

Acceleration noise induced decoherence in Stern-Gerlach interferometers for gravity experiments

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Stern-Gerlach interferometer (SGI) is a kind of matter-wave interferometer driven by magnetic field and has been proposed for various gravity experiments. Stochastic noises can lead to decoherence problems of SGI via various mechanisms. In this paper, I will theoretically study several mechanisms including dephasing, contrast loss, and position localization decoherence. I will first present a rigorous proof that the dephasing behaves as a linear response to the noise, with a transfer function given by the Fourier transform of the unperturbed classical trajectories. Then I will demonstrate that stochastic acceleration noise only induces dephasing to the witness operator constructed in spin space, while it does not lead to contrast loss or position localization decoherence due to common mode cancellation. In contrast, higher-order noise can induce both contrast loss and position localization decoherence, contributing a decay factor proportional to the noise power spectrum density at the intrinsic frequency, which can be physically interpreted as the resonance between the noise and test mass. Based on the result, I apply the framework to analyze two typical noise sources, magnetic field noise and quadratic noise.

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I. INTRODUCTION

Stern-Gerlach interferometer (SGI) is a type of matter-wave interferometer driven by an external magnetic field through the spin of the system [1]. In recent years, SGIs based on nitrogen-vacancy (NV) centres [2] have been proposed as gravity sensors [3] like gravimeters [4,5] and gyroscopes [6,7]. They have also been proposed for exploring the nature of gravity [8]. Compared to the matter-wave interferometers driven by lasers, the test mass of an SGI is usually much heavier. Consequently, the amplitude of the signal of gravitational interactions can be significantly larger and easier to detect.

Robustness is an essential issue for all types of matter-wave interferometers, including the SGI. In real experiments, test masses are affected by acceleration noises arising from various sources, such as mechanical vibrations [9], inertial forces [10], and Coulomb interactions [11]. The test masses also couple to higher-order noises including gravity gradient (tidal force) noise [10], rotation noises [12], and magnetic field noise [13,14]. These noises can disturb the trajectories and velocities of the two arms of the interferometer at the classical level and can cause decoherence of the quantum state. Since noises are usually characterized by their power spectrum densities (PSDs), a fundamental question arises:

How to formulate the decoherence factors caused by noises based on their PSDs?

Numerous previous studies have investigated the decoherence of SGIs, classifying it into several typical mechanisms, including the *dephasing*, the *position localization decoherence* [15–20], and the loss of contrast (known as the *Humpty-Dumpty problem*) [21–23].

The dephasing effect arises from the ensemble average of random phase fluctuations induced by stochastic noise. It can be classified into the phase fluctuation of the spin phase [24–26] and the spatial path integral phase [10–12,27]. In spin space, dephasing is typically characterized by the T_2 time in the Bloch-Redfield equation, which is a linear response to the PSD of the noise [25,26]. As for the spatial dephasing, some researchers have observed that it is also a linear response to the noise, characterized by a transfer function equal to the Fourier transform of the classical unperturbed trajectories [10–12,27]. Nonetheless, several subtle gaps remain to be proved: (1) Why does the dephasing depend solely on the classical trajectories of the center of mass; (2) why does the fluctuation of the trajectories provide no contribution to the dephasing?

Position localization decoherence, which pertains to the decoherence of the spatial wavefunctions, was originally studied to elucidate the collapse model [15] of quantum states caused by the interactions with the ambient environment. This type of decoherence is typically attributed to scattering processes involving particles such as air molecules or photons [15,16,28]. Furthermore, the ensemble average of stochastic noises can also induce a dissipator $\mathcal{D}[\cdot] = -\Lambda[\hat{x}, [\hat{x}, \cdot]]$, leading to a decay factor $e^{-\Lambda(x_1-x_2)^2 t}$ applied to the spatial density matrix [18,20,29,30]. In a scenario involving SGI, the witness operator is usually constructed within the spin space, prompting a question how the position localization decoherence affects the spin witness.

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The Humpty-Dumpty problem is a peculiar issue in SGI [1,31], while the former two effects occur in all types of interferometers. In these types of interferometers, the witness is usually constructed in the spin space, necessitating to trace out of spatial degrees of freedom. Consequently, the contrast loss can lead to a decay factor of the off-diagonal terms in the spin density matrix, known as the Humpty-Dumpty problem. The most fundamental source of the contrast loss arises from the nonclosure of the two arms of the interferometer at the final time [22,31,32]. Other plausible sources have been theoretically studied in recent research works, such as the rotation effect [33], the coupling with phonons [34], and the deformation of the wave packet [35]. Notably, classical noises can also cause contrast loss problem, warranting further investigation to formulate the associated contrast loss.

It is evident that all these mechanisms are intrinsically linked to the PSD of the noise, but previous papers have studied them separately. It underscores the need for systematic research that integrates all these mechanisms via the full dynamics of the test mass, despite mathematical challenges posed by the time dependence and randomness of noises. Consequently, most previous research works involving the full dynamics have focused on the white noise [5,36,37], which is identified by a constant PSD and lacks time correlation. Some authors have noticed the limitations of analyzing white noise, prompting them to consider the time dependence of the noise and to solve the corresponding Schrödinger equation using perturbation methods [38]. However, the connection between the decoherence factor and the PSD remains unestablished.

In light of the aforementioned issues, this paper aims to formulate the decoherence mechanisms outlined above based on the PSD of the external noise. The system will be modeled as a harmonic oscillator under a driven force with fluctuations, and its time evolution can be analytically solved using an elegant method known as the *Lewis-Riesenfeld invariant* [39–42] and *quantum Arnold transformation* [43].

Building upon the exact dynamics, I shall present a systematic analysis on the aforementioned decoherence mechanisms in the presence of an acceleration noise. In particular, the following topics will be addressed:

(1) I begin with a rigorous proof demonstrating that the dephasing is a linear response to the noise, followed by an investigation on mathematical properties of the associated transfer function.

(2) Next, I examine the impact of position localization decoherence and the Humpty-Dumpty problem induced by acceleration noises. As will be demonstrated, this category of noise does not lead to loss of spin witness, which can be interpreted as a form of common mode cancellation [44–46].

(3) Third, I offer a preliminary discussion of the leading-order effects of higher-order noise processes, analyzing the loss of witness arising from these mechanisms.

(4) Finally, I consider two representative noise sources, spin-dependent noise and trap frequency noise (i.e., quadratic noise), as applications of the general framework developed for noise induced decoherence.

This paper is organized as follows. Section II reviews the dynamics of an ideal Stern-Gerlach interferometer in the absence of noise, with Secs. II A and II B addressing the spatial motion and spin-space witness construction,

respectively. Mathematical details are provided in Appendixes A and B. Section III focuses on the dephasing effect, while Sec. IV explores the position localization decoherence and the Humpty-Dumpty problem induced by acceleration noise. Section IV is divided into several subsections, each addressing distinct aspects: the fluctuation of the classical trajectories, the fidelity loss of a single arm, position localization decoherence, and the loss of spin witness. An important integral is computed using the residue theorem in Appendix C, while the master equation for single-arm localization decoherence is derived in Appendix D. Section V presents a preliminary discussion on the witness loss induced by higher-order noises, while Sec. VI applies the results to analyze two typical noises, spin-dependent noise and quadratic noise, each treated in a separate subsection. The analytic solution for the quadratic noise is summarized in Appendix E. Finally, Sec. VII concludes with remarks on the role of randomness (Sec. VII A) and a summary of the paper (Sec. VII B).

II. IDEAL STERN-GERLACH INTERFEROMETER WITHOUT NOISES

A general schematic diagram of a Stern-Gerlach interferometer is shown in Fig. 1. The test mass is trapped in a potential well that can be assumed as a harmonic oscillator for simplicity. Besides, the spin of the test mass has to be split during the experiment, which can be achieved by some intrinsic properties of the test mass or a constant biased external magnetic field (i.e., the Zeeman effect). Then, the Stern-Gerlach interferometer can be created by a nonuniform external magnetic field. For example, the one-dimensional Hamiltonian of the ideal gravimeter without noise based on the NV center can be written as [38]

$$\hat{H} = \frac{\hat{p}^2}{2m} + g_{\text{NV}}\mu_B \frac{\partial B}{\partial x} \hat{S}_z \hat{x} - \frac{\chi V}{2\mu_0} \left(\frac{\partial B}{\partial x} \right)^2 \hat{x}^2 - mg\hat{x} + \hbar D \hat{S}_z^2. \quad (1)$$

The term $\hbar D \hat{S}_z^2$ is the zero-field splitting of the NV center with $D = (2\pi)2.8$ GHz, which physically arises from the electron spin-spin dipole interaction [47]. Under this term, the spin of the test mass splits into eigenstates of $\hat{S}_z$, where z direction is assumed as the direction of the NV axis for simplicity, which can be changed to any axis in a real experiment. The spin-orbit coupling term $g_{\text{NV}}\mu_B (\partial B / \partial x) \hat{S}_z \hat{x}$ arises from the spin-magnetic field interaction, where μ_B is the Bohr magneton and g_{NV} is the Lande g factor of NV center. The quadratic term $\chi V / (2\mu_0) (\partial B / \partial x)^2 \hat{x}^2$ is the diamagnetic energy of the diamond, where $\chi = -2.2 \times 10^{-5}$ and V are the magnetic susceptibility and the volume of the diamond. The magnetic field gradient $\partial B / \partial x$ can be set as a time-independent constant [4], a piecewise function of time [38], several pulses [31], or more complicated functions [35,48]. The term $mg\hat{x}$ accounts for the coupling to gravitational acceleration, which can be substituted with other coupling mechanisms for different experimental contexts. For instance, this sensor can also function as an accelerometer or seismometer by coupling to inertial forces [3,49].

In order to create superpositions by the magnetic field, it is essential to avoid the precession of the spin such that

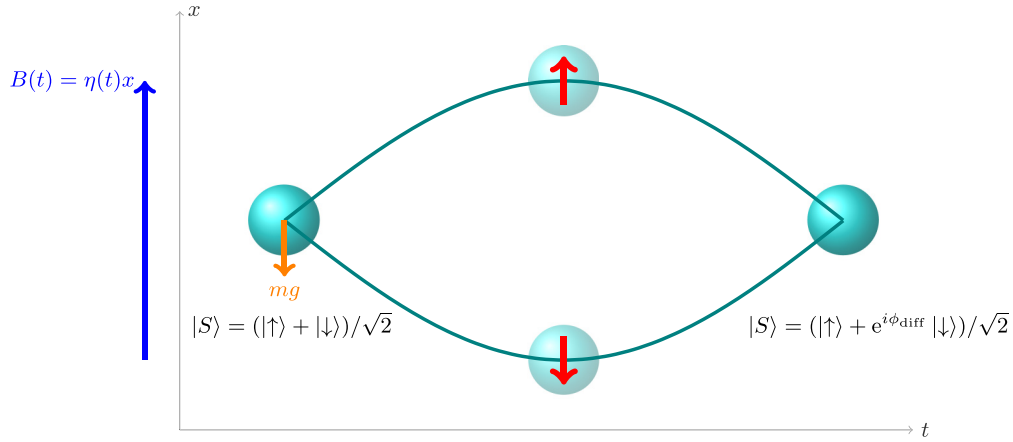


FIG. 1. The schematic diagram of an ideal Stern-Gerlach interferometer for gravity experiments. Under a nonuniform magnetic field $\mathbf{B} = \eta\mathbf{x}$, the test mass splits into two trajectories with respect to different spin states. The magnetic field gradient $\eta \equiv \partial B/\partial x$ can be set as a time-independent constant [4], a piecewise function of time [38], several pulses [31], or more complicated functions [35,48]. The differential phase ϕ_{diff} at the final time encodes the information of gravity, so properties of gravity can be detected by measuring this phase.

spin states are pinned along the direction defined by the splitting term. For the Hamiltonian (1), the $\hat{\mathbf{S}} \cdot \mathbf{B}$ term is supposed smaller than the zero-splitting term to avoid the Larmor precession. Since the energy of the zero-splitting term is $(2\pi)\hbar D \sim 1.8 \times 10^{-24} \text{ J} \sim g\mu_B \times 10^{-1} \text{ T}$, the value of the magnetic field $|\mathbf{B}|$ required is smaller than 0.1 T [33]. In this case, the spin operator $\hat{S}_z$ can be treated as a c number rather than an operator for ease of calculation. Then, $s_z = \pm 1$ of the interferometer corresponds to two paths, and each path of the interferometer is exactly a trajectory of a harmonic oscillator driven by external forces. Thus, the Hamiltonian for a single arm can be written as

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2 - ma(t)\hat{x}, \quad (2)$$

where $ma(t)$ consists of the driven force $g_{\text{NV}}\mu_B(\partial B/\partial x)s_z\hat{x}$ and the signal force $mg\hat{x}$, where the time dependence of the Hamiltonian mainly comes from the time-dependent noise. The corresponding Schrödinger equation can be analytically solved by *Lewis-Riesenfeld invariant* and *quantum Arnold transformation* [42,43], and the math details are summarized in Appendix A. Notably, other interferometers driven by spin-orbit coupling [50] or a superconducting quantum interference device [51] have Hamiltonians of similar forms, so the noise analysis in this paper is also valid for these types of interferometers.

A. Time evolution of single path

As derived in Appendix A, the time-evolution operator for the Hamiltonian (2) is

$$\hat{U}(t) = e^{\frac{i}{\hbar} \int_0^t L[x(t'), \dot{x}(t')] dt'} \hat{D}(\alpha_c(t)) e^{-\frac{i}{\hbar} \hat{H}_0 t} \hat{D}(-\alpha_c(0)), \quad (3)$$

where

$$\begin{aligned} \hat{H}_0 &= \hat{p}^2/2m + m\omega_0^2\hat{x}^2/2, \\ L[x(t), \dot{x}(t)] &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2x^2 + ma(t)x, \\ \hat{D}(\alpha_c(t)) &= \exp[\alpha_c(t)\hat{a}^\dagger - \alpha_c^*(t)\hat{a}], \\ \alpha_c(t) &= \sqrt{m\omega_0/2\hbar}(x_c(t) + ip_c(t)/m\omega_0). \end{aligned} \quad (4)$$

Here, $\hat{H}_0$ is the Hamiltonian of a simple harmonic oscillator without driven forces, $L[x(t), \dot{x}(t)]$ is the Lagrangian corresponding to the Hamiltonian (2), and $\hat{D}(\alpha_c(t))$ is the displacement operator defined by the complex coordinate $\alpha_c(t)$ of the instantaneous equilibrium position in the phase space, which satisfies the canonical equation in complex coordinate

$$\dot{\alpha}_c = -i\omega_0\alpha_c + i\sqrt{\frac{m}{2\hbar\omega_0}}a(t), \quad (5)$$

of which the solution is

$$\alpha_c(t) = e^{-i\omega_0 t} \left(\alpha_c(0) + i\sqrt{\frac{m}{2\hbar\omega_0}} \int_0^t a(t') e^{i\omega_0 t'} dt' \right). \quad (6)$$

The physical interpretation of the time-evolution operator $\hat{U}(t)$ in Eq. (3) is that the system is a simple oscillator without any driven force in the comoving reference frame of the equilibrium position, where the displacement operators $\hat{D}(\alpha_c(t))$ and $\hat{D}(-\alpha_c(0))$ play the role of the reference translation in the classical phase space. Besides, $\hat{U}(t)$ also contains a path integral phase $e^{\frac{i}{\hbar} \int_0^t L[x(t'), \dot{x}(t')] dt'}$ along the classical trajectory of the equilibrium position $x_c(t)$ of the oscillator.

