

Path integral approach to quantum thermalization

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We introduce a quasiclassical Green's function approach describing the unitary yet irreversible dynamics of quantum systems effectively acting as their own environment. Combining a variety of concepts of quantum many-body theory, notably the nonlinear σ model of disordered systems, the $G\Sigma$ formalism for strong correlations, and real-time path integration, the theory is capable of describing a wide range of system classes and disorder models. It extends previous work beyond perturbation theory (in inverse Hilbert-space dimensions), enabling a description of thermalization dynamics from short scattering times, through the onset of ergodicity at an effective “Thouless time,” up to the many-body Heisenberg time. We illustrate the approach with two case studies: (1) a brickwork model of unitarily coupled quantum circuits with and without conserved symmetries and (2) an array of capacitively coupled quantum dots. Using the spectral form factor as a test observable, we find good agreement with numerical simulations. We present our formalism in a self-contained and pedagogical manner, aiming to provide a transferable toolbox for the first-principles description of many-body chaotic quantum systems in regimes of strong entanglement.

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I. INTRODUCTION

Describing the effective irreversible evolution of quantum systems large enough to act as their own environment—in the sense that initially localized information becomes delocalized and effectively inaccessible to local measurements—has been a long-standing challenge of statistical quantum physics. The beginning availability of engineered quantum devices effectively realizing this setting has led to a surge of renewed efforts to understand the quantum thermalization problem (see Refs. [1–3] for recent reviews). A crucial aspect of this type of dynamics is that it manifests itself for system sizes often beyond the scope of exact diagonalization, motivating efforts in the development of analytical approaches. Pioneering contributions in this direction [4–7] semiclassically analyze coupled unitary circuits, employing concepts of statistical mechanics, random matrix theory, and tensor network theory.

However, at this point, a first-principles framework describing thermalization in terms of an effective field theory for a wide range of microscopically different systems and disorder models beyond perturbation theory appears to be missing. In this paper, we introduce such an approach and illustrate its application on two different system classes. To establish

contact with previous work, the first will be realizations of the standard unitary brickwork design (cf. Fig. 10), with and without continuous symmetries. Specifically, we will show how structures beyond perturbation theory in the inverse of local Hilbert-space dimensions, D , essentially affect the dynamics at timescales $t \sim D$. The second are arrays of “quantum dots,” coupled by energy conserving random two-body interactions, where energy diffusion turns out to be the essential bottleneck slowing thermalization.

Modular by design, our approach starts with the identification of system subunits defined to be nonintegrable and sufficiently small that they relax to an ergodic state quasi-instantly. The “thermalizing” systems we are interested in are realized as tensor products of such modules, coupled by local correlations.

Conceptually, our effective description of these systems is a field theory defined over a tensor-product space, as opposed to the more familiar many-body field theories in real space. Its construction draws upon multiple concepts, specifically real-time (Schwinger-Keldysh) path integration [8], quasiclassical Green's functions [9], the nonlinear σ model of disordered systems [10], the $G\Sigma$ [11] (also known as Luttinger-Ward [12]) approach to correlated systems, and insights gained in previous work on thermalization dynamics [4–7]. We have tried to present the material in such a way that it is accessible to readers of different background and interests. The paper opens with two nontechnical sections, which should be regarded as an extended introduction. The first, Sec. II, discusses our qualitative understanding of

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TABLE I. Summary of the most frequently occurring symbols and index conventions used throughout the paper.

Symbol	Meaning
a, b	Causality indices (retarded/advanced, +, -)
c, d	Stage indices (e.g., brickwall model stages, +1, -1)
τ, t, u	Time indices (discrete or continuous)
ϵ, ζ	Energy variables
m, n	Time indices (discrete)
j, k	Site (subsystem) indices
μ, ν	Internal Hilbert-space indices (qudit states)
u, v	Symmetry representation indices ("spin")
a_j	Dimensionless time-translation parameters
q	Momentum (Fourier) mode index
D	Local Hilbert-space dimension (qudit dimension)
L	Number of sites/subsystems
λ	High-energy cutoff
Γ	Interaction damping rate

thermalization dynamics and presents the main results of this paper in comparison to exact diagonalization. The second, Sec. III, introduces conceptual elements of our theory construction still in nontechnical terms. In Sec. IV, we introduce our path integral for a single-qudit building block, and in Secs. V and VI generalize it to the brickwork model with and without continuous symmetries, respectively. Energy conserving Hamiltonian dynamics is discussed in Sec. VII. We conclude in Sec. VIII.

The notation in this paper uses lots of indices, and we aim to be as consistent as possible. To avoid cluttering, we refrain from using primes; i.e., instead of tt' , we write tu , the convention being that second indices are one step up in the alphabet. For the same reason, we avoid commas, i.e., $\psi_{\mu t}$ instead of $\psi_{\mu,t}$. For the convenience of the reader, a list of the most frequently occurring indices is provided in Table I.

II. SUMMARY OF CONCEPTS AND RESULTS

We begin our discussion by reviewing the concept of quasiclassical Green's functions as slowly fluctuating variables describing the late-time stages of chaotic evolution. This will be followed by a summary of our results for systems with discrete and continuous time evolution, respectively.

A. Quasiclassical Green's functions

Consider a D -dimensional quantum system that is strongly chaotic in the sense that an ergodic state is reached after microscopically short timescales. Examples of such building blocks include

- (1) irregularly shaped or disordered quantum dots,
- (2) systems described in terms of unitarily invariant random matrix Hamiltonians, or
- (3) QuDits evolving under Haar-distributed unitaries.

We will summarily refer to these units as *qudits* throughout. Timescales below which qudits become ergodic are not resolved by our theory.

We describe individual qudits in terms of collective variables

$$-\frac{ia}{D} \sum_{\mu} \psi_{\mu t}^a \bar{\psi}_{\mu u}^b \equiv G_{tu}^{ab}, \quad (1)$$

where μ runs over a qudit's D internal states and $\psi_{\mu,t}^a$ are time-dependent fermionic fields of either retarded, $a = +1$, or advanced, $a = -1$, causality [13]. These variables afford different interpretations: Considered as building blocks in correlation functions, and before ensemble averaging, they are defined through their time dependence

$$\langle \psi_{\mu t}^a \bar{\psi}_{\nu u}^b \rangle = -i\Theta(a(t-u))\delta^{ab} \langle \mu | e^{i(t-u)(-H+i\delta s)} | \nu \rangle. \quad (2)$$

(For a system with unitary evolution, replace $e^{-iHt} \rightarrow U^t$, with $t = n \in \mathbb{Z}$.) In this reading, the fields represent wave functions $\psi_{\mu,t}^{\pm} \rightarrow |\mu(\pm t)\rangle$ propagating forward or backward in time. Under the assumed condition of ergodicity, individual ψ_{μ} are rapidly fluctuating amplitudes. However, the summation over μ makes the pairings $G \sim \psi_{\mu} \bar{\psi}_{\mu}$ good candidates for slowly fluctuating effective variables.

When inserted as building blocks in correlation functions, G_{tu} remains slowly fluctuating, including for macroscopic differences $\tau \equiv t - u$ between the readout times. To illustrate this principle, consider the spectral form factor (SFF)

$$K(t) \equiv \langle |\text{tr}(e^{iHt})|^2 \rangle_H,$$

where the angular brackets on the right-hand side denote an average over an ensemble of Hamiltonians. With Eq. (2), it is straightforward to obtain the representation

$$K(t) = D^2 \langle G_{t0}^{++} G_{0t}^{--} \rangle, \quad (3)$$

where the brackets on the right-hand side now include an integral over the ψ variables contained in G [14]. In this case, the Green's functions are evaluated for time differences equaling the observation time of the system. We note that quasiclassical Green's functions have been instrumental in various contexts, including the Luttinger-Ward functional [12] (also known as $G\Sigma$ functional [11]), the nonlinear σ model [10], or in the context of disordered superconductivity [9].

For an L -qudit system, we have as many collective variables, G_j , realized in the tensor-product structure defined by its Hilbert space. For each of them, the time differences $t_j - u_j \equiv \tau_j$ are "soft variables," as schematically indicated in Fig. 1, left. Turning on interqudit correlations, we ask which of these configurations contribute to the path integral in the long time limit. Interactions are instantaneous in time, which will turn out to constrain the freedom to choose the time differences τ_j independently. Rather, we will find that correlations confine the mismatch $\Delta\tau = \tau_j - \tau_k$ between the time variables of neighboring qudits to be microscopically small (Fig. 1, center). For timescales larger than an effective "Thouless time," t_{Th} , even these differences get damped out and the collective amplitudes supporting the system's dynamics assume the form

$$G_{1t_1 u_1}^{a_1 b_1} \cdots G_{Lt_L u_L}^{a_L b_L} \rightarrow G_{1t u}^{ab} \cdots G_{Lt u}^{ab} \\ \equiv (G_1 \odot \cdots \odot G_L)_{tu}^{ab} \equiv G_{tu}^{ab}, \quad (4)$$

i.e., a freezing to one common time set of time and causal indices where the notation on the right-hand side uses the

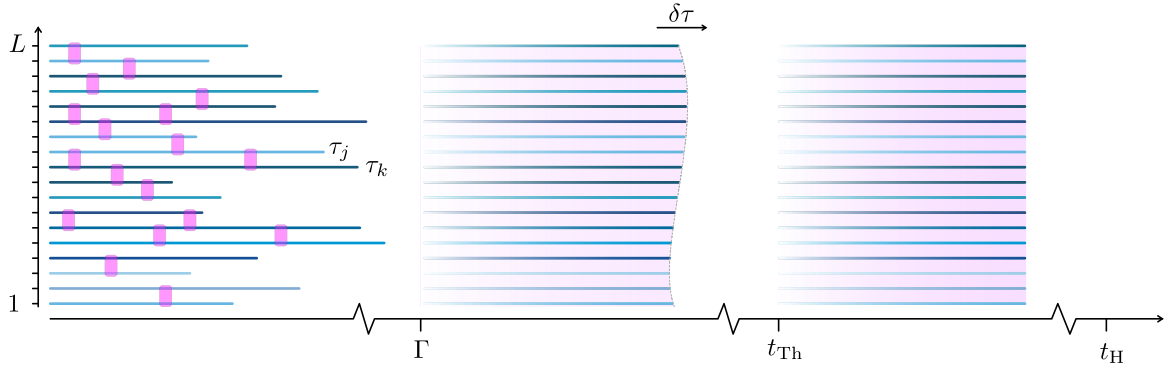


FIG. 1. The three stages of quantum thermalization in an energy conserving medium defined by L chaotic, statistically independent (symbolically indicated by the color scheme) and pairwise entangled subsystems. Left: For times shorter than a characteristic pair interaction scale, Γ , the system supports L dynamical zero modes: the differences in time $\tau_j - \tau_k$ at which subsystem quantum amplitudes propagate in their respective Hilbert spaces. Center: At timescales exceeding $\sim \Gamma \ln(L)$, global phase coherence is maintained only by states with small time differences $\Delta\tau \sim \tau_j - \tau_k$. However, time-translational invariance (due to energy conservation) implies that weak collective fluctuations in $\delta\tau$ remain a soft mode. Right: Beyond an effective “Thouless time” $\tau_{\text{Th}} \sim DL^2$, these modes have relaxed, too, and the system enters a late-time ergodic regime.

elementwise or Hadamard product of matrices

$$(A \odot B)_{tu}^{ab} \equiv A_{tu}^{ab} B_{tu}^{ab}. \quad (5)$$

Physically, this state reduction reflects a “synchronization” of L independently propagating states to a single time-dependent many-body wave function. Their effective quasiclassical Green’s function G then describes the late-time physics in its ergodic (“thermalized”) regime (cf. Fig. 1, right).

B. Discrete time evolution

As a simple model exemplifying this type of dynamics, we consider the brickwork design illustrated in Fig. 10. It contains horizontal rows of L statistically independent qudits that are fast thermalizing in the sense that time evolution $\exp(-iH_j\delta) \equiv U_j$ is statistically equivalent to operators drawn from a Haar distribution. Correlations between neighboring qudits j, k are described by interaction operators $\exp(i\delta H_{j,k})$ with Gaussian distributed interaction operators of variable strength [Eq. (17)].

For this system, we average over realizations of qudit operators to derive a real-time path integral over quasiclassical Green’s functions. To leading order in an expansion in $D \gg 1$ and $t \gg 1$, it yields the SFF as

$$K(t) \simeq (1 + te^{-\Gamma t})^L + t(1 - e^{-\Gamma t})^L, \quad (6)$$

identical to that of a somewhat different brickwork model in Ref. [5]. In this formula, t is discrete time, and $\Gamma = \frac{\Lambda^2}{4}$ a dimensionless golden-rule interaction rate determined by the square of the interaction matrix elements, Λ^2/D^2 , multiplied by the two-qudit spectral density $\sim D^2$. Equation (6) describes a crossover from the semiclassical form factor $K(t) = t^L$ of L independent qudits in the limit of absent interaction $\Gamma t \rightarrow 0$ to that of a single ergodic system $K(t) \simeq t$ for $\Gamma t \gg 1$. Governed by the factors $\exp(-\Gamma t)$, this crossover is exponentially fast and completed after timescales $\sim \Gamma^{-1} \ln L$.

We also note that Eq. (6) affords a statistical mechanics interpretation as the partition sum of a t -state Potts model [5]. Within it, the interaction of t -state “spins” $s_j \in \mathbb{Z}_t$ flags

whether two neighboring discrete time variable (differences) are in synchronization ($s_j = s_k$) or not. Summation over these configurations yields Eq. (6).

However, for weaker interactions $\Gamma t \lesssim 1$ the semiclassical/statistical mechanics approach no longer adequately describes the crossover dynamics. We now need to take into account that the ergodic form factor of n ergodically coupled qudits is not just linear in t , but given by [15]

$$K_n(t) = t \Theta(D^n - t) + D^n \Theta(t - D^n). \quad (7)$$

Within the path integral formalism, information about the late-time stationary values (the “plateau regime”) is contained in saddle point configurations known as Altshuler-Andreev (AA) saddles [16]. Their inclusion leads to the generalized result

$$K(t) = (1 + K_1(t)e^{-\Gamma t})^L + K_L(t)(1 - e^{-\Gamma t})^L. \quad (8)$$

While the departure from linearity in K_L kicks in only at very late times $t > D^L$, the saturation to a plateau value in K_1 takes place at $t = D$; i.e., timescales easily competing with inverse interaction rates $\sim \Gamma^{-1}$. As we are going to discuss next, the inclusion of this structure makes a significance difference in the comparison to exact diagonalization.

1. Numerical validation

Figure 2 shows the results of simulations for $L = 2$, qudit dimensions $D = 8, 16, 32, 64$, and various two-qudit interaction strength, Λ . The agreement with the analytical formula (green) gets better with increasing D , when D^{-1} corrections beyond our approach diminish. For the largest values of D , the one fit parameter used in this comparison, the damping rate $\Gamma = \Lambda^2/4$, equals the analytical prediction; i.e., the comparison with the numerical results is essentially parameter free. Also notice that the difference between the approximation [Eqs. (6) and (8)]—visible in the kinklike structures at the Heisenberg times D of the individual qudits—is essential in obtaining quantitative agreement.

The largest arrays we have analyzed have $L = 4$ with dimension $D = 12$ (Fig. 3). Comparing to the case $L = 2$ with

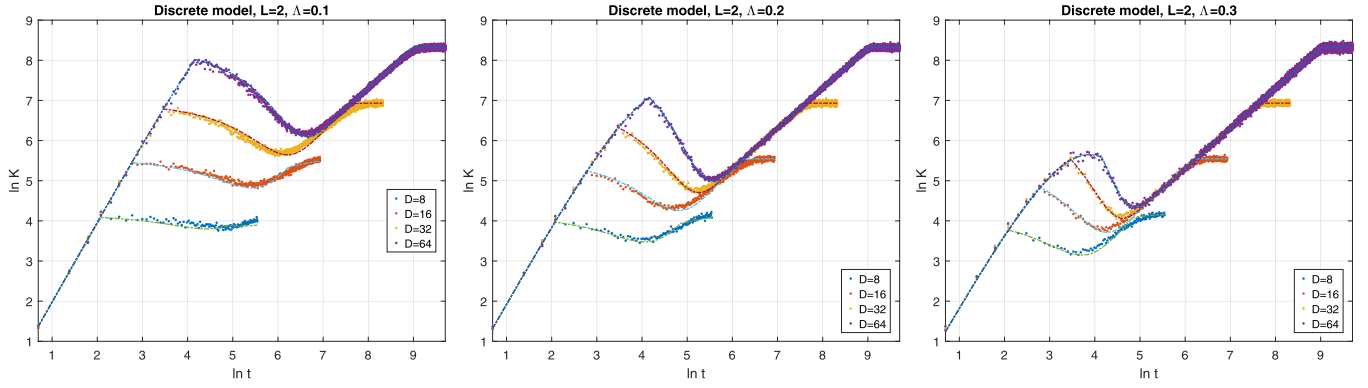


FIG. 2. The SFF of a two-site system, $L = 2$, for three different interaction strengths: $\Lambda = 0.1$ (left), $\Lambda = 0.2$ (middle), $\Lambda = 0.3$ (right). Each dataset is averaged over 10^3 realizations. We fit these results with the effective damping rate Γ as a single parameter, where for qudit dimensions $D = 8, 16, 32, 64$, $\Gamma = 0.0039, 0.0028, 0.0022, 0.0023$ ($\Lambda = 0.1$), $\Gamma = 0.010, 0.0085, 0.0090, 0.0095$ ($\Lambda = 0.2$), and $\Gamma = 0.020, 0.020, 0.0210, 0.0215$ ($\Lambda = 0.3$).

