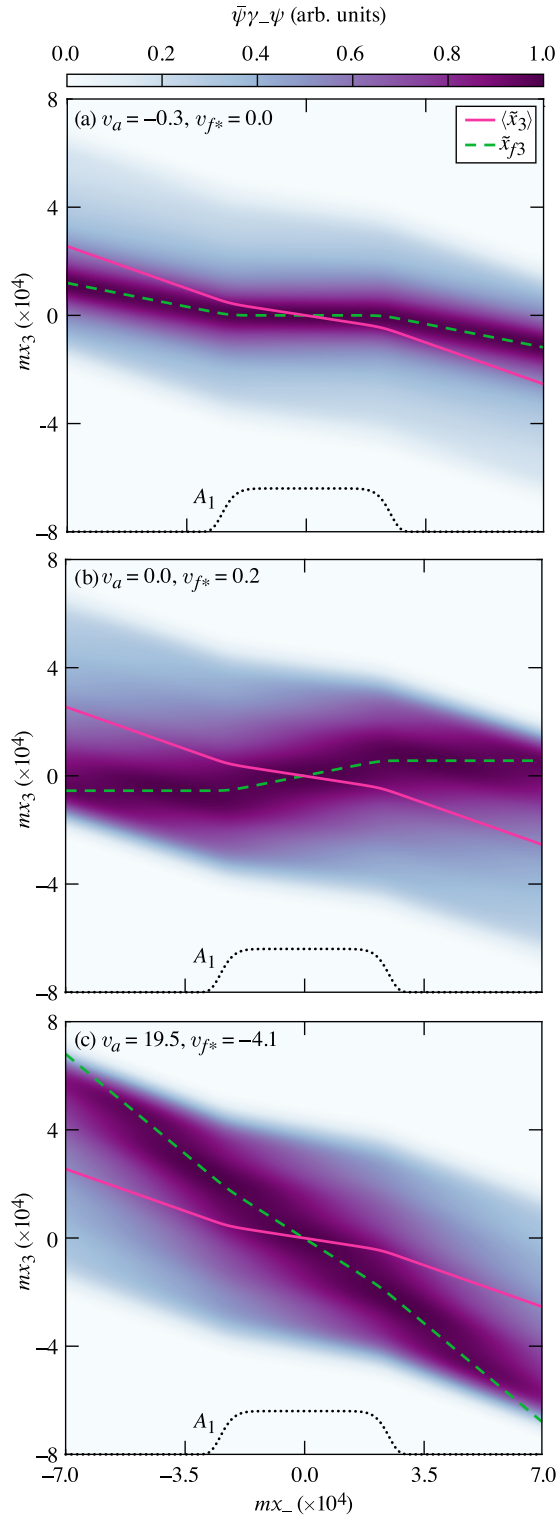


FIG. 2. Propagation of arbitrary-velocity Volkov wave packets in x_- and x_3 at $x_1 = x_{f1}(x_-)$. In panels (a)–(c), the respective out-of-field velocities $v_a = -0.3, 0$, and 19.5 are chosen to yield the desired in-field velocities $v_{f*} = 0, 0.2$, and -4.1 at the maximum potential strength $|e|\xi_*/m = 3$. For each case, the momentum content is peaked at $\mathcal{P}_- = 3m$ and $p_\perp = 0$. The dotted black line shows the location and amplitude of the vector potential. The dashed green line marks the cycle-averaged longitudinal location of the probability-density peak $\bar{x}_{f3}$ [integration of Eq. (32)], and the solid pink line the cycle-averaged expectation value $\langle \bar{x}_3 \rangle$ [integration of the cycle

density and $\xi(x_-)$ are long compared to $1/\omega$ (slowly varying envelope approximation; see Table II), as is the case in Fig. 2, the cycle-averaged trajectory $\bar{x}_{f3}$ captures the motion of the peak; otherwise the unaveraged quantity must be considered. A calculation of the lifetime is presented in Appendix D.

Figure 3(a) compares the three-dimensional trajectories for the case presented in Fig. 2(c) over one cycle $\omega x_- \in [-\pi, \pi]$ near the maximum potential strength $\xi = \xi_*$. In the frame defined by the target velocity $v_{f*} = -4.1$, the peak trajectory (green) traces a figure eight (projection on the bottom wall). This contrasts with the expectation trajectory, which exhibits more complex behavior, diverging from the peak both before and after their coincidence at $\omega x_- = 0$. If instead the frame were defined by the drift velocity v_{1D} [Eq. (38)], the expectation trajectory would trace out a figure eight closely resembling the classic result, whereas the peak would follow the more complex trajectory.

Figure 3(b) presents the numerically computed probability density at $\omega x_- = 0$. For large values of w , the density maintains a transverse Gaussian profile while its maximum traces the figure eight. For smaller values of w , the profile is nearly Gaussian at the beginning of each cycle, but becomes bowed along the trajectory due to terms of $\mathcal{O}(p_1 p_\perp^2)$ in S [Eq. (5)]. These regimes are distinguished by the condition

$$w^2 \gg \left| \frac{e\xi_*}{\omega \mathcal{P}_-^2} \frac{(1 - v_a)}{(\mathcal{P}_3 - v_a \mathcal{E})} \right|^{2/3}. \quad (39)$$

Regardless of this condition, Eqs. (28)–(30) accurately predict the location of the peak as long as the paraxial condition is satisfied.