Consider an arbitrary initial state described by a density matrix $\hat{\rho}_0$ evolving as

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}_0\hat{U}^\dagger(t). \quad (7)$$

From the point of view of Weyl quantization, the Wigner function is the Weyl-Wigner transform of the density matrix [52], i.e.,

$$W(\alpha) = \int \text{Tr}[\hat{\rho}\hat{D}(\alpha')] e^{\alpha'^* \alpha - \alpha' \alpha^*} \frac{d^2\alpha'}{\pi^2}, \quad (8)$$

then the time evolution of the Wigner function of an arbitrary state satisfies

$$\begin{aligned} W(\alpha, t) &= W(\alpha_c(0) + e^{i\omega_0 t}(\alpha - \alpha_c(t)), t=0) \\ &= W\left(e^{i\omega_0 t}\alpha - i\sqrt{\frac{m}{2\hbar\omega_0}} \int_0^t a(t') e^{i\omega_0 t'} dt', t=0\right). \end{aligned} \quad (9)$$

Thus, the shape of the Wigner function of an arbitrary state does not change during time evolution, and its time evolution is only a transformation in the phase space composited by a translation $\alpha_c(t)$ plus a rotation with a constant angular speed ω_0 . Besides, the evolution of an arbitrary quantum state is completely determined by its classical trajectory $x_c(t)$ and momentum $p_c(t)$ satisfying the classical equations of motion (5) in the phase space.

Especially, if the interferometer is initially prepared as a coherent state $|\beta_0\rangle = \hat{D}(\beta_0)|0\rangle$, then it evolves as

$$|\psi(t)\rangle = e^{i\phi(t)} e^{-i\text{Im}(\Delta\alpha_c\beta_0^*)} e^{-\frac{1}{2}\omega_0 t} |\alpha(t)\rangle, \quad (10)$$

where

$$\phi(t) = \frac{1}{\hbar} \int_0^t L[x_c(t'), \dot{x}_c(t')] dt', \quad (11)$$

$$\alpha(t) = \alpha_c(t) + e^{-i\omega_0 t} (\beta - \alpha_c(0)), \quad (12)$$

and the notation $\Delta\alpha_c \equiv \alpha_c(t) - \alpha_c(0)$. So, every single arm remains in a coherent state during the time evolution, of which the wavefunction remains in a Gaussian wave packet with a fixed width. Besides, the time evolution of an initial coherent state can be determined by the classical trajectory. Remarkably, there are additional phases $\text{Im}(\Delta\alpha_c\beta_0^*)$ and $-\omega_0 t/2$ related to the details of the initial quantum state.

B. Time evolution of interferometer and witness in spin space

For an interferometer, the driven forces on its different arms are spin dependent, so the corresponding time-evolution operators are different. In particular, the spatial part of the density matrix related to the spin element $|i\rangle\langle j|$ evolves as $\hat{U}_i(t)\hat{\rho}_0\hat{U}_j^\dagger(t)$ with $i, j = \pm 1$. Suppose the spin state is initially prepared as a pure state

$$|S_z(t=0)\rangle = \frac{|1\rangle + |-1\rangle}{\sqrt{2}}, \quad (13)$$

then the total density matrix of the interferometer at time t is

$$\hat{\rho}_{\text{tot}}(t) = \frac{1}{2} \sum_{i,j=\pm 1} e^{i(\phi_i(t)-\phi_j(t))} |i\rangle\langle j| \otimes \hat{\rho}_{ij}(t), \quad (14)$$

where

$$\hat{\rho}_{ij}(t) \sim \hat{U}_i(t)\hat{\rho}_0\hat{U}_j^\dagger(t), \quad (15)$$

and the phase $\phi_{i,j}(t)$ only contains the spatial path integral along the classical trajectories and the spin dynamics are ignored for simplicity.

At the final time t_f of the experiment, the observable quantity is usually constructed by a witness operator $\hat{W}$ in the spin space, so the spatial wavefunctions have to be traced out, i.e.,

$$\hat{\rho}_{\text{spin}}(t) = \text{Tr}_x[\hat{\rho}(t)]. \quad (16)$$

If the final positions $\alpha_1(t)$ and $\alpha_{-1}(t)$ in the classical phase space are designed the same at the final time t_f , then all the spatial density matrix elements $\hat{\rho}_{ij}(t_f)$ are the same. As a consequence, the spin space and the spatial space are separable, i.e., $\hat{\rho}_{\text{tot}}(t_f) = \hat{\rho}_{\text{spin}}(t_f) \otimes \hat{\rho}(t_f)$, so the spin density matrix is

simply

$$\hat{\rho}_{\text{spin}}(t_f) = \frac{1}{2} \begin{pmatrix} 1 & 0 & e^{-i\phi_{\text{diff}}} \\ 0 & 0 & 0 \\ e^{i\phi_{\text{diff}}} & 0 & 1 \end{pmatrix}, \quad (17)$$

with the notation $\phi_{\text{diff}} \equiv \phi_{-1}(t_f) - \phi_1(t_f)$. A commonly used witness to measure the differential phase ϕ_{diff} is the *Ramsey interferometry* [4,25,53]. It is constructed by measuring the population of $|S_z = 0\rangle$ state after a $\pi/2$ pulse operating on the state. As derived in Appendix B, the witness operator for Ramsey interferometry in the spin-1 system is given by

$$\hat{W} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}. \quad (18)$$

Then, one can obtain the observable result for the Ramsey interferometry as

$$\text{Tr}[\hat{\rho}_{\text{spin}}(t_f)\hat{W}] = \cos^2 \frac{\phi_{\text{diff}}}{2}, \quad (19)$$

which means that the population of the eigenstate $|S_z = 0\rangle$ at the final time after a $\pi/2$ pulse is $\cos^2(\phi_{\text{diff}}/2)$. For a Stern-Gerlach interferometer coupling to the local gravity by $\hat{H}_{\text{int}} = mg\hat{x}$ in Eq. (1), the differential phase ϕ_{diff} is proportional to g , so this device can be used as a gravimeter to measure the gravitational acceleration g through the witness (18) of Ramsey interferometry [4].

III. DEPHASING INDUCED BY ACCELERATION NOISES

In a real experiment, there always exist some acceleration noises $\delta a(t)$ such as the vibration of the mechanical support of the system, the inertial force due to the vibration of the laboratory (known as the residual acceleration noise [10,27]) and electric forces caused by ambient electric particles [11].

In most experimental situations, acceleration noises can be described by a (wide-sense) *stationary Gaussian process* with a zero mean value. In particular, at any time t , the acceleration noise $\delta a(t)$ follows a Gaussian distribution. In addition, the *wide-sense stationarity* requires that the autocorrelation function of $\delta a(t)$ does not vary with respect to time t ; i.e., $\mathbb{E}[\delta a(t)\delta a(t+\tau)]$ only depends on τ , where $\mathbb{E}[\cdot]$ represents the statistical average of stochastic variables. According to the Wiener-Khinchin theorem, the autocorrelation function of $\delta a(t)$ is the Fourier transform of the corresponding *power spectrum density*

$$\begin{aligned} \mathbb{E}[\delta a(t)] &= 0, \quad \forall t, \\ \mathbb{E}[\delta a(t)\delta a(t+\tau)] &= \int S_{aa}(\omega) e^{-i\omega\tau} d\omega, \quad \forall t, \end{aligned} \quad (20)$$

where $S_{aa}(\omega) = \lim_{T \rightarrow \infty} |\delta a_T(\omega)|^2 / T$ and $\delta a_T(\omega)$ is the Fourier transform of $\delta a(t)$ on the finite time domain $0 \sim T$. Note that the second equation is independent of t because the noise is assumed stationary. In addition, let $\tau = 0$, then the autocorrelation function of $\delta a(t)$ gives the variance of $\delta a(t)$ as $\sigma_a^2(t) \equiv \mathbb{E}[(\delta a(t))^2] = \int S_{aa}(\omega) d\omega$, which is also time independent.

The acceleration noise $\delta a(t)$ couples to the test mass directly via the Hamiltonian $\hat{H}_{\text{noise}} = -m\delta a(t)\hat{x}$, then the

Hamiltonian (2) for a single arm becomes to

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2 - ma(t)\hat{x} - m\delta a(t)\hat{x}, \quad (21)$$

which is still of the form as Eq. (2). Hence, the time-evolution operator (3) also holds when the noise is present, although the phase and the complex coordinate $\alpha_c(t)$ fluctuate. This section will concentrate on the dephasing, while the next section will discuss the decoherence of the spatial part of the test mass.

A. Dephasing as a linear response to noise

According to the time-evolution operator (3), the spatial phase is precisely the path integral along the classical trajectories of the center of mass. When the test mass experiences disturbances due to an acceleration noise, this phase is generally fluctuated by two leading-order channels. The first channel involves the path integral of the Lagrangian $m\delta a(t)x$ along the classical trajectory $x_c(t)$, while the second channel arises from the fluctuation of the classical trajectory δx and velocity $\delta\dot{x}$ in the path integral of the ideal Lagrangian. The interplay between these two channels represents a higher-order effect and can be neglected. Thus, the phase fluctuation of a single arm can be expressed as

$$\delta\phi = \frac{1}{\hbar} \int m\dot{x}\delta\dot{x} - m\omega_0^2 x\delta x + ma(t)\delta x + m\delta a(t)x dt. \quad (22)$$

In order to evaluate the phase fluctuation $\delta\phi$, it is notable that the dynamical equation (5) can be written in another form as $\ddot{x} + \omega_0^2 x = a(t)$, then one can find out an identity

$$m\dot{x}\delta\dot{x} - m\omega_0^2 x\delta x + ma(t)\delta x = m \frac{d}{dt}(\dot{x}\delta x). \quad (23)$$

Thus, the phase fluctuation arising from the fluctuation of the trajectory $\delta x(t)$ and velocity $\delta\dot{x}(t)$ is a total derivative that will vanish after the path integral. Remarkably, some authors have already noticed that this channel of the phase fluctuation has to be taken into account in principle, but they neglected this effect by assuming that it is a higher-order effect [10–12,27]. As proved above, the dynamical equation ensures that this effect is strictly negligible without any assumptions [54]. Therefore, the phase fluctuation of a single path is simply

$$\delta\phi(t) = \frac{1}{\hbar} \int_0^t m\delta a(t')x(t')dt'. \quad (24)$$

So, the phase fluctuation $\delta\phi(t)$ of a single path is an Itô integral of the stochastic noise $\delta a(t)$ and can be regarded as a linear response to $\delta a(t)$. Since the acceleration noise $\delta a(t)$ is assumed as a stationary Gaussian process, the phase fluctuation $\delta\phi(t)$ is a Gaussian process with a zero mean value $\mathbb{E}[\delta\phi(t)] = 0$ for arbitrary time t , although it is no longer stationary. The variance of $\delta\phi(t)$ can be computed as

$$\sigma_\phi^2(t) \equiv \mathbb{E}[(\delta\phi(t))^2] = \frac{m^2}{\hbar^2} \int S_{aa}(\omega)|x_t(\omega)|^2 d\omega, \quad (25)$$

where $x_t(\omega)$ is the Fourier transform of the trajectory $x(t')$ in the finite time domain $0 \sim t$, which can be written by a convolution between $x(\omega)$ and the sinc function as $x_t(\omega) =$

$x(\omega) \times \text{sinc}(\omega t)$. Furthermore, $x(\omega)$ plays a role of the *transfer function* (also known as *frequency response function* or *filter function*) of the linear response of $\delta\phi$ to δa [12,27,49,55].

For the interferometer with two paths, the phase fluctuations of both arms are linear responses to the acceleration noises, so the differential phase $\delta\phi_{\text{diff}}$ is also a linear response to δa and its variance is

$$\begin{aligned} \delta\phi_{\text{diff}} &= \frac{m}{\hbar} \int_0^{t_f} \delta a(t)(x_1(t) - x_{-1}(t))dt, \\ \sigma_{\phi_{\text{diff}}}^2 &= \frac{m^2}{\hbar^2} \int S_{aa}(\omega)F(\omega)d\omega, \end{aligned} \quad (26)$$

where

$$F(\omega) = |x_1(\omega) - x_{-1}(\omega)|^2 \quad (27)$$

is the transfer function or filter function of the linear response [12,27,49,55].

B. Properties of transfer function

The transfer function has the following asymptotic behaviors:

(1) $F(\omega)$ tends to a constant in the low-frequency limit. In particular, in the low-frequency limit $\omega \rightarrow 0$, the factor $e^{i\omega t} \rightarrow 1$, then the Fourier transform satisfies $x_i(\omega) = \int x_i(t)e^{i\omega t}dt \approx \int x_i(t)dt$, which is independent of ω , so $F(\omega)$ is approximately constant to ω in the low-frequency limit.

(2) $F(\omega)$ decreases as ω^{-2k} for some index k in the high-frequency limit. In particular, in the high-frequency limit, the trajectories can be expanded as Taylor's series as $x_1(t) - x_{-1}(t) = \sum_{n=k}^{\infty} c_n t^n$ with the leading order as t^k , then the corresponding Fourier transform is $x_1(\omega) - x_{-1}(\omega) = \sum_{n=k}^{\infty} c_n \omega^{-n-1}$, so the transfer function is proportional to ω^{-2k-2} at the leading order in the high-frequency limit.

According to the dynamical equation $\ddot{x}_i + \omega_0^2 x_i = a_i(t)$, which is associated with the Hamiltonian (21), one can obtain an equation $(-\omega^2 + \omega_0^2)x_i(\omega) = a_i(\omega)$ in frequency space. Hence, the transfer function can be written as

$$F(\omega) = \frac{|a_1(\omega) - a_{-1}(\omega)|^2}{(\omega^2 - \omega_0^2)^2}. \quad (28)$$

If $a_i(\omega)$ has no singularity, the transfer function is a meromorphic function with two second-order poles $\pm\omega_0$. Since $S_{aa}(\omega)$ and $a_i(\omega)$ are symmetric in the frequency domain, the variance of the phase fluctuation can be computed by the residue theorem as

$$\begin{aligned} \sigma_{\phi_{\text{diff}}}^2 &= 4\pi \frac{m^2}{\hbar^2} \frac{d}{d\omega} \left[\frac{S_{aa}(\omega)}{(\omega + \omega_0)^2} |a_1(\omega) - a_{-1}(\omega)|^2 \right] \Big|_{\omega=\omega_0} \\ &= \pi \frac{m^2}{\hbar^2 \omega_0^3} S_{aa}(\omega_0) |a_1(\omega_0) - a_{-1}(\omega_0)|^2, \end{aligned} \quad (29)$$

where the math properties of $a_i(\omega)$ and $S_{aa}(\omega)$ are assumed good enough such that their derivatives do not have dominant contributions to $\sigma_{\phi_{\text{diff}}}^2$. In this sense, the transfer function $F(\omega)$ is approximately a delta function, i.e.,

$$F(\omega) \approx A\delta(\omega - \omega_0), \quad (30)$$

with a factor $A = \pi |a_1(\omega_0) - a_{-1}(\omega_0)|^2 / \omega_0^3$.

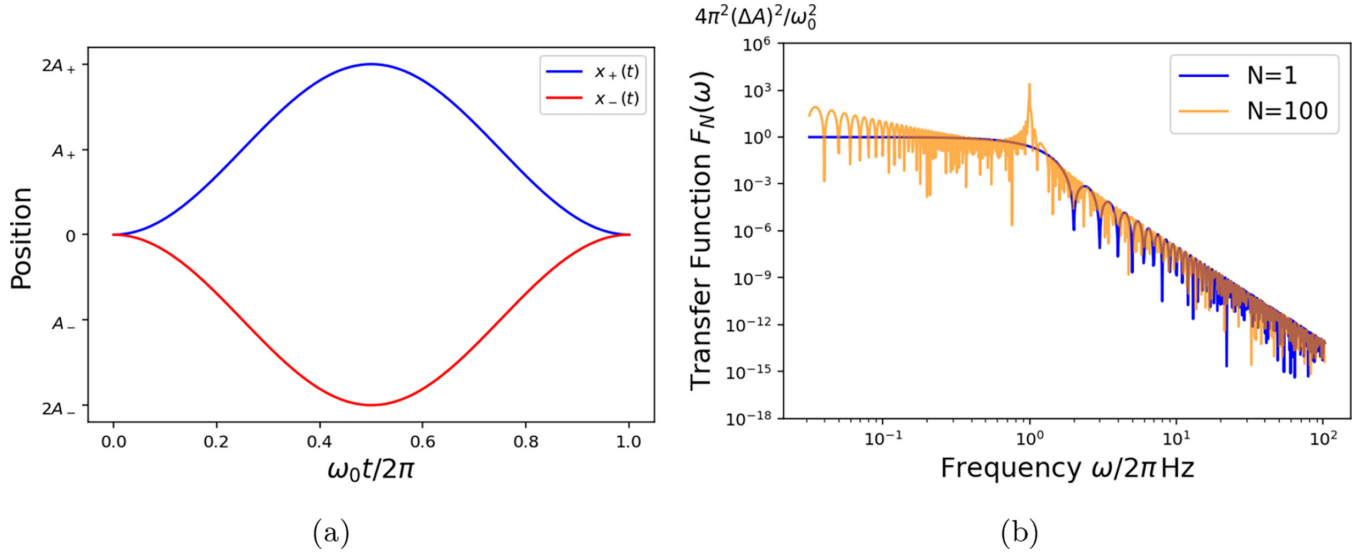


FIG. 2. (a) The unperturbed trajectories of an interferometer under a time-independent magnetic field $B = \eta x$. (b) The corresponding transfer function for the trajectories with N periods.

