

Explosive entrainment in Arnold tongues

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Entrainment refers to the unidirectional synchronization of an oscillator system to external periodic forcing, with phase-locking regions forming Arnold tongues. In some specific models, the hysteresis phenomenon of Arnold tongue has been observed, but methods to actively induce it across models are still lacking. In this study, we show that adaptive responses to periodic forcing can induce explosive entrainment, a phenomenon of sudden locking to an external frequency accompanied by significant hysteresis. In contrast to explosive synchronization, where synchronization emerges suddenly due to internal coupling, explosive entrainment describes the sudden locking to a given frequency caused by external forcing. Therefore, explosive entrainment is more likely to occur in systems where internal couplings are weak or difficult to estimate, but external forcing can be applied. Furthermore, we demonstrate the universal emergence of explosive entrainment in the Kuramoto model, Poincaré model, chaotic Rössler model, and complex Ginzburg-Landau equation. Theoretical hysteresis bounds in the Kuramoto model are determined based on analytical results of the Ott-Antonsen reduction method. This study provides ideas for actively inducing hysteresis through external forcing.

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I. INTRODUCTION

From circadian rhythms to animal migration, organisms have evolved the ability to synchronize individual behavior with periodic environmental changes [1–4]. The phenomenon of synchronizing with an external periodic signal is termed entrainment. It is a unidirectional process, whereby the system is forced to respond to the external forcing, but cannot influence it [5,6]. Entrainment is essential for individuals to adapt to their environment. For circadian rhythms, zeitgebers such as light, temperature, and feeding have been shown to alter endogenous rhythms [7–12]. Furthermore, for the auditory system, it can synchronize with external speech input [13,14]. For bird migration, their collective behavior is driven by seasonal cycles [15–17]. Thus, entrainment reflects the plasticity of biological rhythms in nature [18–20].

To characterize entrainment, the frequency-locked region is studied in the parameter space defined by the external forcing strength and the external frequency. Since this region usually forms a tongue-like structure, it is called the Arnold tongue [21,22]. In some studies, it has been shown that the Arnold tongue can exhibit hysteresis. For example, bistability-induced hysteresis appears within Arnold tongues in the periodically driven saline oscillator [23] and nonlinear electrochemical oscillators [24]. Recently, a modeling study has shown that Arnold tongues exhibit hysteresis in unclock-like oscillators with frequency memory. By introducing a time-dependent memory variable that modulates the oscillator's intrinsic frequency, the system can switch between entrainment and nonentrainment depending on the initial conditions [25]. In addition, other studies have revealed hysteresis in the Arnold tongue, all of which emphasize the importance of initial conditions [26–30].

Although there has been some progress in understanding the hysteresis of the Arnold tongue, existing studies mainly rely on specific models and their specific parameters. Currently, a model that can actively induce hysteresis in various forced oscillators is still lacking. Actively controlling the emergence of hysteresis in nonspecific dynamics is important. On the one hand, since hysteresis often reflects an underlying

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first-order phase transition, regulating hysteresis by external forcing is essential for controlling the resilience of complex networks or preventing the sudden collapse of complex driven systems [31–33]. On the other hand, for those systems where internal interactions are weak or difficult to estimate, it is crucial to enable them to be entrained by adjustable external inputs, for example, biological clocks modulated by periodic light [34], chemical oscillators controlled by reagent concentration [35], or power grids stabilized by voltage signals [36]. Therefore, a theoretical model is needed to actively control the hysteresis in forced systems.

This paper is organized as follows. Section I is the introduction. Section II introduces the Kuramoto model of adaptive response to external forcing and investigates the influence of key parameters on the hysteresis width. Section III provides a theoretical analysis based on the Kuramoto model, including the self-consistent solution of the coherent states and the linear stability conditions of the incoherent states. Section IV reveals explosive entrainment in a forced two-dimensional limit cycle oscillator system and a forced three-dimensional chaotic oscillator system. The last section summarizes the main results of the study and discusses potential research directions.

II. MODEL AND SIMULATIONS

Substantial empirical evidence suggests that the sensitivity of organisms to external periodic inputs is not static but modulated by their internal collective state. For example, in social contagion, cohesive groups are more easily incited than chaotic groups [37,38]. In circadian rhythms, patients with seasonal affective disorder have weakened circadian rhythms and reduced retinal light sensitivity [39]. Long-term sunlight deprivation in mammals leads to weakened circadian rhythm stability and increased sensitivity to blue light [40,41]. In animal behavior, interactions within fish schools enhance their responses to environmental perturbations [42,43]. Recent neuroscience experiments have shown that flickering lights or deep brain stimulation can enhance neural sensitivity and induce lasting sequelae [44,45]. All this evidence suggests that it is necessary to consider adaptive responses to external forcing in theoretical models. Next, we present results in the Kuramoto model of adaptive responses to external forcing.

A. The forced Kuramoto model with adaptive response to periodic forcing

The Kuramoto model describes the phase dynamics of oscillators evolving on a unit circle, where each oscillator evolves according to its intrinsic frequency. The forced Kuramoto model with adaptive response to periodic forcing is given by

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i) + LR^\beta \sin(\Omega t - \theta_i), \quad (1)$$

where $i = 1, \dots, N$. θ_i represents the phase of the i th oscillator. ω_i is its natural frequency and Ω is the forcing frequency. K denotes the coupling strength among oscillators. L quantifies the strength of the system's response to periodic forcing,