As discussed in the context of Eq. (37), the expectation trajectory depends on the value of v_{f*} and, by extension, v_a . This is a direct result of constructing the wave packet with an out-of-field correlation between p_3 and $p_\perp$ [Eq. (14)] to produce an in-field correlation between q_3 and $p_\perp$ [Eq. (25)]. Figure 4 shows that this correlation manifests as curved contours in $|\mathcal{N}(\eta)|^2 |\mathcal{T}(\mathbf{p}_\perp)|^2$, with curvature governed by v_{f*} . The values $v_{f*} = 0.0, -0.5$, and -4.1 were chosen to illustrate distinct regimes. When the second term in Eq. (37) is positive, the curvature is convex, shifting the expected light-front velocity above the 1D result [Fig. 4(a)]; when it is negative, the curvature is concave, shifting the light-front velocity below the 1D value [Fig. 4(b)]. For $v_{f*} = v_{1D}^{-1}$, the curvature vanishes, and the light-front velocity coincides with the 1D result [Fig. 4(c)]. In each case, the expectation velocity was obtained numerically from the cycle average of Eq. (36) under no approximation. Because the influence of the correlation on the expectation velocity scales inversely with w [right-hand side of Eq. (37)], a smaller $w = 4\pi m^{-1}$ was used than in Figs. 2 and 3. The distribution $\mathcal{N}(\eta)$ is the same as in Figs. 2 and 3, but with a larger spread, $\Delta\eta = 10^{-3}m$, for improved visualization.

average of Eq. (36)]. In all cases, the peak trajectory moves independently of the expectation trajectory. At $\xi = \xi_*$ and $\xi = 0$, the cycle-averaged expectation velocities are $\langle \bar{v}_3 \rangle \approx -0.24$ and $\langle \bar{v}_3 \rangle \approx -0.8$, respectively.

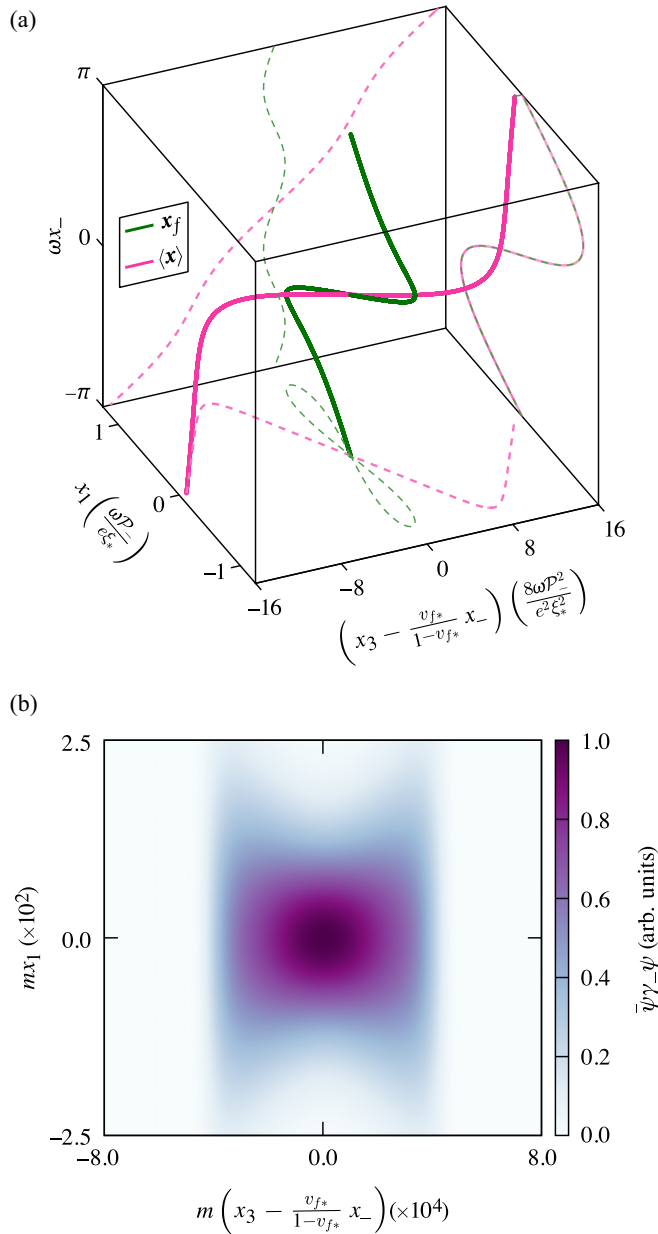


FIG. 3. Three-dimensional trajectory and probability density of the arbitrary-velocity Volkov wave packet presented in Fig. 2(c) with $v_{f*} = -4.1$. The electromagnetic plane wave has a constant strength $|e|\xi_*/m = 3$ and $\omega = 0.01 m$. (a) The peak-probability trajectory [green, integration of Eqs. (28)–(30)] and expectation trajectory [pink, integration of Eqs. (34)–(36)] in a comoving frame defined by v_{f*} , over one cycle $\omega x_- \in [-\pi, \pi]$. In this frame, the peak trajectory traces a figure eight, whereas in the comoving frame defined by v_{ID} , the expectation trajectory traces a figure eight (not shown), consistent with the classic result. (b) The probability density at $\omega x_- = 0$. The transverse profile remains Gaussian while the maximum traces the figure eight. The minimum transverse width of the wave packet is $w = 170 m^{-1}$ and $\mathcal{P}_- = 3 m$.