For example, consider a time-independent magnetic field $B(t) = \eta x$, with $\eta \equiv \partial B / \partial x$. As shown in Fig. 2(a), the unperturbed trajectories are

$$x_{\pm}(t) = A_{\pm}(1 - \cos \omega_0 t), \quad \text{with } A_{\pm} = \pm \frac{g_{NV} \mu_B \eta}{m \omega_0^2}, \quad (31)$$

where the gravitational forces are ignored for simplicity. If the total experiment time is $2\pi N / \omega_0$, i.e., the trajectories repeat N periods in a single run of the experiment, then the corresponding transfer function is

$$F_N(\omega) = \frac{4(\Delta A)^2 \omega_0^4 \sin^2 \frac{N\pi\omega}{\omega_0}}{\omega^2 (\omega^2 - \omega_0^2)^2}, \quad (32)$$

where $\Delta A \equiv A_+ - A_-$. According to Eq. (32), $F_N(\omega)$ tends to a constant $4\pi^2 N^2 (\Delta A)^2 / \omega_0^2$ in the low-frequency limit $\omega \rightarrow 0$ and decreases with respect to ω^{-6} in the high-frequency limit $\omega \rightarrow \infty$. As shown in Fig. 2(b), the transfer function has a peak at $\omega = \omega_0$, so it is approximated as a delta function $\delta(\omega - \omega_0)$, except when the number of periods $N = 1$, in which case this peak is hidden. In fact, as derived in Appendix C, the variance of the phase fluctuation in this system can be evaluated using the residue theorem, yielding

$$\sigma_{\phi_{\text{diff}}}^2 = 4\pi m^2 \frac{N^2 (\Delta A)^2 S_{aa}(\omega_0)}{\omega_0 \hbar^2}. \quad (33)$$

IV. POSITION LOCALIZATION DECOHERENCE AND HUMPTY-DUMPTY PROBLEM

As pointed out at the beginning of the previous section, the time-evolution operator of the interferometer under the noise $\delta a(t)$ has the same form as the ideal situation. Thus, the time evolution (9) of the Wigner function of an arbitrary initial state becomes

$$W(\alpha, t) = W(\alpha_c(0) + e^{i\omega_0 t}(\alpha - \alpha_c(t) - \delta\alpha(t)), t = 0). \quad (34)$$

The first notable conclusion is that the acceleration noise $\delta a(t)$ does not change the shape of the Wigner function of an arbitrary initial state, and it only affects the path integral phase $\delta\phi$ and the classical trajectory $\delta\alpha$ in phase space.

A. Fluctuation of classical phase space $\delta\alpha$ of single arm

For a coherent state given by Eq. (10), the wavefunction under the acceleration noise is still a coherent state, although its parameter $\beta(t) = \alpha_c(t) + e^{-i\omega_0 t}(\beta_0 - \alpha_c(0))$ changes to $\beta'(t) = \beta(t) + \delta\alpha(t)$. According to the time evolution (6) of the complex coordinate α_c in the phase space, its fluctuation $\delta\alpha$ under the acceleration noise $\delta a(t)$ is given by

$$\delta\alpha(t) = i\sqrt{\frac{m}{2\hbar\omega_0}} e^{i\omega_0 t} \int_0^t \delta a(t') e^{-i\omega_0 t'} dt'. \quad (35)$$

So, $\delta\alpha(t)$ is also a linear response to the acceleration noise $\delta a(t)$. Besides, $\delta\alpha(t)$ is also a Gaussian process with a zero mean value $\mathbb{E}[\delta\alpha(t)] = 0$ and its autocorrelation function is

$$\begin{aligned} \mathbb{E}[\delta\alpha(t)\delta\alpha^*(t+\tau)] &= \frac{m}{2\hbar\omega_0} e^{-\frac{i}{2}\omega_0\tau} t(t+\tau) \int S_{aa}(\omega) e^{-\frac{i}{2}\omega\tau} \\ &\times \text{sinc} \frac{(\omega - \omega_0)t}{2} \text{sinc} \frac{(\omega - \omega_0)(t+\tau)}{2} d\omega. \end{aligned} \quad (36)$$

Remarkably, $\mathbb{E}[\delta\alpha(t)\delta\alpha^*(t+\tau)]$ relies on both t and τ , so $\delta\alpha$ is not stationary. Hence, the Wiener-Khinchin theorem does not hold for $\delta\alpha$.

In particular, considering the dynamical equation (5) of α_c , one may obtain a frequency response relation as

$$\delta\alpha(\omega) = \chi(\omega)\delta a(\omega), \quad \text{with } \chi(\omega) = \frac{\sqrt{m/(2\hbar\omega_0)}}{\omega - \omega_0}. \quad (37)$$

Here, the factor $\chi(\omega)$ is a generalized *mechanical susceptibility* of $\alpha_c(\omega)$. Then, the PSD of $\delta\alpha$ is

$$S_{\alpha\alpha}(\omega) = \frac{m}{2\hbar\omega_0} \frac{S_{aa}(\omega)}{(\omega - \omega_0)^2}. \quad (38)$$

Therefore, one may check that the relationship between the autocorrelation function and PSD of $\delta\alpha$ is

$$\mathbb{E}[\delta\alpha(t)\delta\alpha^*(t+\tau)] = \int S_{\alpha\alpha}(\omega)e^{-i\omega\tau}(1 + e^{-i(\omega-\omega_0)\tau} - e^{i(\omega-\omega_0)t} - e^{-i(\omega-\omega_0)(t+\tau)})d\omega. \quad (39)$$

In the autocorrelation function (36) of $\delta\alpha$, let $\tau = 0$, then one may have the variance of $\delta\alpha$ at any time t as

$$\sigma_\alpha^2(t) \equiv \mathbb{E}[|\delta\alpha(t)|^2] = \frac{m}{2\hbar\omega_0}t^2 \int S_{aa}(\omega)\text{sinc}^2\frac{(\omega-\omega_0)t}{2}d\omega. \quad (40)$$

Thus, $\delta\alpha$ can be regarded as a generalized diffusion process. In the short-time and long-time limits, its variance can be approximated as [56]

$$\sigma_\alpha^2(t) \approx \begin{cases} D_1 t^2, & t \ll 1, \\ D_2 t, & t \rightarrow \infty \end{cases}, \quad (41)$$

where the generalized diffusion coefficients are given by $D_1 = m\sigma_\alpha^2/(2\hbar\omega_0)$ and $D_2 = \pi m S_{aa}(\omega_0)/(\hbar\omega_0)$. Notably, in the long-time limit, $\delta\alpha$ exhibits behavior analogous to standard Brownian motion. That is, the diffusion process becomes *Markovian* or *memoryless* as $t \rightarrow \infty$, implying that its long-time diffusion is independent of its initial diffusion history.

Especially, at the final time $t_f = 2N\pi/\omega_0$, the variance is

$$\sigma_\alpha^2(t_f) = \frac{2\pi^2 N^2 m}{\hbar\omega_0^3} \int S_{aa}(\omega) \frac{\sin^2(N\pi\omega/\omega_0)}{N^2\pi^2(\omega/\omega_0 - 1)^2} d\omega, \quad (42)$$

which can be evaluated by the residue theorem, as shown in Appendix C. This yields

$$\sigma_\alpha^2(t_f) = \frac{2\pi^2 N m}{\hbar\omega_0^2} S_{aa}(\omega_0). \quad (43)$$

Physically, it can be interpreted that the test mass only resonates with the noise at frequency ω_0 , which is the intrinsic frequency of the interferometer. Remarkably, this result coincides with the long-time limit expression $\sigma_\alpha^2(t) = D_2 t$, although t_f does not necessarily fall within the asymptotic regime.

B. Fidelity loss of a single arm

Since the trajectory of a single arm is disturbed by the noise, the corresponding quantum state is also affected, which is usually characterized by the *Uhlmann fidelity* [57,58]. In particular, denote the ideal and noisy density matrices of the single arm at an arbitrary time t as

$$\hat{\rho}_{\text{id}}(t) \equiv \hat{U}_{\text{id}}(t)\hat{\rho}_0\hat{U}_{\text{id}}^\dagger(t), \quad \hat{\rho}_n(t) \equiv \hat{U}_n(t)\hat{\rho}_0\hat{U}_n^\dagger(t), \quad (44)$$

where $\hat{U}_{\text{id}}(t)$ and $\hat{U}_n(t)$ are given by Eq. (3) with complex coordinates $\alpha_c(t)$ and $\alpha_c(t) + \delta\alpha(t)$. Then, the corresponding fidelity is defined as

$$F(\hat{\rho}_{\text{id}}(t), \hat{\rho}_n(t)) \equiv \text{Tr}\left[\sqrt{\sqrt{\hat{\rho}_{\text{id}}(t)}\hat{\rho}_n(t)\sqrt{\hat{\rho}_{\text{id}}(t)}}\right]. \quad (45)$$

It is not difficult to prove that [59]

$$F(\hat{\rho}_{\text{id}}(t), \hat{\rho}_n(t)) = F(\hat{\rho}_0, \hat{D}(\delta\alpha(t)e^{i\omega_0 t})\hat{\rho}_0\hat{D}^\dagger(-\delta\alpha(t)e^{i\omega_0 t})). \quad (46)$$

So, the fidelity between $\hat{\rho}_{\text{id}}(t)$ and $\hat{\rho}_n(t)$ is exactly the fidelity of the initial state $\hat{\rho}_0$ and its fluctuation under $\hat{D}(\delta\alpha(t)e^{i\omega_0 t})$. Especially, if the system is initially prepared as a pure state $\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0|$, then the fidelity is simply the contrast

$$F(\hat{\rho}_{\text{id}}(t), \hat{\rho}_n(t)) = |\langle\psi_0|\hat{D}(\delta\alpha(t)e^{i\omega_0 t})|\psi_0\rangle|. \quad (47)$$

In addition, if the system is initially prepared as a coherent state, then the fidelity is

$$F(\hat{\rho}_{\text{id}}(t), \hat{\rho}_n(t)) = \exp\left[-\frac{|\delta\alpha(t)|^2}{2}\right], \quad (48)$$

which is exactly the overlap of two Gaussian wave packets.

C. Position localization decoherence

It is notable that the randomness of the trajectory fluctuation $\delta\alpha(t)$ will cause a *position localization decoherence* [16] on the single arm of the interferometer [18,19]. For an ideal interferometer, the matrix element of the density matrix $\hat{\rho}_{\text{id}}(t) = \hat{U}_{\text{id}}(t)\hat{\rho}_0\hat{U}_{\text{id}}^\dagger(t)$ satisfies

$$\begin{aligned} \langle x_1|\hat{\rho}_{\text{id}}(t)|x_2\rangle &= \langle x'_1(t)|\hat{\rho}_0|x'_2(t)\rangle, \\ x'_i(t) &= \left(x_i - \sqrt{\frac{2\hbar}{m\omega_0}}\text{Re}\alpha(t)\right)\cos\omega_0 t + \sqrt{\frac{2\hbar}{m\omega_0}}\text{Re}\alpha(0). \end{aligned} \quad (49)$$

So, the time evolution of the density matrix element is simply the oscillation defined by the Hamiltonian (2) without noise, which has been concluded by the Wigner function (9). Besides, one can prove that the *purity* of the density matrix $\mathcal{P}_{\text{id}}(t) \equiv \text{Tr}[\hat{\rho}_{\text{id}}^2(t)]$ remains invariant according to the following calculation:

$$\begin{aligned} \mathcal{P}_{\text{id}}(t) &= \int dx_1 dx_2 \langle x_1|\hat{\rho}_{\text{id}}(t)|x_2\rangle \langle x_2|\hat{\rho}_{\text{id}}(t)|x_1\rangle \\ &= \int dx'_1(t) dx'_2(t) \langle x'_1(t)|\hat{\rho}_0|x'_2(t)\rangle \langle x'_2(t)|\hat{\rho}_0|x'_1(t)\rangle \\ &= \text{Tr}[\hat{\rho}_0^2] \equiv \mathcal{P}_0. \end{aligned} \quad (50)$$

On the other hand, for the interferometer under noises, according to the identity $\hat{U}_n(t) = \hat{D}(\delta\alpha(t))\hat{U}_{\text{id}}(t)e^{i\delta\phi(t)}$ with some phase $\delta\phi(t)$, the matrix element of the density matrix $\hat{\rho}_n(t) = \hat{U}_n(t)\hat{\rho}_0\hat{U}_n^\dagger(t)$ is

$$\langle x_1|\hat{\rho}_n(t)|x_2\rangle = \langle x'_1(t)|\hat{D}(\delta\alpha(t))\hat{\rho}_0\hat{D}^\dagger(\delta\alpha(t))|x'_2(t)\rangle. \quad (51)$$

The ensemble average of this matrix element can be computed as [60]

$$\langle x_1|\mathbb{E}[\hat{\rho}_n(t)]|x_2\rangle = \langle x'_1(t)|\hat{\rho}_0|x'_2(t)\rangle e^{-\frac{m\omega_0}{\hbar}\sigma_\alpha^2(t)(x'_1(t)-x'_2(t))^2}. \quad (52)$$

Consequently, in the long-time limit, off-diagonal terms of $\hat{\rho}_n(t)$ decay exponentially and only the diagonal terms of $\hat{\rho}_n(t)$ remains nonzero, so the corresponding purity approaches to zero. Furthermore, since the variance of $\delta\alpha$ is approximately proportional to t in the long-time limit, then there is a

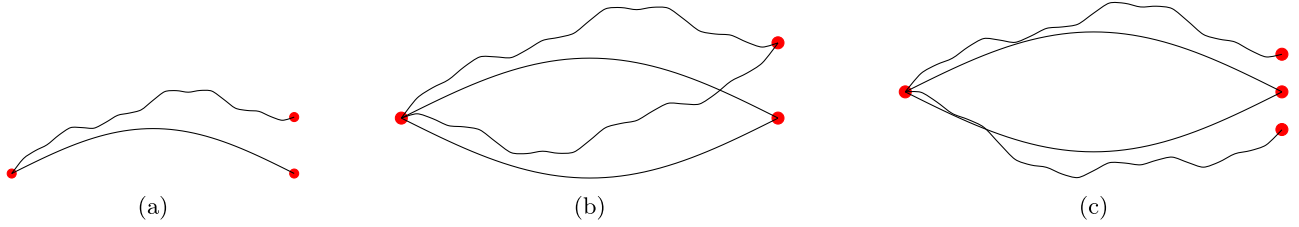


FIG. 3. The trajectories affected by acceleration noises: (a) for a single path, (b) for acceleration noises, and (c) for path-dependent noises. As shown in panel (b), under common mode noises, the final points in phase space of both arms translate the same, so the two paths can be overlapped properly. As shown in panel (c), under path-dependent noises, the points of both arms translate differently, so there is a Loschmidt echo loss.

well-defined decoherence rate

$$\gamma = \frac{\pi m^2}{\hbar^2} (x_1 - x_2)^2 S_{aa}(\omega_0). \quad (53)$$

It is noteworthy that a decoherence rate proportional to $(x_1 - x_2)^2$ is related to the following master equation [16,18]:

$$\frac{d}{dt} \hat{\rho} = \frac{1}{i\hbar} [\hat{H}, \hat{\rho}] - \Lambda [\hat{x}, [\hat{x}, \hat{\rho}]], \quad (54)$$

where $\Lambda = \pi m^2 S_{aa}(\omega_0)/\hbar^2$, which is the same as Ref. [18] up to a factor 2π . This master equation can be derived directly from the Hamiltonian (2), and the math details are summarized in Appendix C. It is noteworthy that the decoherence due to the environmental scattering in the long-wavelength limit also follows the same master equation [16,28], so the scattering of long-wavelength particles like photons can be regarded as a classical stochastic noise in a long-time limit.