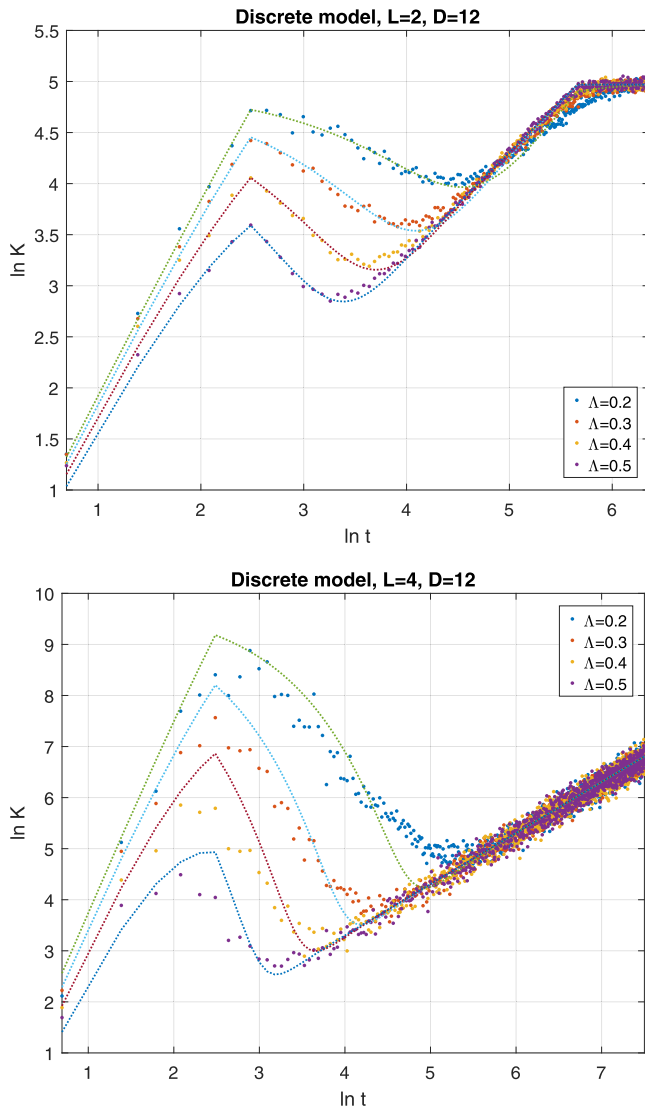


FIG. 3. The SFF for $D = 12$, interaction strengths, $\Lambda = 0.2, 0.3, 0.4, 0.5$, and system sizes $L = 2, 10^3$ runs (top) and $L = 4, 200$ runs (bottom). For increasing interaction strength, we use the fitting parameters $\Gamma = 0.009, 0.019, 0.034, 0.053$ ($L = 2$) and $\Gamma = 0.0135, 0.0310, 0.0550, 0.0900$ ($L = 4$).

the same dimension, shown in the left panel for comparison, we observe sizable deviations from the analytical formula. We also note that for large interactions the best fits are obtained for damping rates Γ , significantly larger than the simple golden rule estimates by factors of $\mathcal{O}(1-2)$, which may hint at an effectively larger scattering phase volume in the multisite array. However, considering the smallness of the dimension $D = 12$ the agreement with a theory having D^{-1} as an expansion parameter is still satisfactory.

2. Discrete time evolution with symmetries

We mentioned in the Introduction that local conservation laws reflecting the presence of a symmetry dramatically slow down the relaxation into ergodic states [6]. The first stage of thermalization outlined above then is followed by a slower second one governed by the (typically diffusive) exchange dynamics of conserved charges.

As a simple model for such a scenario, we consider the previous one, upgraded to a state space $\mu \rightarrow (\mu, a)$, where $a = 1, \dots, N$ are the globally conserved “charges” of the symmetry, think of the z quantum number of a spin, etc. Symmetry transformations with weak site-to-site variations describe the relaxation of the symmetry’s charges. As pointed out in Ref. [6] in the context of $SU(2)$ symmetry, the diffusive nature of this process implies full equilibration on timescales $\sim L^2$. In Sec. VI, we include this mechanism into our path integral in terms of a (nonrandom) $SU(N)$ -spin exchange operator.

As a result, we obtain a form factor at timescales exceeding $\sim \Gamma \ln(L)$ modulated by a factor describing the time evolution of an $SU(N)$ -Heisenberg model [cf. Eq. (58)]. The late-time dependence of the latter is set by the action cost $\sim tL^{-2}$ of its lowest lying magnon excitations, establishing a timescale $t \sim L^2$. (The fact that the magnetic action enters with a purely imaginary coupling is due to the specific modeling of the spin exchange. A generalization to randomly fluctuating spin operators would lead to relaxation.)

C. Continuous time evolution

In Sec. VII, we consider linear arrays of capacitively coupled quantum dots as an example of an energy conserving

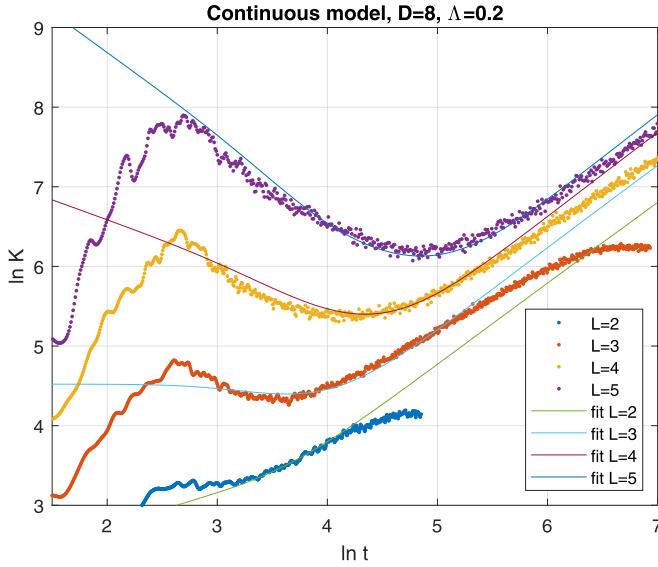


FIG. 4. The product formula in Eq. (9) compared to numerics for systems of size $L = 2, 3, 4, 5$ with $D = 8$. We fit the numerically obtained SFF with $\Gamma \equiv \Lambda^2/2\lambda = 0.024, 0.0210, 0.0178, 0.0152$ for increasing length. A second fit parameter $C = 0.86, 1.45, 2.34, 3.16$ is used to fit the undetermined constant of proportionality in Eq. (9).

system. In this case, time translations $\tau_j \rightarrow \tau + \lambda^{-1}a_j$ controlled by small dimensionless parameters $a_j = \mathcal{O}(1)$ assume the role of symmetry generators, and Eq. (67) assumes a role analogous to that of the Heisenberg action, in the symmetry-enriched discrete time framework. One technical issue is that the deviations in the a parameters from site to site are small, of the order of the inverse of the high-energy cutoff λ of the theory. Our effective low-energy theory lacks precision control over these scales, meaning that the action is obtained with parametric, but not numerical accuracy.

Integrating over a modes, we obtain

$$K(t) \sim Ct \prod_{n=1}^{L-1} F_n, \quad F_n = \max \left(\left(t\Gamma \sin^2 \left(\frac{\pi n}{L} \right) \right)^{-1/2}, 1 \right), \quad (9)$$

in the regime $t \gg \Gamma$, where C is a numerical constant. The qualitative difference to the previously derived Eq. (6) is the soft relaxation toward the ergodic limit on timescales $t \sim L^2$.

Numerical validation

Figure 4 compares Eq. (9) to numerical simulations of arrays of length $L = 2, 3, 4, 5$ with low Hilbert-space dimension $D = 8$. Our lack of control over high-energy scales requires working with two fit parameters. The first is the overall multiplicative constant C , and the second the parameter Γ . We also used a max function softened over λ scales.

III. PATH INTEGRAL ESSENTIALS

Having summarized our main findings, we now turn to their derivation. As mentioned in the Introduction, our principal workhorse will be a path integral over quasiclassical Green's functions describing the evolution of correlated states

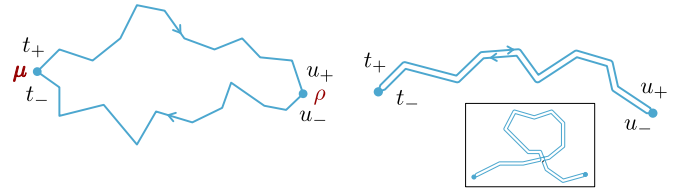


FIG. 5. Left: coherent double sum over retarded (top) and advanced (bottom) scattering path propagators connecting two qudit states μ and ρ . Right: averaging removes the contribution of different paths leading to a semiclassical pair propagator and constraints on the propagation times.

in a tensor-product space. To guide its construction, we begin by discussing the physics of these G variables from two complementary perspectives: semiclassical analysis and symmetries.

A. Semiclassical analysis

Consider the correlation function

$$\begin{aligned} \Pi_{u_+u_-,t_+t_-} &\equiv \langle G_{u_+u_-}^{+-} G_{t_+t_-}^{+-} \rangle \\ &= \frac{1}{D^2} \langle \psi_{\rho u_+}^+ \bar{\psi}_{\mu t_+}^+ \psi_{\mu t_-}^- \bar{\psi}_{\rho u_-}^- \rangle \end{aligned}$$

of two Green's functions [Eq. (2)], where we have reordered field variables to highlight an interpretation as the product of a retarded and an advanced transition amplitude $|\rho\rangle \rightarrow |\mu\rangle$ in time $t_+ - u_+$ and $t_- - u_-$, respectively [Fig. 5, left]. Considering these amplitudes individually expanded as coherent sums over Feynman scattering paths, the majority of path configurations to the double sum (such as the one shown on the left) will average out due to random phase cancellations. To leading order in D^{-1} , the dominant contribution to the product Π is provided by pairs of identical paths, whose retarded and advanced amplitudes are complex conjugate to each other. The argument is “semiclassical” in that quantum interference “loop corrections” (inset) will begin to matter at higher orders in a D^{-1} expansion. They are included in our construction below, but for the moment we concentrate on the leading semiclassical order.

For a system with a well-defined classical limit, this pair propagator defines the semiclassical representation of a classical transition process between different states ρ and μ . Ergodicity implies its independence of both the choice of μ and ρ , and of the propagation time $u_+ - t_+$. At the same time, the rigid locking between the advanced and retarded scattering paths requires the equality of propagation times $u_+ - t_+ = u_- - t_-$ or $u_+ - u_- = t_+ - t_-$. We thus anticipate

$$\Pi_{u_+u_-,t_+t_-} = \langle G_{u_+u_-}^{+-} G_{t_+t_-}^{+-} \rangle = \frac{1}{D\lambda} \delta_{\Delta t - \Delta u} \Theta(u - t). \quad (10)$$

Here, and throughout, δ_t are δ functions smeared over the inverse of the cutoff scale, λ , i.e., $\delta_0 = \lambda$. For discrete time dynamics, $\lambda = 1$ and δ is a Kronecker- δ . The Heaviside function implements the causality conditions $u_s > t_s$ and we introduced center of mass coordinates, $\Delta t = t_+ - t_-$ and $t = (t_+ + t_-)/2$. The normalizing prefactor follows from probability conservation $\Pi_{uu,tt} = 1$.

B. Symmetries

Equation (1) describes the evolution of the $G^{\pm\mp}$ components of the quasiclassical Green's function. To obtain complementary information for the diagonal blocks, $G^{\pm\pm}$, we now turn to symmetry arguments, here formulated in the continuous time framework for concreteness. (All conclusions carry over to discrete time evolution without changes.)

The present approach to quantum chaos understands the ergodic phase as a symmetry broken phase. The symmetry in question is that between the retarded and advanced dynamical generators $H \pm i\delta$, which are identical up to the infinitesimal causality increment $\pm i\delta$. In the energy domain, Fourier transform of Eq. (2) (according to the convention $f_\epsilon = \int dt e^{i\epsilon t} f_t$) yields

$$\langle G_{\epsilon\zeta}^{ab} \rangle = \delta^{ab} \delta_{\epsilon-\zeta} \frac{1}{D} \sum_{\mu} \langle \mu | \frac{1}{\epsilon + ia\delta - H} | \mu \rangle,$$

i.e., an expression that, for energies ϵ, ζ away from the poles of H (a discrete set of measure zero embedded into the continuum of energies), is independent of the sign of a . We may express this independence as the statement that the theory possesses a continuous symmetry under transformations $G \rightarrow TGT^{-1}$, where T is a two-dimensional matrix acting on the a indices.

This symmetry is spontaneously broken by the configuration average in the sense that

$$\begin{aligned} \langle G_{\epsilon\zeta}^{++} - G_{\epsilon\zeta}^{--} \rangle_H &= \delta_{\epsilon-\zeta} 2i \frac{\text{Im}}{D} \sum_{\mu} \left\langle \mu \left| \frac{1}{\epsilon + ia\delta - H} \right| \mu \right\rangle_H \\ &\equiv -\frac{2\pi i}{D} \delta_{\epsilon-\zeta} \rho(\epsilon), \end{aligned}$$

i.e., a nonvanishing difference between the averaged retarded and advanced Green's function, proportional to the average density of states, $\rho(\epsilon) = \Delta(\epsilon)^{-1}$, or the inverse of the level spacing. The average $\langle G \rangle_H$ “spontaneously breaks” the unitary invariance between the two causal sectors [17].

However, as we are in an effectively zero-dimensional framework (ergodicity), we expect Goldstone mode fluctuations to restore the symmetry via strong fluctuations. We also expect these fluctuations to become effective in the limit of times $t \gg t_H \sim \Delta^{-1}$ larger than the Heisenberg time. In this long time regime, the fine-grained pole structure of the spectrum on scales Δ is resolved, and the symmetry between the retarded and advanced sectors of the theory must become visible again.

For times $t \gg \lambda^{-1}$, we expect these Goldstone modes to be the dominant fluctuation degrees of freedom. We include them into the theory by parametrizing the quasiclassical Green's function—now interpreted as an effective integration variable—as

$$G_{tu}^{ab} \rightarrow -\frac{i}{\lambda} (T \tau_3 T^{-1})_{tu}^{ab} \equiv -\frac{i}{\lambda} Q_{tu}^{ab}, \quad (11)$$

(in the discrete case, $\lambda = 1$) where $(\tau_3)_{ab} = a\delta_{ab}$ is a Pauli matrix in causal space. Finally, $T = \{T_{tu}^{ab}\} \in \text{U}(2N)$ are generalizations of the exact two-dimensional symmetry matrices to bilocal matrices exhibiting slow dependence on time indices, where we consider time discretized into N steps. The saddle point $\propto \tau_3$ breaks this symmetry down to $\text{U}(N) \times$



FIG. 6. Left: identification of Goldstone mode generators with the paired fields entering the quasiclassical Green's function. Right: the pairing of Goldstone mode generators identified with the ergodic pair propagator with propagation times $u_+ - t_+ = u_- - t_-$.

$\text{U}(N)$, and the matrices Q span the associated Goldstone mode manifold $\text{U}(2N)/\text{U}(N) \times \text{U}(N)$. In this way, they resemble $2N$ -dimensional “spin” configurations continuously rotated away from the mean field τ_3 .

C. Low-energy fluctuations

In concrete terms, we represent fluctuations away from $T = \mathbb{1}$ in terms of the so-called rational parametrization [10]:

$$T = \begin{pmatrix} \mathbb{1} & B \\ -B^\dagger & \mathbb{1} \end{pmatrix}, \quad (12)$$

where $B = \{B_{tu}\}$. The matrix T^{-1} then assumes the form of a geometric series, and a straightforward resummation leads to

$$Q = \begin{pmatrix} \frac{1-BB^\dagger}{1+BB^\dagger} & -\frac{2B}{1+B^\dagger B} \\ -\frac{2B^\dagger}{1+BB^\dagger} & -\frac{1-B^\dagger B}{1+B^\dagger B} \end{pmatrix}. \quad (13)$$

(Readers familiar with spin coherent states [18] may notice the resemblance with a spin degree of freedom represented in stereographic projection, underscoring the magnetic analogy.)

A truncation at second order in Goldstone modes yields the Green's function as

$$G_{tu} = -\frac{i}{\lambda} \begin{pmatrix} \mathbb{1} - 2BB^\dagger & -2B \\ -2B^\dagger & -\mathbb{1} + 2B^\dagger B \end{pmatrix}_{tu} + \dots, \quad (14)$$

an expression establishing contact to the previously discussed semiclassical picture.

The identifications $G^{+-} \sim \lambda^{-1} B$ and $G^{-+} \sim \lambda^{-1} B^\dagger$ and Eq. (10) indicate that

$$\langle B_{u_+u_-} B_{t_+t_-}^\dagger \rangle = \Delta \delta_{\Delta t - \Delta u} \Theta(u - t), \quad (15)$$

i.e., a Goldstone mode “propagator,” physically identical to the semiclassical propagator, see also Fig. 6. The smallness of the scaling factor $\Delta \propto D^{-1}$ provides *a posteriori* justification for the lowest-order expansion of the Green's function in B .

The above analysis gets us into a position to predict physical observables in the ergodic regime. As an example, consider the spectral form factor (3), where Eqs. (1) and (2) enter, and $\langle \dots \rangle_c$ is the connected part of the ensemble average. Using Eq. (14), this becomes

$$\begin{aligned} K(t) &\sim \rho^2 \langle (BB^\dagger)_{t,0} (B^\dagger B)_{0,t} \rangle_c \\ &\sim \rho^2 \int_{t_+, t_-} \langle (B_{t_+} B_{t_-}^\dagger) (B_{0,t_+}^\dagger B_{0,t_-}) \rangle_c. \end{aligned} \quad (16)$$

Referring to Fig. 7 for a visualization, we finally use Eq. (15) for the B propagators to arrive at

$$K(t) \sim \int_0^t d\tau = t,$$

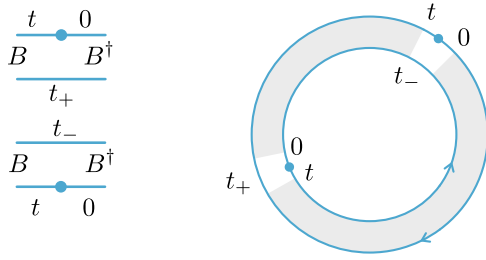


FIG. 7. The pairing of four Goldstone mode generators to two propagators leads to the time constraint $\tau \equiv t_- = t - t_+$, which leaves one free time integration, to be interpreted as the relative time mismatch, τ , at which the retarded and advanced time evolution traverse a common scattering path.

where we noted that the combination of δ and Θ functions leaves the integral over one time coordinate $\tau = t_- = t - t_+$ unconstrained. Physically, this time is the temporal offset between retarded and advanced pair propagation along identical scattering paths. Irrespective of this mismatch, the accumulated dynamical phases cancel out, and the summation over all these contributions yields the familiar linear “ramp.” Note that this result follows without calculation and irrespective of the detailed disorder model from symmetry and semiclassical arguments. (For the fixation of numerical factors, we refer to the quantitative analysis below.)

D. Qudit networks

With the above description of the single-qudit ergodic phase in place, we next discuss the influence of interqudit correlations. Consider neighboring qudits j with qudit states $|\mu\rangle, |\rho\rangle, \dots$, and k with $|\nu\rangle, |\sigma\rangle, \dots$, (from here on, we aim to be consistent in labeling variables pertaining to neighboring sites by adjacent letters of the Greek or Roman alphabet), each described by a low-energy effective theory as discussed above, and coupled by an interaction $V = \{V_{\mu\nu,\rho\sigma}\} = \exp(i\delta H)$ where δ is a timelike parameter chosen infinitesimally/finite in the description of continuous/discrete time dynamics, and a Gaussian distributed generator

$$\langle |H_{\mu\nu,\rho\sigma}|^2 \rangle \equiv \frac{\Lambda^2}{2D^2}, \quad (17)$$

with adjustable strength Λ . (All other products of matrix elements of H vanish upon averaging.)

1. Synchronization dynamics

These interactions are instantaneous in time, and they couple wave function amplitudes of identical causality, retarded, or advanced (see Fig. 8, left). Due to their Gaussian statistics, they must occur pairwise, coupling qudit amplitudes as indicated in Fig. 8, right. To leading order in D^{-1} , these are the dominant correlation processes. Formally, the Gaussian statistics penalizes semiclassical B propagators (indicated by shading) between statistically correlated interaction events by lost index summations. Physically, this suppression represents the loss of correlation under ergodic semiclassical propagation.