IV. DISCUSSION AND CONCLUSION

The probability-density peak of a charged-lepton wave packet evaluated on fixed-light-cone slices can move through the field of an electromagnetic plane wave at an arbitrary,

prescribed velocity. This is achieved by constructing the wave packet with a specific momentum correlation prior to its interaction with the field. Upon entering the field, the initial structure gives rise to correlations between the dressed momenta of the constituent Volkov states. These correlations leave an imprint on the expectation trajectory, providing a potentially measurable signature of the prescribed velocity.

Signatures of the moving probability-density peak could be observed in either the far or near field. In the far field, repeated position measurements of identically prepared wave packets provide the probability density. Time-resolved position measurements at a fixed x_3 would reveal a transverse width that changes at a rate determined by v_f . In the near field, repeated measurements at different transverse locations on a spherical detector would reveal an angularly dependent arrival time determined by v_a . This is a consequence of the wave packet carrying a correlation between energy and transverse momentum, $\langle p_0 \mathbf{p}_\perp \rangle - \langle p_0 \rangle \langle \mathbf{p}_\perp \rangle \neq \mathbf{0}$ when $v_a \neq 0$, which gives rise to an angularly dependent velocity. The interaction with the electromagnetic field adds an additional signature: The arrival times are further modified by a momentum and v_a -dependent phase acquired in the field [e.g., the last term in Eq. (27), which results in the correction in Eq. (37)].

In configuration space and outside of the field, arbitrary-velocity Volkov wave packets have constant spherical or hyperbolic wave fronts with respect to the comoving coordinate $x_3 - v_a x_0$. A potential experimental realization of such a wave packet could involve introducing a time-dependent curvature phase at a fixed value of x_3 . This can be accomplished using the Kapitza-Dirac effect. More specifically, two counterpropagating light waves with time-varying amplitudes form a standing wave that acts as a “lens,” focusing different temporal slices of the wave packet to different locations. This mechanism is directly analogous to optical “flying-focus” techniques [27,28,30,44]. In this context, v_a can be interpreted as the “focal” velocity.

The theory of arbitrary-velocity Volkov wave packets was presented for linearly polarized electromagnetic plane waves, and the examples performed for wave packets with longitudinal momentum antiparallel to the plane-wave propagation direction. However, both the theory and examples generalize to arbitrary polarizations, relative orientations, and peak velocity directions. An in-field velocity vector $\mathbf{v}_{f*}$ can be realized for a general plane-wave four-potential $A(x_-)$. Outside the field, this generalization is achieved by setting $\eta = p_0 - \mathbf{v}_a \cdot \mathbf{p}$. Because the transverse dressed momenta equals the transverse momenta, the in-field and out-of-field transverse velocities are identical. As a result, a specific $\mathbf{v}_{f*}$ can be obtained by following the procedure in Sec. II C with $\xi_*^2/2$ replaced by the cycle average of $-A \cdot A$ at the target strength. As before, the component of $\mathbf{v}_a$ along the propagation direction of the plane wave is chosen to ensure that the target velocity is reached at ξ_* .

More broadly, this work highlights that electromagnetic fields induce spatiotemporal structure in charged-lepton wave packets. Even a conventional wave packet, separable in energy and transverse momentum, acquires a correlation among its dressed momenta during its interaction with an electromagnetic wave. These correlations cause the probability-density peak to follow a trajectory that deviates from the expectation