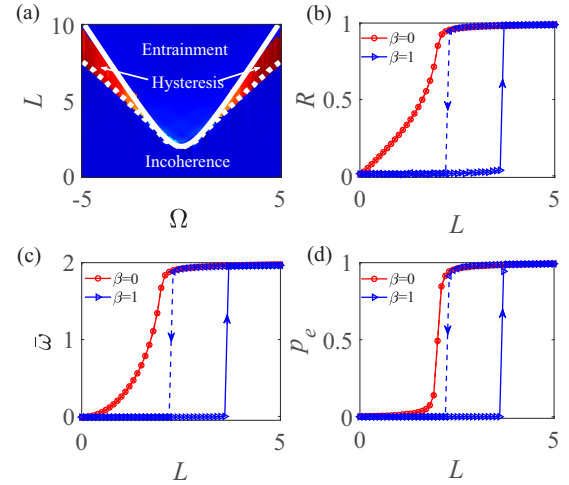


FIG. 1. Explosive entrainment in the forced Kuramoto model. (a) Arnold tongue for Eq. (1), where a clear hysteresis region is observed. In this panel, $\beta = 1$, $K = 0$, and $N = 5000$. (b) Adiabatic evolution of the order parameter R . (c) Adiabatic evolution of the system's average frequency $\bar{\omega}$. (d) Adiabatic evolution of the number of entrained oscillators p_e , determined using a threshold $\chi = 10^{-3}$. In panel (a), the solid and dashed white curves represent the theoretical boundaries of the hysteresis region as given by Eqs. (19) and (28), respectively, with $\Delta = 1$. In panels (b)–(d), the parameters are set to $\Omega = 2$.

which is weighted by R^β . R is the order parameter for measuring synchronization within the system, and it is defined as

$$z = Re^{i\varphi} = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}. \quad (2)$$

R is the magnitude (coherence) and φ is the collective phase for oscillators. $R \rightarrow 1$ indicates that the system tends toward complete synchronization, while $R \rightarrow 0$ indicates that the system tends toward incoherence. Since the order parameter depends on the real-time state of the system, the adaptive response exists when $\beta \neq 0$ [46–49].

Unless otherwise specified, the following parameter settings are used throughout the numerical simulations. The system size is set to $N = 5000$. The natural frequency ω is drawn from a Lorentzian distribution, given by

$$g(\omega) = \frac{\Delta}{\pi[\omega^2 + \Delta^2]}, \quad (3)$$

which is denoted as $\text{Loren}(0, \Delta)$. In this context, the terms forward (FW) transition and backward (BW) transition refer, respectively, to the processes of adiabatically increasing or decreasing L . For instance, in forward transition simulations, the system state at a given L_0 is initialized using the steady state obtained at the previous step, $L_0 - \delta L$, where δL is the step size used to vary L .

B. Hysteresis in Arnold tongues

A key finding is that adaptive responses to external forcing lead to the emergence of hysteresis regions in the Arnold tongue diagram. When $\beta = 0$, Eq. (1) reduces to the classical

forced Kuramoto model, which has been extensively studied in earlier works [46,47]. However, when $\beta \neq 0$, Eq. (1) incorporates an adaptive mechanism. Figure 1 illustrates the Arnold tongue for Eq. (1) with $\beta = 1$. In this phase diagram, the upper region labeled *entrainment* indicates frequency locking between the system and the external forcing, whereas the lower region labeled *incoherence* corresponds to an incoherent state where oscillators fail to entrain. A distinct hysteresis region appears between these two regions.

To characterize the evolution of the system's collective behavior, we examine three macroscopic quantities: (1) the order parameter R , which measures the degree of coherence; (2) the system's average frequency $\bar{\omega}$, which is defined as $\bar{\omega} = \langle |\delta\theta_i/\delta t|_N \rangle_t$ (here, $\delta\theta_i/\delta t$ denotes the instantaneous frequency of oscillator i at the final state, $|\cdot|_N$ represents the average over all oscillators, and $\langle \cdot \rangle_t$ indicates a temporal average); and (3) the fraction of entrained oscillators $p_e = N_e/N$, where N_e denotes the number of oscillators satisfying $|\bar{\omega}_i - \Omega| < \chi$. Here, $\bar{\omega}_i = \langle \delta\theta_i/\delta t \rangle_t$ and χ is the entrained threshold. Figures 1(b)–1(d) show the adiabatic evolution of these quantities as a function of the coupling strength L . In contrast to the continuous and reversible transition observed in the nonadaptive case ($\beta = 0$), the adaptive forced Kuramoto model exhibits a discontinuous, first-order phase transition characterized by a pronounced hysteresis loop. After the abrupt transition from incoherence to entrainment, the system shows three characteristic features: high coherence ($R \rightarrow 1$), frequency alignment with the external drive ($\bar{\omega} \rightarrow \Omega$), and a majority of oscillators becoming entrained ($p_e \rightarrow 1$). This process characterizes the essential nature of the explosive entrainment. In the Supplemental Material [50], we further examine two cases in which the intrinsic frequency distributions follow Gaussian and uniform distributions, respectively. A finding is that, regardless of the specific form of the frequency distribution, the presence of adaptive responses consistently gives rise to explosive transitions between the incoherent and entrained states.

C. Parameter dependence of hysteresis width

Once the hysteresis within the Arnold tongue is revealed, a natural question arises: What determines the width of the hysteresis loop? Four parameters are examined—namely, the external driving frequency Ω , the adaptive response exponent β , the width Δ of the natural frequency distribution, and the internal coupling strength K . Two opposing trends are identified: increasing Ω and β leads to a broadening of the hysteresis width, as demonstrated in Fig. 2; in contrast, increasing Δ and K results in a reduction of the hysteresis width, as shown in Fig. 3. In the next section, we provide a theoretical explanation for the observed parameter dependence of the hysteresis width.

III. THEORETICAL ANALYSIS

We now present both a self-consistent analysis of the coherent state and a linear stability analysis of the incoherent state, based on the Ott-Antonsen reduction method. In the thermodynamic limit, the former yields an exact solution for the order parameter, while the latter provides the instability condition for the incoherent state.