D. Loss of witness

For an interferometer with two arms, the trajectory fluctuations $\delta\alpha(t)$ of both paths are the same according to Eq. (35) as long as both arms couple to the noise in the same way. As proved by Eq. (34), the Wigner functions of the arms are totally determined by their classical trajectories in phase space, so the two arms can meet with each other properly at the final time; that is, the contrast of the two arms remains 1, although the fidelity of each arm gets lost, as shown in Fig. 3(b). In addition, the position localization decoherence also has no influence on decoherence in spin space, because $\hat{\rho}_{\text{spin}}$ only relies on the diagonal terms of $\hat{\rho}_n(t)$, which do not decay during evolution according to Eq. (52).

Thus, the spin space and the coordinate space are still separable, and the noise only contributes a fluctuation of the differential phase $\delta\phi_{\text{diff}}$ to the ideal density matrix (17) in the spin space. It can be regarded as a type of common mode noise cancellation, which has been applied in atomic interferometers (AIs) [44–46]. Notably, this effect is usually used to reduce dephasing in AIs, while it mitigates position localization decoherence and fidelity loss in SGIs.

As discussed in Sec. III A, the phase fluctuations of both arms follow Gaussian distributions, so the ensemble average of the random phase factor contributes an exponential decay factor $\mathbb{E}[e^{i\delta\phi_{\text{diff}}}] = e^{-\sigma_{\phi_{\text{diff}}}^2/2}$. Then, the density matrix in spin

space under the ensemble average is

$$\mathbb{E}[\hat{\rho}_{\text{spin}}(t_f)] = \frac{1}{2} \begin{pmatrix} 1 & 0 & e^{-\frac{\sigma_{\phi_{\text{diff}}}^2}{2}} e^{-i\phi_{\text{diff}}} \\ 0 & 0 & 0 \\ e^{-\frac{\sigma_{\phi_{\text{diff}}}^2}{2}} e^{i\phi_{\text{diff}}} & 0 & 1 \end{pmatrix}. \quad (55)$$

Recall that ideal value of the witness of the Ramsey interferometry is $\text{Tr}[\hat{\rho}_{\text{spin}}(t_f)\hat{W}] = \cos^2(\phi_{\text{diff}}/2)$. For the spin density matrix (55) under noise, the observable result of the witness is

$$\langle W \rangle \equiv \text{Tr}[\mathbb{E}[\hat{\rho}_{\text{spin}}(t_f)]\hat{W}] = \frac{1}{2} [1 + e^{-\frac{\sigma_{\phi_{\text{diff}}}^2}{2}} \cos \phi_{\text{diff}}], \quad (56)$$

and the corresponding loss of witness is

$$\Delta W = \frac{e^{-\frac{\sigma_{\phi_{\text{diff}}}^2}{2}} - 1}{2} \cos \phi_{\text{diff}} \approx -\frac{\sigma_{\phi_{\text{diff}}}^2}{2} \cos \phi_{\text{diff}}. \quad (57)$$

Therefore, the loss of witness is proportional to the variance of the phase fluctuation. Recall that $\sigma_{\phi_{\text{diff}}}^2$ is approximately proportional to $S_{aa}(\omega_0)$, so the loss of witness ΔW should be mitigated by controlling the PSD at the intrinsic frequency ω_0 of the interferometer in a real experiment.

V. LEADING-ORDER EFFECTS OF HIGHER-ORDER NOISES

Apart from acceleration noises, there exist noises coupling to the interferometer in other ways such as $m\delta\omega^2(t)\hat{x}^2$ or $\mu\delta\eta(t)\hat{S}_z\hat{x}$, with $\eta(t) \equiv \partial B(t)/\partial x$. In order to study these noises analytically, it is necessary in principle to solve the corresponding stochastic Schrödinger equation for these coupling Hamiltonians, which can be a great challenge in math. However, by treating some operators as c numbers, these noises can be regarded as path-dependent acceleration noises. Then, the leading-order effects of these noises can be studied through the analytic solution for acceleration noises, although some higher-order properties can be lost.

As shown in Fig. 3(c), the final states of the two paths are usually not able to meet with each other, i.e., $\alpha_1(t_f) \neq \alpha_{-1}(t_f)$, so the total density matrix is not separable. As a consequence, apart from the dephasing effect characterized by a decay factor $e^{-\sigma_{\phi_{\text{diff}}}^2/2}$, the off-diagonal terms of the spin density matrix have an additional decay factor arising from the partial trace of the spatial degrees of freedom. Therefore, the spin density matrix under path-dependent noises is given

by

$$\mathbb{E}[\hat{\rho}_{\text{spin}}(t_f)] = \frac{1}{2} \begin{pmatrix} 1 & 0 & \mathbb{E}[C]e^{-\frac{\sigma_{\text{diff}}^2}{2}}e^{-i\phi_{\text{diff}}} \\ 0 & 0 & 0 \\ \mathbb{E}[C]e^{-\frac{\sigma_{\text{diff}}^2}{2}}e^{i\phi_{\text{diff}}} & 0 & 1 \end{pmatrix}, \quad (58)$$

where

$$C \equiv \text{Tr} \left[\sqrt{\hat{\rho}_1(t_f)} \hat{\rho}_{-1}(t_f) \sqrt{\hat{\rho}_1(t_f)} \right] \quad (59)$$

is the fidelity between the two paths at the final time, which is the generalization of the contrast $C \equiv \langle \psi_1(t_f) | \psi_{-1}(t_f) \rangle$ of pure states to mixed states.

There are two decay factors $e^{-\sigma_{\text{diff}}^2/2}$ and $\mathbb{E}[C]$ in the off-diagonal terms of the density matrix. The first factor characterizes the dephasing effect, while the second term corresponds to the combination of the position localization decoherence and the Humpty-Dumpty problem.

A. Dephasing

For the dephasing effect, the fluctuation of the differential phase of the interferometer is given by

$$\delta\phi_{\text{diff}} = \frac{m}{\hbar} \int_0^{t_f} (\delta a_1(t)x_1(t) - \delta a_{-1}(t)x_{-1}(t)) dt. \quad (60)$$

In this case, the variance of the phase fluctuation is determined by the correlation between $\delta a_1(t)$ and $\delta a_{-1}(t)$, which is characterized by their cross-PSD $S_{a_1 a_{-1}}$. Hence, the variance $\sigma_{\phi_{\text{diff}}}^2$ can be formulated as

$$\sigma_{\phi_{\text{diff}}}^2 = \frac{m^2}{\hbar^2} \int S_{a_1 a_1}(\omega) F_{11}(\omega) + S_{a_{-1} a_{-1}}(\omega) F_{-1-1}(\omega) + S_{a_1 a_{-1}}(\omega) F_{1,-1}(\omega) + S_{a_{-1} a_1}(\omega) F_{-1,1}(\omega) d\omega, \quad (61)$$

where

$$F_{ij}(\omega) = x_i(\omega)x_j^*(\omega), \quad \text{for } i, j = 1, -1. \quad (62)$$

For the diagonal terms, $F_{ii}(\omega) = |x_i(\omega)|^2$ are real-valued functions, which is the same as the linear acceleration case (27). In contrast, the cross-transfer functions $F_{1,-1}(\omega)$ and $F_{-1,1}(\omega)$ have complex values.

Notably, the cross-PSDs and their transfer functions satisfy the conjugate symmetry relations: $S_{a_1 a_{-1}}(\omega) = S_{a_{-1} a_1}^*(\omega)$ and $F_{1,-1}(\omega) = F_{-1,1}^*(\omega)$. As a result, the contribution of cross correlation in Eq. (61) is purely real and given by $2\text{Re}[S_{a_1 a_{-1}}(\omega)F_{1,-1}(\omega)]$.

For a special case that $\delta a_1(t)$ and $\delta a_{-1}(t)$ arise from a noise term $\delta L = mf(t)x^n$ with a Gaussian noise $f(t)$ by defining $\delta a_{\pm 1}(t) \equiv f(t)x_{\pm 1}^{n-1}(t)$, then the dephasing of the single arm becomes

$$\delta\phi_{\text{diff}} = \frac{m}{\hbar} \int_0^{t_f} f(t)x^n(t) dt. \quad (63)$$

Thus, the variance of the differential phase also has a form (26) with a transfer function

$$F(\omega) = |x_1^n(\omega) - x_{-1}^n(\omega)|^2. \quad (64)$$

For the case $n = 2$, the term $m\delta\omega^2(t)\hat{x}^2$ can be used to study various noises such as the gravity gradient noise and rotation

noise, of which the dephasing effect has been studied in previous research works [10,12].

B. Position localization decoherence and Humpty-Dumpty problem

The fidelity defined in Eq. (59) characterizes the position localization decoherence and the Humpty-Dumpty problem, which can be computed as

$$C = \text{Tr}[\hat{D}(\Delta\alpha(t_f)e^{i\omega_0 t_f})\hat{\rho}_0]e^{i\text{Im}(\alpha_1(t_f)\alpha_{-1}^*(t_f))}, \quad (65)$$

where $\Delta\alpha(t_f) \equiv \alpha_1(t_f) - \alpha_{-1}(t_f)$. Especially, if the initial state is prepared as a Gaussian state, then the fidelity is

$$C = \text{Tr}[\hat{D}(\Delta\alpha(t_f)e^{i\omega_0 t_f})\hat{\rho}_0] = e^{-\frac{|\Delta\alpha(t_f)|^2}{2}} = e^{-\frac{(\Delta x)^2}{m\omega_0/\hbar} - \frac{(\Delta p)^2}{m\hbar\omega_0}}. \quad (66)$$

Since both $\delta\alpha_{\pm 1}$ follow Gaussian distribution, then $\delta\alpha_1 - \delta\alpha_{-1}$ also follows Gaussian distribution with a variance $\sigma_{\text{tot}}^2 = \sigma_1^2 + \sigma_{-1}^2$ if they are assumed independent. Note that $\sigma_{\pm 1}$ are exactly the same, then the total variance is simply $\sigma_{\text{tot}}^2 = 2\sigma_\alpha^2$, where σ_α^2 is exactly the variance of a single arm given by Eq. (40), which is a linear response to the acceleration noise $\delta a(t)$ with a transfer function $\text{sinc}^2(\omega - \omega_0)t/2$. Thus, the ensemble average of the contrast is

$$-\mathbb{E}[\ln C] = \frac{1}{2}\mathbb{E}[|\delta\alpha_1 - \delta\alpha_{-1}|^2] = \sigma_\alpha^2. \quad (67)$$

C. Witness loss

To sum up, higher-order noises can cause both dephasing and contrast loss on the interferometer, and both effects are linear responses to $\delta a(t)$. Both effects can contribute a decay factor on off-diagonal terms of the spin space, so the observable result for the Ramsey interferometry becomes

$$\text{Tr}[\mathbb{E}[\hat{\rho}_{\text{spin}}(t_f)]\hat{W}] = \frac{1}{2} \left[1 + e^{-\frac{\sigma_{\text{diff}}^2}{2}} e^{-\sigma_\alpha^2} \cos(\phi_{\text{diff}}) \right], \quad (68)$$

and the corresponding witness loss is

$$\Delta W = \frac{e^{-\sigma_\alpha^2 - \frac{\sigma_{\text{diff}}^2}{2}} - 1}{2} \cos \phi_{\text{diff}} \approx - \left(\sigma_\alpha^2 + \frac{\sigma_{\text{diff}}^2}{2} \right) \cos \phi_{\text{diff}}. \quad (69)$$

Recall that both σ_α^2 and σ_{diff}^2 are determined by the PSD at ω_0 , given by Eqs. (33) and (43). The resulting loss of witness is therefore expressed as

$$\Delta W \approx - \left(\frac{2\pi^2 N m}{\hbar\omega_0^2} + \frac{2\pi m^2 N^2 (\Delta A)^2}{\omega_0 \hbar^2} \right) S_{aa}(\omega_0) \cos \phi_{\text{diff}} = - \left(\frac{\pi m}{\hbar\omega_0} t_f + \frac{m^2 (\Delta A)^2 \omega_0}{2\pi \hbar^2} t_f^2 \right) S_{aa}(\omega_0) \cos \phi_{\text{diff}}, \quad (70)$$

where $\Delta A = 2g_{\text{NV}}\mu_B\eta/(m\omega_0^2)$ is the size of the interferometer.

Here, the first term describes the combined influence of position localization decoherence and the Humpty-Dumpty problem, which is characterized by the diffusion of the classical trajectories and scales linearly with the total time $\Delta W_1 \propto$

t_f . In contrast, the second term accounts for the dephasing effect, which exhibits a quadratic dependence on the total time, i.e., $\Delta W_2 \propto t_f^2$. A further distinction between these contributions lies in their dependence on the interferometric path separation: Only the dephasing relies on the magnitude of the ideal trajectories, characterized by ΔA .

VI. APPLICATION TO TYPICAL NOISES

As applications, I will briefly discuss the dephasing and contrast loss of two typical noises in SGIs: magnetic field noise and trap frequency noise.

A. Magnetic field noise

One of the most significant noise is the fluctuation of the magnetic field. For simplicity, this paper focuses on the magnetic gradient noise $\delta\eta(t)$, coupling to the test mass via the Hamiltonian

$$\hat{H}_m(t) = \mu_B \delta\eta(t) \hat{S}_z \hat{x}. \quad (71)$$

Since this noise couples to both $\hat{S}_z$ and $\hat{x}$, it inherently affects both the spin space and the spatial degrees of freedom. Consequently, a complete analysis of the resulting decoherence requires solving the full spin-position dynamics, which poses a considerable mathematical challenge.

At leading order, however, the spin states can be assumed to remain fixed along $s_z = \pm 1$ states. In this approximation, the spins reduce to c number, and the noise Hamiltonian takes a simplified form

$$\hat{H}_m = \pm \mu_B \delta\eta(t) \hat{x}, \quad (72)$$

which corresponds to an effective acceleration noise of the generic form given in Eq. (21). Thus, the decoherence arises solely from the spatial degrees of freedom and can be analyzed using the previously established results on linear acceleration noise.

The phase fluctuation of each arm is given by Eq. (24), then the differential phase is fluctuated by the magnetic noise as

$$\delta\phi_{\text{diff}} = \frac{\mu_B}{\hbar} \int_0^{t_f} \delta\eta(t) (x_1(t) + x_{-1}(t)) dt. \quad (73)$$

Hence, the variance of the phase fluctuation takes a similar form as the case for a uniform acceleration noise, given by Eq. (26), where the transfer function takes a different form as the Fourier transform of the averaged position of two paths rather than the differential position between them. The variance can be formulated as

$$\sigma_{\phi_{\text{diff}}}^2 = \frac{m^2}{\hbar^2} \int S_{aa}(\omega) |x_1(t) + x_{-1}(t)|^2 d\omega. \quad (74)$$

Especially, for a test mass driven by a constant magnetic gradient without bias [61], i.e., $B = \eta x$, the trajectories are given by Eq. (31). In this case, the trajectories $x_1(t)$ and $x_{-1}(t)$ are opposite to each other, that is,

$$x_1(t) + x_{-1}(t) \equiv 0. \quad (75)$$

Consequently, the phase fluctuations can cancel each other, that is, $\delta\phi_{\text{diff}} \equiv 0$. In other words, the magnetic field noise

does not cause dephasing effect via the spatial path integral phase.