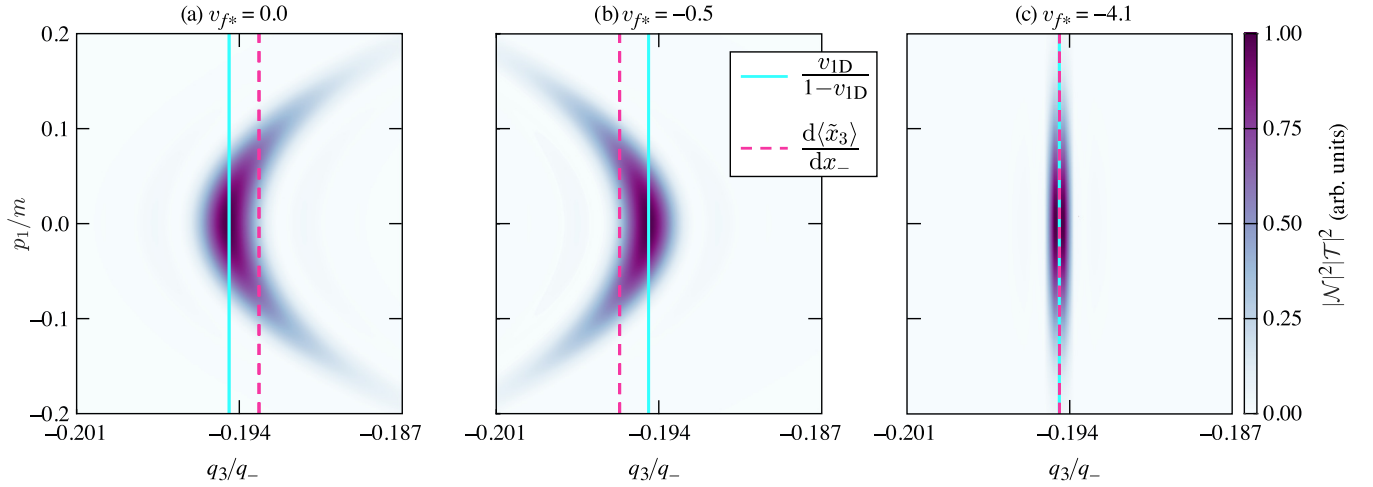


FIG. 4. The momentum density $|\mathcal{N}(\eta)|^2 |\mathcal{T}(\mathbf{p}_\perp)|^2$ as a function of q_3/q_- and p_1 , illustrating the structure of the correlation built into the wave packet for different values of v_{f*} . In panels (a)–(c), the respective values $v_{f*} = 0.0, -0.5$, and -4.1 were chosen to produce distinct curvatures with respect to p_1 . (a) The convex curvature shifts the expectation velocity [dashed pink line, integration of cycle-averaged Eq. (36)] above its one-dimensional value (solid blue line), $v_{1D}/(1 - v_{1D}) = -0.1944$ [see second term on right-hand side of Eq. (37)]. (b) The concave curvature shifts the expectation velocity below its one-dimensional value. (c) The curvature vanishes in the special case $v_{f*} = v_{1D}^{-1} = -4.1$. Here, $|e|\xi_*/m = 3$, $v_{1D} = -0.24$, $\mathcal{P}_- = 3m$, and $w = 4\pi m^{-1}$.

trajectory. Thus, the dynamics of structured wave packets are more than a product of wavefunction engineering; they are fundamental to the interaction of charged particles with electromagnetic fields. In fact, the structure of a conventional wave packet is only preserved during interactions with stationary potentials. These results point to space-time correlations as a general feature of wave packet dynamics, with implications beyond the leptonic systems studied here. For instance, analogous correlations would arise in conventional electromagnetic waves propagating through dynamic and structured media, where the refractive index acts as the potential.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [47], embargo periods may apply.

APPENDIX A: REPARAMETERIZATION AT FIXED x_0

The calculations in this article use the density $\bar{\psi}\gamma_-\psi$, which is evaluated on fixed- x_- slices and is therefore parametrized by x_- . Reparameterization on fixed- x_0 slices requires computing the density $\bar{\psi}\gamma_0\psi$. The construction of $\bar{\psi}\gamma_0\psi$ follows the same steps as in Sec. II. Specifically, Eqs. (1), (3), (5), and (15) are combined to build $\bar{\psi}\gamma_0\psi$ outside of the field [44]. Assuming the conventional normalization $\bar{u}\gamma_0 u = 2p_0$, the derivation relies on the same two ingredients as before: (1) that $\mathcal{N}(\eta)$ is narrowly peaked about some characteristic value of η and (2) that $\langle \eta \mathbf{p}_\perp \rangle - \langle \eta \rangle \langle \mathbf{p}_\perp \rangle = \mathbf{0}$, where expectation values are now computed on fixed- x_0 slices.

When the lepton is dressed by the field, there are two fundamental differences between $\bar{\psi}\gamma_0\psi$ and $\bar{\psi}\gamma_-\psi$ on fixed- x_0 slices. First, $\bar{\psi}\gamma_0\psi$ is built from the momentum-space product $\bar{\mathcal{V}}(p', x_-)\gamma_0\mathcal{V}(p, x_-)$, which retains explicit dependence on $A_1(x_-)$, unlike $\bar{\mathcal{V}}(p', x_-)\gamma_-\mathcal{V}(p, x_-)$. As a result, $\int \bar{\psi}\gamma_0\psi d^3\mathbf{x}$ varies with x_0 . The presence of $A_1(x_-)$ in the bispinor product gives rise to a modulation of the peak density with period π/ω along x_3 . As a result, the mapping

$$\frac{d\mathbf{x}_f}{dx_0} = \frac{d\mathbf{x}_f(x_-)/dx_-}{1 + dx_{f3}(x_-)/dx_-} \quad (\text{A1})$$

describes the location of the minimum transverse width of $\bar{\psi}\gamma_0\psi$, which only periodically coincides with its maximum. Second, at a fixed x_0 , a wave packet simultaneously samples

different potential strengths due to its nonzero length. Since Eq. (30) depends on the local potential strength and phase, the wave packet stretches and compresses along x_3 , causing a longitudinal asymmetry about its minimum transverse width.