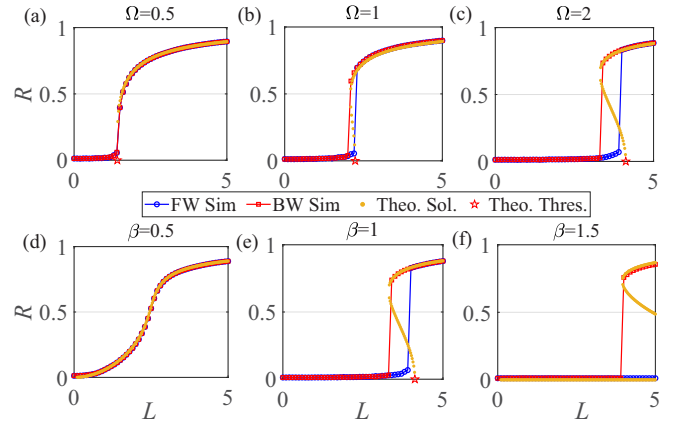


FIG. 2. Expansion of hysteresis width with increasing Ω and β , respectively. (a)–(c) Ω takes values of 0.5, 1, and 2, respectively. $\Delta = 0.5$, $\beta = 1$, $K = 0$, and $N = 5000$ are fixed. (d)–(f) β takes values of 0.5, 1, and 1.5, respectively. Here, $\Delta = 0.5$, $\Omega = 2$, and $K = 0$ are fixed. FW Sim. and BW Sim. denote the numerical simulations of the forward and backward transitions, respectively. Theo. Sol. refers to the self-consistent solution of the order parameter, and Theo. Thres. indicates the instability threshold of the incoherent state, as derived from Eqs. (19) and (28), respectively.

A. Self-consistent analysis of coherence

By introducing $\phi_i = \theta_i - \Omega t$, Eq. (1) yields

$$\frac{d\phi_i}{dt} = (\omega_i - \Omega) + \frac{K}{N} \sum_{j=1}^N \sin(\phi_j - \phi_i) - LR^\beta \sin \phi_i. \quad (4)$$

In the thermodynamic limit ($N \rightarrow \infty$), the state of the system is described by a density function $f(\phi, \omega, t)$, which satisfies

$$\int_{-\infty}^{+\infty} \int_0^{2\pi} f(\phi, \omega, t) d\phi d\omega = \int_{-\infty}^{+\infty} g(\omega) d\omega = 1. \quad (5)$$

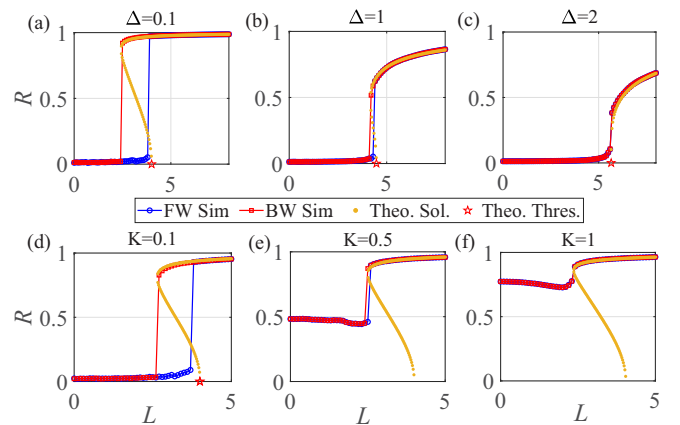


FIG. 3. Expansion of hysteresis width with decreasing Δ and K , respectively. (a)–(c) Δ takes values of 0.1, 1, and 2, respectively. $\Omega = 2$, $\beta = 1$, $K = 0$, and $N = 5000$ are fixed. (d)–(f) K takes values of 0.1, 0.5, and 1, respectively. Here, $\Delta = 0.2$, $\Omega = 2$, and $\beta = 1$ are fixed. The definitions of FW Sim., BW Sim., Theo. Sol., and Theo. Thres. are consistent with those in Fig. 2.

The complex order parameter z becomes

$$z(t) = \int_{-\infty}^{+\infty} \int_0^{2\pi} e^{i\phi} f(\phi, \omega, t) d\phi d\omega. \quad (6)$$

The evolution of f is given by the continuity equation, namely,

$$\frac{\partial f}{\partial t} + \frac{\partial(fv)}{\partial \phi} = 0, \quad (7)$$

which describes the conservation of oscillators of frequency ω . From Eqs. (4) and (6), the velocity field on the unit circle, that is, $v(\phi, \omega, t)$, can be written as

$$\begin{aligned} v &= (\omega - \Omega) + K \int_{-\infty}^{+\infty} \int_0^{2\pi} f(x, y, t) \sin(x - \phi) dx dy \\ &\quad - LR^\beta \sin \phi \\ &= (\omega - \Omega) + \frac{1}{2i} [(Kz + LR^\beta) e^{-i\phi} - (Kz + LR^\beta)^* e^{i\phi}], \end{aligned} \quad (8)$$

where the asterisk $*$ denotes complex conjugation. Here, we can expand $f(\phi, \omega, t)$ as a Fourier series in ϕ , namely,

$$f(\phi, \omega, t) = \frac{g(\omega)}{2\pi} \left[1 + \sum_{n=1}^{\infty} f_n(\omega, t) e^{in\phi} + \text{c.c.} \right], \quad (9)$$

where $\text{c.c.} = \sum_{n=1}^{\infty} f_n^*(\omega, t) e^{-in\phi}$. Here, we assume that $f_n(\omega, t)$ are not independent and they are determined by $f_1(\omega, t)$, namely,

$$f_n = f_1^n. \quad (10)$$

This ansatz was first proposed by Ott and Antonsen [47]. By substituting Eqs. (8) and (9) into Eq. (7), and comparing the coefficients of term $e^{in\phi}$, the evolution of f_1 can be expressed as follows:

$$\frac{df_1}{dt} = \frac{1}{2} (Kz + LR^\beta)^* - i(\omega - \Omega) f_1 - \frac{1}{2} (Kz + LR^\beta) f_1^2. \quad (11)$$