Specifically, the retarded *out* process shown on top is a fast succession of two interaction events between forward

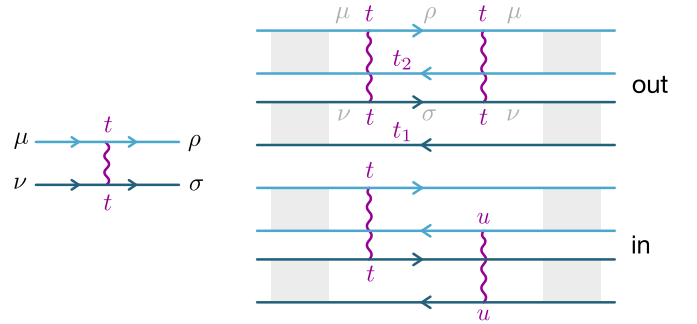


FIG. 8. Interaction vertex coupling the quantum states of two subsystems, here distinguished by differently hued line colors. Note that a physical interaction couples amplitudes of identical causality, retarded, or advanced, as indicated by the arrows.

propagating states at time t (the corresponding advanced analog is not shown). The times t_1 and t_2 of the two backward propagating states in the concerned semiclassical modes are arbitrary. Physically, this second-order interaction process describes the mutually decohering influence of the two subsystems onto each other: From the perspective of system 1, system 2 is an “environment” and coherence is lost via the *out* scattering process. As detailed below, the summation over these processes leads to an exponential damping of the pair of interacting semiclassical modes, at a golden-rule rate $\Gamma \sim \text{var}(V)\rho_2$, where $\rho_2 \sim D^2$ is the density of states of the two-qudit Hilbert space.

Now consider the *in* process, a pair of interaction events, one between two retarded, the other between two advanced amplitudes, as indicated at the bottom of the figure. The times of these two interactions, t and u , can be macroscopically different. However, unlike with the *out* process, where $t_2 - t_1$ remained unconstrained, the fixation of t and u removes all freedom in the relative propagation times of the two subsystems. As a consequence, the *in* process contributes only to two-subsystem quantum amplitudes for which the corresponding pair propagators are dynamically *synchronized* to realize these time constraints.

The *in* process competes with the *out* process in that it enters with opposite sign. (We label these two competitors by *in* and *out* because terms of this structure feature as in and out terms in the kinetic or quantum master equation [19] describing energy-resolved quantum transport.) The two processes cancel for the subset of synchronous particle-hole modes, implying that these do not suffer dynamical damping. Physically, these modes describe the joint propagation of two-state amplitudes $(\psi_{1t}^+ \psi_{2t}^+)$ and $(\psi_{1u}^+ \psi_{2u}^+)$ in the two-body chaotic Hilbert space; i.e., correlation by interaction collapses a system described by “independent” quasiparticle degrees of freedom $\psi_{1t_1}^+$ and $\psi_{2t_2}^+$ to one described by single two-body amplitudes $(\psi_{1t}^+ \psi_{2t}^+)$.

The iteration of this construction leads to a system-wide synchronization, with propagating L -particle modes $\psi_t^+ = (\psi_{1t}^+ \cdots \psi_{Lt}^+)$ and $\psi_u^- = (\psi_{1u}^- \cdots \psi_{Lu}^-)$. The single relative time coordinate $t - u$ remains unconstrained and enters the description of the L -body ergodic mode.

2. Symmetries versus interactions

We conclude this section with a discussion of symmetries in the interacting context. This point will become important when we turn to the nonperturbative physics of thermalization. It can be skipped at first reading.

The causal symmetry-breaking principle discussed in Sec. III B applies to any quantum system in a late-time ergodic phase. One may even consider it as a *definition* of quantum ergodicity. However, it is not easily exposed in many-body theory, with practical consequences for the description of quantum chaos at timescales approaching the Heisenberg time. To understand the problem, consider a Hamiltonian describing the correlation of just two qudits A and B , $H = H_A \otimes \mathbb{1} + \mathbb{1} \otimes H_B + H_{AB}$. Suppose we wanted to describe this system in a path or coherent state field integral language [8]. We then have the choice between two complementary approaches, each highlighting different aspects of the problem.

The first would consider $H = \{H_{\mu\nu,\rho\sigma}\}$ as a matrix in the space of two-qudit states $|\mu\rangle \otimes |\nu\rangle$. Defining (fermionic) annihilation operators $\{C_{\mu\nu}\}$ of these states, one may represent the Hamiltonian as $H = C_{\mu\nu}^\dagger H_{\mu\nu,\rho\sigma} C_{\rho\sigma}$. The path integral action in this representation reads

$$S[\Psi] = \int dt \bar{\Psi}(i\delta\tau_3 + \partial_t - H)\Psi,$$

where $\Psi_{\mu\nu} \leftrightarrow C_{\mu\nu}$ are Grassmann variables representing the two-state operators. A state doubling $\Psi = (\Psi^+, \Psi^-)$ accommodates branches of retarded and advanced evolution, distinguished by an infinitesimal shift $\pm i\delta$. In this language, the causal symmetry is represented as $\Psi \rightarrow T\Psi$, where $T \in U(2t)$ are the symmetry transformations discussed in Sec. III B. Ensemble averaging causes symmetry breaking, the Q -matrices representing its Goldstone modes.

However, in *many-body* quantum chaos, it is more natural to describe a Hamiltonian in a quasiparticle operator formulation: $H = a_\mu^\dagger H_{A\mu\rho} a_\rho + b_\nu^\dagger H_{B\nu\sigma} b_\sigma + a_\mu^\dagger b_\nu^\dagger H_{AB\mu\nu,\rho\sigma} b_\sigma a_\rho$, leading to a path integral with action

$$S[\Psi] = \int dt (\bar{\chi}(i\delta\tau_3 + \partial_t - H_A)\chi + \bar{\psi}(i\delta\tau_3 + \partial_t - H_B)\psi + \bar{\chi}\bar{\psi}H_{AB}\chi\psi). \quad (18)$$

Although our symmetry principle must still be present and physically effective, it is no longer visible in this representation. The origin of the problem lies in the application of single-particle coherent states in the construction of the many-body path integral. Tensor products of these states define separable many-body quantum states, i.e., states void of entanglement. It has been noticed before [20,21] that the representation of state histories in a basis of nonentangled states—which is technically possible due to the overcompleteness of coherent states—obstructs the description of strongly correlated phases. We here face this problem in the context of the ergodic phase of quantum chaos.

However, the alternative formulation in terms of two-body states outlined above, too, has its issues: In the separable limit $H_{AB} = 0$, there are *two* copies of causal symmetries, individually realized in the sectors A and B . Equation (18) implements these symmetries via continuous transformations $\chi \rightarrow T\chi$ and $\psi \rightarrow U\psi$ with independent T and U . However, the sym-

metry operation $T \otimes U$ is not represented in a manifest way in the language of two-state variables $\Psi = \{\Psi_{\mu\nu}^a\}$ containing only a single causal index. Again, we have the situation that a symmetry is present and physical (it leads to the factorization of the form factor $K(t) = t^2$ in the noninteracting limit), but not manifestly so.

We conclude that the starting point of either approach misses out on crucial symmetries. The best we can do is choose one of them and aim to make all symmetries emergent at some stage of the construction. Starting with the second quantized formulation, this will be our strategy in the rest of the paper.

IV. SINGLE QUDIT

In this section, we introduce the path integral formalism for a single qudit evolving under repeated application of a Haar-distributed Floquet operator U^n . As a test observable, we will consider its form factor $K(t) = \langle |\text{tr}(U^t)|^2 \rangle_U$ with the known result (7). The description of interacting systems will then only require gradual adjustments of the formalism introduced here.

A. Path integral

We begin with the formulation of a real-time path integral effectively trotterizing the evolution $|\text{tr}(U^t)|^2 = \text{tr}(UU \dots U)\text{tr}(U^\dagger U^\dagger \dots U^\dagger)$ in terms of $2t$ resolutions of unity between consecutive Floquet operators U ($a = +$) and adjoints $U^\dagger$ ($a = -$). Referring to Appendix A for details, we achieve this by introducing a set of Grassmann variables $\{\psi_{\mu n}^a\}$ (which in the framework of the coherent state path integral [8] would represent annihilation operators of states $|\mu\rangle$ evolving forward or backward in discrete time). Next, define the “partition sum”

$$1 = Z \equiv \int D\psi e^{-S[U,\psi]},$$

$$S[U, \psi] = \bar{\psi}^+(1 - T_- U)\psi^+ + \bar{\psi}^-(1 - T_+ U^\dagger)\psi^-, \quad (19)$$

where $\int D\psi \equiv \prod_{\mu n} \int d\bar{\psi}_{\mu n}^a d\psi_{\mu n}^a$, all indices are summed over, and $T_\pm$ are one-time step translation operators, $(T_\pm f)_n = f_{n\pm 1}$. Noting that the operator $1 - T_- U$ has t -dimensional unit matrices on the diagonal and U on the next-leading diagonal, we conclude that it has unit determinant, explaining the normalization $Z = 1$. Introducing preexponential source terms, we obtain the SFF as

$$K(t) = \int D\psi \int DU e^{-S[U,\psi]} (\psi_{t\mu}^+ \bar{\psi}_{0\mu}^+) (\psi_{0\rho}^- \bar{\psi}_{t\rho}^-),$$

conveniently without the need for normalizing denominators. To understand this relation without reference to its construction, just note that the Gaussian integral over ψ^+ yields the matrix element $\langle \mu t | (1 - T_- U)^{-1} | \mu 0 \rangle$. A straightforward matrix inversion gives $\langle \mu | U^t | \mu \rangle$ as required.

Note that the current functional integral construction, uncharacteristically for a theory with disorder, requires no supersymmetry, replicas, or related overhead to cancel unwanted determinants. The reason is that the discrete time functional integral [Eq. (19)] produces unit-normalized partition sums out of the box, for each realization of the disorder.

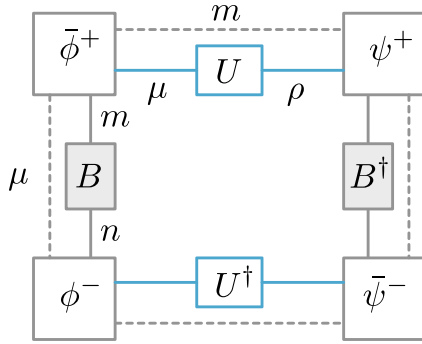


FIG. 9. Color-flavor transform visualized: Bilinears of field variables transforming under U 's acting on their Hilbert space (“color”) indices and contracted over the internal (“flavor”) indices get mapped onto dual bilinears where the role of color and flavor indices is exchanged, flavor matrices B assuming the role of the U 's.

In this regard, it resembles the real-time Keldysh formalism of quantum transport [19]. However, unlike with the infinite continuous time domains of Keldysh theory, our current formalism is defined for a finite dimensional field space, which makes for a significant technical simplification.

B. Haar average

The next construction step is the integral over the Haar distribution of the matrices U . The result can be expressed as the integral transform

$$\int dU e^{\bar{\phi}^+ U \psi^+ + \bar{\psi}^- U^\dagger \phi^-} = \int dB e^{-D \text{tr} \ln(1 + BB^\dagger) - \bar{\phi}^+ B \phi^- + \bar{\psi}^- B^\dagger \psi^+}, \quad (20)$$

where the integral on the right-hand side is over complex t -dimensional matrices $B = \{B_{mn}\}$. Equation (20) is a variant of the color-flavor transformation (CFT) introduced in Ref. [22]. This exact identity plays a crucial role in our further analysis—it is to Haar random unitary operators what the celebrated Hubbard-Stratonovich transform is to Gaussian Hermitian operators.

Unfortunately, the mathematical proof of Eq. (20) is not easy. The original derivation of Ref. [22] relies on the concept of coherent states associated with Lie groups and algebras [18], while more elementary derivations [23] are lengthy. We therefore refer to these references for the proof, and instead discuss the intuitive interpretation of Eq. (20).

The meaning of the CFT is visualized in Fig. 9 in a tensor network representation. Before the transform, the field variables are contracted through the horizontal links: They carry a nontrivial representation of the U 's in Hilbert space, indexed by the μ “color” indices. Their internal m “flavor” indices are contracted trivially.

The Haar average over U 's eliminates all such configurations, except for those where the color indices are contracted to form singlet states. This passage to (an integral over) singlet states is described by the right-hand side of the equation. The transform effectively reverses the role of the bonds: a trivial singlet representation in color space in exchange for a non-

trivial one in flavor space, where the B -flavor matrices assume the role of the U -color matrices. The advantage of the passage to the dual representation is that the B -matrices turn out to be slowly fluctuating in the space of physical time-flavor indices, and hence are candidate degrees of freedom of an effective low-energy theory.

We conclude our discussion of the CFT with a bit of technical fine print: The measure $dB = \prod_{mn} dB_{mn} d\bar{B}_{mn} \det(1 + BB^\dagger)^{-2t}$ in Eq. (20) is normalized in such a way that for $\psi, \phi = 0$ the integral on the right-hand side computes the volume of the unitary group, $\int dU$. Its specific form identifies the B integral as one over the symmetric space $U(2t)/U(t) \times U(t)$ —heuristically, the t -dimensional generalization of a sphere.

In practical terms, the role of the measure is to cancel “disconnected terms” in the functional integral. These show as spurious contributions to its perturbative expansion containing free running dummy time summations. (They are analogous to terms containing excessive replica summations, vanishing in the zero replica limit.) As we are going to demonstrate below, such contributions are easily identified, and as long as we exclude them, the presence of the measure can be ignored.

With the identifications $\bar{\phi}^+ = \bar{\psi}^+ T_-$ and $\phi^- = T_+ \psi^-$, the application of the CFT gets us to the representation

$$Z = \int DBD\psi e^{-S[B, \psi]},$$

$$S[B, \psi] = D \text{tr} \ln(1 + BB^\dagger) + \bar{\psi} \underbrace{\begin{pmatrix} 1 & B_T \\ -B^\dagger & 1 \end{pmatrix}}_{G^{-1}[B]} \psi, \quad (21)$$

where $\bar{\psi} = (\psi^+, \psi^-)$ and

$$(B_T)_{mn} \equiv (T_- B T_+)_{mn} = B_{(m-1)(n-1)}.$$

The merit of this transformation is that it trades the average over high-dimensional random color fields, U , for the simpler integration over flavor degrees of freedom, where the denotation “ B ” indicates that the flavor fields will assume the role of the Goldstone mode generators discussed in Sec. III. To understand this point, execute the Gaussian ψ integral to obtain

$$Z = \int DB e^{-S[B, \psi]},$$

$$S[B, \psi] = D \text{tr} \ln(1 + BB^\dagger) - D \text{tr} \ln(1 + B_T B^\dagger). \quad (22)$$

This representation demonstrates that fields $B_T \simeq B$ with slow time dependence are soft modes with vanishing action. In other words, they are the Goldstone modes of our symmetry-breaking phenomenon.

C. Luttinger-Ward functional

However, to prepare for the subsequent extension to the interacting context, we take the detour to not integrate over the ψ 's right away. Instead, we first introduce the quasiclassical Green's functions G as integration variables to obtain an integral representation interchangeably known as Luttinger-Ward [12] or $G\Sigma$ functional [11]. We wish to enforce the

identification

$$-\frac{ia}{D} \sum_{\mu} \psi_{\mu m}^a \bar{\psi}_{\mu n}^b \equiv G_{mn}^{ab}, \quad (23)$$

i.e., Eq. (1), for discrete time indices. To this end, we introduce a Lagrange multiplier,

$$S[B, \psi] \rightarrow S[B, G] + D \operatorname{tr} \left(\Sigma \left(G - \frac{i}{D} \tau_3 (\psi \bar{\psi}) \right) \right), \quad (24)$$

where $\Sigma = \{\Sigma_{mn}^{ab}\}$ are matrices sharing the index structure of G . The quadratic integration over ψ then produces the “ $G\Sigma$ ” action

$$S[G, \Sigma, B] = D (\operatorname{tr} \ln(1 + BB^\dagger) - \operatorname{tr} \ln(\mathcal{G}^{-1}(B) - i\Sigma \tau_3) + \operatorname{tr}(\Sigma G)). \quad (25)$$

In the $G\Sigma$ representation, the SFF assumes the form

$$K(t) = D^2 \langle G_{t1}^{++} G_{1t}^{--} \rangle_{B, G, \Sigma},$$

where $t = 1$ is set to the starting time step for the discrete model.

D. Stationary phase analysis

To make progress with this expression, we turn to a stationary phase analysis, stabilized by the largeness of D . Variation of the action yields the saddle point equations

$$\begin{aligned} 0 &= \delta_\Sigma S[\bar{G}, \bar{\Sigma}, B] = D(\bar{G} + i\tau_3(\mathcal{G}^{-1}(B) - i\bar{\Sigma}\tau_3)^{-1}), \\ 0 &= \delta_G S[\bar{G}, \bar{\Sigma}, B] = D\bar{\Sigma} = 0. \end{aligned} \quad (26)$$

Their solutions

$$\bar{\Sigma} = 0, \quad \bar{G} = -i\tau_3 \mathcal{G}(B), \quad (27)$$

have a status similar to the self-consistent Born equations describing single Green’s functions in the presence of Hermitian disorder. In that case, averaging over configurations generates an effective self-energy describing the attenuation of an averaged propagator due to impurity scattering. However, for random unitary operators, the vanishing of all moments $\langle U^m \dots \rangle_U = 0$ implies the absence of average self-energies, i.e., the first of the two equations (26). The second equation is solved by inversion of the block matrix $\mathcal{G}^{-1}$ defined in Eq. (21) (e.g., by geometric series summation):

$$\bar{G} = \frac{i}{2}(Q_T - \tau_3), \quad (28)$$

where the matrix Q_T is defined in Eq. (13), except that the matrix $B \rightarrow B_T$ is to be replaced by the one-step time translated B_T . (The matrix $B^\dagger$ remains as it is.)

At this point, we have established that the Green’s function is governed by the Goldstone modes introduced in Sec. III. Comparing to the previously discussed matching $G = i\lambda Q$, the presence of the constant matrix τ_3 in Eq. (28) and the absence of a dimensionful energy scale λ are consequences of the discrete time definition of the present model.

E. Fluctuation action

We next need to construct a fluctuation action describing the long time dynamics of our system. In principle, the term

“fluctuations” refers to deviations $(\delta\Sigma, \delta G)$ around the mean field $(0, \bar{G})$, as well as to fluctuations of B on the saddle point manifold parametrized by $\bar{G}$. It is straightforward to see that fluctuations of the former type do not exist in the present simple model. (Formally, this follows from the fact that the action (25) contains G only in linear order.) Turning to the more interesting B fluctuations, the substitution of $\mathcal{G}^{-1}(B)$ into the “ $\operatorname{tr} \ln$ ” leads to Eq. (22), i.e., a representation of the functional that a straightforward integration over ψ would have produced without the need for a stationary phase program. (However, the merit of the $G\Sigma$ representation will become evident once we turn to interactions.)