Apart from inducing dephasing, the magnetic noise can also cause a nonclosure problem, as shown in Fig. 3 (c). Since the magnetic noise does not distort the shape of wavefunction, the contrast loss is entirely determined by the displacement in the classical phase space, given by Eq. (66), i.e.,

$$C = \exp \left[-\frac{|\Delta\alpha(t_f)|^2}{2} \right], \quad (76)$$

where $\Delta\alpha(t_f) \equiv \alpha_+(t_f) - \alpha_-(t_f)$ and the complex coordinates $\alpha_{\pm}(t_f)$ are defined by Eq. (6). Because the ideal trajectories are designed to overlap perfectly, the displacement $\Delta\alpha(t_f)$ arises solely from the noise contribution, given by

$$\Delta\alpha(t_f) = 2ig_{\text{NV}}\mu_B \sqrt{\frac{m}{2\hbar\omega_0}} e^{-i\omega_0 t_f} \int_0^{t_f} \delta\eta(t) e^{i\omega_0 t} dt. \quad (77)$$

For the stationary noise, the ensemble-averaged contrast can be evaluated using the Wiener-Khinchin theorem as

$$-\mathbb{E}[\ln C] = \frac{mg_{\text{NV}}^2\mu_B^2}{\hbar\omega_0} S_{\eta\eta}(\omega_0), \quad (78)$$

which is solely determined by the PSD $S_{\eta\eta}$ of the magnetic noise at the intrinsic frequency ω_0 of the test mass.

B. Quadratic noise

Another common noise coupling mechanism involves a quadratic interaction with the test mass, represented by the Hamiltonian $\hat{H}_{\text{noise}} = m\delta\omega^2(t)\hat{x}^2$. This form of coupling can arise from fluctuations of the trapping potential, gravity gradients, or rotational motion [12].

The associated dephasing can be formulated using Eq. (63), which again yields a linear response relation with the transfer function given by Eq. (64) with $n = 2$, as previously analyzed in Refs. [10,12].

In contrast to the linear-coupling case, this type of noise can induce a Humpty-Dumpty problem via both fluctuation of the trajectories and the wavefunction shapes. As derived in Appendix D, the fluctuation of each trajectory can be computed by the perturbation method, yielding the leading-order result

$$\delta\alpha_c(t) = -\frac{i}{\omega_0} e^{-i\omega_0 t} \int_0^t \delta\omega^2(\tau) \text{Re}[\alpha_c^{(0)}(\tau)] e^{i\omega_0 \tau} d\tau, \quad (79)$$

where $\alpha_c^{(0)}(\tau)$ is the ideal phase-space trajectory defined in Eq. (6).

In addition, the noise distorts the shape of the Wigner function of the test mass via squeezed operators. As derived in Appendix D, the squeezed operator at the leading order is given by

$$\hat{S}(\rho) \approx \exp \left[\frac{i}{2\hbar} \delta\rho(t) (\hat{x}\hat{p} + \hat{p}\hat{x}) \right] \exp \left[\frac{i}{2\hbar} m\delta\dot{\rho}(t) \hat{x}^2 \right], \quad (80)$$

where the first exponential corresponds to a squeezing operation in phase space and the second represents a shearing transformation. The scaling factor is $\rho(t) = 1 + \delta\rho(t)$, with the perturbative correction $\delta\rho(t)$ and its time derivative given

by

$$\begin{aligned}\delta\rho(t) &= -\frac{1}{2\omega_0} \int_0^t \delta\omega^2(\tau) \sin[2\omega_0(t-\tau)] d\tau, \\ \delta\dot{\rho}(t) &= -\int_0^t \delta\omega^2(\tau) \cos[2\omega_0(t-\tau)] d\tau.\end{aligned}\quad (81)$$

However, due to the unitarity of the squeezed operator $\hat{S}(\rho)$, it does not affect the overlap of the final states [62]. Therefore, the contrast loss of the final state is solely determined by the displacement difference $\Delta\alpha(t) = \delta\alpha_+(t) - \delta\alpha_-(t)$, analogous to the magnetic noise case. The resulting contrast is

$$C = \exp\left[-\frac{|\Delta\alpha(t_f)|^2}{2}\right]. \quad (82)$$

The displacement difference at the final time is

$$\Delta\alpha(t_f) = -\frac{i}{\omega_0} e^{-i\omega_0 t_f} \int_0^{t_f} \delta\omega^2(\tau) \text{Re}[\Delta\alpha^{(0)}(\tau)] e^{i\omega_0 \tau} d\tau, \quad (83)$$

where $\Delta\alpha^{(0)}(\tau) \equiv \alpha_+^{(0)}(\tau) - \alpha_-^{(0)}(\tau)$ represents the ideal differential trajectory, and the complex coordinates $\alpha_{\pm}(t_f)$ are defined by Eq. (6), which can be computed as

$$\text{Re}[\Delta\alpha^{(0)}(\tau)] = \sqrt{\frac{m}{2\hbar\omega_0}} \int_0^\tau \Delta a(\tau') \sin(\omega_0(\tau - \tau')) d\tau'. \quad (84)$$

This result indicates that the displacement $\Delta\alpha(t_f)$ is a linear response to the noise $\delta\omega^2$. Consequently, the logarithm of the contrast is also a linear response, determined by the PSD $S_{\omega^2\omega^2}$ of $\delta\omega^2$ given by

$$-\mathbb{E}[\ln C] = \frac{1}{2} \int S_{\omega^2\omega^2}(\Omega) F(\Omega) d\Omega, \quad (85)$$

with a transfer function

$$F(\Omega) = \left| \frac{1}{\omega_0} \int_0^{t_f} \text{Re}[\Delta\alpha^{(0)}(\tau)] e^{i\Omega\tau} d\tau \right|^2. \quad (86)$$

Here, the frequency variable in the PSD domain is denoted by the capital letter Ω to distinguish it from the intrinsic frequency ω_0 of the test mass and its fluctuation $\delta\omega$. Note that the quantity $\text{Re}[\Delta\alpha^{(0)}(\tau)]$ is exactly proportional to $x_1(t) - x_{-1}(t)$ in Eq. (27), so this transfer function is essentially the same as the transfer function of the dephasing induced by linear acceleration noise, up to some numerical factors.

For the constant driven force $\Delta a(t) \equiv \Delta A$ as given in Eq. (31), and for the final time $t_f = 2N\pi/\omega_0$, the transfer function reduces to

$$F(\Omega) = \frac{2m(\Delta A)^2}{\hbar\omega_0} \times \frac{\sin^2(N\pi\Omega/\omega_0)}{\Omega^2(\Omega^2 - \omega_0^2)^2}, \quad (87)$$

which is the same as the transfer function (32) of the dephasing induced by linear acceleration noise as expected. Hence, as derived in Appendix C, the contrast loss can also be evaluated using the residue theorem, yielding

$$-\mathbb{E}[\ln C] = \frac{mN^2\pi^2(\Delta A)^2}{\hbar\omega_0^6} S_{\omega^2\omega^2}(\omega_0). \quad (88)$$

Comparing to the contrast loss induced by the quadratic noise and magnetic noise, one finds that both are proportional to the noise PSD evaluated at the intrinsic frequency ω_0 , reflecting a resonance between the noise and the test mass.

However, the contrast loss due to magnetic field noise is independent of the ideal trajectories, whereas the quadratic noise couples to the ideal trajectories and scales with the number of periods N . This property arises from their coupling Hamiltonian: magnetic field noise couples linearly to $\hat{x}$ and is therefore insensitive to the differential trajectory, whereas quadratic noise couples through $\hat{x}^2$, making the contrast loss proportional to Δx .

VII. DISCUSSION AND SUMMARY

A. Role of randomness

In the end, I draw a remark on the role of randomness. This distinction can be quantitatively captured by the *purity* $\text{Tr}[\hat{\rho}^2]$ or the *von Neumann entropy* $S = -\text{Tr}[\hat{\rho} \ln \hat{\rho}]$. The purities with and without the ensemble average are [63]

$$\begin{aligned}\text{Tr}[\hat{\rho}_{\text{spin}}^2(t_f)] &= \frac{1}{2}(1 + C^2) \approx 1 - (-\ln C), \\ \text{Tr}[\mathbb{E}[\hat{\rho}_{\text{spin}}(t_f)]^2] &= \frac{1}{2}(1 + \mathbb{E}[C]^2 e^{-\sigma_{\phi_{\text{diff}}}^2}) \approx 1 - \frac{\sigma_{\phi_{\text{diff}}}^2}{2} - \sigma_\alpha^2,\end{aligned}\quad (89)$$

and the corresponding entropies are

$$\begin{aligned}S[\hat{\rho}_{\text{spin}}] &\approx -\ln C, \\ S[\mathbb{E}[\hat{\rho}_{\text{spin}}]] &\approx \frac{\sigma_{\phi_{\text{diff}}}^2}{2} + \sigma_\alpha^2.\end{aligned}\quad (90)$$

The expressions for purity and entropy show that differential phase fluctuation under a deterministic fluctuation does not affect the purity or the entropy. However, a stochastic noise leads to an exponential decay factor due to the ensemble average, resulting in dephasing, characterized by purity loss and entropy growth $\sigma_{\phi_{\text{diff}}}^2/2$.

Likewise, spatial fluctuation under external deterministic perturbation results in contrast loss, where the total information remains but the spin information becomes harder to extract. In contrast, a stochastic noise leads to genuine position localization decoherence after ensemble average, where quantum information is irreversibly lost.

From the point of view of information, ensemble averaging of the random noise transforms reversible contrast loss into irreversible information loss. The information of the quantum system gets lost even if the environment is classical as long as the environment is stochastic. Thus, the information loss indicates a deep connection between the classical randomness and the quantumness [64,65].

B. Summary and outlook

This paper presents a comprehensive theoretical analysis on the decoherence of a Stern-Gerlach interferometer induced by classical stochastic acceleration noise. Based on the exact dynamics of the spatial motion, I analyze the various decoherence effects in an SGI.

Section III presents a rigorous proof on the linear-response relationship between the spatial dephasing and the external noise, where the associated transfer function is the Fourier transform of the unperturbed trajectories. This section further demonstrates that the transfer function behaves approximately as a delta function at the intrinsic frequency ω_0 of the test mass, which can be physically interpreted as the resonance between the test mass and the external noise.

Section IV investigates additional potential decoherence mechanisms, including the perturbations of the Wigner function, nonclosure of the classical trajectories and velocities, and the position localization decoherence. This section concludes that, for a linear acceleration noise, these effects can cancel each other and thus do not affect the witness result.

For higher-order noise, the leading-order decoherence effects can be treated as a path-dependent acceleration noise. From this perspective, the spatial dephasing remains a linear response to the noise, although the transfer function is modified according to the noise Hamiltonian. In this case, the nonclosure problem, together with the position localization decoherence, leads to a contrast loss that affects the spin witness, characterized by the PSD at ω_0 , which is also a resonance between the test mass and the noise.

Based on the general discussion, Sec. VI applies the theoretical framework on two representative noise sources, magnetic field noise and quadratic noise. For magnetic field noise, the spatial dephasing cancels out, while the contrast loss under ensemble average is characterized by the PSD at ω_0 . For a quadratic noise, the spatial dephasing is still a linear response with a modified transfer function, and again the contrast loss under ensemble average is characterized by the PSD at ω_0 .

In summary, this paper develops a theoretical framework for linear effects of generic stochastic classical noise. The theoretical analysis methods presented here are readily adaptable to experiments based on matter-wave interferometry, and several of the results and the associated physical intuition can be directly applied in such setups. Future theoretical investigation may explore the coupling between the spatial and spin degrees of freedom, and study effects such as spin dephasing, relaxation, and Landau-Zener transition (Majorana flip) [36,66] induced by spatial noise.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: ANALYTICAL SOLUTION FOR THE SCHRÖDINGER EQUATION WITH ACCELERATION NOISES

In this Appendix, I will show the math details to solve the Schrödinger equation with the time-dependent Hamiltonian

(2) by the *Lewis-Riesenfeld invariant* and *quantum Arnold transformation* method [39,40,42,43,67].

The basic idea [41] is to find a so-called *Lewis-Riesenfeld operator* $\hat{I}(t)$ such that all the eigenvalues $\{\lambda_n\}$ of $\hat{I}(t)$ are time independent and all the instantaneous eigenstates $\{|n(t)\rangle\}$ of $\hat{I}(t)$ are solutions of Schrödinger equation. Then, the time-evolution operator is given by $\hat{U}(t) = \sum_n e^{i\alpha_n(t)} |n(t)\rangle \langle n(0)|$, where $\alpha_n(t) = \int_0^t \langle n(t') | (i\partial/\partial t' - \hat{H}(t'))/\hbar | n(t') \rangle dt'$ is known as the *Lewis-Riesenfeld phase* [39,41]. From the point view of differential geometry, the instantaneous eigenstates $\{|n(t)\rangle\}$ form a moving frame of the Hilbert space as a manifold, and the Lewis-Riesenfeld phase is a kind of geometric phase arising from the connection 1-form of the manifold.

Such time-dependent operator $\hat{I}(t)$ can be generally constructed by a reference frame transformation on the Hamiltonian from the laboratory reference to the comoving frame [42], known as the *quantum Arnold transformation* [43]. In particular, for a driven oscillator with a Hamiltonian (2), consider a translation operator in the phase space of the oscillator:

$$\hat{U}(t) = \exp \left[-\frac{i}{\hbar} (\hat{x}p(t) - \hat{p}x(t)) \right] = \hat{D}(-\alpha(t)),$$

$$\alpha(t) = \sqrt{\frac{m\omega_0}{2\hbar}} (x(t) + ip(t)/m\omega_0). \quad (A1)$$

Under the unitary transformation, the wavefunction changes as $|\psi\rangle \rightarrow |\psi'\rangle = \hat{U}(t)|\psi\rangle$, while the Hamiltonian changes as

$$\hat{H}(t) \rightarrow \hat{U}\hat{H}\hat{U}^\dagger - i\hbar\hat{U}\frac{d}{dt}\hat{U}^\dagger. \quad (A2)$$

The term $-i\hbar\hat{U}(d\hat{U}^\dagger/dt)$, often referred as the *counterdiabatic term* $\hat{H}_{CD}$ [68], mathematically arises from the connection 1-form to keep the Schrödinger equation covariant under the transformation $\hat{U}(t)$. If the trajectory $x(t)$ and momentum $p(t)$ are chosen satisfying the canonical equation defined by the Hamiltonian (2)

$$\dot{x}_c = \frac{p_c}{m},$$

$$\dot{p}_c = -m\omega_0^2 x_c + ma(t), \quad (A3)$$

which is the real-valued form for Eq. (5), then the Hamiltonian in the comoving reference is

$$\hat{U}\hat{H}\hat{U}^\dagger - i\hbar\hat{U}\frac{d}{dt}\hat{U}^\dagger = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2 \hat{x}^2 + \delta(t), \quad (A4)$$

where the phase $\delta(t)$ is given by

$$\delta(t) = -\frac{1}{2}ma(t)x_c(t). \quad (A5)$$

Therefore, the Hamiltonian in the comoving reference frame is time independent, except for a pure phase term $\delta(t)$. So, the time evolution of the wavefunction under the comoving reference is simply

$$|\psi'(t)\rangle = e^{-\frac{i}{\hbar} \int_0^t \delta(t') dt'} e^{-\frac{i}{\hbar} \left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2 \hat{x}^2 \right) t} |\psi'(0)\rangle. \quad (A6)$$

Then, the wavefunction in the original reference frame is

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar} \int_0^t \delta(t') dt'} \hat{U}^\dagger(t) e^{-\frac{i}{\hbar} \left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2 \hat{x}^2 \right) t} \hat{U}(0) |\psi(0)\rangle. \quad (A7)$$

Furthermore, the equation of motion (A3) allows to rewrite the phase $\delta(t)$ in another form

$$\delta(t) = -\left(\frac{1}{2}m\dot{x}_c^2(t) - \frac{1}{2}m\omega_0^2 x_c^2(t) + ma(t)x_c(t)\right) + \frac{1}{2}m\frac{d}{dt}(\dot{x}_c x_c). \quad (\text{A8})$$

This means that the $\delta(t)$ phase term is exactly the minus Lagrangian $L[x_c(t), \dot{x}_c(t)]$ of the system (2) along the classical trajectory up to a total derivative term. Thus, the global phase in Eq. (A7) is exactly the path integral along the classical trajectory $\phi = \int_0^t L[x_c(t'), \dot{x}_c(t')]/\hbar dt'$, so Eq. (A7) presents the time-evolution operator (3). Remarkably, since the $\delta(t)$ phase arises from the gauge term $-i\hbar\hat{U}(d\hat{U}^\dagger/dt)$, the Feynman's path integral phase in this system has a physical interpretation of geometric phase.