By introducing Eq. (9) into Eq. (6), then

$$z(t) = \int_{-\infty}^{+\infty} \int_0^{2\pi} \frac{1}{2\pi} f_1^*(\omega, t) g(\omega) d\phi d\omega + \Gamma, \quad (12)$$

where Γ is an integral containing the sum of non-zero-order Fourier harmonics. Here, $\Gamma = 0$ since

$$\int_0^{2\pi} e^{in\phi} d\phi = 0, \quad \text{for } n \neq 0, \quad (13)$$

and then

$$z(t) = \int_{-\infty}^{+\infty} f_1^*(\omega, t) g(\omega) d\omega. \quad (14)$$

Introducing the expression of $g(\omega)$ into Eq. (14),

$$g(\omega) = \frac{\Delta}{\pi(\omega^2 + \Delta^2)}. \quad (15)$$

Assuming that f_1 satisfies appropriate analytical conditions in the complex ω plane, the following exact evolution

equation for z can be derived via contour integration,

$$\frac{dz}{dt} = \frac{1}{2} (Kz + LR^\beta) - (\Delta + i\Omega)z - \frac{1}{2} (Kz + LR^\beta)^* z^2. \quad (16)$$

By separating the real and imaginary parts, Eq. (16) can be rewritten as

$$\begin{aligned} \frac{dR}{dt} &= \frac{1}{2} KR + \frac{1}{2} LR^\beta \cos \varphi - \Delta R - \frac{1}{2} R^2 (KR + LR^\beta \cos \varphi), \\ \frac{d\varphi}{dt} &= -\Omega - \frac{1}{2} LR^{\beta-1} \sin \varphi (1 + R^2). \end{aligned} \quad (17)$$

When we solve $\frac{dR}{dt} = 0$ and $\frac{d\varphi}{dt} = 0$, then we can obtain

$$\begin{aligned} \sin \varphi &= -\frac{2\Omega}{LR^{\beta-1}(1 + R^2)}, \\ \cos \varphi &= \frac{2\Delta R - KR(1 - R^2)}{LR^\beta(1 - R^2)}. \end{aligned} \quad (18)$$

Using the Pythagorean identity, an implicit self-consistent solution for the phase-locked state can be obtained as follows:

$$\left(\frac{2\Omega}{LR^{\beta-1}(1 + R^2)} \right)^2 + \left(\frac{2\Delta R - KR(1 - R^2)}{LR^\beta(1 - R^2)} \right)^2 = 1. \quad (19)$$

Equation (19) provides a condition for the existence of a synchronized (phase-locked) solution with finite coherence $R > 0$. Its solutions reveal whether entrainment can persist given the parameters Ω, L, Δ, K , and β .

B. Stability analysis of incoherence

Before the forward transition occurs, all oscillators rotate predominantly within the unit circle according to their natural frequencies [51]. In this state, the system is incoherent, characterized by $R \approx 0$. In the thermodynamic limit, that is, $N \rightarrow \infty$, the density function $f(\phi, \omega, t)$ for the incoherent state is given by

$$f_0(\phi, \omega, t) = \frac{g(\omega)}{2\pi}. \quad (20)$$

Introducing a small perturbation to the incoherent solution [Eq. (20)],

$$f(\phi, \omega, t) = f_0(\phi, \omega, t) + \varepsilon \xi(\phi, \omega, t) g(\omega), \quad (21)$$

with $\varepsilon \ll 1$. According to Eq. (6), the order parameter in continuum form becomes

$$z(t) = \varepsilon \int_{-\infty}^{+\infty} \int_0^{2\pi} e^{i\phi} \xi(\phi, \omega, t) g(\omega) d\phi d\omega. \quad (22)$$

Since $z(t) = R(t) e^{i\varphi(t)}$,

$$z(t) = \varepsilon z_1(t) = \varepsilon R_1(t) e^{i\varphi(t)}. \quad (23)$$

Substituting Eq. (23) into Eq. (16) yields

$$\frac{dz_1}{dt} = -(\Delta + i\Omega)z_1 + \frac{1}{2} Kz_1 + \frac{1}{2} \varepsilon^{\beta-1} LR_1^\beta + o(\varepsilon), \quad (24)$$

where $o(\varepsilon)$ denotes a higher-order infinitesimal of ε .

When $\beta < 1$, the third term on the right-hand side of the equation approaches infinity as ε tends to zero. This implies that the linear perturbation z_1 diverges over time.

Consequently, for $\beta < 1$, any incoherent state with $L > 0$ is unstable.