1. Nonlinear σ model

Introducing the discrete derivative in the center time dependence,

$$dB_{nm} \equiv (B - B_T)_{nm} = B_{nm} - B_{(n-1)(m-1)},$$

the next step is an expansion of the logarithmic action to leading order in this difference:

$$S_0[B] \equiv D \operatorname{tr} \left(\frac{1}{1 + B^\dagger B} B^\dagger dB \right). \quad (29)$$

To establish contact with the literature, we note that a straightforward reordering of terms (always to leading order in the difference operator d) brings the action into the form

$$S_0[T] = \frac{D}{2} \operatorname{tr}(dT \tau_3 T^{-1}), \quad (30)$$

which is a discrete time version of the nonlinear σ model [19].

While Eq. (29) will be the representation primarily used in our further construction of the theory, Eq. (30) has the advantage of making the symmetries of the model manifest: Transformations $T \rightarrow T_0 T$, where $(T_0)_{nm} = T_0 \delta_{nm}$ does not depend on time indices, leave the action invariant.

A closer analysis [24] reveals the existence of entire families of saddle points in addition to the standard configuration $Q = \tau_3$, i.e., configurations of the Q -matrix around which the expansion in fluctuations starts already at quadratic order. Mathematically, these saddles are obtained by permutation of a finite number of the t eigenvalues $+1$ of τ_3 with eigenvalues -1 . They are perturbatively inaccessible in terms of the B expansion. In the simplest case $t = 1$, where the symmetric space $U(2t)/U(t) \times U(t)$ is literally a two-sphere, the single flipped saddle corresponds to the south poles, opposite to the τ_3 north pole, and is known as the AA saddle [16]. Within the framework of a supersymmetric variant of the model, Ref. [16] showed that Gaussian integration around this saddle point gave rise to the late-time plateau of the form factor [Eq. (49)]. Subsequently, Ref. [24] demonstrated that an analogous construction, generalized to the combinatorially large number of AA saddles present for $t > 1$, leads to the same saturation of the late-time form factor.

In the present work, however, we will largely sidestep the complications associated with the AA saddles. We focus primarily on the regime $t < D$, where they do not contribute, and only comment on their role in situations where the late-time dynamics becomes relevant.

2. Quadratic action

To establish contact to the semiclassical principles discussed in Sec. III, consider the quadratic expansion

$$S_0^{(2)}[B] \equiv D \text{tr}(B^\dagger dB). \quad (31)$$

This Gaussian weight defines the discrete time propagator of the theory,

$$\langle B_{n_+, n_-} B_{m_+, m_-}^\dagger \rangle \equiv \Pi_{n, m}, \quad (32)$$

as the inverse

$$D(d\Pi)_{n, m} = D(\Pi_{n, m} - \Pi_{n-1, m}) = \delta_{n, m},$$

where $n \equiv (n_+, n_-)$, $n-1 = (n_+ - 1, n_- - 1)$, and $\delta_{n, m} = \delta_{n_+, m_+} \delta_{n_-, m_-}$. It is straightforward to check that this relation is satisfied by the solution

$$\Pi_{n, m} = \frac{1}{D} \delta_{\Delta n, \Delta m} \Theta(\bar{n} - \bar{m}), \quad (33)$$

where $\Delta n = n_+ - n_-$ and $\bar{n} = (n_+ + n_-)/2$, and likewise for the m indices. $\Theta(0) = 1$ is used. In other words, Π is the discrete time version of the semiclassical ergodic propagator [Eq. (15)].

F. Spectral form factor

The computation of physical observables now proceeds along the lines of a straightforward protocol: (1) express their coherent state ψ representation through the quasiclassical Green's function, (2) expand the latter to any required order in B fluctuations (including saddle points beyond the standard saddle $G = -i\tau_3$ if nonperturbative information is required), and (3) do the B integrals.

Considering the spectral form factor as an example, step (1) leads to the representation of Eq. (3), and step (2) to the discrete time analog of Eq. (16),

$$K(t) = D^2 \langle (BB^\dagger)_{t1} (B^\dagger B)_{t1} \rangle. \quad (34)$$

With the matrix products resolved as $(BB^\dagger)_{t1} = B_{tn} B_{n1}^\dagger$, the Gaussian contraction according to Eq. (32) yields [25]

$$K(t) = D^2 \sum_{n, m} \Pi_{tn, 1m} \Pi_{mt, n1} = t.$$

As mentioned above, the extension to times $t > D$ requires integration over nonstandard saddle points. For the execution of this program in the Fourier conjugate energy representation, we refer to Ref. [24] and for the adaption of this approach to the time domain to a forthcoming publication.

V. BRICKWALL MODEL

With the path integral of the single qudit in place, the extension to the network shown in Fig. 10 is straightforward. There, the blue boxes define an array of L qudits of dimension D , governed by statistically independent unitaries, each repeated periodically in vertical discrete time. Neighboring qudits are coupled by tensor products of two-qudit unitaries $V = V_{\mu\rho, \nu\sigma}$, with $V = \exp(iH)$ and H distributed according to Eq. (17) (with dimensionless Λ).

We label each product $UV_{\text{pink/green}}$, by discrete time steps m , with pink/green interactions for even/odd m . The two

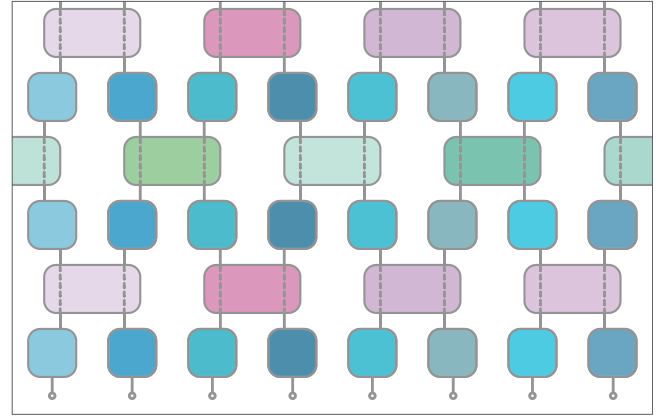


FIG. 10. A brickwall design describing the alternating application of random one- and two-qudit unitary operators. Statistically independent realizations are indicated by color coding. Note the repeated application of identical operators in vertical (time) direction, defining a four-step Floquet dynamics. The limiting case of absent correlations is indicated by dashed lines.

stages of each step are represented by a “stage index” $c = \pm 1$, where the passage from $(m-1, +1)$ to $(m, -1)$ is governed by the tensor product of single-qudit operators U , and that from $(m, -1)$ to $(m, +1)$ by interactions, V . On the retarded contour, the sequence is traversed in reverse order,

$$\begin{aligned} + \quad & \dots \rightarrow (m-1, +1) \xrightarrow{U} (m, -1) \xrightarrow{V} (m, +1) \rightarrow \dots \\ - \quad & \dots \leftarrow (m-1, +1) \xleftarrow{U^\dagger} (m, -1) \xleftarrow{V^\dagger} (m, +1) \leftarrow \dots \end{aligned}$$

A. Path integral

We obtain the path integral for this protocol by straightforward generalization of the previous construction. Apart from site indices $j = 1, \dots, L$ and the stage index c , it contains the two-audit interaction as a new element. With Grassmann variables $\psi_{j\mu mc}^a$, its unit-normalized partition sum assumes the form $Z = \int D\psi \exp(-S[\psi])$, where the action is given by

$$\begin{aligned} S[\psi] &= \bar{\psi} \psi + S[U, \psi] + S[V, \psi], \\ S[U, \psi] &= \bar{\psi}_{j-1}^+ T_- U_j \psi_{j1}^+ + \bar{\psi}_{j1}^- U_j^\dagger T_+ \psi_{j-1}^-, \\ S[V, \psi] &= -i \bar{\psi}_a^a \psi_{-a}^a - \bar{\psi}_{ja}^a \bar{\psi}_{ka}^a \\ &\quad \times \left(-iaH + \frac{1}{2}H^2 \right)_{jk} \psi_{k-a}^a \psi_{j-a}^a. \end{aligned} \quad (35)$$

Referring to Appendix A for details, $\bar{\psi} \psi \equiv \sum \bar{\psi}_{j\mu nc}^a \psi_{j\mu nc}^a$ is the Gaussian weight resulting from the insertion of Grassmann Kronecker- δ 's at each time step. The action $S[U, \psi]$ describes single-qudit evolution at steps $(m-1, 1) \rightarrow (m, -1)$, where $(T_\pm f)_m = f_{m\pm 1}$ as before. (Note that in this action, the stage indices are determined by the $a = \pm 1$ causal index; hence, the double appearance of a as super- and subscript.) Finally, $S[V, \psi]$ describes interactions $(m, -1) \rightarrow (m, 1)$ with $V = \exp(-iH) \simeq \mathbb{1} - iH - H^2/2$ expanded to second order in H . The second term in S_V is the “normal-ordered” quartic Grassmann representation of the two-body operators H and H^2 , where the expansion to second order is required

to describe the time-local second-order interaction processes discussed in Sec. III, and in more detail below.

1. Averaging over randomness

Our system contains two sources of randomness, the Haar-distributed unitaries U and the Gaussian distributed H . Averaging over U yields the generalization of Eq. (21) to multiple sites,

$$S[U, \psi] \rightarrow S[B, \psi] = D \sum_j \text{tr} \ln(1 + B_j B_j^\dagger) + \sum_j (\bar{\psi}_{j-1}^+ B_{Tj} \psi_{j-1}^- - \bar{\psi}_{j1}^- B_j^\dagger \psi_{j1}^+). \quad (36)$$

Averaging over H amounts to a straightforward Gaussian integral with variance (17). The result reads

$$S[V, \psi] \rightarrow S_{\text{int}}[\psi] \equiv S_o[\psi] + S_i[\psi],$$

$$S_o[\psi] = \frac{\Gamma}{2} \sum_{(j,k),m,a} ((\psi \bar{\psi})_k \odot (\psi \bar{\psi})_j)_{m-a,ma}^{a,a},$$

$$S_i[\psi] = \frac{\Gamma}{2D^2} \sum_{(j,k),mn,ab} ab ((\psi \bar{\psi})_j \odot (\psi \bar{\psi})_k)_{m-a,nb}^{a,b} \times ((\psi \bar{\psi})_j \odot (\psi \bar{\psi})_k)_{n-b,ma}^{b,a}, \quad (37)$$

where we use the Hadamard product notation [Eq. (5)] for field bilinears

$$(\psi \bar{\psi})_j \equiv \psi_{j\mu} \bar{\psi}_{j\mu},$$

$$((\psi \bar{\psi})_j)_{mc,nd}^{a,b} \equiv \psi_{j\mu mc}^a \bar{\psi}_{j\mu nd}^b, \quad (38)$$

and the sum is over pairs of nearest-neighbor sites (j, l) subject to the condition that on odd/even time slices m different site pairings are being summed over. We also defined

$$\Gamma \equiv \frac{\Lambda^2}{4}. \quad (39)$$

Note the golden-rule-like structure of the interaction constant as a product of the effective dimensionless interaction strength, $(\Lambda)^2/2D^2$, multiplied by the two-body density of states D^2 of final states.

While the first contribution to the interaction, S_o , results from the averaging of the exponent $\frac{1}{2} \langle \bar{\psi} H^2 \psi \rangle$, the second, S_i , is obtained as $\frac{1}{2} \langle (\bar{\psi} H \psi)^2 \rangle$. A graphical interpretation of these two interaction vertices is indicated in Fig. 11, in a color coding matching that of the earlier Fig. 8. The “out” vertex S_o , visualized on top, describes virtual two-body scattering from two-qudit (j, k) pair states $|\mu, \nu\rangle$ into the transient states $|\rho, \sigma\rangle$ and back, where the identity of initial and final states is required by the Gaussian correlation of matrix elements. This process is instantaneous in the discrete time index m . By contrast, the “in” vertex represents the statistical correlation of scattering events taking place at different times m, n . These two scattering events may either involve propagators of opposite (left) or identical (right) causality. The latter will turn out to be negligible to leading order in the D^{-1} expansion.

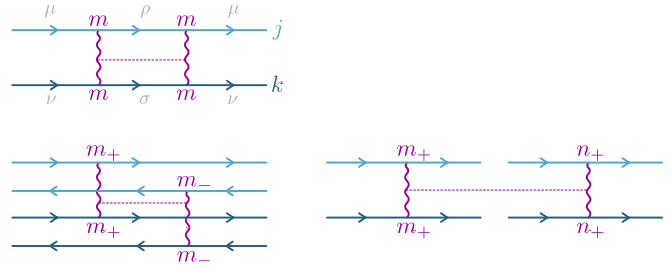


FIG. 11. Top: The vertex S_o in Eq. (37). The dashed line indicates the statistical average with second moment (17), enforcing a Hilbert-space structure as indicated. The bottom-left block represents the instantaneous second-order interaction event at discrete time m . Only fields of identical causality (indicated by the direction of arrows) contribute to this “self-energy vertex.” Bottom-right: Eighth order (in ψ) vertex S_i describing the correlation of scattering processes involving states of opposite (left) or identical (causality) at different discrete times as indicated. (Our subsequent analysis will show that the vertex shown in the bottom-left does not contribute at leading order in the D^{-1} expansion.)

2. Luttinger-Ward functional

The nonlinearity of S_i in ψ implies that we can no longer integrate over these variables in closed form. However, at this point the advantage of the $G\Sigma$ construction becomes evident: The interaction contains ψ in the form of color singlets [Eq. (38)]. Lagrange multipliers [Eq. (24)] implemented locally at each site allow us to trade all of these for quasiclassical Green’s functions (23). The result of this replacement reads as

$$S_o[G] = -\Gamma D^2 \sum_{(j,k),n,a} (G_k \odot G_j)_{n-a,na}^{aa},$$

$$S_i[G] = \Gamma D^2 \sum_{(j,k),nm,ab} ab (G_j \odot G_k)_{m-b,na}^{ba} (G_j \odot G_k)_{n-a,mb}^{ab}. \quad (40)$$

Adding to these terms, the noninteracting part of the action, comprising the B -matrices and the $G\Sigma$ -Lagrange multiplier structure, we obtain

$$S[G, \Sigma, B] = D \sum_j (\text{tr} \ln(1 + BB^\dagger) - \text{tr} \ln(\mathcal{G}^{-1}(B) - i\Sigma\tau_3) + D \text{tr}(\Sigma G))_j + S_{\text{int}}[G]. \quad (41)$$

Here, $\mathcal{G}^{-1}(B)$ is a 4×4 block matrix in the tensor product of causal and stage index space. It is implicitly defined through the quadratic ψ -action in Eqs. (35) and (36), and explicitly in Appendix B [Eq. (B1)]. The straightforward inversion of this matrix leads to

$$\mathcal{G}(B) = \frac{1}{2}(\tau_3 Q - 1) \otimes J_2 + \text{const}, \quad (42)$$

where $Q(B)$ is defined in Eq. (13) and $(J_2)_{cd} = 1$ in stage space. The constant matrix $\text{const} = -\frac{1}{2}((1 + \tau_3) \otimes \sigma^+ + (1 - \tau_3) \otimes \sigma^-)$ with $\sigma^\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ in stage space will not feature in our further discussion.

For later reference, we note that in microscopic path integral variables the form factor generalized to L qudits assumes

the form

$$K(t) \equiv \left\langle \prod_j \psi_{j1}^+ \bar{\psi}_{j1-1}^+ \psi_{j1-1}^- \bar{\psi}_{j1}^- \right\rangle,$$

where $c = \pm 1$ are stage indices. By design of the path integral, this correlation function computes the discrete time evolution between states $|\mu_1\rangle \otimes \cdots |\mu_L\rangle$ and the same final state, summed over all Hilbert-space amplitudes, in retarded (first) and advanced (second factor) order. The corresponding G representation is given by

$$K(t) \equiv (-D^2)^L \left\langle \prod_j G_{j,t1}^{++} G_{j,1t}^{--} \right\rangle. \quad (43)$$

An expansion of the Green's functions to leading quadratic order in B fluctuations generalizes the single qudit [Eq. (16)] to

$$K(t) \approx (-D^2)^L \prod_j \langle (B_{in} B_{n1}^\dagger B_{1m}^\dagger B_{mj})_j \rangle_B, \quad (44)$$

with discrete time indices independently summed for each j .

3. Stationary phase analysis

Following the same protocol as in the single-qudit case, we proceed by stationary phase analysis, where the generalization of equations (26) to the action [Eq. (41)] reads as

$$0 = \delta_{\Sigma_j} S[\bar{G}, \bar{\Sigma}, B] = D(\bar{G}_j + i\tau_3(\mathcal{G}^{-1}(B_j) - i\bar{\Sigma}_j \tau_3)^{-1}),$$

$$0 = \delta_{G_j} S[\bar{G}, \bar{\Sigma}, B] = D\bar{\Sigma}_j + \delta_{G_j} S_{\text{int}}[\bar{G}].$$

At first sight, the presence of S_{int} appears to significantly complicate these equations. However [cf. the single site solution, Eq. (27)],

$$\bar{\Sigma} = 0, \quad \bar{G}_j = -i\tau_3 \mathcal{G}(B_j) \quad (45)$$

at $B_j = 0$ continue to be stationary configurations of the theory. Fluctuations around this configuration start at quadratic order, as we are going to demonstrate momentarily. The physical reason for the robustness of this saddle point is easiest to see within the continuous time framework, and we postpone its discussion to Sec. VII. However, unlike with the noninteracting theory, large B fluctuations away from the semiclassical limit, $B = 0$, do not trivially leave the interaction invariant. Identifying which of these are admissible in the interacting context will be an essential part of our analysis below. For the time being, we proceed by expansion around the saddle point defined above.

B. Interactions

In this central section, we quantitatively describe the synchronization dynamics discussed qualitatively in Sec. II. In the absence of interactions, the form factor [Eq. (43)] factorizes and the execution of L decoupled path integrals yields $K(t) = (K_1(t))^L$. To include interactions, we substitute Eq. (45) into the interaction [Eq. (40)] and expand to leading, quartic order in the B generators. Referring to Appendix B for details, we obtain

$$S_{\text{int}}^{(4)}[B] = \Gamma D^2 \sum_{(j,k),n,m} (B_{Tm+m-} B_{m-}^\dagger)_j (B_{Tn+n-} B_{n-}^\dagger)_k \times (\delta_{n+m+} + \delta_{n-m-} - 2\delta_{n+m+} \delta_{n-m-}), \quad (46)$$

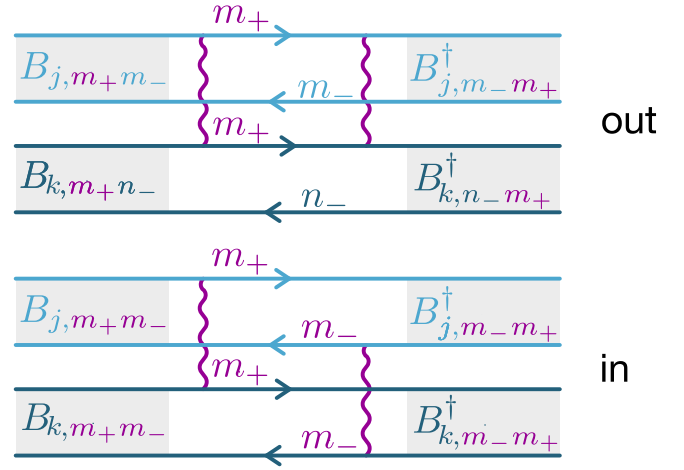


FIG. 12. The interaction vertex expanded to lowest order in the Goldstone mode generators of two neighboring sites. The locking of temporal indices enforced by the instantaneous nature of the interaction is indicated in color code. Note that the out term (top) leaves two time variables unconstrained, while the in term leads to full synchronization.

where the first two (the third) Kronecker- δ originate from the out (in) term (cf. Fig. 12).

