Generalized squeezing as a witness of various quantum properties

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Quantum systems can be prepared in an infinite continuum of states, but only some of them can be used as resources for quantum technologies. Discerning whether a specific quantum state falls into this class is often a challenging task. We show that it can be performed by looking at the squeezing of the quantum states—a scenario in which the variance of some observable operator is suppressed below the threshold given by the nonuseful states. We discuss the general concept first; then, we illustrate the approach by evaluating cubic nonlinear squeezing in the system of collective atomic spins.

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I. INTRODUCTION

The applications of quantum physics are built upon specific quantum states and their properties. Quantum computing relies on quantum states that have both the computational capacity and potential for fault tolerance [1–8], and quantum metrology employs probes that have heightened sensitivity to some specific form of disturbance [9–19]. The specific property required for the protocol varies; in quantum computation, and more generally for the field of quantum information processing, the desired quantum states arise from the assumptions of the envisioned theoretical model—squeezed states [20–22] and their superpositions [1,3,23,24], specific superpositions of Fock states [6,25], or quantum states produced by some currently unfeasible experimental processes [26–29]. In contrast, quantum metrology is fundamentally concerned with minimizing the uncertainty of specific observables to enhance the sensitivity of measurement protocols. In such applications, the variance of a quantum state with respect to a chosen operator directly dictates the achievable precision, as captured by the quantum Cramér-Rao bound [9,17]. The most prominent example is the use of quadrature-squeezed light in gravitational wave detection [11,30,31], where reduced variance in one quadrature significantly improves the signal-to-noise ratio. Other notable instances include the employment of Fock states [18,32,33] and their engineered superpositions [34,35], which are specifically tailored to measure phase shifts and ultrasmall displacements with precision beyond classical limits. In these contexts, identifying and certifying states with optimized variances, particularly in nonlinear observables, becomes essential.

Interestingly, these two approaches are not disjunctive as the quantum states required for quantum information processing can often be defined as eigenstates of some observables [2,3,21,36,37]. The variance of these operators can often directly determine the amount of noise that is introduced in measurement-induced implementations of the protocols [21,26,27,38]. Consequently, the variance of these operators can be used to characterize the quality of approximative realizations of the states. The reduced variance in the specific observable, its squeezing, can also be seen as a quantum resource [39,40]—a feature of quantum states necessary for the applications, which cannot always be feasibly generated, the most prominent examples being quantum entanglement [41] and quantum non-Gaussianity [42,43]. Interestingly, the variance reduced below some threshold can be taken as a witness of those resources being present in the state.

This is nicely demonstrated in the case of cubic nonlinearity. For bosonic modes of light, this single deterministic non-Gaussian nonlinear operation is sufficient for optical quantum computation [3,29,44] together with Gaussian operation. In collective spins, the cubic operation can serve as a resource for quantum metrological protocols [17,45–47] and counterdiabatic driving for fast preparation of Dicke states [48]. The cubic operation can be deterministically implemented with the help of Gaussian operations and a single non-Gaussian ancillary state [3,28,49,50]. The ideal resource state is an eigenstate of the cubic operator, a nonphysical state that can be prepared only as an approximation [51–53]. For these approximations, the variance of the cubic operator depends on many things, one of them being the stellar rank of the approximation [54,55]. This is because the stellar rank serves as an upper bound for achievable nonlinear squeezing [2,37,51,54–56]. As a result, the variance of the nonlinear operator, which reflects the degree of squeezing, can also serve as an indirect witness of stellar rank and, more broadly, of non-Gaussian features of the state. Stellar rank itself is a witness for non-Gaussianity [56]; however, in contrast to stellar rank, our approach additionally captures other important characteristics of the tested state, such as specific noise reduction. This added sensitivity to squeezing is especially valuable in the preparation and analysis of states for quantum metrology and related applications.

Our work contributes directly to the ongoing development of practical quantum technologies by introducing a versatile, variance-based framework centered on nonlinear squeezing,

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which enables both the evaluation and preparation of quantum states with enhanced metrological utility. We extend the concept of squeezing beyond the linear regime and introduce a normalized nonlinear squeezing measure that functions as a witness of quantum features—guaranteeing the presence of a relevant quantum resource. In this paper, we explore the implications of this approach for state preparation and verification, and demonstrate its applicability through the example of cubic squeezing in systems of collective spins.

II. SQUEEZING

In quantum physics, squeezing refers to the process of producing quantum states in which variance with respect to a specifically chosen observable is reduced, at the cost of increasing variance in the conjugate (noncommuting) observables. In practice, the variance of the tested (squeezed) state $\hat{\rho}_T$ is often compared to some unsqueezed *benchmark state* $\hat{\rho}_B$, often called a *free state* $\hat{\rho}_F$, and the *specific* squeezing is given by the relative quantity

$$\xi = \frac{\mathrm{var}_{\hat{\rho}_T}[\hat{\mathcal{O}}]}{\mathrm{var}_{\hat{\rho}_B}[\hat{\mathcal{O}}]}, \tag{1}$$

where $\mathrm{var}_{\hat{\rho}}[\hat{\mathcal{O}}] = \mathrm{Tr}[\hat{\rho}\hat{\mathcal{O}}^2] - (\mathrm{Tr}[\hat{\rho}\hat{\mathcal{O}}])^2$.