Next, we consider the case where $\beta = 1$. Neglecting terms of order ε , Eq. (24) can be written as

$$\frac{dz_1}{dt} = -(\Delta + i\Omega)z_1 + \frac{1}{2}Kz_1 + \frac{1}{2}LR_1. \quad (25)$$

The above equation can be decomposed into its real and imaginary parts as

$$\begin{aligned} \frac{dR_1}{dt} &= \frac{LR_1}{2} \cos \varphi + \left(\frac{1}{2}K - \Delta\right)R_1, \\ \frac{d\varphi}{dt} &= -\frac{L}{2} \sin \varphi - \Omega. \end{aligned} \quad (26)$$

Clearly, $R_1 = 0$ and $\sin \varphi = -2\Omega/L$ correspond to solutions of Eq. (26). The stability of the solutions can be determined by calculating the Jacobian matrix. For Eq. (26), its Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} \frac{1}{2}K - \Delta + \frac{1}{2}LT & 0 \\ 0 & -\frac{1}{2}LT \end{bmatrix}, \quad (27)$$

where $T = \sqrt{1 - \frac{4\Omega^2}{L^2}}$. The eigenvalues of the Jacobian matrix are $\lambda_1 = \frac{1}{2}K - \Delta + \frac{1}{2}LT$ and $\lambda_2 = -\frac{1}{2}LT$, respectively. The condition for an incoherent state to remain stable is that all λ_i have real parts less than zero. Manifestly, λ_2 is perpetually negative for $L > 0$. For λ_1 , its value greater than zero indicates that the incoherent state loses stability. Thus, by setting $\lambda_1 > 0$, the relationship between L and Ω is as follows:

$$L > \sqrt{(2\Delta - K)^2 + 4\Omega^2}. \quad (28)$$

Equation (28) provides the instability condition of the incoherent state, revealing the critical point at which a sudden transition from incoherence to coherence occurs in the thermodynamic limit. This theoretical threshold, marked by red pentagrams, corresponds to the forward transition points shown in Figs. 2 and 3.

Finally, the case of $\beta > 1$ is also considered. Neglecting terms of order ε , Eq. (24) reduces to

$$\frac{dz_1}{dt} = -(\Delta + i\Omega)z_1 + \frac{1}{2}Kz_1. \quad (29)$$

By separating the real and imaginary parts, its Jacobian matrix can be written in the form

$$\mathbf{J} = \begin{bmatrix} \frac{1}{2}K - \Delta & 0 \\ 0 & 0 \end{bmatrix}. \quad (30)$$

When $K \rightarrow 0$ and $\Delta > 0$, the Jacobian matrix has one negative eigenvalue and one eigenvalue that is identically zero. This implies that the incoherent state is neutrally stable.

C. Theoretical explanation of hysteresis

According to the above derivation, Eqs. (19) and (28) provide solutions to determine the critical strength for the phase transition. By substituting specific parameters, the upper and lower boundaries of hysteresis shown in Fig. 1(a) can be determined. For the backward transition, Eq. (19) provides accurate prediction of the order parameter, as shown in Figs. 2 and 3. In contrast, Eq. (28) only provides a conditional estimate of

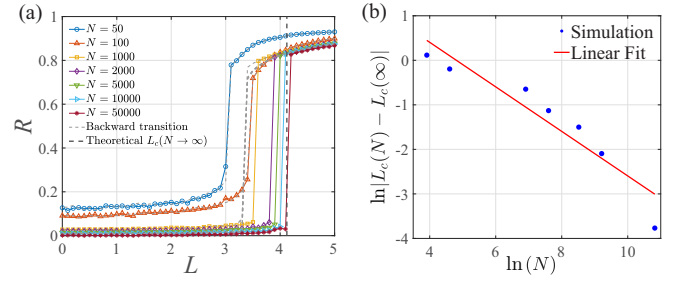


FIG. 4. Finite-size scaling of the forward phase transitions in the forced Kuramoto model. (a) Phase transition curves for various system sizes N . The solid colored lines represent forward transitions, and the dashed gray lines represent backward transitions. (b) Visualization of the finite-size scaling analysis. The linear fit to the data on a log-log scale yields a slope of -0.5 , which implies the scaling relation $|L_c(N) - L_c(\infty)| \propto N^{-0.5}$. Here, $\beta = 1$, $\Delta = 0.5$, $\Omega = 2$, and $K = 0$. $L_c(\infty)$ is the theoretical critical value for the forward transition, derived from Eq. (28), and $L_c(N)$ is the numerically determined critical value for a finite system of size N .

the threshold for the forward transition, which depends on the system size and the coupling strength. The reasons for this limitation are as follows. First, due to finite-size fluctuations, the theoretical predictions are consistent with the simulation results only in the large N limit. This is confirmed by the finite-size scaling analysis in Fig. 4. In Fig. 4(a), the transition point in simulations approaches the theoretical value (dashed line) as N increases. This convergence follows a clear power law, as demonstrated in Fig. 4(b). The linear fit on the log-log plot reveals a slope of -0.5 , corresponding to the scaling relation $|L_c(N) - L_c(\infty)| \propto N^{-0.5}$. Second, strong internal coupling will cause the system to no longer be in the incoherent state, so the analysis of the incoherent state is not applicable. When both K and L are strong, their competition leads to more complicated phase diagrams, as studied by Childs and Strogatz [27]. However, this case is beyond the scope of this study.

IV. EXPLOSIVE ENTRAINMENT IN DIVERSE MODELS

This section demonstrates that adaptive response to periodic forcing can induce explosive entrainment in other dynamical systems, including the Poincaré oscillator, the chaotic Rössler system, and the complex Ginzburg-Landau equation, highlighting its model-independent nature.