The crucially important feature of this vertex is its vanishing on temporally synchronized configurations $(B_j \odot B_k)_{m+m-} = B_{j,m+m-} B_{k,m+m-}$. In Appendix D2, we show that this vanishing extends to all orders in the expansion in Goldstone mode generators. The conclusion of this discussion is that

$$X_{mn} = (\odot_{j=1}^L B_j)_{mn} = \prod_{j=1}^L B_{j,mn} \quad (47)$$

are collective field amplitudes commuting with all interaction vertices. The single generator $X = \{X_{mn}\}$ then parametrizes a Q -matrix describing the ergodic phase of the coupled system.

In the following, we describe the crossover dynamics governing the passage from the short time limit of decoupled form factors $K_1^L(t)$ to the late-time limit with $K_L(t)$. To this end, we focus on the interaction vertex expanded to quartic order in B . Higher-order contributions are suppressed in inverse powers of D (cf. the discussion of Sec. III). The goal is to sum over an infinite series of vertex insertions, physically representing the competition of in and out scattering processes discussed above. While this may look like a complicated task, the triviality of the ergodic propagator [Eq. (33)] provides sufficient tailwind to execute it in closed form. In concrete terms, we proceed along a succession of four steps, executed in detail in Appendix C:

(1) Hubbard-Stratonovich decoupling of the interaction by introduction of a time-bilocal auxiliary field, $\phi_{jn+n-} \equiv \phi_{jn}$.

(2) The quadratic B -action including this field then reads (symbolic notation) $\sum_{jn} B_{jn}^\dagger (d - \phi_{j,n}) B_{jn}$, with the discrete time derivative $dB = B - B_T$. Reminiscing a scalar potential coupling, it suggests removing ϕ from the action by a time-dependent gauge transformation. With the phase factor $\Theta_n = \sum_m \phi_{n-m}$, this is achieved by the change of variables $B_n \rightarrow \exp(\Theta_n) B_n$.

(3) This transformation removes the interaction potential from the action. However, it reappears in operators representing correlation functions in the form of gauge-phase factors containing discrete time integrals over Θ_n .

(4) The final Gaussian integration over these variables then effectively sums over interaction processes to infinite order in perturbation theory.

C. Form factor (semiclassical)

Applied to the form factor [Eq. (44)], the protocol above leads to

$$K(t) \rightarrow D^{2L} \sum_{\{m,n\}} \prod_j \langle (B_{tn} B_{n1}^\dagger B_{1m}^\dagger B_{mt})_j \rangle_B \times \langle e^{\sum_j (\Theta_{tm} - \Theta_{1m} - \Theta_{n1} + \Theta_{nt})_j} \rangle_\phi, \quad (48)$$

with the abridged notation $(B_{tn})_j \equiv B_{jtn_j}$ or $(\Theta_{tn})_j = \Theta_{jtn_j}$, etc. This expression exemplifies how the gauge-phase factors factorize correlation functions into an interaction contribution times a factor describing the ergodic dynamics of the individual systems. As in our discussion of the noninteracting qudits (cf. Sec. IV F), the role of the latter is to enforce constraints $t - n_j = m_j$ on the intermediate time variables. The final integration over the Hubbard-Stratonovich phases detailed in Appendix C then leads to the result

$$K(t) = \sum_{\{n\}} e^{-S[n]}, \quad S[n] = \Gamma t \sum_{\langle j,k \rangle} (1 - \delta_{n_j, n_k}), \quad (49)$$

obtained earlier in Ref. [5] for systems of qudits correlated by diagonal matrices of randomly fluctuating phases. It was there interpreted as the partition sum of a t -state Potts model, i.e., a system of L fictitious spins, n_j of size t , subject to nearest-neighbor interaction $S[n]$. In this reading, the t configurations of the fully synchronous/magnetized state $n = n_1 = \dots = n_L$ with action $S[n] = 0$ contribute the long time asymptotic $K(t) \rightarrow t$. More generally, the summation over all configurations can be performed using the representation $K(t) = \text{tr}(T^L)$, where T is the transfer matrix of the corresponding Potts model. In the present case, T is a t -dimensional matrix with unit diagonal entries and uniform off-diagonal entries equal to $e^{-\Gamma t}$. Such a matrix has a single nondegenerate eigenvalue $\lambda_0 = 1 + (t-1)e^{-\Gamma t}$, associated with the uniform eigenvector, and a $(t-1)$ -fold degenerate eigenvalue $\lambda_1 = 1 - e^{-\Gamma t}$, associated with eigenvectors orthogonal to it. Substituting the eigenvalues into $K(t) = \lambda_0^L + (t-1)\lambda_1^L$ directly yields

$$K(t) = (1 + (t-1)e^{-\Gamma t})^L + (t-1)(1 - e^{-\Gamma t})^L, \quad (50)$$

which for $t \gg 1$ simplifies to Eq. (6), interpolating between $K(t) \approx t^L$ for $\Gamma t \ll 1$ and $K(t) \approx t$ for $\Gamma t \gg 1$.

D. Form factor (nonperturbative)

Equation (6) is obtained in lowest-order perturbation theory in the D^{-1} expansion of the theory. There are two timescales at which these results need refinement. The first are times $t \sim D$ comparable to the Heisenberg time of individual qudits, where their dynamics is no longer adequately described by lowest-order expansion in the qudit pair propagators [Eq. (33)]. Both in applications and numerical

simulations, D need not be particularly large (we use $D = 12$ in the largest systems $L = 5$ of our numerical analysis), implying that substantial corrections at $t \sim D$ timescales can be practically relevant. Below, we will reason that the extension of the analysis to $t \sim D$ nonperturbative fluctuations requires a replacement $t \rightarrow K_1$ in the first bracket on the right-hand side of Eq. (6).

The second is the much larger Heisenberg time of the full system, $t \sim D^L$. This scale corresponds to the smallest energy of the system, its many-body level spacing, and for sizable interaction scales, is embedded deeply into the ergodic regime, $D^L \gg \Gamma^{-1}$. The generalization of the analysis to nonperturbative fluctuations at this timescale amounts to a replacement $t \rightarrow K_L(t)$ in the second bracket at the right-hand side of Eq. (6), so that we arrive at the result [Eq. (8)]. In view of the fact that this generalization of Eq. (6) is relatively obvious (but requires substantially more computational overhead), readers primarily interested in the generalization to other system classes are invited to proceed to Sec. VI. For all others, the following two subsections outline the nonperturbative extension of the so-far analysis. The full technical execution of this program, with a focus on the regime of ultraweak interactions $\Gamma^{-1} \lesssim D^{-1}$, where mechanisms of Fock- or many-body localization begin to play a role, will be the subject of a forthcoming publication.

1. Timescales $\sim D$

At the time $t = D$, the form factor $K_1(t)$ of a single qudit levels off at the value $K_1(t) = D$. The representation $K_1(t) = t - (t-D)\Theta(t-D)$ shows that the termination of the semiclassical ramp $K_{1,\text{semicl.}} = t$ is caused by contributions adding to the path integral for times $t > D$. They originate from stationary points of the nonlinear sigma-model action, the AA saddles [16], mentioned previously in Sec. IV E 1.

The real-time (Keldysh) nonlinear σ model admits AA-type saddles obtained by exchange of retarded and advanced entries of the $2t$ -dimensional matrix $\tau_3 \otimes \mathbb{1}_t$. The analysis of these saddles is best carried out in a frequency Fourier basis conjugate to the real-time framework used here [24]. One there finds that only AA saddles obtained by the exchange of a single pair of ± 1 values [rather than the totality of all $\binom{2t}{t}$ permutation saddles] contribute to the form factor. The interplay of these relevant AA saddles with the interaction operator has not been analyzed in detail at this point. However, the fact that they are identical to the standard saddle up to an $\mathcal{O}(t^{-1}) \lesssim \mathcal{O}(D^{-1})$ correction indicates that they are subject to nearly identical interaction-induced damping. This motivates the replacement $t \rightarrow K_1(t)$ in the semiclassical expression $t, e^{-\Gamma t}$, yielding $K_1(t)e^{-\Gamma t}$, in agreement with numerical results. A complete derivation within the present real-time framework is left for future work.

2. Timescales $\sim D^L$

The analysis of nonperturbative effects at late scales requires a different strategy, which we here outline with additional details provided in Appendix D.

Semiclassical exactness. To prepare the analysis of the survivor modes, we notice that in the noninteracting theory all nonlinear contributions to the noninteracting actions (29)

cancel out. This mechanism is known as “semiclassical exactness.” In Appendix D, we discuss how semiclassical exactness extends to the interacting theory. The result of this discussion is that the sum over quadratic actions [Eq. (31)] with the quadratic reduction of the functional expectation value (44) describes the expansion of the form factor to all order in perturbation theory in D^{-1} .

Interactions. The semiclassical exactness principle does not extend to the expansion of the interaction vertex in B fluctuations. We proceed by splitting the interaction as $S_{\text{int}} = S_{\text{int}}^{(4)} + \delta S_{\text{int}}$ into a leading, quartic contribution plus rest. The quartic term is then processed via the strategy outlined in Sec. VB. At an intermediate stage, we obtain a theory in which the functional expectation value [Eq. (C6)] as well as nonsynchronized contributions to δS_{int} are weighed by phase factors containing the interaction field Θ . Since integration over these fields produces damping at scales $\sim \Gamma^{-1}$, we ignore all contributions to the path integral carrying Θ dependence.

Late-time effective theory. At this point, the theory is reduced to an expectation value

$$\begin{aligned} K(t) &\rightarrow D^{2L} \sum_{mn} \prod_j \langle (B_{in} B_{n1}^\dagger B_{1m}^\dagger B_{mt})_j \rangle_B \\ &= D^{2L} \langle (XX^\dagger)_{t1} (X^\dagger X)_{1t} \rangle_B, \end{aligned} \quad (51)$$

involving the globally synchronized “Hadamard mode” [Eq. (47)]. It is now straightforward to establish the latter as an effective integration variable, replacing the integration over multiple B fields. As a result of a quick calculation detailed in Appendix E, we obtain

$$K(t) \rightarrow D^{2L} \int DX e^{-S_0[X]} (XX^\dagger)_{t1} (X^\dagger X)_{1t}, \quad (52)$$

with [cf. Eq. (31)]

$$S_0^{(2)}[X] = D^L \text{tr}(X^\dagger dX), \quad (53)$$

i.e., the semiclassical matrix action governing an ergodic quantum system with spectral density D^L .

Causal symmetry. This is as far as (infinite order) perturbation theory around the standard saddle point will get us. To complete the construction, we need to invoke a symmetry argument. As discussed in Sec. VI, the late-time ergodic phase of our system is a symmetry broken phase in which invariance under a large $U(2t)$ is spontaneously broken to $U(t) \times U(t)$, by the time-translation operator d in the action. The Goldstone mode generator of this system-wide global symmetry are precisely the matrices $X, X^\dagger$, combined into a rotation matrix [cf. Eq. (12)]

$$U = \begin{pmatrix} \mathbb{1} & X \\ -X^\dagger & \mathbb{1} \end{pmatrix}.$$

The minimal theory extending Eq. (53) to one incorporating the symmetry principle is defined by

$$S_0[X] = \frac{D^L}{2} \text{tr}(dU \tau_3 dU^{-1}), \quad (54)$$

i.e., the universal ergodic theory [cf. Eq. (30)] of a chaotic system with D^L levels. This theory retracts to Eq. (53) to

all orders in perturbation theory around the standard saddle τ_3 , but also includes the AA saddle point structure required to describe the plateau of the form factor at the many-body Heisenberg time D^L . The identification of Eq. (54) as the effective late-time theory of the coupled system establishes the equivalence between its late-time physics with that of a circular unitary ensemble of dimension D^L , i.e., the maximum-entropy ensemble of unitary time evolution.

VI. THERMALIZATION AND SYMMETRIES

The previous analysis demonstrated exponential relaxation toward an ergodic state on a timescale determined by the rate of local correlations. However, the situation changes when unitary symmetries come into play [6]. In this case, the continuity equations expressing the conservation of local charges lead to locally diffusive dynamics, which takes a minimum time $\sim L^2$ to reach relaxation. In the architecture we consider, motivated by quantum computation, there is no transport of particles between local qudit units. Consequently, the only conserved local quantities entering the continuity equations are internal ones, such as energy (in continuous time models) or spin. In the following, we consider the slightly more general symmetry under the action of a continuous unitary transformation group, G . We begin by exploring the consequences of such symmetries within the framework of the discrete time model considered previously. The important case of energy conservation (where G = group of time translations) is addressed in Sec. VII.

Consider a Lie group with Lie algebra generators $\{T^\alpha\}$ and structure constants defined by $[T^\alpha, T^\beta] = f^{\alpha\beta\gamma} T^\gamma$. We wish to describe evolution allowing for local exchange of the symmetry’s conserved charges, at conserved total charge. With the local symmetry generators defined as

$$T_j^\alpha \equiv \mathbb{1}, \dots, T^\alpha \otimes, \dots, \mathbb{1} \quad (55)$$

acting on qudit j , this condition is satisfied by the nearest-neighbor exchange operators $T_j^\alpha \otimes T_k^\alpha$, where summation over α is implied. These operators do not commute with the locally represented symmetry, but do commute with the total generators,

$$T^\alpha \equiv \sum_j T_j^\alpha,$$

as a straightforward consequence of the antisymmetry of structure constants. Assuming that the symmetry is acting on the single-qudit level in a K -dimensional representation, we tensor our fields $\psi_\mu \rightarrow \{\psi_\mu^u\}$, $u = 1, \dots, K$ and define the single-qudit path integral representation of the generators as $\bar{\psi}_{j,\mu}^u T_{uv}^\alpha \psi_{j,\mu}^v$.

We now declare that the interaction Hamiltonians acting as $\exp(i\delta H_{\text{int}})$ during the two-qudit evolution stretches of our network dynamics include a symmetry exchange contribution as

$$H_s = \frac{\gamma}{2} \sum_{(j,k)} (\bar{\psi}_j T^\alpha \psi_j) (\bar{\psi}_k T^\alpha \psi_k). \quad (56)$$

When included into our discrete time coherent state action [Eq. (35)], this operator appears as

$$S_s[\psi] = -\frac{i\Gamma_s}{2} \sum_{n,a} a \sum_{\langle j,k \rangle} (\bar{\psi}_{jn-a}^a T^\alpha \psi_{jna}^a) (\bar{\psi}_{kn-a}^a T^\alpha \psi_{kna}^a),$$

with $\Gamma_s \equiv \gamma_s \delta$. Note that there is no randomness involved here, so we do have a contribution at first order in the time increment δ .

Following our previous protocol, we next implement the locking (23). This is done by reordering the two factors in the action as $(\bar{\psi}_{j-a}^a T^\alpha \psi_{ja}^a) = -\text{tr}(\psi_{ja}^a \bar{\psi}_{j-a}^a T^\alpha)$, where the trace is over the u representation space indices of the symmetry. This structure requires an upgrade $G \rightarrow \{G^{uv}\}$ of the quasiclassical Green's function to a matrix u space, the intuition being that G^{uv} describes interfering trajectories carrying symmetry labels u and v , respectively. The vertex in G representation then reads

$$S_s[G] = -\frac{i\Gamma_s D^2}{2} \sum_{n,a} a \sum_{\langle j,k \rangle} \text{tr}(G_{j,n-a,na}^{aa} T^\alpha) \text{tr}(G_{k,na,n-a}^{aa} T^\alpha). \quad (57)$$

We process this perturbation similarly to the interaction in Sec. V B: an algorithm consisting of the expansion to leading (quartic) order in B generators, decoupling by a Hubbard-Stratonovich field, and removal of the latter from the action via a—now non-abelian—gauge transformation involving the T^α generators. Since this procedure will not introduce conceptually new elements to our discussion, its detailed execution is relegated to Appendix F. As a result, we obtain the expression

$$K(t) = t \left| \text{tr} \left(e^{it\Gamma_s \sum_{\langle j,k \rangle} T_j^\alpha T_k^\alpha} \right) \right|^2. \quad (58)$$

Equation (58) multiplies the bare dependence $K(t) \sim t$ with the squared time evolution operator of an $SU(N)$ -Heisenberg model, illustrating how a local conservation law can qualitatively alter the SFF. The fact that the symmetry factor features as a unitary phase (rather than as irreversible damping) is owed to a peculiarity of the present model, namely, the commutativity of the Hamiltonian [Eq. (56)], with the Floquet operator describing the unperturbed dynamics. It implies that the factorization of the time evolution operators visible in Eq. (58) is an exact feature. However, it is straightforward to extend the solution strategy to cases where this feature is absent; see, e.g., Sec. VII or Ref. [6] for the case of spin 1/2 rotation symmetry, $N = 2$. The modifying factors then assume the form of traces over genuine damping factors, i.e., partition sums rather than time evolution operators.

Either way, symmetries imply that local charge exchange processes become the bottlenecks for thermalization. To quantify their effect at large timescales, one may expand the corresponding partition sums in terms of their lowest lying excitation, e.g., spin- N magnons. In the generic case of quadratic dispersion, the corresponding excitation gaps $\Delta \sim k^2$ scale quadratic in wave numbers, and the modes of lowest nonvanishing spatial variation are gapped as $\Delta \sim L^{-2}$, implying a Thouless time $t_{\text{Th}} L^{-2} \sim 1$ quadratic in system size. (However, there can be exceptions to this rule. For example, antiferromagnetic coupling realized in a one-dimensional system tuned to have interaction couplings alternating as $\Gamma_s \sim (-)^l$ would

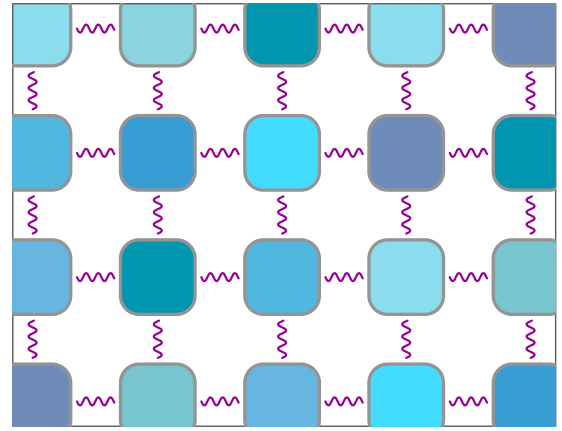


FIG. 13. An array of random quantum dots coupled by pair interaction. The color shading indicates statistical independence of the on-site Hamiltonians.

lead to a shorter relaxation time $t_{\text{Th}} \sim L$, reflecting the linear dispersion of antiferromagnetic couplings. Another scenario, realized for $K = 3$ chains with antiferromagnetic coupling, is the presence of an excitation gap in the system. In this case, we should expect relaxation toward the ground state at a timescale independent of system size.)