In some situations, especially when targeting specific eigenstates of an observable, it is sufficient to evaluate squeezing purely through the second moment form of the operator [2].

States in which $\xi = 0$ are eigenstates of the operator $\hat{\mathcal{O}}$. Consequently, the ratio in Eq. (1) quantifies the deviation of the tested state from the ideal target eigenstate and is often directly related to the amount of quantum fluctuations relevant for metrological or quantum information tasks.

Squeezing can, in principle, be observed in any quantum system, but it plays the most important role in systems with large Hilbert space dimensions, where the preparation of eigenstates of specific observables is often not straightforward.

For the harmonic oscillator with quadrature operators $\hat{x}$ and $\hat{p}$ satisfying $[\hat{x}, \hat{p}] = i$, squeezing of a given state $\hat{\rho}_T$ in the quadrature $\hat{x}$ is defined by Eq. (1), where $\hat{\mathcal{O}} = \hat{x}$. The maximally squeezed state is a nonphysical quadrature eigenstate with infinite energy, and the benchmark state is typically the classical vacuum ground state, $\hat{\rho}_B = |0\rangle\langle 0|$, for which $\mathrm{var}_{|0\rangle\langle 0|}\hat{x} = 1/2$. The normalization ensures that values $\xi < 1$ indicate nonclassicality of the tested state. As a metric, squeezing in quantum harmonic oscillators is often more practical and widely applicable than, for example, fidelity or distance to the target state [57,58], which are difficult to evaluate when the target state is not normalized.

Squeezing can also be generalized to multimode systems. For instance, consider a two-mode squeezed state, which in the idealized case may be a nonphysical, maximally entangled state. In this scenario, squeezing can still be characterized by an appropriately chosen operator. For example, we consider the operator $\hat{\mathcal{O}} = \sqrt{(\hat{x}_1 - \hat{x}_2)^2 + (\hat{p}_1 + \hat{p}_2)^2}$ ($\hat{x}_1$, $\hat{x}_2$ and $\hat{p}_1$, $\hat{p}_2$ are the quadrature operators of the individual modes). Its second moment is equal to zero only for the nonphysical maximally entangled state. Minimizing it over the set of local operations with classical communication [40] and comparing it to the bound obtained by the separable states allows us to recreate the Duan criterion [59] for the detection of quantum entanglement. In this sense, our generalized squeezing framework naturally extends to serve as a witness of separability in bipartite systems, further illustrating that squeezing, when properly defined, can reveal a wide range of quantum properties beyond non-Gaussianity.

A similar situation exists in the system of collective spins of N atoms [60–62], which is discrete but approaches continuous nature for large N. In such a system, the angular momentum operators $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ can be used for both theoretical description and practical experimental measurements [63–67]. In these scenarios, the spin of the system is aligned into a single direction, meaning, for example, $\langle\hat{J}_x\rangle = \langle\hat{J}_y\rangle = 0$, and the metrological properties depend on second moments of these operators, $\mathrm{var}(\hat{J}_x)$, for example. The natural choice of the normalization benchmark is the ground state of the operator $\hat{J}_z$, an analogy of the vacuum state in a harmonic oscillator, for which $\mathrm{var}(\hat{J}_x) = N/4$.

Squeezed states are used as resources in quantum protocols. Furthermore, operations manipulating the system can be divided into those that can generate squeezing and those that can only transform the state without improving the squeezing [68]. Unitary operators $\hat{U}_F$ that perform such transformations are called *free operations* in quantum resource theories [39,40,69,70] and allow us to define *resource* squeezing as

$$\xi_{\hat{\mathcal{O}}}(\hat{\rho}_T) = \frac{\min_{\hat{U}_F} \mathrm{var}_{\hat{\rho}_T}(\hat{U}_F^\dagger \hat{\mathcal{O}} \hat{U}_F)}{\mathrm{var}_{\hat{\rho}_B}(\hat{\mathcal{O}})}. \tag{2}$$

The free operations always depend on the physical system. For the quadrature squeezing, the sets of free operations are composed of displacement and phase shift in harmonic oscillator and rotation in systems of collective spins, while the benchmark states are coherent states. For squeezing in general observables that can be expressed as nonlinear combination of quadrature or angular momentum operators, the sets of free operations and benchmark states have the largest practical impact when driven by specific experimental limitations. In particular, selecting Gaussian states and Gaussian operations as benchmark states and free operations allows the quantities in Eqs. (1) and (2) to witness non-Gaussianity present in the tested states. Note that selecting a class of states, such as Gaussian states, as benchmark states necessitates performing some kind of minimization, but the derived thresholds are then immutable and can be used as straightforward numeric benchmarks.