A. Explosive entrainment in the Poincaré model

The forced Poincaré model represents a two-dimensional limit-cycle oscillator that encapsulates both frequency and amplitude dynamics [52,53]. Its mathematical expression with adaptive response to periodic forcing is given by

$$\begin{aligned} \frac{dx_i}{dt} &= x_i(A_i - r_i) - \omega_i y_i + \frac{K}{N} \sum_{j=1}^N x_j + LR^\beta \sin \Omega t, \\ \frac{dy_i}{dt} &= y_i(A_i - r_i) + \omega_i x_i, \end{aligned} \quad (31)$$

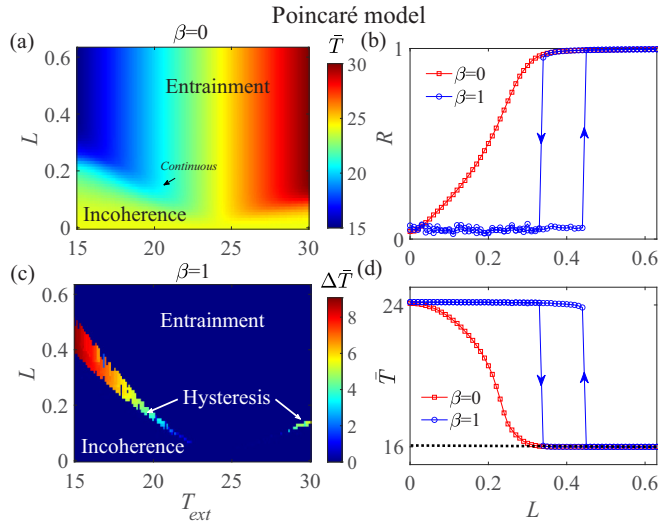


FIG. 5. Explosive entrainment in the forced Poincaré model. Arnold tongues are shown for (a) $\beta = 0$ and (c) $\beta = 1$, where $\delta = 0.02$. Evolution of R and $\bar{T}$ are shown in panels (b) and (d), respectively, under the adiabatic variation of L , when $T_{\text{ext}} = 16$. In all panels, there are $N = 1000$, $K = 0.03$, $A = 1$, and $\omega_i \sim \text{norm}(\frac{2\pi}{24}, 0.02^2)$.

where $i = 1, \dots, N$. x_i and y_i are state variables of i th oscillator in Cartesian coordinates. A_i denotes its intrinsic amplitude. r_i is the coupled amplitude, which satisfies $r_i = \sqrt{x_i^2 + y_i^2}$. The order parameter R is defined by Eq. (2).

Here, we consider a case about the biological clock. For the mammalian master clock, when a network composed of heterogeneous neurons is exposed to periodic light-dark cycles, the system exhibits rhythms synchronized with the external cycle [53–55]. In this context, two modeling assumptions are essential. First, the oscillators are coupled, reflecting neural interactions mediated by neurotransmitters. Mean-field coupling among oscillators is assumed here, whose strength is $K = 0.03$. Second, the oscillators are given an average intrinsic period of 24 h to reflect the circadian nature of the system, with individual periods drawn from a Gaussian distribution, i.e., $\text{norm}(\mu, \delta^2)$. Here, the forcing period is denoted by T_{ext} , which satisfies $T_{\text{ext}} = 2\pi/\Omega$, and the intrinsic period of the oscillator is denoted by T_i , which satisfies $T_i = 2\pi/\omega_i$.

Figure 5 demonstrates the critical role of β in the emergence of explosive entrainment in the Poincaré model. There are hysteresis loops in the evolution of both the order parameter R and the system's period $\bar{T}$ when $\beta = 1$. In the phase diagram on the (L, T_{ext}) parameter plane, these hysteresis loops correspond to regions of hysteresis within the Arnold tongue. As shown in panel (c), the width of the hysteresis increases as T_{ext} deviates further from the center of the intrinsic period distribution. In addition, we observe that the width of the frequency distribution δ determines the continuity of entrainment as shown in Fig. 6. At a fixed forcing period, i.e., $T_{\text{ext}} = 20$, for $\delta = 0.02$, explosive entrainment emerges, manifested as first-order transitions in both R and $\bar{T}$ as the coupling strength L is varied adiabatically. As δ increases, the first-order transition disappears, giving way to continuous transitions in R and $\bar{T}$, as shown in the phase diagram on the (L, δ) parameter plane. In panels (a) and (c), the

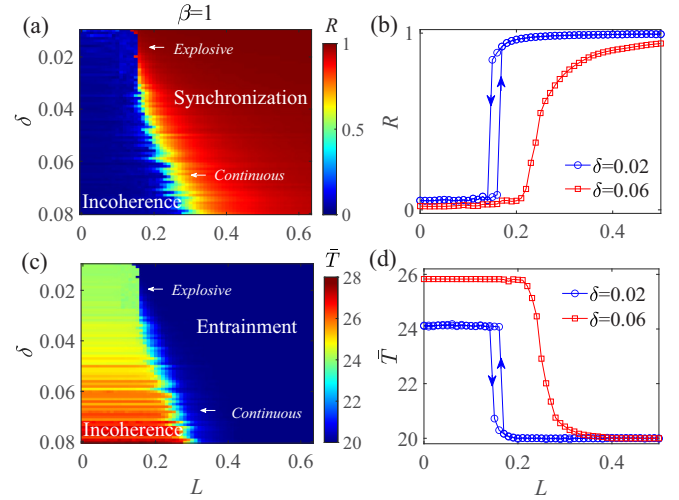


FIG. 6. The continuity of explosive entrainment is affected by the frequency width δ in the forced Poincaré model. Phase diagrams of R and $\bar{T}$ are shown in panels (a) and (c), respectively. Evolutions of R and $\bar{T}$ are shown in panels (b) and (d), respectively, under the adiabatic variation of L . In all panels, there are $N = 1000$, $K = 0.03$, $A = 1$, $\omega_i \sim \text{norm}(\frac{2\pi}{24}, \delta^2)$, and $T_{\text{ext}} = 20$.

annotations *explosive* and *continuous*, respectively, indicate explosive and continuous entrainment corresponding to different values of δ . More numerical simulation details for the Poincaré model are shown in the Supplemental Material [50].