In the next section, we investigate how this principle works in the case of time-translational symmetry.

VII. HAMILTONIAN SYSTEMS

In this section, we extend our approach to Hamiltonian systems. Instead of the previously discussed array of unitary circuits, we consider a system of L “quantum dots” with D -dimensional Hilbert space governed by independently drawn Hamiltonians h_l with variance

$$\langle |h_{\mu\rho}|^2 \rangle = \frac{\lambda^2}{2D},$$

coupled by the two-body interaction [Eq. (17)] (cf. Fig. 13). Most of the changes relative to the previously considered system amount to taking straightforward continuum time limits. However, the nonuniform spectral density of Hamiltonian systems also leads to a few complications. They relate to structures at the scale of the energy cutoff λ of individual sites, over which the theory has parametric but no numerical control. Where possible, we will fix numerical factors by consistency arguments, but otherwise leave them unspecified.

A. Luttinger-Ward functional

We start by adapting elements of the previous discrete time formalism to the present setting. For example, with $U = \exp(iH\delta)$, and δ an infinitesimal time step, the continuum version of the action in Eq. (19) reads as $S[\psi] = \int dt \bar{\psi}(\partial_t - iH)\psi$ and the interacting theory is described by [cf. Eq. (35)]

$$S[\psi] = \sum_a \int_t \left(\sum_j \bar{\psi}_j^a (\partial_t - ih_j) \psi_j^a - i \sum_{\langle j,k \rangle} \bar{\psi}_j^a \bar{\psi}_k^a H_{\text{int}} \psi_k^a \psi_j^a \right),$$

with continuum fields $\psi = \{\psi_{j,t}^a\}$, i.e., by an orthodox coherent state field integral action. The subsequent averaging over the ensemble generates quartic (h) and eighth-order (H_{int}) fermion terms, to which we apply the replacement rule [Eq. (1)], and then integrate over Grassmann fields. As a result, we obtain the continuum version of the $G\Sigma$ -action,

$$S[G, \Sigma] = -D \sum_j \left(\text{tr} \ln(i\partial_t - \Sigma) + \text{tr} \left(\Sigma G - \frac{\lambda^2}{2} G^2 \right) \right)_j + \frac{\Lambda^2 D^2}{2} \sum_{\langle j,k \rangle} \text{tr}((G_j \odot G_k) \tau_3 (G_j \odot G_k) \tau_3), \quad (59)$$

where the second line defines the interaction action $S_{\text{int}}[G, \Sigma]$ and the traces now include summation over the causal index, and integration over time, i.e., $\text{tr}(X) = \sum_a \int_t X_{tt}^{aa}$.

B. Stationary phase analysis

Varying Eq. (59), we obtain the equations

$$0 = \delta_{\Sigma_j} S[\bar{G}, \bar{\Sigma}] = \bar{G}_j - \frac{1}{i\partial_t - \bar{\Sigma}_j},$$

$$0 = \delta_{G_j} S[\bar{G}, \bar{\Sigma}] = D\lambda^2 \bar{G}_j - D\bar{\Sigma}_j + \delta_{G_j} S_{\text{int}}[\bar{G}].$$

If it were not for the interaction S_{int} , these would define a set of self-consistency equations for a system of decoupled disordered quantum dots, identifying the on-site self-energy $\Sigma_j = \lambda^2 G_j$ with the Green's function. Before turning to the role played by interactions, we solve this equation, temporarily omitting the site index j . Turning to Fourier space [cf. the conventions stated above Eq. (2)], $i\partial_t \rightarrow \epsilon$, the equation assumes the form of a self-consistent Born equation

$$\bar{G} = \frac{1}{\epsilon - \lambda^2 \bar{G}}. \quad (60)$$

We focus on energies $|\epsilon| < \lambda$, for which the equation is approximately solved by $\bar{G}^a \simeq -ia\lambda^{-1}$, where causality determines the retarded and advanced choice of sign a . Putting ϵ back in, we obtain $\bar{G}^a \simeq \frac{1}{\epsilon + ia\lambda}$, and Fourier transformation back to the time domain leads to

$$\bar{G}_{tu}^a = -ia\Theta(a(t-u))e^{-\lambda|t-u|}. \quad (61)$$

We next demonstrate that Eq. (61) defines a solution of the stationary phase equations, including in the presence of interactions. To this end, consider $S_{\text{int}}[\bar{G}] \equiv \frac{\Lambda^2 D^2}{2} \sum_{\langle j,k \rangle} \sum_a \int_{tu} ((\bar{G}_j^a \odot \bar{G}_k^a)_{tu} (\bar{G}_j^a \odot \bar{G}_k^a)_{ut})$, where we noted that the mean-field solution is diagonal in causal a space. The time integral extends over products of Green's functions of identical causality evaluated at positive and negative time differences, implying the vanishing of both, this expression and of the derivative $\delta_{\bar{G}} S_{\text{int}}$. The physical principle for this vanishing of the self-energy is easy to understand: For a generic interacting many-particle system, the second-order (in interaction amplitudes) contribution to the self-energy would be interpreted as a particle in dot j creating a particle-hole excitation in dot k at time t that is getting reabsorbed at time u . Here, the hole is equivalent to a particle propagating backward in time, and no issues with causality arise. However, in our present system, the self-energy of the j propagator is computed relative to the

vacuum state of the k dot. (In our system, each dot is occupied by precisely one-particle in an otherwise empty Hilbert space, and the one-particle “self”-energy considers these states in isolation.) Its state space does not admit a propagating hole, hence the vanishing of the interaction self-energy of single Green's function propagators.

C. Continuum action

As discussed in Sec. III, the presence of Goldstone modes in our system is a consequence of the symmetry breaking between $a = \pm 1$ sectors on the mean-field level. Formally, the Goldstone mode manifold is generated by continuous transformations of the mean field $\bar{G} \rightarrow T \bar{G} T^{-1}$, where $T = \{T_{tu}^{ab}\}$ are matrices in causal and time domain. In order to avoid “empty integrations,” it is customary to represent these modes as $T = \mathbb{1} + W + \dots$, where the generator matrices W are chosen to anticommute with the mean field $\bar{G}$.

Differently from the discrete model, the mean field $\bar{G}_{t-u}$ carries explicit time dependence, making the anticommutativity condition difficult to realize exactly. At the same time, $\bar{G}_{t-u} \approx -i\tau_3 \delta_{t-u}$ decays rapidly as a function of distance, and fields W_{tu} with time variation on scales $> \lambda^{-1}$ are oblivious to the finite range of $\bar{G}$. Imposing this slowness condition, we continue to work with the representation in Eq. (12), where the B -dependent generator anticommutes with $\bar{G}$ in the δ -function approximation. At the same time, we must not forget about the time dependence of $\bar{G}$ entirely; the causality condition $\bar{G}_{t-u}^a \propto \Theta(a(t-u))$ remains an essential element in the derivation of the theory.

Single site action and form factor. The computation of the continuous time effective action, detailed in Appendix G, parallels our previous discrete time construction and yields

$$S_0[T] = -\frac{D}{\lambda} \sum_j \int_{tu} \text{tr}(\tau_3 T_{tu}^{-1} (\partial_u + \partial_t) T_{ut})_j, \quad (62)$$

for the noninteracting action. Upon insertion of the representation in Eq. (12), this becomes the continuous time version of Eq. (29). (The role of the discrete derivative d in the center coordinate is now taken by the sum of two time derivatives.) For our present purposes, it will be sufficient to consider the quadratic expansion in B 's, which is readily obtained as

$$S_0^{(2)}[B] = \frac{2D}{\lambda} \sum_j \int_{tu} B_{tu}^\dagger (\partial_t + \partial_u) B_{ut} = \frac{2D}{\lambda} \sum_j \int_{t\Delta t} B_{t\Delta t}^\dagger \partial_t B_{t\Delta t}, \quad (63)$$

and now assumes the role of Eq. (31). In the second line, we switched to a center $[t = (t+u)/2]$ and difference time $[\Delta t = t-u]$ representation $B_{tu} \rightarrow B_{t\Delta t}$, $B_{ut}^\dagger \rightarrow B_{t\Delta t}^\dagger$, which will be convenient throughout. The continuum time version of the propagator in Eq. (33) is given by Eq. (15), and that of the SFF (44) by

$$K(t) \sim \left(\frac{D}{\lambda} \right)^{2L} \prod_k ((B B^\dagger)_{k0} (B^\dagger B)_{k0})_B, \quad (64)$$

where the factor λ^{-2L} , formally introduced by the replacement rule (11), balances the dimension of field matrix elements, $[(BB^\dagger)_{tu}] = (\text{energy})$.

Interaction vertex. Expanding the interaction $S_{\text{int}}[G]$ to quartic order in fluctuation generators B , we obtain the continuous time variant of Eq. (46) (cf. Appendix G):

$$S_{\text{int}}^{(4)}[B] = \frac{2\Gamma D^2}{\lambda} \sum_{(j,k)} \int B_{jt_+t_-} B_{ku_+u_-} B_{jt_-t_+}^\dagger B_{ku_-u_+}^\dagger (\lambda^{-1} \delta_{t_+,u_+} + \lambda^{-1} \delta_{t_-,u_-} - 8\lambda^{-2} \delta_{t_+,u_+} \delta_{t_-,u_-}), \quad (65)$$

where $\Gamma = 4\Lambda^2/\lambda$ has dimension energy and $\int$ is shorthand notation for the integration over all four time variables. Here, the δ functions are to be interpreted as smeared over scales λ^{-1} . On this basis, we observe the synchronization principle at work again: For field configurations $B_{jt_+t_-} B_{lt_+t_-}$ with identical time arguments (always with λ^{-1} uncertainty), the two terms cancel out, stabilizing a global ergodic mode.

However, differently from the discrete model, uniform time translations now realize a continuous symmetry, which leads to qualitative effects in the late-time dynamics. Comparing to our earlier discussion in Sec. VI, the role of the symmetry generators T^a is now taken by the time-translation operator, ∂_t , energy being the globally conserved charge.

D. Energy diffusion

In view of the structures outlined above, we expect the emergence of an effective action playing a role analogous to the previously discussed $SU(N)$ -Heisenberg action. Time-translational invariance is microscopically realized as $\psi_t \rightarrow (e^{T^a} \psi)_t = \psi_{t+a}$, the generators $T^a = a\partial_t$ assuming the role of the previously considered $SU(N)$ generators.

To identify the action cost of weakly position-dependent symmetry operations, $a \rightarrow a_j$, we follow a continuum time version of the discrete time protocol outlined in Sec. VB. Referring to Appendix G 3 for its detailed execution, it yields the form factor as

$$K(t) = \lambda^L \int D\tau e^{-S[\tau]}, \quad S[\tau] = -4\Gamma t \sum_{(j,l)} F(\tau_j - \tau_l), \quad (66)$$

where the integration $\int D\tau = \prod \int d\tau_j$ extends over the relative time coordinates of the individual dots. The weight function $F = 1 - f$, where $f(\tau)$ is a Gaussian implicitly defined by $f(0) = 1$, and $f(\tau \gg \lambda^{-1}) = 0$.

Equation (66) is the continuous time version of Eq. (49). The only difference is that the cost function for dots out of temporal synchronization is now continuous, instead of step-function like. In the short time limit, $\Gamma t \rightarrow 0$; this leaves us with L independent time integrations and the product form factor $K(t) \sim (\lambda t)^L$. For $\Gamma t \gg 1$, the action enforces near-synchronization to time configurations $\tau_l = \tau + \lambda^{-1} a_l$, where the dimensionless time-mismatch parameters a_l are defined to have zero mean, $\sum_l a_l = 0$. To describe the manifestation of time-translational invariance in the SFF, we note that for $\Gamma t \gg 1$, the synchronization mismatches $|a_j - a_l| = \mathcal{O}(1)$ must be small. A second-order Taylor expansion $F(a) \sim a^2$ then gets

us to the approximation

$$S[a] \sim t\Gamma \sum_{(j,l)} (a_j - a_l)^2 \sim t\Gamma \int_0^L dx (\partial_x a)^2, \quad (67)$$

where the continuum limit introducing a dimensionless length parameter is meaningful in the case where the number of sites is large.

Equation (67) describes the relaxation of the system at large timescales. To estimate the contribution of its lowest lying excitations to the SFF, we recall that the integration $\int Da$ extends over fluctuations excluding the constant mode $\sum_l a_l = 0$. We implement this condition by passing to an integration over momentum modes,

$$a_q = \frac{1}{L^{1/2}} \sum_j e^{-iqj} a_j, \quad a_j = \frac{1}{L^{1/2}} \sum_q e^{iqj} a_q,$$

where the normalization prefactor is defined in such a way that the passage $\int Da = \int \prod_j a_j \rightarrow \int \prod_q da_q$ has unit determinant, and the product extends over all q values, $q = (2\pi/L)(1, \dots, L-1)$, excluding the constant mode $q = 0$. In the q representation, the action assumes the form

$$S[a] \sim t\Gamma \sum_q \sin^2\left(\frac{q}{2}\right) |a_q|^2, \quad (68)$$

and for the modes $q = \pm 2\pi/L \bmod 2\pi$ with lowest action. A linearization of the sin yields

$$S[a_{2\pi/L}] \sim \Gamma \frac{t}{L^2} |a_{2\pi/L}|^2,$$

demonstrating the scaling $\sim t/L^2$ characteristic for a diffusive relaxation process. To describe its influence on the form factor, we integrate over q modes, to obtain Eq. (9), where the cutoff imposed by the max function reflects the lack of resolution of the lowest timescales $a_q \sim 1$ in our theory: For large values of t , the exponentials $\exp(-\Gamma t \sin^2(q) |a_q|^2)$ appear to suppress the integration to windows of width $\Delta a^2 \sim (\sin^2(q) t \Gamma)^{-1} \lesssim 1$. However, these lie below the time uncertainty implied by the presence of microscopic fine structures, notably the mean-field Green's functions, of range λ . In fact, one can show by a somewhat elaborate diagrammatic analysis not reproduced here that fluctuations of $\mathcal{O}(\lambda^{-1})$ in the a -variables of the preexponential sources do not significantly affect the value of the form factor. We therefore implicitly assume a small tolerance window in the definition of the weight function in Eq. (66).

As discussed in Sec. II, the approach to the ergodic form factor described by Eq. (9) is softer than the exponential relaxation in the model without symmetries. Presently, the long-lived modes responsible for the slow decay are the weakly inhomogeneous time-translation modes, as witnessed by the action [Eq. (68)].

VIII. SUMMARY AND DISCUSSION

In this paper, we introduced a microscopic path integral formulation for the evolution of complex quantum systems whose unitary dynamics leads—consistent with the eigenstate thermalization hypothesis—to thermal behavior. In this regime, information is scrambled across the system, and

late-time states effectively realize maximum-entropy thermal ensembles.

Our construction begins with the identification of strongly chaotic, D -dimensional subsystems that relax quasi-instantaneously into ergodic states when isolated. In the controlled limit of large local Hilbert-space dimension, $D \gg 1$, we describe their dynamics in terms of quasiclassical Green's functions that track the propagation of pair amplitudes through an ergodic background.

Interactions between subsystems generate correlations that progressively lock initially independent amplitudes into a single effective many-body state evolving in the full tensor-product Hilbert space. For interaction rate Γ , this synchronization occurs on a short, subextensive timescale $\sim \Gamma^{-1} f(L, D)$, with f growing logarithmically in system size and Hilbert-space dimension. This behavior follows directly from Eq. (8): nonsynchronized contributions scale linearly with L but are exponentially damped in time as $e^{-\Gamma t}$, leading to a crossover time $t \sim (1/\Gamma) \ln L$. We further note that for $\Gamma \lesssim D^{-1}$, resolving this crossover requires access to fluctuations nonperturbative in D , beyond the scope of simple semiclassical approximations.

In systems with local symmetries, subsequent relaxation is governed by conserved charges that equilibrate via local exchange processes. The resulting hydrodynamic modes induce diffusive dynamics, with full ergodization occurring only on timescales $\sim L^2$. We illustrated this mechanism using circuit arrays with $U(N)$ symmetry and Hamiltonian systems with time-translation symmetry.

The path integral developed here is formulated over fields in Fock space rather than conventional $(d+1)$ -dimensional spacetime. This perspective exposes an open conceptual challenge: causality enforces two distinct symmetry-breaking mechanisms—one at the subsystem level and one at the global many-body level—which break and are restored on parametrically different timescales set by the Hilbert-space dimensions D and D^L , respectively. Two alternative path integral formulations based on many-body coherent states, corresponding to first- and second-quantized descriptions, capture one mechanism but not both (see Sec. III D 2). This points to a fundamental difficulty in representing strongly entangled quantum states using separable coherent-product constructions.

Despite this open issue, the present framework provides an efficient and microscopically controlled description of quantum relaxation in a broad class of systems with integrability-breaking randomness. Systems with finite single-particle relaxation times can be incorporated through straightforward extensions of the construction introduced in Sec. V, allowing access to microscopic relaxation dynamics beyond purely random unitary settings. Moreover, replicated versions of the present path integral framework offer a route to information-theoretic observables such as entanglement growth and operator spreading. These extensions will be presented elsewhere.

We therefore expect this framework to develop into a versatile toolbox for addressing problems in many-body quantum chaos that lie beyond the reach of numerical simulations and perturbative semiclassical methods.

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DATA AVAILABILITY

The data that support the findings of this article are available [26].

APPENDIX A: GRASSMANN PATH INTEGRAL

We here discuss the construction of the path integrals discussed in the main text, beginning with the case of single-qudit evolution (Sec. IV).

1. Single-qudit path integral

For a single Grassmann variable, ψ , [8] consider the definition of the integral

$$\int d\psi \psi^m = \delta_{m,1}.$$

It implies the vector integral generalization

$$\delta_{\mu\rho} = \int D\psi e^{-\bar{\psi}\psi} \psi_\mu \bar{\psi}_\rho, \quad (\text{A1})$$

where $\psi = \{\psi_\mu\}$ now is a D -component Grassmann vector, $D\psi = \prod_\mu d\bar{\psi}_\mu d\psi_\mu$ and $\bar{\psi}\psi = \sum_\mu \bar{\psi}_\mu \psi_\mu$.