In contrast to Eq. (1), which requires prior knowledge of the specific operator, Eq. (2) quantifies the amount of squeezing in any observable that can be revealed by the free nonsqueezing operations if one has full knowledge of the quantum state. The quantity in Eq. (2) also cannot increase under such operations and is, therefore, better suited for the description of squeezing as a quantum resource.

The *specific* squeezing in Eq. (1) is practical for optimizing the state preparation protocols, in which the goal is to obtain quantum states best approximating the desired eigenstate of $\hat{\mathcal{O}}$ to be used as a resource [2,70]. In this case, the preparation procedure is optimized to minimize over the set of preparable states to obtain $\tilde{\xi}_{\min} = \min_{\hat{\rho}} \xi_{\hat{\mathcal{O}}}(\hat{\rho})$. The *resource* squeezing in Eq. (2) can be used as a general quantifier for the chosen property. For example, it can be applied to an experimentally generated state with the goal of determining whether it possesses the required property, either directly or in an extractable form. In some scenarios, it is also conceivable to aim for maximal resource squeezing by scrutinizing some set of preparable quantum states to obtain $\xi_{\min} = \min_{\hat{\rho}} \xi_{\hat{\mathcal{O}}}(\hat{\rho})$, even though this approach is computationally the most demanding. In all cases, value $\xi = 0$ confirms the investigated state as the perfect eigenstate of the target operator, best suited for any application. On the other hand, $\xi \geqslant 1$ implies that the state is no better than a free state and is therefore unsuitable for further applications. For any quantum state, the intermediate value $0 < \xi < 1$ can then serve as a quantifier for the given resource, which can be expressed either directly or in the dB scale, $\xi[\mathrm{dB}] = 10 \log_{10} \xi$.

III. EXAMPLE: CUBIC SQUEEZING IN COLLECTIVE SPIN SYSTEMS

It was mentioned in the previous section that the nature of generalized squeezing is strongly informed by the specific choice of not only the target operator but also the sets of free states and free operations. Let us demonstrate this in a particular example: the cubic squeezing in a system of collective spins. The cubic operation [3] and the related cubic squeezing [36,37,52] were originally investigated for the quantum harmonic oscillator but can also be expanded to the collective spin systems [48,71].

The collective spin systems can be described with help of collective spin operators, $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$, with commutator $[\hat{J}_k, \hat{J}_l] = i\epsilon_{klm}\hat{J}_m$ (where $k, l, m = \{x, y, z\}$). We can use Schwinger's representation equivalent to two bosonic modes with annihilation (creation) operators $\hat{a}_{1,2}$ ($\hat{a}^\dagger_{1,2}$) satisfying commutator $[\hat{a}_k, \hat{a}^\dagger_l] = \delta_{kl}$. The collective spin operators have the form $\hat{J}_x = \frac{1}{2}(\hat{a}^\dagger_1\hat{a}_2 + \hat{a}_1\hat{a}^\dagger_2)$, $\hat{J}_y = (1/2i)(\hat{a}^\dagger_1\hat{a}_2 - \hat{a}_1\hat{a}^\dagger_2)$, and $\hat{J}_z = \frac{1}{2}(\hat{a}^\dagger_1\hat{a}_1 - \hat{a}^\dagger_2\hat{a}_2)$. Since $\hat{J}_{x,y,z}$ commutate with $\hat{N} \equiv \hat{a}^\dagger_1\hat{a}_1 + \hat{a}^\dagger_2\hat{a}_2$, one can consider a finite-dimensional Hilbert space with a fixed number of particles N.

The cubic operation can be implemented by a unitary operator

$$\hat{U}_c(\chi) = \exp\left(i\frac{\chi}{3}\left(\frac{N}{2}\right)^{-3/2}\hat{J}_z^3\right), \tag{3}$$

where the factor $(N/2)^{-3/2}$ represents the scaling of the effective strength arising from the relation between the commutation relations and dimension of the system. Having this scaling in this definition allows the effective interaction strength χ to be independent of the system size, which allows for better comparison. The nonlinear cubic squeezing is then tied to variance of operator

$$\hat{\mathcal{O}}_c(\chi) = \hat{U}_c(\chi)\hat{J}_y\hat{U}_c^\dagger(\chi). \tag{4}$$

Note that the choice of operators $\hat{J}_y$ and $\hat{J}_z$ in Eqs. (3) and (4) is not unique—they could be swapped with no bearing on generality. To complete the definition of squeezing, both specific and resource, we need to define the free states and the free operations. In harmonic oscillators, the goal of cubic operation is to go beyond the Gaussian domain and cubic squeezing can be taken as a witness of non-Gaussianity. It is not as straightforward in the systems of collective spins, but we can separate the target state from a class of experimentally feasible states defined as ground states of Hamiltonian. We define Gaussian-like states as ground states of the Hamiltonian

$$\begin{aligned}\hat{H}(g) =&\hat{U}_F\left[\left(\frac{g^2}{1+g^2}\right)\hat{J}_z^2 + \left(\frac{1}{1+g^2}\right)\hat{J}_y^2\right.\\ &\left.+\left(\frac{g}{1+g^2}\right)\hat{J}_x\right]\hat{U}_F^\dagger.\end{aligned} \tag{5}$$

This Hamiltonian, based on the general twist and turn Hamiltonian [72], represents two types of operations depending on the choice of the g parameter: two-axis countertwisting operation for $g \in (0, \infty) \setminus \{1\}$ and one-axis twisting operation for $g \in \{0, 1\}$. The free operations $\hat{U}_F$ represent a rotation of the Bloch sphere along an arbitrary axis—an operation that cannot create any nontrivial property of the state.