B. Explosive entrainment in the Rössler model

In 1976, Rössler proposed a three-dimensional nonlinear model to study chaotic dynamics, which is referred to as the Rössler model [56,57]. The forced Rössler model with an adaptive response to periodic forcing reads

$$\begin{aligned} \frac{dx_i}{dt} &= -\omega_i y_i - z_i + \frac{K}{N} \sum_{j=1}^N x_j + L R^\beta \sin \Omega t, \\ \frac{dy_i}{dt} &= \omega_i x_i + a y_i, \\ \frac{dz_i}{dt} &= b + z_i(x_i - c), \end{aligned} \quad (32)$$

where a , b , and c are parameters that control the chaotic behavior. For a single Rössler oscillator, the flow rotates around a specific reference point and can be projected onto a plane with well-defined rotational properties. As a result, the phase can be defined as $\theta = \arctan(\frac{y}{x})$. Accordingly, the order parameter R is defined by Eq. (2).

Figure 7 shows explosive entrainment and continuous entrainment in the forced Rössler model. Consistent with the results of the previous two models, a distinct hysteresis region is observed in Arnold tongue when $\beta = 1$, whereas the hysteresis boundary disappears when $\beta = 0$. For $\Omega = 0.94$, panels (b) and (d) illustrate the details of explosive entrainment in the Rössler model. Here, the parameters are set to $a = 0.04$, $b = 0.2$, and $c = 10$. In the Rössler model, different parameter settings can lead to distinct chaotic behaviors. Zaks *et al.* introduce an alternative set of chaotic

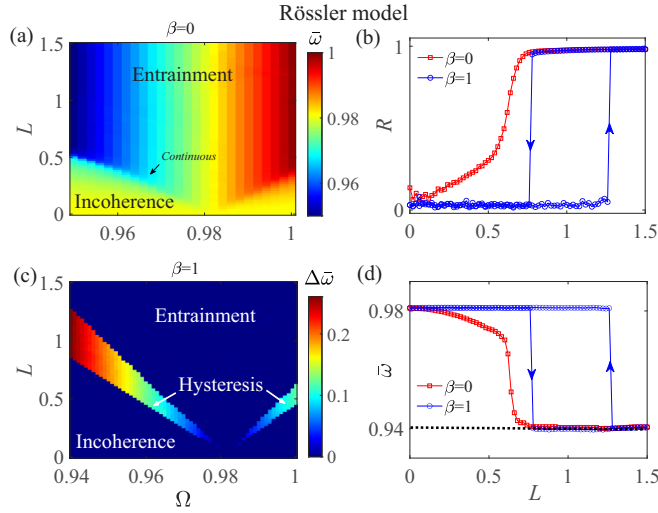


FIG. 7. Explosive entrainment in the forced Rössler model. Arnold tongues are shown for (a) $\beta = 0$ and (c) $\beta = 1$. Evolutions of the order parameter and the mean frequency of the system are shown in panels (b) and (d), respectively, when $\Omega = 0.94$. In all panels, $N = 1000$, $a = 0.04$, $b = 0.2$, $c = 10$, and $\omega_i \sim \text{norm}(0.98, 0.001^2)$.

parameters, namely, $a = 0.165$, $b = 0.2$, and $c = 10$ [58]. As shown in Fig. 8, explosive entrainment still emerges under this alternative parameter configuration. The evolution of the x component for two randomly selected oscillators is shown in panels (b) and (c). Before the phase transition, that is, at $L = 0.43$, the two oscillators evolve asynchronously, and the order parameter of system is equal to 0.1258. When L increases slightly to 0.431, the order parameter abruptly jumps to 0.6744, and the evolutions of the two oscillators nearly coincide. More numerical simulation details for the Rössler model are shown in the Supplemental Material [50].

C. Explosive entrainment in the complex Ginzburg-Landau equation

Finally, we introduce the complex Ginzburg-Landau equation (CGLE), which is a partial differential equation

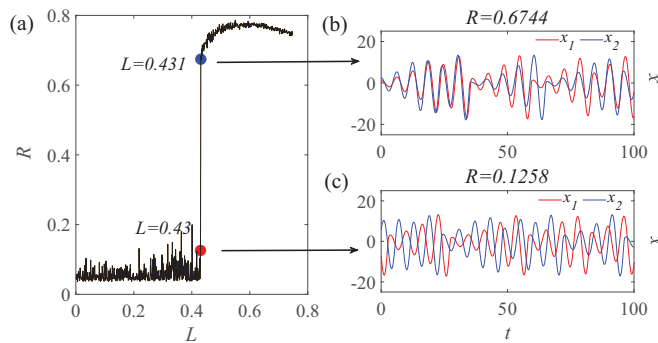


FIG. 8. Microscopic state of the system before and after the explosive entrainment in the forced Rössler model. (a) Phase transitions when $\beta = 1$. The evolutions of the x component of two randomly selected oscillators are shown in panel (b) when $L = 0.431$ and in panel (c) when $L = 0.43$, respectively. In all panels, $a = 0.165$, $b = 0.2$, $c = 10$, $\Omega = 1.05$, $N = 500$, and $\omega_i \sim \text{norm}(1, 0.005^2)$.