A final remark is that a total derivative term in $\delta(t)$ is usually tolerant in an interferometry experiment because the initial and final positions and momenta of different trajectories are the same and they will cancel each other. It is notable that such a total derivative term is related to the choice of the unitary transform operator $\hat{U}(t)$. In particular, the operator $\hat{U}(t)$ allows an arbitrary phase and can be chosen as, for example [69],

$$\hat{U}'(t) = e^{\frac{i}{\hbar}\hat{p}x_c(t)} e^{-\frac{i}{\hbar}\hat{x}p_c(t)} = \hat{U}(t) e^{\frac{i}{2\hbar}p_c(t)x_c(t)}. \quad (\text{A9})$$

The corresponding pure phase term $\delta(t)$ for $\hat{U}'(t)$ is [70]

$$\delta'(t) = \frac{1}{2}m\dot{x}_c^2(t) - \frac{1}{2}m\omega_0^2 x_c^2(t). \quad (\text{A10})$$

Based on the equation of motion (A3), one can easily check the equivalence between $\delta(t)$ and $\delta'(t)$ up to a total derivative

term $\frac{1}{2}m\frac{d}{dt}(x_c\dot{x}_c)$, i.e.,

$$-\frac{1}{2}ma(t)x_c(t) = \frac{1}{2}m\dot{x}_c^2(t) - \frac{1}{2}m\omega_0^2 x_c^2(t) - \frac{1}{2}m\frac{d}{dt}(x_c\dot{x}_c). \quad (\text{A11})$$

APPENDIX B: WITNESS FOR RAMSEY INTERFEROMETRY IN SPIN-1 SYSTEM

In this Appendix, the witness for Ramsey interferometry in spin-1 system will be constructed. A Ramsey interferometry is to measure the population of the spin state $|S_z = 0\rangle$ after acting an external microwave pulse on it. The Hamiltonian for the Rabi oscillation of the spin-1 system under the microwave pulse is given by $\hat{H}_p = \hbar\Omega_p(|1\rangle\langle 0| + |-1\rangle\langle 0| + \text{H.c.})$, where H.c. stands for “Hermitian conjugate” and Ω_p is the frequency of the microwave. The Hamiltonian can be written in a matrix form and can be diagonalized as

$$\hat{H}_p = \hbar\Omega_p \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \hbar\Omega_p \hat{T}^\dagger \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix} \hat{T}, \quad (\text{B1})$$

where the transform matrix $\hat{T}$ is

$$\hat{T} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}. \quad (\text{B2})$$

Then, the time-evolution operator $\hat{U}_p(t) = e^{-i\hat{H}_p t/\hbar}$ for the pulse acting on the spin-1 system is given by

$$\hat{U}_p(t) = \hat{T}^\dagger \begin{pmatrix} e^{-i\sqrt{2}\Omega_p t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\sqrt{2}\Omega_p t} \end{pmatrix} \hat{T} = \begin{pmatrix} \cos^2 \frac{\sqrt{2}\Omega_p t}{2} & -\frac{i}{\sqrt{2}} \sin \sqrt{2}\Omega_p t & -\sin^2 \frac{\sqrt{2}\Omega_p t}{2} \\ -\frac{i}{\sqrt{2}} \sin \sqrt{2}\Omega_p t & \cos \sqrt{2}\Omega_p t & -\frac{i}{\sqrt{2}} \sin \sqrt{2}\Omega_p t \\ -\sin^2 \frac{\sqrt{2}\Omega_p t}{2} & -\frac{i}{\sqrt{2}} \sin \sqrt{2}\Omega_p t & \cos^2 \frac{\sqrt{2}\Omega_p t}{2} \end{pmatrix}. \quad (\text{B3})$$

If the time of the microwave pulse is set as $t_p = \pi/(2\sqrt{2}\Omega_p)$, then the corresponding time-evolution operator is

$$\hat{U}_{\pi/2} \equiv \hat{U}_p\left(t_p = \frac{\pi}{2\sqrt{2}\Omega_p}\right) = \begin{pmatrix} \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{i}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}. \quad (\text{B4})$$

After the $\pi/2$ pulse, the witness to measure the population of $|S_z = 0\rangle$ is simply $\hat{W}' = |S_z = 0\rangle\langle S_z = 0|$. Since the $\pi/2$ pulse makes a transformation on $\hat{\rho}_{\text{spin}}$ to $\hat{U}_{\pi/2}\hat{\rho}_{\text{spin}}\hat{U}_{\pi/2}^\dagger$, the observable result for the witness $\hat{W}'$ is $\text{Tr}[\hat{W}'\hat{U}_{\pi/2}\hat{\rho}_{\text{spin}}\hat{U}_{\pi/2}^\dagger]$. Therefore, the witness for Ramsey interferometry is given by

$$\hat{W} = \hat{U}_{\pi/2}^\dagger \hat{W}' \hat{U}_{\pi/2} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}. \quad (\text{B5})$$

APPENDIX C: COMPUTATION OF INTEGRAL VIA RESIDUE THEOREM

In this paper, integrals of the form

$$I_0 = A \int S(\omega) \frac{\sin^2(N\pi\omega/\omega_0)}{\omega^2(\omega^2 - \omega_0^2)^2} d\omega \quad (\text{C1})$$

or similar form appear repeatedly. Examples include the dephasing (26) induced by linear acceleration noise in Sec. III, the phase-space diffusion (42) of a single arm under noise in Sec. IV, and the contrast loss (85) induced by quadratic noise in Sec. VI. This Appendix aims to evaluate this integral by the residue theorem.

First of all, ignoring the amplitude factor A , this integral is exactly the real part of the integral

$$I = \int S(\omega) \frac{1 - e^{2\pi i N \omega / \omega_0}}{\omega^2(\omega^2 - \omega_0^2)^2} d\omega \equiv \int f(\omega) d\omega, \quad (\text{C2})$$

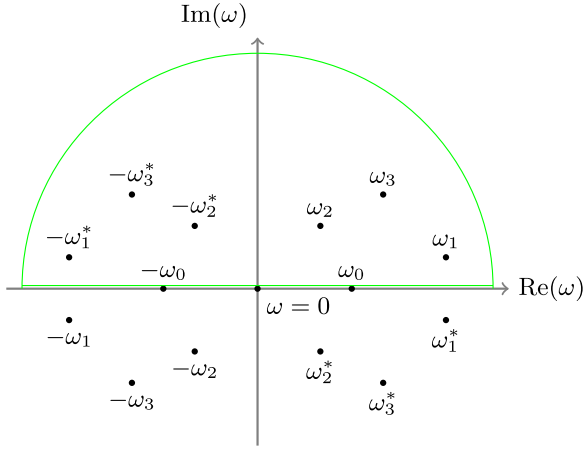


FIG. 4. Poles of the integrand in Eq. (C2) and the integral path in the complex plane. The poles $\pm\omega_0$ arise from the transfer function $F(\omega)$, while other poles arise from the PSD $S(\omega)$.

where the integrand is denoted as $f(\omega)$ for ease of writing. The poles of $f(\omega)$ consist of 0, $\pm\omega_0$, and the poles of $S(\omega)$. Noticing that $S(\omega)$ is a real-valued even function on the real axis, the poles of $S(\omega)$ are symmetric to both the real axis and the imaginary axis, as shown in Fig. 4. The proof of this statement is summarized at the end of this Appendix.

According to the residue theorem, this integral equals the sum of the residues in the upper half of the complex plane, including those on the real axis. As shown in Fig. 4, ω_j and $-\omega_j^*$ are in the upper half of the complex plane, while 0 and $\pm\omega_0$ are located on the real axis, so the integral (C2) is equal to

$$I = 2\pi i \left[\sum_j (\text{Res}_{\omega=\omega_j} f(\omega) + \text{Res}_{\omega=-\omega_j^*} f(\omega)) \right] + \pi i (\text{Res}_{\omega=0} f(\omega) + \text{Res}_{\omega=\omega_0} f(\omega) + \text{Res}_{\omega=-\omega_0} f(\omega)). \quad (\text{C3})$$

It is not difficult to check that $f(-\omega^*) = [f(\omega)]^*$ [71], then the residue of $f(\omega)$ at the pole $-\omega_j^*$ satisfies

$$\text{Res}_{\omega=-\omega_j^*} f(\omega) = \text{Res}_{\omega=-\omega_j^*} [f(-\omega^*)]^* = [\text{Res}_{\omega=\omega_j} f(\omega)]^*. \quad (\text{C4})$$

Therefore, $2\pi i (\text{Res}_{\omega=\omega_j} f(\omega) + \text{Res}_{\omega=-\omega_j^*} f(\omega))$ contributes a pure imaginary number. Consequently, the summation of all the poles ω_j and $-\omega_j^*$ has no contribution to the original integral $I_0 \propto \text{Re } I$. In other words, I_0 can be completely determined by the poles $\omega = 0$ and $\pm\omega_0$.

According to the symmetry of $f(\omega)$, the residue values of the poles $\pm\omega_0$ are equal. Since both of them are second-order poles, their contribution to the integral (C2) can be directly evaluated as

$$2\pi i \text{Res}_{\omega=\omega_0} f(\omega) = 2\pi i \frac{d}{d\omega} \left[S(\omega) \frac{1 - e^{2\pi i N \omega / \omega_0}}{\omega^2 (\omega + \omega_0)^2} \right] \Big|_{\omega=\omega_0} = \frac{\pi^2 N^2}{\omega_0^5} S(\omega_0). \quad (\text{C5})$$

As for the other second-order pole $\omega = 0$, it contributes to the integral (C2) as

$$\begin{aligned} \pi i \text{Res}_{\omega=0} f(\omega) &= \pi i \frac{d}{d\omega} \left[S(\omega) \frac{1 - e^{2\pi i N \omega / \omega_0}}{(\omega^2 - \omega_0^2)^2} \right] \Big|_{\omega=0} \\ &= \frac{2\pi^2 N^2}{\omega_0^5} S(\omega = 0). \end{aligned} \quad (\text{C6})$$

In summary, the original integral (C1) is

$$I_0 = \text{Re } I = \frac{\pi^2 N^2}{\omega_0} [S(\omega_0) + 2S(\omega = 0)]. \quad (\text{C7})$$

However, the PSD at zero frequency is usually vanished (like a Lorentzian PSD) or divergent (like a $1/f$ noise); both cases do not have observable effect on the witness of the test mass. Hence, the terms related to $S(\omega = 0)$ are dropped off in the main text.

1. Proof of lemma on symmetry of poles

Now I prove the lemma that $-\omega_j$ and $\pm\omega_j^*$ are poles of $S(\omega)$ if ω_j is a pole. Without loss of generality, suppose ω_j is a k th-order pole in the first quadrant of the complex plane, then $S(\omega)$ can be expanded as a Laurent series at ω_j as

$$S(\omega) = \frac{S_0}{(\omega - \omega_j)^k} + \dots \quad (\text{C8})$$

Since $S(-\omega) = S(\omega)$, then one can obtain another Laurent series of $S(\omega)$ as

$$S(\omega) = S(-\omega) = \frac{(-1)^k S_0}{(\omega + \omega_j)^k} + \dots, \quad (\text{C9})$$

which indicates that $-\omega_j$ is also a k th-order pole of $S(\omega)$. On the other hand, noticing that $S(\omega)$ takes real values on the real axis, i.e., $S(\omega) \in \mathbb{R}$, $\forall \omega \in \mathbb{R}$, the Schwarz reflection principle states that $S(\omega^*) = [S(\omega)]^*$. This property implies another Laurent series of $S(\omega)$ as

$$S(\omega) = [S(\omega^*)]^* = \frac{S_0^*}{(\omega - \omega_j^*)^k} + \dots, \quad (\text{C10})$$

which indicates that ω_j^* is also a k th-order pole of $S(\omega)$. Now that both $-\omega_j$ and ω_j^* are poles of $S(\omega)$, one can directly obtain that $-\omega_j^*$ is also a pole of $S(\omega)$.

APPENDIX D: MASTER EQUATION OF POSITION LOCALIZATION DECOHERENCE OF SINGLE PATH

In this Appendix, I will derive the master equation for the position-localization decoherence of a single path of the interferometer. The procedure is similar to the derivation of the *Redfield master equation*, which can be found in textbooks such as Ref. [72].

Let us work under the interaction picture where the total Hamiltonian (2) of a single arm can be divided into $\hat{H}_0$ and $\hat{H}_{\text{int}}(t) = -ma(t)\hat{x}$, where $\hat{H}_0$ is of a simple oscillator and $\hat{H}_{\text{int}}(t)$ contains all force terms. So, the driven force and the gravity signal perform as the mean value of $ma(t)$, while the acceleration noise affects the autocorrelation of $ma(t)$. The

dynamical equation for $\hat{\rho}^I(t)$ is

$$\frac{d}{dt}\hat{\rho}^I = -\frac{i}{\hbar}[\hat{H}_{\text{int}}^I(t), \hat{\rho}^I(t)]. \quad (\text{D1})$$

This differential equation can be directly written in an integral form as

$$\hat{\rho}^I(t) = \hat{\rho}^I(0) - \frac{i}{\hbar} \int_0^t [\hat{H}_{\text{int}}^I(t'), \hat{\rho}^I(t')] dt'. \quad (\text{D2})$$

The basic idea is to do recursion for $\hat{\rho}^I(t)$, that is, to insert Eq. (D2) on the right side of Eq. (D1), then one may obtain

$$\frac{d}{dt}\hat{\rho}^I = -\frac{1}{\hbar^2} \int_0^t [\hat{H}_{\text{int}}^I(t), [\hat{H}_{\text{int}}^I(t'), \hat{\rho}^I(t')]] dt', \quad (\text{D3})$$

where $\hat{\rho}^I(0)$ is assumed to commute with $\hat{H}_{\text{int}}^I(t)$, i.e., $[\hat{H}_{\text{int}}^I(t), \hat{\rho}^I(0)] = 0$. Besides, the decoherence is usually assumed memoryless, known as the *Markovian* assumption [28,72], which means that $\hat{\rho}^I(t)$ can be assumed to only depend on t ; i.e., the $\hat{\rho}^I(t')$ in the integral can be substituted by $\hat{\rho}^I(t)$. In addition, one can substitute t' by $t - \tau$ in the integral and take the upper limit of the integral to infinite and then obtain

$$\frac{d}{dt}\hat{\rho}^I = -\frac{1}{\hbar^2} \int_0^\infty [\hat{H}_{\text{int}}^I(t), [\hat{H}_{\text{int}}^I(t - \tau), \hat{\rho}^I(t)]] d\tau. \quad (\text{D4})$$

Further, by substituting the interaction Hamiltonian $\hat{H}_{\text{int}}^I(t) = -ma(t)\hat{x}^I(t)$ and taking the ensemble average, the dynamical equation of the density matrix becomes

$$\begin{aligned} \frac{d}{dt}\mathbb{E}[\hat{\rho}^I(t)] &= -\frac{m^2}{\hbar^2} \int_0^\infty \mathbb{E}[a(t)a(t - \tau)] \\ &\times [\hat{x}^I(t), [\hat{x}^I(t - \tau), \mathbb{E}[\hat{\rho}^I(t)]]] d\tau. \end{aligned} \quad (\text{D5})$$

This is a deformation of the Redfield master equation for a classical stochastic noise. In the original Redfield equation, the environment is assumed as a quantum bath, so the density matrix of the test mass is obtained by tracing out the degrees of freedom of the quantum bath. For the classical random noise, the ensemble average plays a similar role as tracing out, which intimates a correspondence between randomness and quantumness [64,65].