The ground states of Eq. (5) cover the entire class of squeezed Gaussian-like states, including the coherent state—the linear term in the Hamiltonian in Eq. (5) takes off the degeneracy, so the eigenstates are only on one hemisphere of the Bloch sphere. The reasoning for these states becomes apparent when we consider the Holstein-Primakoff approximation [73] for a large number of atoms. In this approximation, the Gaussian-like states on a Bloch sphere turn into Gaussian states in a two-dimensional plane of a harmonic oscillator. From now on, we will use the term Gaussian and non-Gaussian states to mean Gaussian-like and non-Gaussian-like states.

We can now discuss some specific applications of the defined cubic operator, in the increasing order of numerical cost.

(1) Direct evaluation of specific squeezing. For any given state, the squeezing in a given cubic nonlinear operator can be evaluated by calculating the variance of Eq. (4). This is checked against the minimal value obtainable by the free states to see whether this state is relevant for potential applications.

$$\xi(\chi) = \frac{\min_{\hat{U}_F}[\mathrm{var}_{\hat{\rho}}(\hat{U}_F\hat{\mathcal{O}}_c(\chi)\hat{U}_F^\dagger)]}{\min_{\hat{U}_F}\min_{\hat{\rho}_F}[\mathrm{var}_{\hat{\rho}_F}(\hat{U}_F\hat{\mathcal{O}}_c(\chi)\hat{U}_F^\dagger)]}. \tag{6}$$

In some cases, it is possible to directly measure the squeezing [2,36,37,74,75]. This practical advantage can be complemented by theoretical techniques, specifically by analyzing the eigenstates of the squeezing operator restricted to a relevant subspace. Such an approach allows for a rapid identification of the best approximation to the ideal state [2,51,55].

(2) Deeper characterization of *resource* squeezing. In experiments, it is often important to witness whether some broader quantum feature is present in the state reconstructed from the experimental data in case there was some previously unaccounted influence. In this case, we could be interested in cubic squeezing with any parameter χ, prompting a modification of Eq. (2). To search for which cubic parameter is the variance of the tested state minimal, we minimize the entire

fraction over the possible nonlinear interaction strength χ. The resulting relation has the form

$$\xi = \min_{\chi} \frac{\min_{\hat{U}_F}[\mathrm{var}_{\hat{\rho}}(\hat{U}_F\hat{O}_c(\chi)\hat{U}_F^{\dagger})]}{\min_{\hat{U}_F}\min_{\hat{\rho}_F}[\mathrm{var}_{\hat{\rho}_F}(\hat{U}_F\hat{O}_c(\chi)\hat{U}_F^{\dagger})]}. \tag{7}$$

(3) State preparation. Finally, we can be interested in optimizing the state preparation circuit with the goal of obtaining the quantum state with the desired property. Rather than searching in the full set of quantum states, we might be searching in a subset of states given by a specific preparation procedure [36]. In this case, we need to minimize the cubic variance for any given state from the set and then minimize over the states from the set:

$$\bar{\xi}_{\chi} = \frac{\min_{\hat{\rho}}\min_{\hat{U}_F}[\mathrm{var}_{\hat{\rho}}(\hat{U}_F\hat{O}_c(\chi)\hat{U}_F^{\dagger})]}{\min_{\hat{U}_F}\min_{\hat{\rho}_F}[\mathrm{var}_{\hat{\rho}_F}(\hat{U}_F\hat{O}_c(\chi)\hat{U}_F^{\dagger})]}. \tag{8}$$

In cases (2) and (3), the formula contains unitary operations $\hat{U}_F$ through which we minimize. We always solve this step by centering the tested state in one preselected axis (in the given axis, we rotate the state so that its variance matrix is diagonal), in which we calculate the given variance. Details of this calibration are provided in Appendix A.

Example of a superposition of two Dicke states

As an illustrative example, we chose to test a state that is defined as a superposition of two Dicke states with a free parameter γ:

$$|\psi\rangle = \sqrt{\gamma}\,|0\rangle + \sqrt{(1-\gamma)}\,|1\rangle\,, \tag{9}$$

where the Dicke states are defined with the help of collective spin states as $|0\rangle = |N/2, -N/2\rangle$ and $|1\rangle = |N/2, -(N/2)+1\rangle$. This state has been used as an approximation of a cubically squeezed state [26,37,51]. The reason is simple; this state is consistent with different numbers of atoms both in the system and in a harmonic oscillator; this is demonstrated in Appendix B.