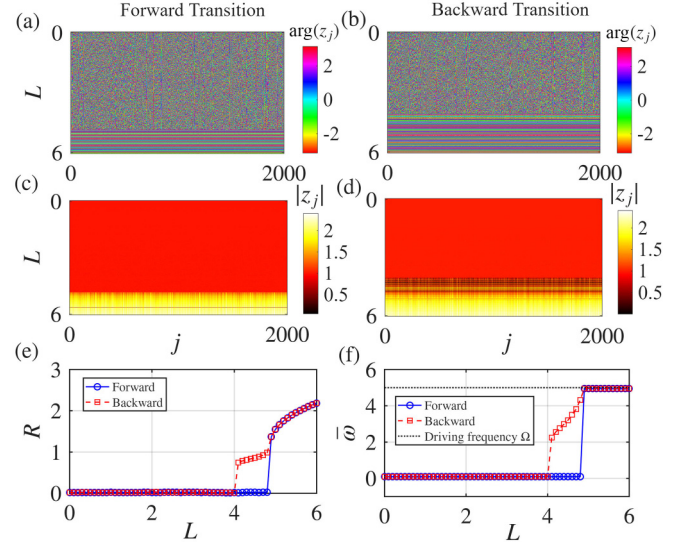


FIG. 9. Explosive entrainment in the forced CGLE. (a)–(d) Evolution of the oscillators' phases $\arg(z_j(t))$ and amplitudes $|z_j(t)|$ during forward and backward transitions. (e) The evolution of R as L varies. (f) The evolution of $\bar{\omega}$ as L varies. In all panels, $\omega_j \sim \text{norm}(0, 0.5^2)$, $\Omega = 5$, $\alpha = 0.1$, $\beta = 1$, $K = 0$, and $N = 2000$.

describing a spatially extended system near a Hopf bifurcation point [59,60]. A globally coupled version of CGLE, which has been studied previously [61], is given by

$$\partial_t z_j = (1 + i\omega_j)z_j - (1 + i\alpha)|z_j|^2 z_j + K(\bar{z} - z_j), \quad (33)$$

where $j = 1, 2, \dots, N$. z_j is the complex state for the j th oscillator, ω_j is its natural frequency, α is a parameter of nonlinear saturation effect, and K is the coupling strength to the mean field $\bar{z}$ of system, i.e., $\bar{z} = \frac{1}{N} \sum_j z_j$. Here, in order to determine the heterogeneity of each frequency, it is assumed that z is independent of the spatial coordinates. To quantify the degree of synchronization, a global order parameter R is defined as $R = \langle \bar{z} \rangle_t$.

Introducing adaptive response to periodic forcing into Eq. (33), its mathematical expression is [62]

$$\partial_t z_j = (1 + i\omega_j)z_j - (1 + i\alpha)|z_j|^2 z_j + K(\bar{z} - z_j) + LR^\beta e^{-i\Omega t}. \quad (34)$$

Figure 9 illustrates the scenario for $\beta = 1$. Panels (a)–(d) show the evolution of the phase $\arg(z_j(t))$ and amplitude $|z_j(t)|$ of all oscillators during the adiabatic increase L (forward transition) and the adiabatic decrease L (backward transition). Here, the state of the j th oscillator is defined in its polar form, i.e., $z_j(t) = |z_j(t)|e^{i\arg(z_j(t))}$. To depict abrupt phase transitions in the system, each panel consists of slices corresponding to the steady-state evolution of L , each with a time length of 100 time units. Panel (e) shows the evolution of the macroscopic order parameter R as L varies, and panel (f) shows the corresponding evolution of the system's collective frequency $\bar{\omega}$. These two panels reveal that the process of system frequency locking to the external drive is a first-order phase transition with a hysteresis loop, namely, explosive entrainment.

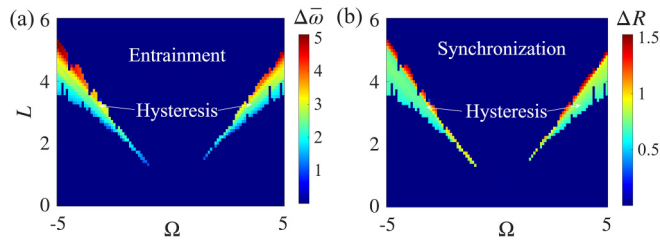


FIG. 10. Hysteresis region within the Arnold tongue of the forced CGLE. (a) Phase diagram of the difference in the collective frequency, $\Delta\bar{\omega} = |\bar{\omega}_{FW} - \bar{\omega}_{BW}|$. (b) Phase diagram of the difference in the order parameter, $\Delta R = |R_{FW} - R_{BW}|$. In all panels, $\omega_j \sim \text{norm}(0, 0.5^2)$, $\alpha = 0.1$, $\beta = 1$, $K = 0$, and $N = 2000$.

Furthermore, the hysteresis region within the Arnold tongue in CGLE is investigated. Figure 10 illustrates this region by calculating the difference in system parameters between the forward and backward transitions. Panel (a) shows the difference in collective frequency, $\Delta\bar{\omega} = |\bar{\omega}_{FW} - \bar{\omega}_{BW}|$, and panel (b) shows the difference in order parameter, $\Delta R = |R_{FW} - R_{BW}|$. Consistent with previous model results, the hysteresis region appears and expands with increasing $|\Omega|$.

V. CONCLUSION

In this study, we introduce the concept of explosive entrainment in oscillator populations under periodic forcing. Unlike explosive synchronization, explosive entrainment is driven by external forcing rather than internal coupling. Thus, explosive synchronization reflects the convergence of the collective frequency to a value that depends on the state of the system [63–66], while explosive entrainment reflects the alignment of the collective frequency with a determined external frequency. This external controllability makes explosive entrainment particularly suitable for those systems where the internal coupling is weak or hard to estimate, but the periodic input signal is easy to alter. For instance, in chemical oscillators, electronic circuits, and circadian neural networks

[67–69], the ability to actively induce or suppress hysteresis can help achieve entrainment regulation, accelerate or delay rhythmic recovery, and prevent system collapse [70–72].

Explosive entrainment has only been studied in coupled oscillator systems, which are ordinary differential equations. In spatially extended systems, which are governed by partial differential equations, multistability, chaotic transients, and hysteresis can also be exhibited under periodic forcing [73]. In addition, recent studies have shown that Arnold tongues can coexist with shrimplike structures. Shrimplike structures were first reported by Gallas [74] and later observed in various physical systems [75,76]. In the future, extending our model to such systems is a research direction.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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