APPENDIX B: VELOCITY LIMITS OF EQ. (14)

This Appendix examines three special cases of the velocity parameter: $v_a = \pm 1$ and $v_a = \mathcal{P}_3/\mathcal{E}$, with the latter corresponding to the singularity in Eqs. (24), (27), and (37). For $v_a = \pm 1$, $\eta = p_{\mp}$, and Eq. (14) has one finite branch [44]

$$p_3(\eta, p_{\perp}) = \mp \frac{\eta^2 - m^2 - p_{\perp}^2}{2\eta}, \quad (\text{B1})$$

which is real for all values of $p_{\mp}$ and $p_{\perp}$. However, care must still be taken to exclude counterpropagating longitudinal content in the transverse distribution $\mathcal{T}(\mathbf{p}_{\perp})$ by enforcing $p_{\perp}^2 < \eta^2 - m^2$. The special case of wave packets in an electromagnetic plane wave with $v_a = 1$ and $\eta = p_{-}$ was investigated in Ref. [45].

For $v_a = \mathcal{P}_3/\mathcal{E}$, the on-shell longitudinal momentum evaluated at $\eta = \langle \eta \rangle$ is given by

$$p_3(\langle \eta \rangle, p_{\perp}) = \mathcal{P}_3 \pm i \frac{p_{\perp} \mathcal{E}}{m}. \quad (\text{B2})$$

This marks the threshold at which p_3 acquires an imaginary component for any nonzero $p_{\perp}$. To exclude these evanescent modes, the product $\mathcal{N}(\langle \eta \rangle) f(p, \langle \eta \rangle)$ must be zero for all $p_{\perp} > 0$. This can be accomplished in two ways: Either $f(p, \eta) \propto \delta(\mathbf{p}_{\perp})$ or $\mathcal{N}(\eta)$ vanishes at $\eta = \langle \eta \rangle$ with local maxima about this value. In the latter case, the expansions performed in the main text must be taken about each maxima to accurately characterize the spatiotemporal evolution of the wave packet.

APPENDIX C: NUMERICAL DETAILS

The densities displayed in Figs. 2–4 are numeric evaluations of Eqs. (1), (3), (5), and (15), with the v_a predicted by Eq. (23) used as an input. While the theory is three dimensional, all numerical examples use a transverse distribution with reduced dimensionality for computational efficiency. The considered distribution is

$$|\mathcal{T}(\mathbf{p}_{\perp})|^2 = \frac{w}{(2\pi)^{1/2}} \exp\left(-\frac{w^2 p_{\perp}^2}{2}\right) \delta(p_2), \quad (\text{C1})$$

where w is the minimum transverse width (with units of space, m^{-1}), such that $\int |\mathcal{T}(\mathbf{p}_{\perp})|^2 d\mathbf{p}_{\perp} = 1$. For Figs. 2 and 3, $w = 170 m^{-1}$; for Fig. 4, $w = 4\pi m^{-1}$.

In each numerical calculation, the function $\mathcal{N}(\eta)$ is selected so that its Fourier transform in time at fixed $x_3 - v_a x_0$, $\hat{\mathcal{N}}(x_0)$, has an extended region of constant amplitude. This ensures that the overall wave packet is longer than the moving probability peak (see Appendix D). Specifically, for the calculations,

$$\hat{\mathcal{N}}(x_0) = C \exp[-(x_0 \Delta\eta)^{10}], \quad (\text{C2})$$

where $\Delta\eta$ characterizes the spread in η and C is a constant. The corresponding $\mathcal{N}(\eta)$ does not have a simple closed form

and is therefore obtained by taking the numerical Fourier transform of Eq. (C2). The coefficient C is then computed so that $\int |\mathcal{N}(\eta)|^2 d\eta = 1$. With these normalizations for $|\mathcal{N}(\eta)|^2$ and $|\mathcal{T}(\mathbf{p}_{\perp})|^2$, $\int \psi \gamma_{\perp} \psi d^3\mathbf{x} = 1$ is guaranteed.

The length of the wave packet is given by $L \approx 2|v_a - \mathcal{P}_3/\mathcal{E}|/\Delta\eta$ (see Appendix D). In Figs. 2 and 3, the value of $\Delta\eta$ is chosen to give the same length $L = 1.6 \times 10^5 m^{-1}$ (FWHM $1.54 \times 10^5 m^{-1}$) for the different v_a values. The specific values are $\Delta\eta = 5.9 \times 10^{-6}$, 1.0×10^{-5} , $2.6 \times 10^{-4} m$ for $v_a = -0.3, 0, 19.5$, respectively. In all panels of Fig. 4, $\Delta\eta = 1 \times 10^{-3} m$, which was used for better visualization of the distributions.