Deeper characterization of the resource squeezing. We chose a particular state in Eq. (9), one with $\gamma = \frac{1}{2}$ and $N = 80$, and found its optimal cubic squeezing by minimizing the squeezing parameter in Eq. (7) over the possible nonlinear interaction strengths χ. As can be seen in Fig. 1(a), the state has $\xi < 1$ and is therefore cubically squeezed and non-Gaussian for $4\times10^{-4} < \chi' < 15\times10^{-4}$ ($\chi' = [\chi/3][N/2]^{-3/2}$). The maximal cubic squeezing corresponds to the minimum $\xi = 0.748$ (-1.26 dB), and it is found for $\chi' = 7.96\times10^{-4}$. The difficulty of numerical simulation is sensitive to the size of the chosen system, and the difficulty increases with increasing dimension.

State preparation. For this task, we try to optimize the resource squeezing for a state specified by some constraints on preparation. This can be a set of defined operations [36], but in this instance, we will simply consider an arbitrary state with the form in Eq. (9). The goal now is to find γ, for which the resource squeezing has minimal value. The results are shown in Fig. 1(b). The value for $\gamma = \frac{1}{2}$ corresponds to the minimum in Fig. 1(a), and we can immediately see

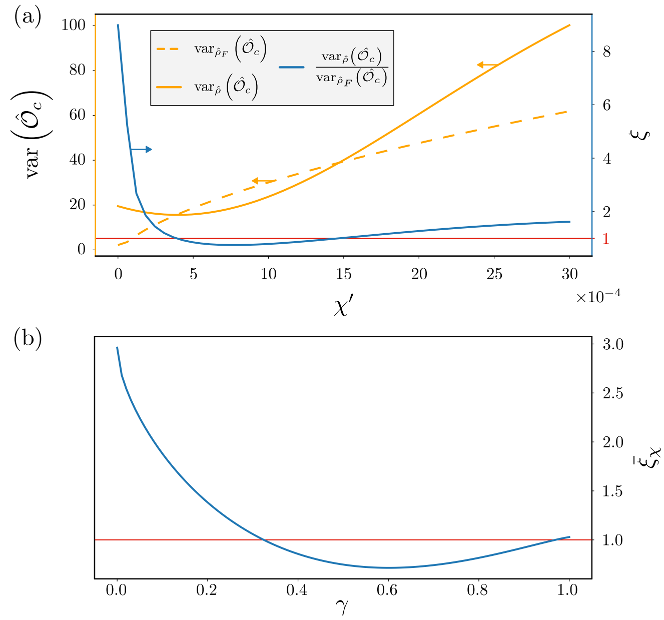

FIG. 1. Numerical evaluation of cubic squeezing for superposition of the first two Dicke states in Eq. (9) in systems with $N = 80$. (a) The variance of the cubic nonlinear operator in Eq. (7) relative to the effective cubicity χ'. The yellow dashed curve corresponds to the variance of the benchmark state; the yellow full line corresponds to the variance of the analyzed state in Eq. (9) with $\gamma = \frac{1}{2}$ (the yellow curves are connected to the left y axis, which is also yellow). The blue curve describes the resulting parameter of Eq. (7) (the values of the blue curve correspond to the right y axis, which is also blue). The red line is the limit corresponding to $\xi = 1$; i.e., when we are below this limit, the state is nonlinearly cubically squeezed, and when above the limit, we achieve better values with the benchmark states. (b) Optimal nonlinear squeezing in Eq. (8) relative to the parameter γ for the state described by Eq. (9). The blue curve is the optimal nonlinear squeezing parameter $\bar{\xi}_{\chi}$ obtainable for the particular state. For $\gamma = \frac{1}{2}$, the value of $\bar{\xi}_{\chi}$ corresponds to the minimum of the blue curve in panel (a). The red line denotes $\bar{\xi}_{\chi} = 1$ that delineates the boundary between Gaussian and non-Gaussian.

that it is not the full optimum. The optimum is $\gamma = 0.551$ [as can be seen in Fig. 1(b)], and we achieve squeezing $\xi_c = 0.715$, which corresponds to -1.459 dB. χ' parameter was set as $\chi' = 7.959\times10^{-4}$ to match the previously used optimal value. The difficulty of the numerical simulation is similar to that encountered in the deeper characterization example; however, in this case, the complexity increases due to the need to optimize parameters associated with state preparation. The more intricate the preparation process, the greater the number of parameters to be optimized, leading to higher computational demands.

IV. CONCLUSION

We present a different perspective on state evaluation, which focuses on the use of variance-based squeezing to observe specific state properties. By selecting an appropriate operator for variance analysis, we can reveal distinct characteristics of quantum states. In addition, for any class of benchmarking states we chose, the variance is bounded. This means that we can use this variance to verify whether the examined state can be a part of this class or whether it has to be excluded. This is of particular interest when the states from the benchmarking class are unsuitable for some task, as is the case for, for example, separable, classical, or Gaussian states. We first introduce our approach in a general framework and subsequently demonstrate its utility on a discrete-variable system comprising collective spins. In particular, we analyze cubic squeezing in superpositions of Dicke states $|0\rangle$ and $|1\rangle$. This example illustrates how the nonlinear squeezing can be leveraged both for characterizing quantum resources and for guiding state preparation. The proposed framework offers a robust and operationally accessible method for identifying and quantifying nonlinear squeezing, which is directly relevant for enhancing precision in quantum metrology. Owing to its generality and computational feasibility, we expect this approach to contribute significantly to experimental analysis and the optimization of nonclassical states for metrological and quantum information tasks.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [76], embargo periods may apply.