Note that the time evolution of the position operator $\hat{x}^I(t)$ of a harmonic oscillator satisfies $\hat{x}^I(t - \tau) = \hat{x}^I(t) \cos \omega_0 \tau - \hat{p}^I(t)/(m\omega_0) \sin \omega_0 \tau \sim \hat{x}^I(t)e^{i\omega_0 \tau}$, so Eq. (D5) becomes

$$\begin{aligned} \frac{d}{dt}\mathbb{E}[\hat{\rho}^I(t)] &= -\frac{m^2}{\hbar^2} [\hat{x}^I(t), [\hat{x}^I(t), \mathbb{E}[\hat{\rho}^I(t)]]] \\ &\times \int_0^\infty \mathbb{E}[a(t)a(t - \tau)] e^{i\omega_0 \tau} d\tau. \end{aligned} \quad (\text{D6})$$

Note that the integral is nothing but $2\pi S_{aa}(\omega_0)/2$ according to the Wiener-Khinchin theorem and the symmetry of $S_{aa}(\omega)$, i.e., $S_{aa}(-\omega) = S_{aa}(\omega)$. So, this master equation describes a position localization decoherence with a dissipator $\Lambda = \pi m^2 S_{aa}(\omega_0)/\hbar^2$, and it becomes the master equation (54) by switching back to the Schrödinger picture.

APPENDIX E: PERTURBATION ANALYSIS ON QUADRATIC NOISE

This Appendix presents a detailed perturbative analysis on quadratic noise term $\hat{H}_{\text{noise}} = m\delta\omega^2(t)\hat{x}^2$, as discussed in Sec. V.

The total Hamiltonian can be written as

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(t)\hat{x}^2 - ma(t)\hat{x}, \quad (\text{E1})$$

where $\omega^2(t) = \omega_0^2 + \delta\omega^2(t)$. The corresponding Lewis-Riesenfeld invariant is expressed as [40,67,73]

$$\begin{aligned} \hat{I}(t) &= \frac{(\rho(t)(\hat{p} - m\dot{x}_c(t)) - m\dot{\rho}(\hat{x} - x_c(t)))^2}{2m} \\ &+ \frac{1}{2}m\omega_0^2 \left(\frac{\hat{x} - x_c(t)}{\rho(t)} \right)^2, \end{aligned} \quad (\text{E2})$$

where the auxiliary functions $\rho(t)$ and $x_c(t)$ satisfy the dynamical equations

$$\begin{aligned} \ddot{x}_c + \omega^2(t)x_c(t) &= a(t), \\ \ddot{\rho} + \omega^2(t)\rho(t) &= \frac{\omega_0^2}{\rho^3(t)}. \end{aligned} \quad (\text{E3})$$

Based on the Lewis-Riesenfeld invariant, the time-evolution operator takes the form

$$\hat{U}(t) = e^{i\phi(t)} \hat{S}(\rho(t)) \hat{D}(\alpha_c(t)) e^{-\frac{i}{\hbar} \hat{H}_0 t} \hat{D}^\dagger(\alpha_c(0)) \hat{S}^\dagger(\rho(0)), \quad (\text{E4})$$

where $\alpha_c(t) = \sqrt{m\omega_0/2\hbar}(x(t) + i\dot{x}(t)/\omega_0)$ is the complex coordinate of the classical phase space and $\hat{S}$ is the squeeze operator given by

$$\hat{S}(\rho) = \exp \left[\frac{i}{2\hbar} \ln \rho(\hat{x}\hat{p} + \hat{p}\hat{x}) \right] \exp \left[\frac{i}{2\hbar} m\rho\hat{p}\hat{x}^2 \right]. \quad (\text{E5})$$

Compared to the solution of acceleration noise (3), this time-evolution operator has an additional squeeze operator $\hat{S}$.

1. Perturbative solution of $x_c(t)$ and $\alpha_c(t)$

The dynamical equation for $x_c(t)$ is given by the first equation of Eq. (E3). The perturbative solving method is similar to solving the Pinney equation (E12). In particular, the solution can be written as a perturbative form

$$x_c(t) = x_c^{(0)}(t) + x_c^{(1)}(t), \quad (\text{E6})$$

where the zeroth-order $x_c^{(0)}(t)$ is exactly the real part of $\alpha_c(t)$ of an ideal interferometer, given by Eq. (6), i.e.,

$$x_c^{(0)}(t) = \frac{1}{\omega_0} \int_0^t \sin[\omega_0(t - \tau)] a(\tau) d\tau. \quad (\text{E7})$$

Then, the first-order perturbation $x_c^{(1)}(t)$ satisfies

$$\ddot{x}_c^{(1)} + \omega_0^2 x_c^{(1)}(t) = -\delta\omega^2(t)x_c^{(0)}(t). \quad (\text{E8})$$

Then, the first-order perturbation solution is

$$x_c^{(1)}(t) = -\frac{1}{\omega_0} \int_0^t \sin[\omega_0(t - \tau)] \delta\omega^2(\tau) x_c^{(0)}(\tau) d\tau. \quad (\text{E9})$$

Then, one may compute the fluctuation of the complex coordinate $\delta\alpha_c(t)$ of classical phase space at the leading order as

$$\delta\alpha_c(t) = -i\sqrt{\frac{m}{2\hbar\omega_0}} e^{-i\omega_0 t} \int_0^t \delta\omega^2(\tau) x_c^{(0)}(\tau) e^{i\omega_0 \tau} d\tau, \quad (\text{E10})$$

which is the same as Eq. (35).

2. Perturbative solution of $\rho(t)$

The second equation in the auxiliary dynamical equation (E3) is known as the *Ermakov equation* [74]. Its analytic solution takes the general form

$$\rho(t) = \sqrt{Ay_1^2(t) + 2By_1(t)y_2(t) + Cy_2^2(t)}, \quad (\text{E11})$$

where $y_1(t)$ and $y_2(t)$ are two linear independent solutions of the so-called *Pinney equation* [75]

$$\ddot{y} + \omega^2(t)y(t) = 0. \quad (\text{E12})$$

The constants A , B , and C in Eq. (E11) are arbitrary but must satisfy the constraint

$$AC - B^2 = \frac{\omega_0^2}{(y_1\dot{y}_2 - y_2\dot{y}_1)^2}. \quad (\text{E13})$$

Since the Pinney equation (E12) does not admit a closed-form analytic solution in general, I will solve it by the perturbation method. Suppose the solution of the Pinney equation (E12) can be expanded as

$$y(t) = y^{(0)}(t) + y^{(1)}(t), \quad (\text{E14})$$

where the zeroth-order $y^{(0)}(t)$ satisfies the static oscillator equation

$$\ddot{y}^{(0)} + \omega_0^2(t)y^{(0)}(t) = 0, \quad (\text{E15})$$

of which the two independent solutions are

$$y_1^{(0)}(t) = \cos \omega_0 t, \quad y_2^{(0)}(t) = \sin \omega_0 t. \quad (\text{E16})$$

Substituting this expansion into the Pinney equation, the first-order correction $y^{(1)}(t)$ satisfies

$$\ddot{y}^{(1)} + \omega_0^2 y^{(1)}(t) = -\delta\omega^2(t)y^{(0)}(t). \quad (\text{E17})$$

This inhomogeneous equation can be solved by the Green's function method. The solution is simply the convolution between the inhomogeneous term $-\delta\omega^2(t)y^{(0)}(t)$ and the

Green's function $\sin \omega_0 t$, given as

$$y^{(1)}(t) = -\frac{1}{\omega_0} \int_0^t \delta\omega^2(\tau) \sin \omega_0(t - \tau) y^{(0)}(\tau) d\tau. \quad (\text{E18})$$

Hence, the perturbative solutions of the Pinney equation (E12) are

$$\begin{aligned} y_1(t) &= \cos \omega_0 t - \frac{1}{\omega_0} \int_0^t \delta\omega^2(\tau) \sin \omega_0(t - \tau) \cos \omega_0 \tau d\tau, \\ y_2(t) &= \sin \omega_0 t - \frac{1}{\omega_0} \int_0^t \delta\omega^2(\tau) \sin \omega_0(t - \tau) \sin \omega_0 \tau d\tau. \end{aligned} \quad (\text{E19})$$

Using these two linearly independent perturbative solutions, the Wronskian at first order becomes

$$y_1\dot{y}_2 - y_2\dot{y}_1 = \omega_0. \quad (\text{E20})$$

Consequently, the constraint on the constants in Eq. (E11) reduces to

$$AC - B^2 = 1. \quad (\text{E21})$$

Moreover, in the unperturbed case $\delta\omega^2(t) = 0$, the physically relevant solution of the Ermakov equation is the constant function $\rho(t) \equiv 1$. This fixes the auxiliary constants to

$$A = C = 1, \quad B = 0. \quad (\text{E22})$$

Therefore, the solution of the Ermakov equation (E11) can be simplified as

$$\rho(t) = \sqrt{y_1^2(t) + y_2^2(t)}. \quad (\text{E23})$$

Substituting the perturbative solutions (E19) into this equation, retaining only first-order terms [discarding those quadratic in the integral of $\delta\omega^2(\tau)$], and simplifying the trigonometric functions, the perturbed solution is finally obtained as

$$\begin{aligned} \rho(t) &\approx \sqrt{1 - \frac{1}{\omega_0} \int_0^t \delta\omega^2(\tau) \sin 2\omega_0(t - \tau) d\tau} \\ &\approx 1 - \frac{1}{2\omega_0} \int_0^t \delta\omega^2(\tau) \sin[2\omega_0(t - \tau)] d\tau. \end{aligned} \quad (\text{E24})$$

It indicates that the fluctuation of $\rho(t)$ is exactly a convolution between the input noise and the kernel $\sin 2\omega_0 t$, i.e., $\delta\rho(t) = [\delta\omega^2(\tau) * \sin 2\omega_0 \tau](t)$, which indicates the linear response of $\rho(t)$ to the noise $\delta\omega^2(t)$.

Substituting the perturbative result (E24) of $\rho(t)$ into the squeezed operator (E5), the perturbative squeezed operator is given by Eq. (80).

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- [1] M. Keil, S. Machluf, Y. Margalit, Z. Zhou, O. Amit, O. Dobkowski, Y. Japha, S. Moukouri, D. Rohlich, Z. Binstock, Y. Bar-Haim, M. Givon, D. Groswasser, Y. Meir, and R. Folman, *Stern-Gerlach Interferometry with the Atom Chip* (Springer International Publishing, Cham, 2021), pp. 263–301.
- [2] M. W. Doherty, N. B. Manson, P. Delaney, F. Jelezko, J. Wrachtrup, and L. C. L. Hollenberg, The nitrogen-vacancy colour centre in diamond, *Phys. Rep.* **528**, 1 (2013).

- [3] L. Zhao, X. Shen, L. Ji, and P. Huang, Inertial measurement with solid-state spins of nitrogen-vacancy center in diamond, *Adv. Phys.: X* **7**, 2004921 (2022).
- [4] M. Scala, M. S. Kim, G. W. Morley, P. F. Barker, and S. Bose, Matter-wave interferometry of a levitated thermal nano-oscillator induced and probed by a spin, *Phys. Rev. Lett.* **111**, 180403 (2013).

- [5] X.-Y. Chen and Z.-Q. Yin, High-precision gravimeter based on a nano-mechanical resonator hybrid with an electron spin, *Opt. Express* **26**, 31577 (2018).
- [6] M. P. Ledbetter, K. Jensen, R. Fischer, A. Jarmola, and D. Budker, Gyroscopes based on nitrogen-vacancy centers in diamond, *Phys. Rev. A* **86**, 052116 (2012).
- [7] A. Ajoy and P. Cappellaro, Stable three-axis nuclear-spin gyroscope in diamond, *Phys. Rev. A* **86**, 062104 (2012).
- [8] S. Bose, A. Mazumdar, G. W. Morley, H. Ulbricht, M. Toroš, M. Paternostro, A. A. Geraci, P. F. Barker, M. S. Kim, and G. Milburn, Spin entanglement witness for quantum gravity, *Phys. Rev. Lett.* **119**, 240401 (2017).
- [9] M.-Z. Wu, M. Toroš, S. Bose, and A. Mazumdar, Destructive interference and inertial noise in matter-wave interferometry, *Phys. Rev. D* **112**, 062006 (2025).
- [10] M. Toroš, T. W. van de Kamp, R. J. Marshman, M. S. Kim, A. Mazumdar, and S. Bose, Relative acceleration noise mitigation for nanocrystal matter-wave interferometry: Applications to entangling masses via quantum gravity, *Phys. Rev. Res.* **3**, 023178 (2021).
- [11] M. Schut, H. Bosma, M.-Z. Wu, M. Toroš, S. Bose, and A. Mazumdar, Dephasing due to electromagnetic interactions in spatial qubits, *Phys. Rev. A* **110**, 022412 (2024).
- [12] M.-Z. Wu, M. Toroš, S. Bose, and A. Mazumdar, Inertial torsion noise in matter-wave interferometers for gravity experiments, *Phys. Rev. D* **111**, 064004 (2025).
- [13] T. Zhou, R. Rizaldy, M. Schut, and A. Mazumdar, Spin contrast, finite temperature, and noise in matter-wave interferometers, *Phys. Rev. A* **112**, 012613 (2025).
- [14] S. N. Moorthy, A. Geraci, S. Bose, and A. Mazumdar, Magnetic noise in macroscopic quantum spatial superpositions, *Phys. Rev. A* **112**, 022416 (2025).
- [15] E. Joos and H. D. Zeh, The emergence of classical properties through interaction with the environment, *Z. Phys. B* **59**, 223 (1985).
- [16] O. Romero-Isart, Quantum superposition of massive objects and collapse models, *Phys. Rev. A* **84**, 052121 (2011).
- [17] O. Romero-Isart, Coherent inflation for large quantum superpositions of levitated microspheres, *New J. Phys.* **19**, 123029 (2017).
- [18] H. Pino, J. Prat-Camps, K. Sinha, B. P. Venkatesh, and O. Romero-Isart, On-chip quantum interference of a superconducting microsphere, *Quantum Sci. Technol.* **3**, 025001 (2018).
- [19] M. Roda-Llordes, A. Riera-Campeny, D. Candoli, P. T. Grochowski, and O. Romero-Isart, Macroscopic quantum superpositions via dynamics in a wide double-well potential, *Phys. Rev. Lett.* **132**, 023601 (2024).
- [20] A. Riera-Campeny, M. Roda-Llordes, P. T. Grochowski, and O. Romero-Isart, Wigner analysis of particle dynamics and decoherence in wide nonharmonic potentials, *Quantum* **8**, 1393 (2024).
- [21] B.-G. Englert, J. Schwinger, and M. O. Scully, Is spin coherence like humpty-dumpty? I. Simplified treatment, *Found. Phys.* **18**, 1045 (1988).
- [22] J. Schwinger, M. O. Scully, and B.-G. Englert, Is spin coherence like Humpty-Dumpty? II. General theory, *Z. Phys. D* **10**, 135 (1988).
- [23] M. O. Scully, B.-G. Englert, and J. Schwinger, Spin coherence and Humpty-Dumpty. III. The effects of observation, *Phys. Rev. A* **40**, 1775 (1989).
- [24] J. Jing, T. Yu, C.-H. Lam, J. Q. You, and L.-A. Wu, Control relaxation via dephasing: A quantum-state-diffusion study, *Phys. Rev. A* **97**, 012104 (2018).
- [25] P. Krantz, M. Kjaergaard, F. Yan, T. P. Orlando, S. Gustavsson, and W. D. Oliver, A quantum engineer's guide to superconducting qubits, *Appl. Phys. Rev.* **6**, 021318 (2019).
- [26] L. Chirrolli and G. Burkard, Decoherence in solid-state qubits, *Adv. Phys.* **57**, 225 (2008).
- [27] M.-Z. Wu, M. Toroš, S. Bose, and A. Mazumdar, Quantum gravitational sensor for space debris, *Phys. Rev. D* **107**, 104053 (2023).
- [28] M. Schlosshauer, Decoherence: The quantum-to-classical transition, *The Frontiers Collection* (Springer Berlin, Heidelberg, 2007).
- [29] N. G. Van Kampen, A cumulant expansion for stochastic linear differential equations. II, *Physica* **74**, 239 (1974).
- [30] N. G. Van Kampen, Stochastic differential equations, *Phys. Rep.* **24**, 171 (1976).
- [31] Y. Margalit, O. Dobkowski, Z. Zhou, O. Amit, Y. Japha, S. Moukouri, D. Rohrich, A. Mazumdar, S. Bose, C. Henkel, and R. Folman, Realization of a complete Stern-Gerlach interferometer: Toward a test of quantum gravity, *Sci. Adv.* **7**, eabg2879 (2021).
- [32] Y. Japha, Unified model of matter-wave-packet evolution and application to spatial coherence of atom interferometers, *Phys. Rev. A* **104**, 053310 (2021).
- [33] Y. Japha and R. Folman, Quantum uncertainty limit for stern-gerlach interferometry with massive objects, *Phys. Rev. Lett.* **130**, 113602 (2023).
- [34] C. Henkel and R. Folman, Internal decoherence in nano-object interferometry due to phonons, *AVS Quantum Sci.* **4**, 025602 (2022).
- [35] R. Zhou, Q. Xiang, and A. Mazumdar, Spin-dependent force and inverted harmonic potential for rapid creation of macroscopic quantum superpositions, *Phys. Rev. A* **111**, 052207 (2025).
- [36] Y. B. Band and Y. Japha, Tuning the adiabaticity of spin dynamics in diamond nitrogen vacancy centers, *J. Phys.: Condens. Matter* **34**, 255503 (2022).
- [37] X. Rong, J. Geng, F. Shi, Y. Liu, K. Xu, W. Ma, F. Kong, Z. Jiang, Y. Wu, and J. Du, Experimental fault-tolerant universal quantum gates with solid-state spins under ambient conditions, *Nat. Commun.* **6**, 8748 (2015).
- [38] J. S. Pedernales, G. W. Morley, and M. B. Plenio, Motional dynamical decoupling for interferometry with macroscopic particles, *Phys. Rev. Lett.* **125**, 023602 (2020).
- [39] W. B. Riesenfeld, An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field, *J. Math. Phys.* **10**, 1458 (1969).
- [40] H. R. Lewis and P. G. L. Leach, A direct approach to finding exact invariants for one-dimensional time-dependent classical Hamiltonians, *J. Math. Phys.* **23**, 2371 (1982).
- [41] S. S. Mizrahi, The geometrical phase: An approach through the use of invariants, *Phys. Lett. A* **138**, 465 (1989).