The amplitude of the vector potential used for the numerical calculations is given by

$$\xi(x_{-}) = \xi_{*} \exp(-x_{-}^8/\tau^8), \quad (\text{C3})$$

with $|e|\xi_{*}/m = 3$ and $\tau = 2.5 \times 10^4 m^{-1}$. The corresponding FWHM is $4.8 \times 10^4 m^{-1}$, which spans 70 wavelengths ($\lambda = 2\pi/\omega$). The selected profile provides a long region where $\xi \approx \xi_{*}$, making it easier to observe that the velocity v_{f*} is achieved at ξ_{*} . Equation (C3) is input into Eq. (5), and the integrals over x_{-} are evaluated numerically.

APPENDIX D: LENGTH AND LIFETIME OF THE PEAK-PROBABILITY TRAJECTORY

In general, the probability-density peak and overall envelope of the wave packet travel at different velocities. The peak moves at v_a or v_f , depending on whether it is inside or outside the field, whereas the envelope propagates at the expectation velocity. Because of this mismatch, the peak remains visible only while it overlaps the envelope. Its lifetime is therefore determined by the length of the envelope,

$$L \approx 2 \left| v_a - \frac{\mathcal{P}_3}{\mathcal{E}} \right| \frac{1}{\Delta\eta}, \quad (\text{D1})$$

where $\Delta\eta$ is the spectral width. The corresponding envelope duration is $\mathcal{E}L/|\mathcal{P}_3|$. Throughout, $\Delta\eta$ is selected so that for $|v_a| < 1$, L exceeds the length of the moving probability peak,

$$\mathcal{Z} \equiv \frac{1}{2} |\mathcal{P}_3 - v_a \mathcal{E}| w^2, \quad (\text{D2})$$

while for $|v_a| > 1$, $\mathcal{E}L/|\mathcal{P}_3|$ exceeds the duration of the peak, $\mathcal{Z}/|v_a|$. The length $\mathcal{Z}$ is directly analogous to a Rayleigh range. Note that in the reduced transverse dimensionality used for the numerical calculations, $\mathcal{Z}$ is elongated by a factor of $3^{1/2}$, i.e., $\mathcal{Z} \rightarrow 3^{1/2} \mathcal{Z}$.

To leading order, the rising and falling edges of the envelope follow the trajectories

$$\begin{aligned} \langle x_3 \rangle_{\text{edge}} &\approx \frac{\mathcal{P}_3}{\mathcal{P}_{-}} x_{-} + \frac{e^2}{2\mathcal{P}_{-}^2} \int_0^{x_{-}} A_1^2(x'_{-}) dx'_{-} \\ &\pm \frac{\frac{1}{2}L}{1 - \mathcal{P}_3/\mathcal{E}}, \end{aligned} \quad (\text{D3})$$

while the probability-density peak follows [Eq. (30)]

$$x_{f3} = \frac{v_a}{1 - v_a} x_{-} + \frac{e^2}{2\mathcal{P}_{-}^2} \int_0^{x_{-}} A_1^2(x'_{-}) dx'_{-}. \quad (\text{D4})$$

The intersections of these trajectories,

$$x_{f3} - \langle x_3 \rangle_{\text{edge}} = 0, \quad (\text{D5})$$

determine the duration Δx_- over which the peak persists, and the corresponding distance traversed along x_3 . If Δx_- is large compared to $1/\omega$, the cycle-averaged trajectory $\tilde{x}_{f3}$ is a meaningful description of the dynamics of the peak; otherwise, the unaveraged trajectory $\mathbf{x}_f$ is needed to accurately describe the dynamics. Generally, these intercepts must be computed numerically. However, when $\Delta x_- \gg 1/\omega$ and for a constant potential strength $\xi = \xi_*$, the lifetime in x_0 can be computed

analytically by cycle averaging Eq. (D5), providing

$$\Delta x_0 = \frac{L}{|1 - \mathcal{P}_3/\mathcal{E}|} \left| \frac{1 - v_{\text{ID}}}{v_{f*} - v_{\text{ID}}} \right|, \quad (\text{D6})$$

where v_{ID} is evaluated at $\xi = \xi_*$ [Eq. (38)]. In the absence of an electromagnetic field, $\xi_* = 0$, $v_{\text{ID}} = \mathcal{P}_3/\mathcal{E}$, $v_{f*} = v_a$, and Eq. (D6) reduces to

$$\Delta x_0 = \frac{L}{|v_a - \mathcal{P}_3/\mathcal{E}|}, \quad (\text{D7})$$

consistent with the out-of-field result in Ref. [44].