Using these equations, we represent matrix elements of the time evolution operator with redundantly introduced Kronecker- δ 's:

$$\begin{aligned} \langle \rho | U^t | \mu \rangle &= \delta_{\mu_{t+1}\rho} \delta_{\mu_{t+1}\rho_t} U_{\rho_t \mu_t} \delta_{\mu_t \rho_{t-1}} \cdots U_{\rho_1 \mu_1} \delta_{\mu_1 \rho_0} \delta_{\rho_0 \mu} \\ &= \delta_{\mu_{t+1}\rho} \int D\psi e^{-\bar{\psi}\psi} \psi_{t\mu_{t+1}} \bar{\psi}_{t\rho_t} U_{\rho_t \mu_t} \psi_{(t-1)\mu_t} \bar{\psi}_{(t-1)\rho_{t-1}} \\ &\quad \cdots U_{\rho_1 \mu_1} \psi_{0\mu_1} \bar{\psi}_{0\rho_0} \delta_{\rho_0 \mu} \\ &= \int D\psi e^{-\bar{\psi}\psi} \psi_{t\rho} e^{\sum_{n=1}^t \bar{\psi}_n U \psi_{n-1}} \bar{\psi}_{0\mu} \\ &= \int D\psi e^{-\bar{\psi}\psi + \bar{\psi} U T_- \psi} \psi_{t\rho} \bar{\psi}_{0\mu}. \end{aligned}$$

Here, the individual Kronecker tensors are resolved by Grassmann integrals over $\{\psi_{n,\mu}\}$. Introducing an analogous representation for $\text{tr}(U^{\dagger t})$, we obtain the result [Eq. (19)].

2. Qudit network

The interaction contribution can be treated analogously, by insertion of the qudit generalization of Eq. (A1),

$$\delta_{\mu\nu,\rho\sigma} = \int D\psi e^{-\bar{\psi}\psi} \psi_{2\nu} \psi_{1\mu} \bar{\psi}_{1\rho} \bar{\psi}_{2\sigma}. \quad (\text{A2})$$

(Notice the “antinormal ordering,” grouping creation and annihilation “operators” together.) If we now place such Kronecker- δ representations at the interaction time slices $(m, -1) \rightarrow (m, 1)$ on the bonds of the tensor network defined by Fig. 10, we encounter Grassmann contractions of the type

$$\begin{aligned} & \bar{\psi}_{j1\rho} \bar{\psi}_{k1\sigma} V_{\rho\sigma,\mu\nu} \psi_{k-1\nu} \psi_{j-1\mu} \\ & \approx \bar{\psi}_{j1\rho} \bar{\psi}_{k1\sigma} (\delta_{\rho\mu} \delta_{\sigma\nu} + iX_{\rho\sigma,\mu\nu}) \psi_{k-1\nu} \psi_{j-1\mu} \\ & \rightarrow e^{\bar{\psi}_{j1} \psi_{j-1} + \bar{\psi}_{k1} \psi_{k-1} + i\bar{\psi}_{j1} \bar{\psi}_{k1} X \psi_{k-1} \psi_{j-1}}, \end{aligned}$$

where $X \equiv H + \frac{i}{2}H^2$. Combining all these factors into a single exponent, we arrive at the interaction action $S[V, \psi]$.

APPENDIX B: DISCRETE TIME INTERACTION VERTEX

In this Appendix, we provide some details on the interaction vertex of the discrete time theory.

1. The matrix $\mathcal{G}(B)$

The construction of the path integral starts with the integration over ψ variables, Gaussian after all nonlinear terms have been Lagrange multiplier locked to G . Collecting the quadratic terms in Eqs. (35) and (36), we obtain the bilinear

form $\bar{\psi} \mathcal{G}^{-1} \psi$ with

$$\mathcal{G}^{-1}(B) = \begin{pmatrix} 1 & & B_T \\ -1 & 1 & \\ & -B^\dagger & 1 \\ & & & -1 \\ & & & & 1 \end{pmatrix}, \quad (\text{B1})$$

where the matrix structure refers to a representation of the integration variables as $\psi = (\psi_{-1}^+, \psi_1^+, \psi_{-1}^-, \psi_1^-)^T$, and the site indices j are left implicit. The straightforward inversion of $\mathcal{G}^{-1}$, e.g., by series expansion around the unit-matrix diagonal, yields Eq. (B1).

Adding to this bilinear form the quadratic term from Eq. (24), we obtain $\bar{\psi}(\mathcal{G}^{-1}(B) - i\Sigma\tau_3)\psi$. The integral over ψ then leads to the second $\text{tr} \ln$ in Eq. (41).

2. Interaction vertex

In this Appendix, we discuss the expansion of the discrete time interaction vertex in B fluctuations. Our starting point is the configurations in Eq. (45). With Eqs. (B1) and (13), a breakdown of the individual components we need in the following is given by (discrete time indices omitted for brevity)

$$\begin{aligned} \bar{G}_{-1,1}^{++} & \rightarrow \frac{i}{2}(Q^{++} - 1) = -\frac{iB_TB^\dagger}{1 + B_TB^\dagger}, \\ \bar{G}_{1,-1}^{--} & \rightarrow \frac{i}{2}(Q^{--} + 1) = \frac{iB^\dagger B_T}{1 + B^\dagger B}, \\ \bar{G}_{-1,-1}^{+-} & \rightarrow \frac{i}{2}Q^{+-} = -\frac{iB_T}{1 + B^\dagger B_T}, \\ \bar{G}_{1,1}^{-+} & \rightarrow \frac{i}{2}Q^{-+} = -\frac{iB^\dagger}{1 + B_TB^\dagger}. \end{aligned} \quad (\text{B2})$$

We substitute these terms into the interaction [Eq. (40)], temporarily dropping the $\langle j, k \rangle$ summation over neighboring sites and relabeling the order parameter fields as $Q_j \rightarrow Q$, $Q_k \rightarrow R$. Their Goldstone mode generators will be denoted by $B_{Tj} \rightarrow B$, $B_{Tk} \rightarrow C$. In this notation, the interaction vertex assumes the form

$$\begin{aligned} S_0[B, C] & = \frac{\Gamma D^2}{4} \text{tr}((Q^{++} - 1) \odot (R^{++} - 1) + (+ \leftrightarrow -)) = \Gamma D^2 \sum_n \sum_{q,r=1}^{\infty} (-)^{q \cdot r} ((BB^\dagger)_{nn}^q (CC^\dagger)_{nn}^r + (B^\dagger B)_{nn}^q (C^\dagger C)_{nn}^r), \\ S_i[B, C] & = \frac{\Gamma D^2}{16} \text{tr}([(Q^{++} - 1) \odot (R^{++} - 1)][(Q^{++} - 1) \odot (R^{++} - 1)] - [Q^{+-} \odot R^{+-}][Q^{-+} \odot R^{-+}] + (+ \leftrightarrow -)) \\ & = D^2 \sum_{n,m} \sum_{p,q,r,s=0}^{\infty} (-)^{p+q+r+s} ((BB^\dagger)_{mn}^{p+1} (CC^\dagger)_{mn}^{q+1} (BB^\dagger)_{nm}^{r+1} (CC^\dagger)_{nm}^{s+1} + (B^\dagger B)_{mn}^{p+1} (C^\dagger C)_{mn}^{q+1} (B^\dagger B)_{nm}^{r+1} (C^\dagger C)_{nm}^{s+1} \\ & \quad - (B(B^\dagger B)^p)_{mn} (C(C^\dagger C)^q)_{mn} (B^\dagger (BB^\dagger)^r)_{nm} (C^\dagger (CC^\dagger)^s)_{nm} - (B^\dagger (BB^\dagger B)^p)_{mn} (C^\dagger (CC^\dagger)^q)_{mn} (B(B^\dagger B)^r)_{nm} (C(C^\dagger C)^s)_{nm}). \end{aligned} \quad (\text{B3})$$

Extracting the terms with $p = q = r = s = 0$ and turning back to the original notation $B \rightarrow B_{Tj}$, $C \rightarrow B_{Tk}$, we obtain the fourth order vertex [Eq. (46)] discussed in the main text. For the discussion of terms beyond quartic perturbation theory, we refer to Appendix D 2.

APPENDIX C: HUBBARD-STRATONOVICH DECOUPLING OF THE INTERACTION

In this Appendix, we discuss the decoupling of the interaction according to the procedure outlined in Sec. V B. To keep the notation slim, we interchangeably use notation

$B_{n_+n_-} \equiv B_n$ and $B_{n_-n_+}^\dagger = B_n^\dagger$ throughout. We start with a compactified representation of the interaction [Eq. (46)]:

$$S_{\text{int}}^{(4)}[B] = \Gamma D^2 \sum_{(j,k),nm} (B_T B^\dagger)_{jm} F_{mn} M_{jk} (B_T B^\dagger)_{kn}, \quad (\text{C1})$$

where $(B_T B)_{jm} \equiv B_{Tjm} B_{jm}^\dagger = B_{Tjm_+m_-} B_{jm_+m_-}^\dagger$,

$$M_{jk} = \begin{cases} 1, & (j,k) \text{ nearest neighbors,} \\ 0, & \text{else,} \end{cases} \quad (\text{C2})$$

and

$$F_{nm} \equiv \delta_{n_+m_+} + \delta_{n_-m_-} - 2\delta_{n_+m_+}\delta_{n_-m_-}. \quad (\text{C3})$$

Next, we introduce an auxiliary field $\phi \equiv \{\phi_{jn}\} = \{\phi_{jn_+n_-}\}$ carrying the same index structure as our B fields, and the quadratic action

$$S[\phi] = -\frac{1}{4\Gamma} \phi_{jm} F_{mn}^{-1} M_{jk}^{-1} \phi_{kn},$$

defining the second moment

$$\langle \phi_{jm} \phi_{kn} \rangle_\phi = -2\Gamma M_{jk} F_{mn}. \quad (\text{C4})$$

(As we need only this relation, there will be no need to compute the matrix inverses M^{-1} and F^{-1} explicitly.) A variable shift

$$\phi_{jn} \rightarrow \phi_{jn} + 2\Gamma D \sum_{km} (M_{jk} F_{n,m})(B_T B^\dagger)_{km}$$

generates a quartic term of the structure $(B_T B^\dagger)M \otimes F(B_T B^\dagger)$ cancelling the interaction vertex (46), alongside the linear coupling $\sum_{j,n} \phi_{jn} (B_T B^\dagger)_{jn}$. As a consequence, the full quadratic action now reads

$$S_0^{(2)}[B, \phi] = -D \sum_j \text{tr}(B_n^\dagger d B_n - B_n^\dagger B_{Tn} \phi_n)_j,$$

with the discrete time derivative $d B = B - B_T$, the “potential” ϕ . This structure suggests removing the field from the action by a gauge transformation of the B -field variables, for which we try the ansatz

$$B_n \rightarrow e^{\Theta_n} B_n, \quad B_n^\dagger \rightarrow B_n^\dagger e^{-\Theta_n}. \quad (\text{C5})$$

This transformation leaves the first term in the action invariant and changes the second as

$$\begin{aligned} B_n^\dagger B_{Tn} &\rightarrow B_n^\dagger B_{Tn} e^{-d\Theta_n} \\ &\approx B_n^\dagger B_{Tn} (1 - d\Theta_n), \end{aligned}$$

where $d\Theta_n = \Theta_n - \Theta_{n-1}$ or $d\Theta_{n_+n_-} = \Theta_{n_+n_-} - \Theta_{(n_+-1)(n_-1)}$ in an index-resolved notation. Note the analogy to the (symbolical) continuum formula $B^\dagger \partial_t B \rightarrow B^\dagger (\partial_t - \partial_t \Theta) B$ representing a time-dependent gauge transformation. In the second equality, we assumed smallness of the interaction per time slice, $\text{var}(\phi) \sim \Gamma \sim \Lambda^2$, implying the smallness of $d\Theta_n$. On the same basis, $\phi \sim \Theta \ll 1$, allowing us to ignore the ϕ -dependence of the last term. We conclude that the ϕ -term is eliminated by solutions of the discrete time difference equation $d\Theta = \phi$ (the analog of the continuum $\partial_t \Theta = \phi$), which we solve as

$$\Theta_n = \Theta_{n_+n_-} = \sum_{s=-\infty}^0 \phi_{n_++s, n_-+s},$$

under the assumption that $\phi_{n_+, n_-} = 0$, for negative discrete time arguments. At this point, the interaction has been decoupled from the action, and we turn to the preexponential terms in the SFF (44), which transform as $(\Theta_{n_+n_-}$ notation from now on)

$$\begin{aligned} K(t) &\rightarrow D^{2L} \sum_{\{m,n\}} \prod_j \langle (B_{tn} B_{n0}^\dagger B_{0m}^\dagger B_{mt})_j \rangle_B \\ &\times \langle e^{\sum_j (\Theta_{tm} - \Theta_{0m} - \Theta_{n0} + \Theta_{mt})_j} \rangle_\phi. \end{aligned} \quad (\text{C6})$$

Note the shorthand notation $(B_{tn})_j \equiv B_{jtn_j}$ or $(\Theta_{tm})_j = \Theta_{jtn_j}$, etc. Using that Eq. (33) requires $t - n_j = m_j$ or $t - n = m$, in shorthand notation, the exponent evaluates to

$$\Theta_{tm} - \Theta_{0n} - \Theta_{m0} + \Theta_{mt} \quad (\text{C7})$$

$$= \sum_{s=-t+1}^0 \phi_{(t+s)(n+s+t\Theta(s+n))} \equiv X[n, \phi]. \quad (\text{C8})$$

To understand the structure of this expression, consider the diagram shown in Fig. 7 with the discrete time identifications $t_+ = t - n$ and $t_- = n$. Along the stretches $0 \rightarrow t - n$ and $t - n \rightarrow t$ of duration $t - n$ and n , respectively, the time arguments of the individual scattering amplitudes equal $(t + s, t + n + s)$ and $(t + s, n + s)$. The exponents sum the corresponding time-dependent interaction fluctuation amplitudes, ϕ .

In the final step, we need to integrate over ϕ in $\langle e^{\sum_j X[n_j, \phi_j]} \rangle_\phi$. This Gaussian integration yields

$$\begin{aligned} \langle e^{\sum_j X[n_j, \phi_j]} \rangle_\phi &= e^{\frac{1}{2} \sum_{j,k} \langle X[n_j, \phi_j] X[n_k, \phi_k] \rangle_\phi} \\ &= e^{-\Gamma \sum_{(j,k)} \sum_{s,u=-t+1}^0 F_{(t+s)(n_j+s+t\Theta(s+n_j)), (t+u)(n_k+u+t\Theta(u+n_k))}} \\ &= e^{-\Gamma t \sum_{(j,k)} (1 - \delta_{n_j n_k})}. \end{aligned}$$

In the last line, we noted that the function F requires the pairwise equality of at least two of its indices. For each of the s values in the double sum, the first two Kronecker- δ lock in for precisely one u value, leading to a contribution $\sum_s 2$. In that remaining sum over s , we need to take into account that for each pair of nearest neighbors $\langle j, k \rangle$, and a given value of t , the brickwork architecture implies $t/2$ two-qudit interaction processes; i.e., the summand is nonvanishing for $t/2$ values of s , leading to the damping rate Γt . Finally, for synchronous configurations $n_j = n_k$, the product of two Kronecker δ 's in the function F balances the damping, as indicated in the exponent. Summing over all configurations $\{n_j\}$, we arrive at the result (49).

APPENDIX D: BEYOND THE SEMICLASSICAL APPROXIMATION

In this Appendix, we provide details concerning the analysis of the discrete time path integral beyond leading order in perturbation theory.

1. Semiclassical exactness

Consider the form factor of a single qudit [Eq. (3)] as described by an integral over $G = -iT \tau_3 T^{-1}$ with the action [Eq. (30)]. A series expansion of both the functional

observable and the action in the generators B defined through Eq. (12) will yield the integral as a series in powers of D^{-1} , but fall short of reaching perturbatively inaccessible stationary points on the integration manifold. (Note the similarity of Eq. (13) with the stereographic projection of a sphere: An expansion around the north pole $Q = \tau_3$ or $B = 0$ will fail to reach the south pole, $Q = -\tau_3$ or $B = \infty$.) “Semiclassical exactness” means that all contributions to the expansion around individual stationary points beyond Gaussian order vanish. In the following, we review this feature for the expansion around the standard saddle; however, the same principle works for all others. While discussions of semiclassical exactness can be found in the literature (cf., e.g., Ref. [15]), we here wish to demonstrate that the corresponding “higher-order diagram cancellations” survive the presence of interaction terms.

Demonstrating semiclassical exactness for the form factor as represented by Eq. (3) requires a separate expansion of the preexponential observables and the action in generators, which is cumbersome. To avoid this complication, we begin by switching to a generating function representation.

Instead of working with “open boundary conditions” on the discrete time interval $[1, t]$ as before, let us switch to one, still discrete, with periodic boundary conditions, $t + 1 = 1 \bmod t$. We also introduce phase factors, $U \rightarrow U e^{i\phi_+}$, $U^\dagger \rightarrow U^\dagger e^{-i\phi_-}$. With $\psi_{t+1} = \psi_1$, the path integral in Eq. (19) then no longer yields unity but

$$Z(\theta) = \det(1 - U^t e^{i\theta_+}) \det(1 - U^{\dagger t} e^{-i\theta_-}),$$

with $\theta_\pm = t\phi_\pm$. Expanding the determinants in these phase factors, we obtain the representation

$$K(t) = |\text{tr}(U^t)|^2 = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta_+ d\theta_- e^{-i(\theta_+ - \theta_-)} Z(\theta). \quad (\text{D1})$$

Our task thus is to compute the generating functional in an expansion to leading order in the factor $\exp(i(\theta_+ - \theta_-))$.

Tracing the fate of the phases, we find that they lead to a modification $T_- B T_+ \rightarrow e^{i\phi} T_- B T_+ \equiv B_T$ in Eq. (29), where $\phi = \phi_+ - \phi_-$. The inverse of the quadratic action defined by the phase-decorated time-translation operator is given by [cf. Eq. (33)]

$$\Pi_{n,m} = \frac{1}{D} \delta_{\Delta n, \Delta m} e^{i\phi(\bar{n} - \bar{m})} (\Theta(\bar{n} - \bar{m}) + \dots), \quad (\text{D2})$$

where the ellipses denote contributions with multiple windings around the time circle, $\propto \exp(ik\theta)$, which do not contribute to the observable. The strategy now is to expand in nonlinear contributions to the action, compute terms in the ensuing series by Wick’s theorem, and extract the contribution $\propto e^{i\theta}$.

This task is simplified by the rule that contractions between neighboring matrices $\text{tr}(\dots B_T B^\dagger \dots)$ vanish by causality:

$$\langle \text{tr}(\dots B_{n_+-1, n_--1} B_{n_+ m_+}^\dagger \dots) \rangle_B = \Pi_{n_+-1, n_+}(\dots) = 0,$$

where we used that Eq. (33) enforces the incompatible conditions $n_+ - m_+ > 2$ and $n_+ = m_+$, and the vanishing of the propagation one step backward in time is prohibited by the temporal Θ function. (We here ignore causality-allowed prop-

agation around the periodic time cycle, $n \rightarrow n - 1$, as they are penalized by a damping factor $\exp(i\theta)$ and do not contribute to the result at the required order in these factors.)