- [42] J. Guerrero and F. F. López-Ruiz, On the Lewis–Riesenfeld (Dodonov–Man’ko) invariant method, *Phys. Scr.* **90**, 074046 (2015).
- [43] V. Aldaya, F. Cossío, J. Guerrero, and F. F. López-Ruiz, The quantum Arnold transformation, *J. Phys. A: Math. Theor.* **44**, 065302 (2011).
- [44] M. J. Snadden, J. M. McGuirk, P. Bouyer, K. G. Haritos, and M. A. Kasevich, Measurement of the Earth’s gravity gradient with an atom interferometer-based gravity gradiometer, *Phys. Rev. Lett.* **81**, 971 (1998).
- [45] F. Sorrentino, Q. Bodart, L. Cacciapiuoti, Y.-H. Lien, M. Prevedelli, G. Rosi, L. Salvi, and G. M. Tino, Sensitivity limits of a Raman atom interferometer as a gravity gradiometer, *Phys. Rev. A* **89**, 023607 (2014).
- [46] Yu.-P. Wang, J.-Q. Zhong, X. Chen, R.-B. Li, D.-W. Li, L. Zhu, H.-W. Song, J. Wang, and M.-S. Zhan, Extracting the differential phase in dual atom interferometers by modulating magnetic fields, *Opt. Commun.* **375**, 34 (2016).
- [47] V. Ivády, T. Simon, J. R. Maze, I. A. Abrikosov, and A. Gali, Pressure and temperature dependence of the zero-field splitting in the ground state of NV centers in diamond: A first-principles study, *Phys. Rev. B* **90**, 235205 (2014).
- [48] R. J. Marshman, A. Mazumdar, R. Folman, and S. Bose, Constructing nano-object quantum superpositions with a Stern-Gerlach interferometer, *Phys. Rev. Res.* **4**, 023087 (2022).
- [49] C. L. Degen, F. Reinhard, and P. Cappellaro, Quantum sensing, *Rev. Mod. Phys.* **89**, 035002 (2017).
- [50] Y. Xu, Y. Chen, and X. Chen, Fast transport and splitting of spin-orbit-coupled spin-1 Bose-Einstein condensates, *Phys. Rev. A* **109**, 063310 (2024).
- [51] M. T. Johnsson, G. K. Brennen, and J. Twamley, Macroscopic superpositions and gravimetry with quantum magnetomechanics, *Sci. Rep.* **6**, 37495 (2016).
- [52] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, 1997).
- [53] N. F. Ramsey, A new molecular beam resonance method, *Phys. Rev.* **76**, 996 (1949).
- [54] This conclusion can be generalized to a generic matter-wave interferometer under a general noise. Suppose the interferometer is built up through a Lagrangian $L[x_i(t), \dot{x}_i(t)]$ and the noise is described by a Lagrangian $\delta L[x_i(t), \dot{x}_i(t)]$. Denote the ideal trajectory as $x_i^0(t)$, and the ideal phase is $\phi^0 = \int L[x_i^0, \dot{x}_i^0]/\hbar dt$. Then, the phase under noise at the leading order is

$$\begin{aligned} \phi^0 + \delta\phi &= \frac{1}{\hbar} \int L[x_i^0 + \delta x_i, \dot{x}_i^0 + \delta \dot{x}_i] + \delta L[x_i^0, \dot{x}_i^0] dt \\ &= \frac{1}{\hbar} \int L[x_i^0, \dot{x}_i^0] + \left. \frac{\partial L}{\partial x_i} \right|_{(x_i^0, \dot{x}_i^0)} \delta x_i \\ &\quad + \left. \frac{\partial L}{\partial \dot{x}_i} \right|_{(x_i^0, \dot{x}_i^0)} \delta \dot{x}_i + \delta L[x_i^0, \dot{x}_i^0] dt. \end{aligned}$$

According to the Euler-Lagrangian equation $\partial L/\partial x_i = d(\partial L/\partial \dot{x}_i)/dt$, one can prove an identity

$$\left. \frac{\partial L}{\partial x_i} \right|_{(x_i^0, \dot{x}_i^0)} \delta x_i + \left. \frac{\partial L}{\partial \dot{x}_i} \right|_{(x_i^0, \dot{x}_i^0)} \delta \dot{x}_i = \frac{d}{dt} \left(\left. \frac{\partial L}{\partial \dot{x}_i} \right|_{(x_i^0, \dot{x}_i^0)} \delta x_i \right).$$

Thus, the phase fluctuation arising from the fluctuation of the trajectory δx_i and the velocity $\delta \dot{x}_i$ is the integral of a total deriva-

tive. Then, one may read $\delta\phi = \int \delta L[x_i^0, \dot{x}_i^0]/\hbar dt$ by neglecting this total derivative term. Note that the same conclusion has been proved by using Feynman’s path integral method [76].

- [55] J. Junca, A. Bertoldi, D. O. Sabulsky, G. Lefèvre, X. Zou, J.-B. Decitre, R. Geiger, A. Landragin, S. Gaffet, P. Bouyer, and B. Canuel, Characterizing Earth gravity field fluctuations with the MIGA antenna for future gravitational wave detectors, *Phys. Rev. D* **99**, 104026 (2019).
- [56] Here, asymptotic behaviors of the sinc function have been used. That is, $\text{sinc}^2((\omega - \omega_0)t/2) = 1 + \mathcal{O}(t^2)$ for small t , and $\text{sinc}^2((\omega - \omega_0)t/2) = 2\pi\delta(\omega - \omega_0)/t$ for large t .
- [57] A. Uhlmann, The “transition probability” in the state space of a $*$ -algebra, *Rep. Math. Phys.* **9**, 273 (1976).
- [58] L. Lami, J. S. Pedernales, and M. B. Plenio, Testing the quantumness of gravity without entanglement, *Phys. Rev. X* **14**, 021022 (2024).
- [59] Here, the following identities are used:

$$\begin{aligned} \sqrt{\hat{\rho}_{\text{id}}(t)} &= \hat{U}_{\text{id}}(t) \sqrt{\hat{\rho}_0} \hat{U}_{\text{id}}^\dagger(t), \\ \hat{U}_{\text{id}}^\dagger(t) \hat{U}_n(t) &= \hat{D}(\delta\alpha(t) e^{i\omega_0 t}) e^{i\delta\phi(t)}. \end{aligned}$$

- [60] To compute the ensemble average of $\langle x_1 | \hat{\rho}_n(t) | x_2 \rangle$, one can consider the Glauber’s P -representation [52] of the initial density matrix

$$\hat{\rho}_0 = \int P(\beta) |\beta\rangle \langle \beta| d^2\beta.$$

According to the identity

$$\hat{D}(\delta\alpha(t)) |\beta\rangle = e^{i\phi(\beta, \delta\alpha)} |\beta + \delta\alpha(t)\rangle,$$

the matrix element of $\hat{\rho}_n(t)$ becomes

$$\langle x_1 | \hat{\rho}_n(t) | x_2 \rangle = \int P(\beta) \langle x_1'(t) | \beta + \delta\alpha(t) \rangle \langle \beta + \delta\alpha(t) | x_2'(t) \rangle d^2\beta.$$

Based on the wavefunction of a coherent state, one can obtain the ensemble average of the matrix element in the P -representation satisfying

$$\begin{aligned} \mathbb{E}[\langle x_1'(t) | \beta + \delta\alpha(t) \rangle \langle \beta + \delta\alpha(t) | x_2'(t) \rangle] \\ = \langle x_1'(t) | \beta \rangle \langle \beta | x_2'(t) \rangle e^{-\frac{m\omega_0}{\hbar} \sigma_a^2(t) (x_1'(t) - x_2'(t))^2}. \end{aligned}$$

Then, one can obtain the result (52).

- [61] If there is a biased magnetic field $B_0 \neq 0$, then the amplitudes $A_\pm$ defined in Eq. (31) are not opposite to each other. As a result, the trajectories $x_1(t)$ and $x_{-1}(t)$ do not cancel each other. In this case, there is still a dephasing effect, and the transfer function takes a similar form as Eq. (32) but the factor ΔA is replaced by $A_+ + A_-$.
- [62] Suppose the final states before squeezing are $|\psi_+\rangle$ and $|\psi_-\rangle$. After applying the squeezing operator, their overlap becomes

$$\langle \psi_+ | \hat{S}^\dagger(\rho) \hat{S}(\rho) | \psi_- \rangle = \langle \psi_+ | \psi_- \rangle,$$

which remains unchanged because $\hat{S}(\rho)$ is unitary.

- [63] The trick to compute the purity and the von Neumann entropy is to diagonalize $\hat{\rho}_{\text{spin}}$. In particular, suppose the nonzero eigenvalues of $\hat{\rho}_{\text{spin}}$ are $\{\lambda_i\}$, then the purity and the von Neumann entropy of $\hat{\rho}_{\text{spin}}$ are $\sum_i \lambda_i^2$ and $-\sum_i \lambda_i \ln \lambda_i$, respectively. For the density matrix $\hat{\rho}_{\text{spin}}$ with a decay factor $e^{-\Gamma_{\text{tot}}}$ on the off-diagonal terms, its eigenvalues are $\lambda_{1,2} = (1 \pm e^{-\Gamma_{\text{tot}}})/2$ and $\lambda_3 = 0$. Especially, when the total decoherence factor Γ_{tot} is

very small, then the entropy can be approximated as

$$\begin{aligned} S &= - \left[\frac{1 + e^{-\Gamma_{\text{tot}}}}{2} \ln \frac{1 + e^{-\Gamma_{\text{tot}}}}{2} + \frac{1 - e^{-\Gamma_{\text{tot}}}}{2} \ln \frac{1 - e^{-\Gamma_{\text{tot}}}}{2} \right] \\ &\approx \frac{1 + e^{-\Gamma_{\text{tot}}}}{2} \times \frac{1 - e^{-\Gamma_{\text{tot}}}}{2} + \frac{1 - e^{-\Gamma_{\text{tot}}}}{2} \times \frac{1 + e^{-\Gamma_{\text{tot}}}}{2} \\ &= \frac{1}{2} (1 - e^{-2\Gamma_{\text{tot}}}) \approx \Gamma_{\text{tot}}, \end{aligned}$$

where the first approximated equation used an approximation $\ln((1 \pm e^{-\Gamma_{\text{tot}}})/2) \approx (-1 \pm e^{\Gamma_{\text{tot}}})/2$.

- [64] J. A. Barandes, The stochastic-quantum correspondence, *Philos. Phys.* **3**, 8 (2025).
- [65] J. A. Barandes, The stochastic-quantum theorem, [arXiv:2309.03085](https://arxiv.org/abs/2309.03085).
- [66] O. V. Ivakhnenko, S. N. Shevchenko, and F. Nori, Nonadiabatic Landau–Zener–Stückelberg–Majorana transitions, dynamics, and interference, *Phys. Rep.* **995**, 1 (2023).
- [67] A. K. Dhara and S. V. Lawande, Feynman propagator for time-dependent Lagrangians possessing an invariant quadratic in momentum, *J. Phys. A: Math. Gen.* **17**, 2423 (1984).
- [68] M. Kolodrubetz, D. Sels, P. Mehta, and A. Polkovnikov, Geometry and non-adiabatic response in quantum and classical systems, *Phys. Rep.* **697**, 1 (2017).
- [69] J. M. Hogan, D. M. S. Johnson, and M. A. Kasevich, Light-pulse atom interferometry, in *Atom Optics and Space Physics*, edited by E. Arimondo, W. Ertmer, W. P. Schleich, and E. M. Rasel (IOS Press, Amsterdam, 2009), pp. 411–447.
- [70] E. H. Kerner, Note on the forced and damped oscillator in quantum mechanics, *Can. J. Phys.* **36**, 371 (1958).
- [71] One can directly write
- $$f(-\omega^*) = S(-\omega^*) \frac{1 - e^{-2\pi i N \omega^* / \omega_0}}{\omega^{*2} (\omega^{*2} - \omega_0^2)^2}.$$
- It is obvious that $1 - e^{-2\pi i N \omega^* / \omega_0} = (1 - e^{2\pi i N \omega / \omega_0})^*$ and $\omega^{*2} (\omega^{*2} - \omega_0^2)^2 = [\omega^2 (\omega^2 - \omega_0^2)^2]^*$. Besides, according to the properties $S(-\omega) = S(\omega)$ and $S(\omega^*) = [S(\omega)]^*$, one can obtain $S(-\omega^*) = [S(\omega)]^*$. Combining these equations, one may find $f(-\omega^*) = [f(\omega)]^*$.
- [72] H.-P. Breuer and F. Petruccione, *Quantum Master Equations*, in *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2007).
- [73] E. Torrontegui, S. Ibáñez, X. Chen, A. Ruschhaupt, D. Guéry-Odelin, and J. G. Muga, Fast atomic transport without vibrational heating, *Phys. Rev. A* **83**, 013415 (2011).
- [74] V. P. Ermakov, Second-order differential equations. Conditions of integrability in finite form, *Appl. Anal. and Discrete Math.* **2**, 123 (2008).
- [75] E. Pinney, The nonlinear differential equation $y'' + p(x)y + c/y^3 = 0$, *Proc. Am. Math. Soc.* **1**, 681 (1950).
- [76] P. Storey and C. Cohen-Tannoudji, The Feynman path integral approach to atomic interferometry: A tutorial, *J. Phys. II* **4**, 1999 (1994).