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- [1] D. M. Volkov, Über eine klasse von lösungen der diracschen gleichung, *Z. Physik* **94**, 250 (1935).
 - [2] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics* (Butterworth-Heinemann, Oxford, UK, 1982), Vol. 4.
 - [3] W. Furry, On bound states and scattering in positron theory, *Phys. Rev.* **81**, 115 (1951).
 - [4] A. Di Piazza, C. Müller, K. Hatsagortsyan, and C. H. Keitel, Extremely high-intensity laser interactions with fundamental quantum systems, *Rev. Mod. Phys.* **84**, 1177 (2012).
 - [5] A. Gonoskov, T. Blackburn, M. Marklund, and S. Bulanov, Charged particle motion and radiation in strong electromagnetic fields, *Rev. Mod. Phys.* **94**, 045001 (2022).
 - [6] A. Fedotov, A. Ilderton, F. Karbstein, B. King, D. Seipt, H. Taya, and G. Torgrimsson, Advances in QED with intense background fields, *Phys. Rep.* **1010**, 1 (2023).
 - [7] M. Piccardo, M. O. Cernaianu, J. P. Palastro, A. Arefiev, C. Thaur, J. Vieira, D. H. Froula, and V. Malka, Trends in relativistic laser-matter interaction: The promises of structured light, *Optica* **12**, 732 (2025).
 - [8] A. F. Abouraddy, M. Yessenov, I. Divliansky, and A. T. Watnik, New frontiers in spatiotemporally structured light, *Opt. Photon. News* **36**, 38 (2025).
 - [9] H. Okamoto, T. Latychevskaia, and H.-W. Fink, A quantum mechanical scheme to reduce radiation damage in electron microscopy, *Appl. Phys. Lett.* **88**, 164103 (2006).
 - [10] B. J. McMoran, A. Agrawal, I. M. Anderson, A. A. Herzing, H. J. Lezec, J. J. McClelland, and J. Unguris, Electron vortex beams with high quanta of orbital angular momentum, *Science* **331**, 192 (2011).
 - [11] S. Lloyd, M. Babiker, and J. Yuan, Quantized orbital angular momentum transfer and magnetic dichroism in the interaction of electron vortices with matter, *Phys. Rev. Lett.* **108**, 074802 (2012).
 - [12] R. Shiloh, Y. Lereah, Y. Lilach, and A. Arie, Sculpturing the electron wave function using nanoscale phase masks, *Ultramicroscopy* **144**, 26 (2014).
 - [13] I. Kaminer, J. Nemirovsky, M. Rechtsman, R. Bekenstein, and M. Segev, Self-accelerating Dirac particles and prolonging the lifetime of relativistic fermions, *Nat. Phys.* **11**, 261 (2015).
 - [14] J. Harris, V. Grillo, E. Mafakheri, G. C. Gazzadi, S. Frabboni, R. W. Boyd, and E. Karimi, Structured quantum waves, *Nat. Phys.* **11**, 629 (2015).
 - [15] L. J. Wong, N. Rivera, C. Murdia, T. Christensen, J. D. Joannopoulos, M. Soljačić, and I. Kaminer, Control of quantum electrodynamical processes by shaping electron wavepackets, *Nat. Commun.* **12**, 1700 (2021).
 - [16] M. C. Chirita Mihaila, P. Weber, M. Schneller, L. Grandits, S. Nimmrichter, and T. Juffmann, Transverse electron-beam shaping with light, *Phys. Rev. X* **12**, 031043 (2022).
 - [17] A. G. Campos, K. Z. Hatsagortsyan, and C. H. Keitel, Non-spreading relativistic electron wavepacket in a strong laser field, *Phys. Rev. Res.* **6**, 023040 (2024).
 - [18] C. I. Velasco and F. J. García de Abajo, Free-space optical modulation of free electrons in the continuous-wave regime, *Phys. Rev. Lett.* **134**, 123804 (2025).
 - [19] A. Sainte-Marie, O. Gobert, and F. Quere, Controlling the velocity of ultrashort light pulses in vacuum through spatio-temporal couplings, *Optica* **4**, 1298 (2017).
 - [20] D. H. Froula, D. Turnbull, A. S. Davies, T. J. Kessler, D. Haberberger, J. P. Palastro, S.-W. Bahk, I. A. Begishev, R. Boni, S. Bucht, *et al.*, Spatiotemporal control of laser intensity, *Nat. Photon.* **12**, 262 (2018).
 - [21] H. E. Kondakci and A. F. Abouraddy, Optical space-time wave packets having arbitrary group velocities in free space, *Nat. Commun.* **10**, 929 (2019).
 - [22] J. Palastro, J. Shaw, P. Franke, D. Ramsey, T. Simpson, and D. Froula, Dephasingless laser wakefield acceleration, *Phys. Rev. Lett.* **124**, 134802 (2020).
 - [23] C. Caizergues, S. Smartsev, V. Malka, and C. Thaur, Phase-locked laser-wakefield electron acceleration, *Nat. Photon.* **14**, 475 (2020).
 - [24] T. T. Simpson, D. Ramsey, P. Franke, N. Vafaei-Najafabadi, D. Turnbull, D. H. Froula, and J. P. Palastro, Nonlinear spatiotemporal control of laser intensity, *Opt. Express* **28**, 38516 (2020).
 - [25] S. W. Jolly, O. Gobert, A. Jeandet, and F. Quéré, Controlling the velocity of a femtosecond laser pulse using refractive lenses, *Opt. Express* **28**, 4888 (2020).
 - [26] M. Yessenov, L. A. Hall, K. L. Schepler, and A. F. Abouraddy, Space-time wave packets, *Adv. Opt. Photon.* **14**, 455 (2022).
 - [27] T. T. Simpson, D. Ramsey, P. Franke, K. Weichman, M. V. Ambat, D. Turnbull, D. H. Froula, and J. P. Palastro, Spatiotemporal control of laser intensity through cross-phase modulation, *Opt. Express* **30**, 9878 (2022).
 - [28] D. Ramsey, A. Di Piazza, M. Formanek, P. Franke, D. Froula, B. Malaca, W. Mori, J. Pierce, T. Simpson, J. Vieira, *et al.*, Exact solutions for the electromagnetic fields of a flying focus, *Phys. Rev. A* **107**, 013513 (2023).