APPENDIX A: DETAILED PROCEDURE FOR EVALUATING NONLINEAR SQUEEZING-FREE OPERATIONS

In this Appendix, we address a key technical challenge in the evaluation of cubic squeezing for collective spin systems: the optimization over free operations. This optimization is, in general, computationally demanding and can significantly slow down the evaluation of squeezing or preparation squeezed state for large systems. One can see the formulas for evaluating nonlinear squeezing:

$$\xi = \min_{\chi} \frac{\min_{\hat{U}_F}[\mathrm{var}_{\hat{\rho}}(\hat{U}_F\hat{\mathcal{O}}_c(\chi)\hat{U}_F^{\dagger})]}{\min_{\hat{U}_F}\min_{\hat{\rho}_F}[\mathrm{var}_{\hat{\rho}_F}(\hat{U}_F\hat{\mathcal{O}}_c(\chi)\hat{U}_F^{\dagger})]}, \tag{A1}$$

and for state preparation:

$$\bar{\xi}_{\chi} = \frac{\min_{\hat{\rho}}\min_{\hat{U}_F}[\mathrm{var}_{\hat{\rho}}(\hat{U}_F\hat{\mathcal{O}}_c(\chi)\hat{U}_F^{\dagger})]}{\min_{\hat{U}_F}\min_{\hat{\rho}_F}[\mathrm{var}_{\hat{\rho}_F}(\hat{U}_F\hat{\mathcal{O}}_c(\chi)\hat{U}_F^{\dagger})]}. \tag{A2}$$

As we can see, in both cases, it is minimized through Gaussian operations—$\min_{\hat{U}_F}$.

Instead of performing a full numerical minimization over all possible free operations, we present a method that reduces the complexity of this task by analytically constructing a specific class of free operations based on the alignment of the mean spin vector. By preselecting the axis along which squeezing is to be evaluated, specifically the J_x axis, we effectively eliminate an entire optimization layer. The approach involves determining a unitary rotation that aligns the mean spin along this chosen axis and then applying an additional rotation to diagonalize the covariance matrix in the orthogonal subspace. This method enables a direct evaluation of the squeezing without the need for iterative numerical minimization. The following procedure yields analytical expressions that streamline the numerical evaluation of the resource squeezing.

The first step in the evaluation of the state is always to center the state on a predefined axis. In this case, it is the J_x axis. Below is an analytical prescription for calculating the unitary rotation operation from knowledge of the mean values.

Suppose we have a collective spin state described by the mean spin vector

$$\mathbf{J} = (\langle\hat{J}_x\rangle, \langle\hat{J}_y\rangle, \langle\hat{J}_z\rangle).$$

Our goal is to find a unitary rotation $\hat{U}_{F_1}$ generated by collective spin operators that rotates the state such that the mean spin aligns with the J_x axis.

First, we normalize the mean spin vector to obtain a unit vector indicating the current orientation on the Bloch sphere:

$$\mathbf{n} = \frac{\mathbf{J}}{\|\mathbf{J}\|} = \frac{(\langle\hat{J}_x\rangle, \langle\hat{J}_y\rangle, \langle\hat{J}_z\rangle)}{\sqrt{\langle\hat{J}_x\rangle^2 + \langle\hat{J}_y\rangle^2 + \langle\hat{J}_z\rangle^2}}.$$

We choose the target axis as the J_x axis:

$$\mathbf{x} = (1, 0, 0).$$

The rotation axis $\mathbf{a}$ is perpendicular to both $\mathbf{n}$ and $\mathbf{x}$, given by the normalized cross product:

$$\mathbf{a} = \mathbf{n} \times \mathbf{x} = \frac{(0, J_z, -J_y)}{\|\mathbf{J}\|}, \quad \bar{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{(0, J_z, -J_y)}{\sqrt{J_y^2 + J_z^2}}.$$

The angle θ of rotation is the angle between $\mathbf{n}$ and $\mathbf{x}$:

$$\theta = \arccos(\mathbf{n}\cdot\mathbf{x}) = \arccos\left(\frac{\langle\hat{J}_x\rangle}{\|\mathbf{J}\|}\right).$$

Using the collective spin operators $\hat{\mathbf{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)$, the unitary operator implementing this rotation is

$$\hat{U}_{F_1} = \exp(-i\theta\,\bar{\mathbf{a}}\cdot\hat{\mathbf{J}}) = \exp[-i\theta(a_x\hat{J}_x + a_y\hat{J}_y + a_z\hat{J}_z)].$$

This unitary rotation aligns the mean spin vector of the state along the J_x axis. If the mean spin is already along J_x, then $\theta = 0$ and $\hat{U}_{F_1}$ is the identity operator.