- [29] J. R. Pierce, J. P. Palastro, F. Li, B. Malaca, D. Ramsey, J. Vieira, K. Weichman, and W. B. Mori, Arbitrarily structured laser pulses, *Phys. Rev. Res.* **5**, 013085 (2023).
- [30] R. Almeida, D. Ramsey, A. F. Abouraddy, J. P. Palastro, and J. Vieira, Universal structure of propagation-invariant optical pulses, *Opt. Lett.* **50**, 3393 (2025).
- [31] M. Yessenov and A. F. Abouraddy, Optical spatiotemporal Fourier synthesis: Tutorial, *J. Opt. Soc. Am. A* **42**, 1295 (2025).
- [32] D. Ramsey, P. Franke, T. Simpson, D. Froula, and J. Palastro, Vacuum acceleration of electrons in a dynamic laser pulse, *Phys. Rev. E* **102**, 043207 (2020).
- [33] J. Palastro, B. Malaca, J. Vieira, D. Ramsey, T. Simpson, P. Franke, J. Shaw, and D. Froula, Laser-plasma acceleration beyond wave breaking, *Phys. Plasmas* **28**, 013109 (2021).
- [34] A. Liberman, A. Golovanov, S. Smartsev, S. Tata, I. A. Andriyash, S. Benracassa, E. Y. Levine, E. Kroupp, and V. Malka, First direct observation of a wakefield generated with structured light, *Nat. Commun.* **16**, 10957 (2025).
- [35] A. Howard, D. Turnbull, A. Davies, P. Franke, D. Froula, and J. Palastro, Photon acceleration in a flying focus, *Phys. Rev. Lett.* **123**, 124801 (2019).
- [36] D. Ramsey, B. Malaca, A. Di Piazza, M. Formanek, P. Franke, D. Froula, M. Pardal, T. Simpson, J. Vieira, K. Weichman, *et al.*, Nonlinear Thomson scattering with ponderomotive control, *Phys. Rev. E* **105**, 065201 (2022).
- [37] A. Kabacinski, E. Oliva, F. Tissandier, J. Gautier, M. Kozlová, J.-P. Goddet, I. A. Andriyash, C. Thauray, P. Zeitoun, and S. Sebban, Spatio-temporal couplings for controlling group velocity in longitudinally pumped seeded soft X-ray lasers, *Nat. Photon.* **17**, 354 (2023).
- [38] T. T. Simpson, J. J. Pigeon, M. V. Ambat, K. G. Miller, D. Ramsey, K. Weichman, D. H. Froula, and J. P. Palastro, Spatiotemporal control of two-color terahertz generation, *Phys. Rev. Res.* **6**, 013041 (2024).
- [39] S. Fu, B. Groussin, Y. Liu, A. Mysyrowicz, V. Tikhonchuk, and A. Houard, Steering laser-produced THz radiation in air with superluminal ionization fronts, *Phys. Rev. Lett.* **134**, 045001 (2025).
- [40] D. Ramsey, B. Malaca, T. Simpson, M. Formanek, L. Mack, J. Vieira, D. Froula, and J. Palastro, X-ray free-electron lasing in a flying-focus undulator, *Commun. Phys.* **8**, 113 (2025).
- [41] M. Formanek, J. P. Palastro, D. Ramsey, S. Weber, and A. Di Piazza, Signatures of vacuum birefringence in low-power flying focus pulses, *Phys. Rev. D* **109**, 056009 (2024).
- [42] M. S. Formanek, J. P. Palastro, D. Ramsey, and A. Di Piazza, Enhanced quantum radiation with flying-focus laser pulses, *Phys. Rev. A* **112**, L051102 (2025).
- [43] L. A. Hall, M. Yessenov, M. A. Romer, and T. A. F. Abouraddy, Long-distance axial spectral encoding using space-time wave packets, *Opt. Lett.* **50**, 4698 (2025).
- [44] J. Palastro, D. Ramsey, M. Formanek, J. Vieira, and A. Di Piazza, Superluminal matter waves, *Phys. Rev. Res.* **6**, 043075 (2024).
- [45] A. Di Piazza, M. Formanek, D. Ramsey, and J. P. Palastro, Radiation by a laser-driven flying-focus electron wave packet, *Phys. Rev. D* **110**, 116010 (2024).
- [46] D. Seipt, Volkov states and non-linear Compton scattering in short and intense laser pulses, [arXiv:1701.03692](https://arxiv.org/abs/1701.03692).
- [47] D. Ramsey, Arbitrary-Velocity Volkov Wavepackets (DATA) [Dataset], Zenodo, 2026, <https://doi.org/10.5281/zenodo.19558295>.