After rotating the mean spin vector $\mathbf{J}$ to align with the J_x axis by

$$\hat{U}_{F_1} = \exp(-i\theta\bar{\mathbf{a}}\cdot\hat{\mathbf{J}}),$$

we consider the covariance matrix of the orthogonal components $\hat{J}_y, \hat{J}_z$:

$$\Gamma' = \begin{pmatrix} \mathrm{Var}(\hat{J}_y') & \mathrm{Cov}(\hat{J}_y', \hat{J}_z') \\ \mathrm{Cov}(\hat{J}_y', \hat{J}_z') & \mathrm{Var}(\hat{J}_z') \end{pmatrix},$$

where $\hat{J}_y' = \hat{U}_{F_1}^{\dagger}\hat{J}_y\hat{U}_{F_1}$, $\hat{J}_z' = \hat{U}_1^{\dagger}\hat{J}_z\hat{U}_1$.

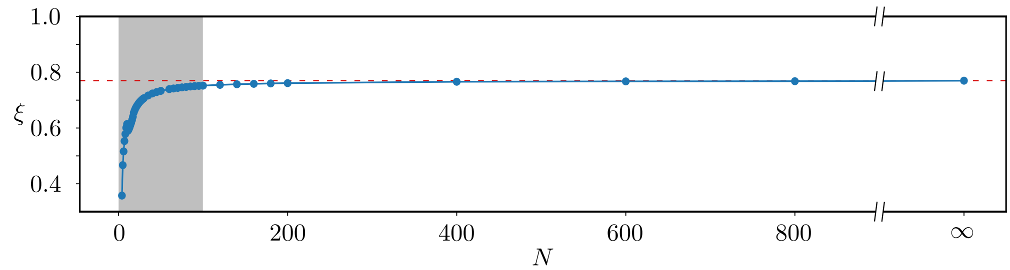

FIG. 2. Results of numerical simulation of minimizing nonlinear squeezing parameter with test state in Eq. (B4) in the interval of N, where the gray area corresponds to $N \in [4, 100]$, and ∞ point in the x axis corresponds to the transition to the harmonic oscillator phase space.

To diagonalize Γ' while preserving the alignment of the mean spin along J_x, we apply a further rotation around the J_x axis:

$$\hat{U}_{F_2} = \exp(-i\phi \hat{J}_x),$$

with the angle ϕ determined by

$$\tan(2\phi) = \frac{2\,\mathrm{Cov}(\hat{J}'_y, \hat{J}'_z)}{\mathrm{Var}(\hat{J}'_y) - \mathrm{Var}(\hat{J}'_z)}.$$

The final rotation is thus

$$\hat{U}_F = \hat{U}_{F_2}\hat{U}_{F_1} = \exp(-i\phi J_x)\exp(-i\theta \hat{\mathbf{a}} \cdot \mathbf{J}),$$

which aligns the mean spin along the x axis and diagonalizes the covariance matrix of J_y, J_z.

This unitary rotation aligns the mean spin vector of the state along the J_x axis. If the mean spin is already along x, then $\theta = 0$ and $\hat{U}_F$ is the identity operator.

The procedure described above describes minimization via free operations, i.e., the correct placement and rotation of the studied state. Since it can be expressed analytically in terms of the mean values and the covariance matrix, no minimization needs to be performed, and it is a minimally time-consuming operation.

All calculations leading to the results presented in the publication and Appendix were performed on a standard desktop computer. The optimization for the search for χ, shown in Fig. 1(a), took 404 s to compute for the system $N = 80$, 2634 s for the system $N = 200$, 7444 s for the system $N = 400$, and 14 126 s for the system $N = 600$. It should be noted that, in cases of $N > 200$, the Holstein-Primakoff approximation is considered, but not in the case of our numerical simulations. The code is available online [76].

APPENDIX B: BEHAVIOR OF NONLINEAR SQUEEZING DEPENDING ON THE SIZE OF THE STUDIED SYSTEM

In the following, we will show that the cubic nonlinear squeezing, which manifests as reduced variance of operator

$$\hat{\mathcal{O}}_c(\chi) = \hat{U}_c(\chi)\hat{J}_y\hat{U}_c^\dagger(\chi), \tag{B1}$$

where

$$\hat{U}_c(\chi) = \exp\left(i\frac{\chi}{3}\left(\frac{N}{2}\right)^{-3/2}\hat{J}_z^3\right), \tag{B2}$$

is consistently described across the different sizes of the investigated collective spin systems. The size of a system is given by the number of atomic spins, N, and the relevant range, in which we can try to observe that the nonlinear squeezing lies between $N = 4$ and $N \to \infty$ when the systems converge into the Holstein-Primakoff approximation of continuous variable linear harmonic oscillator (LHO). It is not meaningful to evaluate cubic squeezing for $N \leqslant 3$ when investigating the non-Gaussian-like properties given by the parameter

$$\xi = \min_{\chi} \frac{\min_{\hat{U}_F}[\mathrm{var}_{\hat{\rho}}(\hat{U}_F\hat{\mathcal{O}}_c(\chi)\hat{U}_F^\dagger)]}{\min_{\hat{U}_F}\min_{\hat{\rho}_F}[\mathrm{var}_{\hat{\rho}_F}(\hat{U}_F\hat{\mathcal{O}}_c(\chi)\hat{U}_F^\dagger)]}, \tag{B3}$$

because the Gausssian-like operations are expressed as eigenstates of Hamiltonian quadratic in the momentum operators. For $N \leqslant 3$, the space is spanned by only four Dicke states $|0\rangle, |1\rangle, |2\rangle, |3\rangle$, and the third power of angular momentum operators is not meaningfully different from the second power.

For the illustration, we have evaluated the nonlinear squeezing in Eq. (B3) for a test state given by superposition of the first two Dicke states,

$$|\psi\rangle = \sqrt{\frac{1}{2}}\,|0\rangle + \sqrt{\frac{1}{2}}\,|1\rangle, \tag{B4}$$

which is an approximation of the cubically squeezed state. The results are shown in Fig. 2. We can see that the value of nonlinear squeezing parameter ξ is lowest for the smallest system with $N = 4$ and that it steadily grows until it reaches the limit for the harmonic oscillator. The largest changes happen in the interval $N \in [4, 100]$, which is marked by the gray area in the figure. Beyond this point, the value of squeezing differs by only 0.015 from the LHO case (which corresponds to 1.95% difference).

The reason why the nonlinear squeezing of the test state is so good for the small sizes of the system is that in those dimensions, the first two Dicke states of the test state span 40% of the total Hilbert space of the system, and the test state can therefore best match the optimal state, which is the minimal value eigenstate of Eq. (4). As the dimension of the system increases, this influence vanishes. This behavior is illustrated in Fig. 3, where we can see differences between the test state and the ideal state for different dimensions of the system.

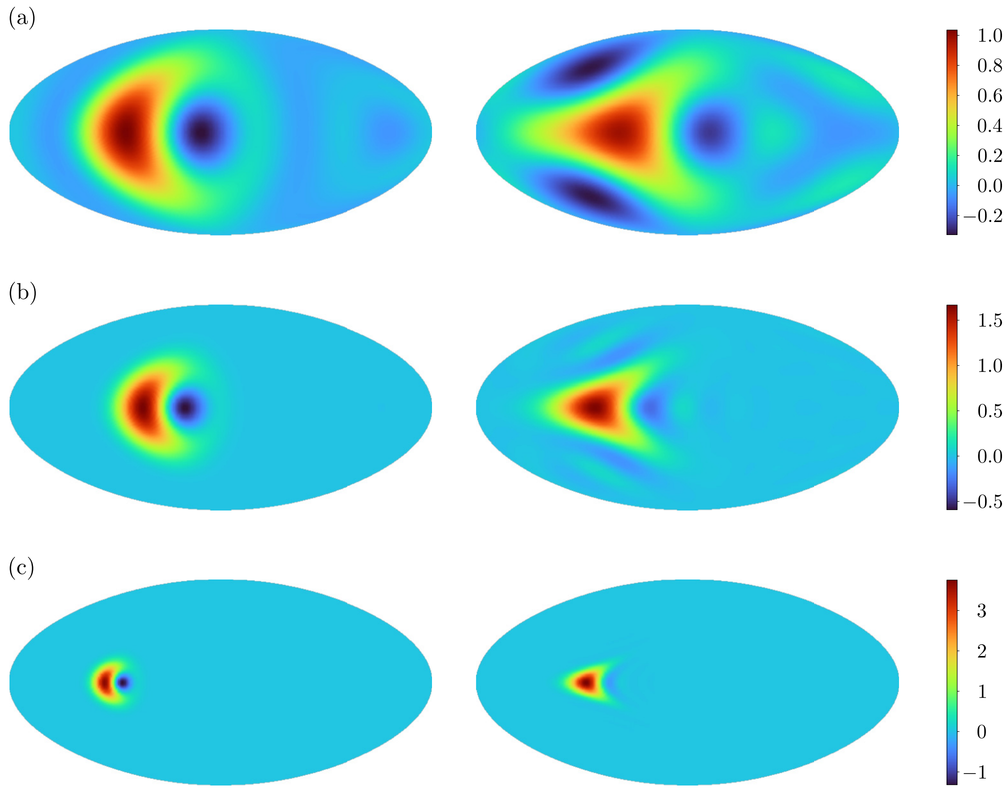

FIG. 3. Bloch spheres in Hammer's projection. The left column corresponds to the tested state in Eq. (B4), i.e., the superposition of $|0\rangle$ and $|1\rangle$ for different numbers of atoms in the systems (where $\gamma = 1/2$). The right column shows the ground states of the operator $\hat{\mathcal{O}}$ in Eq. (B1) for systems with different numbers of atoms. (a) $N = 4$, (b) $N = 10$, and (c) $N = 50$.

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