Inspection of action (29) shows that some of our contractions will involve the fields dB , acted upon by the discrete time derivative (with phase factors included, i.e., $dB = B - e^{i\phi} T_- B T_+$). Since d is the operator inverse of Π [cf. Eq. (33)] and the subsequent displayed equation, $D(d\Pi)_{n,m} = \delta_{n,m}$, such Wick contractions yield the simple result

$$\begin{aligned} \langle \text{tr}(A(dB)CB^\dagger) \rangle_B &= \frac{1}{D} \text{tr}(A) \text{tr}(C), \\ \langle \text{tr}(A(dB)) \text{tr}(CB^\dagger) \rangle_B &= \frac{1}{D} \text{tr}(AC). \end{aligned} \quad (\text{D3})$$

Physically, $d\Pi_{n,m}$ is vanishing, unless $n_+ - m_+ = n_- - m_- = 0$. These are operator stretches of vanishing duration (see Ref. [27] for their in-depth discussion), which in an interacting context will not be affected by interaction corrections. This observation establishes the effectiveness of local semiclassical exactness, including in correlated multiqudit systems.

The contraction rules [Eq. (D3)] are instrumental in demonstrating semiclassical exactness. To understand how, consider the action [Eq. (29)] expanded

$$S_0[B] = D \sum_{l=0}^{\infty} (-)^l \text{tr}((B^\dagger B)^l B^\dagger dB) \equiv \sum_l S^{(2l+2)}[B].$$

The expansion of these vertices out of the exponent leads to a series of vertices that all contain a single operator dB . To each of these vertices $S^{(2l+2)}$, we may associate a definite D power in the perturbative expansion. The structure of the Gaussian weight $S^{(2)}[B] = D \text{tr}(B^\dagger dB)$ implies $B \sim D^{-1/2}$, and hence $S^{(2l+2)} \sim D D^{-(2l+2)/2} = D^{-l}$.

As an example, consider the product

$$\begin{aligned} \langle (S^{(4)}[B])^2 \rangle &= D^2 \langle (\text{tr}(B^\dagger B B^\dagger dB))^2 \rangle \\ &\rightarrow 2D \text{tr}((B^\dagger B)^2 B^\dagger dB) = -2S^{(6)}[B], \end{aligned}$$

of two D^{-1} operators, where the arrow means that we picked an operator dB for contraction using the second of Eqs. (D3). This one-step reduction exemplifies a general principle: The linkage of vertices contributing to the expansion at a given order in D^{-1} (presently two D^{-1} operators contributing at D^{-2} level) leads to other operators of the same order (a single D^{-2} operator), showcasing “perturbative renormalizability” of the model.

The instantaneous (in time) cancellation of all fluctuation diagrams in the expansion of the functional can be demonstrated order by order in powers of D^{-1} . Referring to Ref. [27] for the general proof, Fig. 14 illustrates the cancellation for the orders D^{-2} (first line) and D^{-3} (second line). In this way, the expansion of the B functional around the standard saddle point, $B = 0$, reduces to the quadratic functional with action $S^{(2)}[B]$.

For completeness, we outline how this reduction leads to the semiclassical SFF. The final integration over the reduced generating functional yields $\det(d)^{-t}$, i.e., the t -th power of the determinant of an operator containing 1 on the diagonal, and $\exp(i\phi)$ on the next-leading diagonal. (The prefactor D

$$\frac{1}{2} \left(\text{diamond} \right)^2 - \text{hexagon} = 0$$

$$\frac{1}{6} \left(\text{diamond} \right)^3 - \text{diamond} - \text{hexagon} + \text{octagon} = 0$$

FIG. 14. The cancellation of contributions to the D^{-1} expansion of the functional illustrated for the two lowest orders, D^{-2} (top) and D^{-3} bottom. Polygons represent $\text{tr}(BB^\dagger \dots B^\dagger dB)$ traces with $2l$ corners matching the added number of B - and $B^\dagger$ -matrices. The vertex representing dB is indicated by a red dot, and contractions are only possible between vertices of different number parity relative to the dotted vertex, i.e., between B 's and $B^\dagger$'s. The contraction between the dotted B in a polygon of degree $2l$ and any one of the k $B^\dagger$ -matrices in a $2k$ -polygon yields k identical $2(k + l - 1)$ polygons, explaining the cancellation in the two lines of the figure.

multiplying d in the action cancels out due to the normalization of the integral, $\int dB \exp(-D \text{tr}(B^\dagger B)) = 1$.) A straightforward calculation shows that $\det(d)^{-t} = \exp(-t \ln(1 - e^{i\phi t})) = \exp(t e^{i\theta} + \mathcal{O}(e^{2i\theta})) = 1 + t e^{i\theta} + \mathcal{O}(e^{2i\theta})$. Substituting this result into Eq. (D1), we obtain $K(t) = t$.

2. Interaction vertex beyond leading order perturbation theory

We here demonstrate the vanishing of the interaction vertex on field configurations satisfying the synchronization condition. To this end, consider the expansion [Eq. (B3)] of the interaction between two neighboring qudits with Goldstone modes denoted by B and C , respectively. We have seen [cf. the discussion below Eq. (46)] how the leading quartic contribution to the interaction vanishes on the Hadamard mode $X_{mn} = (B \odot C)_{mn}$.

In the remainder of this Appendix, we show that this feature generalizes to all contributions to the sum above, and in this way demonstrate that the Hadamard mode is a zero mode of the interaction vertex. To this end, we decouple the quartic interaction via the construction outlined in Sec. VB. The removal of the interaction via a discrete time-dependent gauge transformation leads to the appearance of coherence-inducing phase factors in all contributions to the action that cannot be expressed solely in terms of the X mode. Focusing on late-time physics, we ignore all these in the following, thus effecting field reductions such as

$$(BB^\dagger)_{nn}(CC^\dagger)_{nn} = B_{no}B_{on}^\dagger C_{np}C_{pn}^\dagger \rightarrow B_{no}B_{on}^\dagger C_{no}C_{on}^\dagger$$

$$= X_{no}X_{on}^\dagger = (XX^\dagger)_{nn},$$

$$(BB^\dagger)_{mn}^p(CC^\dagger)_{mn}^q \rightarrow \delta^{pq}(XX^\dagger)_{mn}^p.$$

Using these rules, the interaction vertex reduces to

$$S_{i,1}[X] = \frac{\Gamma D^2}{2} \sum_n \sum_{q=1}^{\infty} \text{tr}((XX^\dagger)_{nn}^q + (X^\dagger X)_{nn}^q) = \Gamma D^2 \text{tr} \left(\frac{XX^\dagger}{1 - XX^\dagger} \right),$$

$$S_{i,2}[X] = \frac{\Gamma D^2}{2} \sum_{n,m} \sum_{p,r=0}^{\infty} \text{tr}((XX^\dagger)_{mn}^{p+1}(XX^\dagger)_{nm}^{r+1} + (X^\dagger X)_{mn}^{p+1}(X^\dagger X)_{nm}^{r+1}$$

$$- (X(X^\dagger X)^p)_{mn}(X^\dagger(XX^\dagger)^r)_{nm} - (X^\dagger(XX^\dagger)^p)_{mn}(X(X^\dagger X)^r)_{nm})$$

$$= \Gamma D^2 \text{tr} \left(XX^\dagger(XX^\dagger - 1) \left(\frac{1}{1 - XX^\dagger} \right)^2 \right).$$

It is easily checked that $S_{i,1}[X] + S_{i,2}[X] = 0$, confirming the statement above.

APPENDIX E: VARIABLE CHANGE TO THE HADAMARD MODE

To pass from an integration over L synchronized B -field variables to that over a single X in Eq. (51), we introduce a Lagrange multiplier action

$$S_{\text{LM}}[B, X, \Lambda] = \text{tr}(\Lambda(X - \odot_j B_j)) + \text{tr}((X^\dagger - \odot_j B_j^\dagger)\Lambda^\dagger),$$

where the trace is over discrete time space. This locking leads to the representation

$$K(t) \rightarrow D^{2L} \int DX D\Lambda \langle e^{-S_{\text{LM}}[B,X,\Lambda]} \rangle_B (XX^\dagger)_{t0} (X^\dagger X)_{0t},$$

enabling the Gaussian integral over the B_j 's with action [Eq. (29)]. As a result, we obtain

$$\langle e^{-S_{\text{LM}}[B,X,\Lambda]} \rangle_B = e^{-\text{tr}(X\Lambda + \Lambda^\dagger X) + \sum_{n,m} \Lambda_n \Pi_{nm}^L \Lambda_m^\dagger},$$

where the propagator is defined in Eq. (33), and we use the shorthand notation $n = (n_+, n_-)$. Crucially, the product of L single-qudit propagator matrix elements [Eq. (33)] equals $\Pi_{nm}^L \equiv \Pi_{Lnm}$, where Π_L is defined by Eq. (33) with the replacement $D \rightarrow D^L$, i.e., the propagator of a single ergodic quantum system with D^L levels. This identification enables us to perform the integration over Λ , leading to Eq. (52).

APPENDIX F: INCLUDING SYMMETRY EXCHANGE CORRELATIONS

We here describe how to advance from the symmetry exchange interaction represented in quasiclassical Green's functions [Eq. (57)] to the final result [Eq. (58)] (without

relying on the commutativity feature discussed in the main text). We start by using Eq. (B2), where $Q = \{Q^{uv}\}$ and $B = \{B^{uv}\}$ carry symmetry space structure, followed by leading order expansion in generators to identify the quartic symmetry exchange vertex

$$S_s^{(4)}[B] \equiv -\frac{i\Gamma_s}{2} \sum_{\langle j,k \rangle} \text{tr}(B_T B^\dagger T^a)_{jnn} \text{tr}(B_T B^\dagger T^a)_{knn} - (B_T \leftrightarrow B^\dagger).$$

We next decouple the two traces contributing to the vertex by a Hubbard-Stratonovich field $\theta = \{\theta_{jn}^{sa}\}$, in analogy to our discussion of the likewise quartic qudit correlation in Appendix C:

$$S_s^{(4)}[B] \rightarrow \frac{1}{2i\Gamma_s} \sum_n \left(\sum_s s \theta_n^{saT} M \theta_n^{sa} + D \theta_n^{+a} \text{tr}(B_T B^\dagger T^a)_{nn} + D \theta_n^{-a} (B^\dagger B_T T^a)_{nn} \right), \quad (\text{F1})$$

where the matrix M is defined in Eq. (C2). The second line couples the symmetry generators into the quadratic B -action, which now reads

$$S_0^{(2)}[B, \theta] = D \sum_j \text{tr}(B^\dagger (B - B_T) + B^\dagger (T^a \theta^{+a}) B_T + B^\dagger B_T (T^a \theta^{-a}))_j.$$

Remembering that $\theta = \{\theta_n\}$ carries structure in discrete time space, this is again a case for elimination by discrete gauge transformation. We change our fields in a manner resembling Eq. (C5),

$$B_{mn} \rightarrow U_m^+ B_{mn} U_n^-, \quad B_{nm}^\dagger \rightarrow (U_n^-)^{-1} B_{nm}^\dagger (U_m^+)^{-1},$$

where $U = \{U_j^{uv}\}$ are matrices in symmetry space and carry a site index. Substituting this ansatz into the action and requiring the elimination of θ to first order in a small- θ expansion leads to the condition

$$dU_n^+ = -U_n^+ (\theta_n^{+a} T^a), \quad dU_n^- = -(\theta_n^{-a} T^a) U_n^-,$$

where we temporarily removed the site index for transparency, and $dU_n = U_n - U_{n-1}$. This is the discrete version of a non-abelian differential equation, involving operators not commuting at different instances of time, n . It is solved by the likewise discrete version of an (anti)time ordered exponential,

$$U_n^s = T_s \exp \left(- \sum_{m=0}^n \theta_m^{sa} T^a \right), \quad (\text{F2})$$

where the time ordering protocol is defined as

$$T_+ \exp \left(\sum_{m=0}^n X_m \right) \equiv \sum_k \sum_{m_1=0}^n \sum_{m_2=0}^{m_1} \cdots \sum_{m_k=0}^{m_{k-1}} \begin{cases} X_{m_k} X_{m_{k-1}} \cdots X_{m_1}, & s = +1, \\ X_{m_1} X_{m_2} \cdots X_{m_k}, & s = -1. \end{cases}$$

We are now in the same situation as before, the action has become θ -independent, while the source terms pick up a gauge

factor. Specifically, the SFF transforms as

$$K(t) \rightarrow D^{2L} \prod_j \langle \text{tr}(U_t^+ B B^\dagger)_{j,t,0} \text{tr}(U_t^- B^\dagger B)_{j,0,t} \rangle_{B,\theta}.$$

Assuming that we are considering timescales $t \gg \Gamma^{-1}$, where relaxation to the globally synchronized mode $X = \bigodot_j B_j$ has taken place, the integral over B yields

$$K(t) = t \left\langle \prod_j \text{tr}(U_{j,t}^+) \text{tr}(U_{j,t}^-) \right\rangle_\theta.$$

Here, the product is over $2L$ independent traces over $\mathbb{C}^K$, where the matrices $U_{j,t}^\pm = U_{j,t}^\pm(T^a)$ depend on the generators represented in $\mathbb{C}^k$. However, for our purposes it is more natural to turn to a tensor space representation

$$\prod_j \text{tr}(U_{j,t}^\pm(T^a)) = \text{tr} \left(\prod_j U_{j,t}^\pm(T_j^a) \right),$$

where the trace on the right-hand side now is over the tensor space $(\mathbb{C}^K)^{\otimes L}$, and T_j^a [cf. Eq. (55)] are mutually commuting operators acting in this space. With the evolution operators given by Eq. (F2), we obtain

$$K(t) = t \text{tr} \left(T e^{-\sum_j \sum_{m=0}^t \theta_{jm}^{+a} T_j^a} \right)_{\theta^+} \times (+ \leftrightarrow -),$$

where the time ordering protocol applies to each of the L exponentials individually. Expanding the time ordered exponential functions, and doing the contractions of the θ -variables according to the second moment

$$\langle \theta_{jm}^{ra} \theta_{kn}^{sb} \rangle = is \Gamma_s \delta^{rs} \delta^{ab} \delta_{mn} M_{jk}, \quad (\text{F3})$$

implied by Eq. (F1), this yields Eq. (58).

APPENDIX G: CONTINUOUS TIME THEORY

In this Appendix, we detail the construction of various elements of the continuous time theory.

1. Noninteracting theory

To obtain the noninteracting part of the continuous time action, we substitute the stationary phase configurations $G = T \bar{G} T^{-1}$ and $\Sigma = \lambda^2 G$ into the action (59). Noting that the T rotations do not couple to the second and third terms in the first line due to cyclic invariance, we focus on the first term. A leading order gradient expansion then yields

$$\begin{aligned} S_0[T] &\equiv -D \sum_j \text{tr} \ln(i\partial_t - \lambda^2 T \bar{G} T^{-1})_j \\ &= -D \sum_j \text{tr} \ln(i\partial_t - \lambda^2 \bar{G} - iT^{-1}[\partial_t, T])_j \\ &\approx -\frac{D}{\lambda} \sum_j \text{tr}(\tau_3 T^{-1}[\partial_t, T])_j, \end{aligned}$$

where in the second line we used the cyclic invariance of $\text{tr} \ln$ and in the third expanded in the symmetry-breaking operator $T^{-1}[\partial_t, T]$ using the stationary phase equation (60) with the solution $\bar{G}_{tt} = -is\tau_3$ [Eq. (61)]. Representing the trace through time integrals, we arrive at Eq. (62).

2. Interaction vertex

Next, consider the interaction

$$\begin{aligned} S_i[G] &= \frac{\Lambda^2 D^2}{2} \sum_{\langle j,k \rangle} \text{tr}((G_j \odot G_k) \tau_3 (G_j \odot G_k) \tau_3) \\ &= \frac{\Lambda^2 D^2}{2} \sum_{\langle j,l \rangle} \sum_s \int_{t'} ((G_j \odot G_k)_{tt'}^{ss} (G_j \odot G_k)_{t't}^{ss} \\ &\quad - (G_j \odot G_k)_{tt'}^{s\bar{s}} (G_j \odot G_k)_{t't}^{s\bar{s}}). \end{aligned}$$

$$\begin{aligned} S_i^{(4)}[B] &\approx 2\Lambda^2 D^2 \sum_{\langle j,k \rangle} \int_{t'} \\ &\quad \times (((B[B^\dagger \bar{G}^+ - \bar{G}^- B^\dagger])_{j,tt'} (B[B^\dagger \bar{G}^+ - \bar{G}^- B^\dagger])_{l,tt'}) \bar{G}_{t't}^{+2} + ((B^\dagger [B \bar{G}^- - \bar{G}^+ B])_{j,tt'} (B^\dagger [B \bar{G}^- - \bar{G}^+ B])_{k,tt'}) \bar{G}_{t't}^{-2} \\ &\quad - (B \bar{G}^- - \bar{G}^+ B)_{jtt'} (B \bar{G}^- - \bar{G}^+ B)_{ktt'} (B^\dagger \bar{G}^+ - \bar{G}^- B^\dagger)_{t't} (B^\dagger \bar{G}^+ - \bar{G}^- B^\dagger)_{kt't}), \end{aligned}$$

where the first and second lines represent the contribution of the term with identical and opposite indices, i.e., the out and the in term, respectively. We simplify these expressions by assuming slow variation of the B fields, which allows us to approximate their products with the quickly decaying mean-field Green's functions as, e.g.,

$$(B \bar{G}^-)_{t_+, t_-} = \int dt B_{t_+, t} \bar{G}_{t, t_-}^- \approx -\frac{i}{\lambda} B_{t_+, t_-},$$

i.e., each G^s in a noncausality violating expression enters as a factor is/λ . Similarly, products of two Green's functions featuring in the out term $\bar{G}_{t't}^{\pm 2} \propto \exp(-2\lambda|t - t'|) \rightarrow 1/2\lambda$ upon time integration. In this way, we obtain the simplified structure [Eq. (65)].

3. Summing over interaction processes

In this Appendix, we adapt our treatment of interactions in Appendix C to the continuous time framework. Essentially, this amounts to taking continuum limits of the formulas discussed there. We display them here for the sake of reusability of the formalism. Our starting point is the continuum action (65) represented in a manner analogous to Eq. (C1):

$$S_{\text{int}}^{(4)}[B] = \frac{2\Gamma D^2}{\lambda} \sum_{\langle j,k \rangle} \int_{tu} (BB^\dagger)_{jt} F_{tu} M_{jk} (BB^\dagger)_{ku},$$

where $(BB^\dagger)_t \equiv B_t B_t^\dagger \equiv B_{t_+, t_-} B_{t_-, t_+}^\dagger$ and

$$F_{tu} = \lambda^{-1} \delta_{t_+, u_+} + \lambda^{-1} \delta_{t_-, u_-} - 8\lambda^{-2} \delta_{t_+, u_+} \delta_{t_-, u_-}.$$

Next introduce the Hubbard-Stratonovich field ϕ_{jt} with action

$$S[\Phi] = -\frac{\lambda}{8\Gamma} \sum_{\langle j,k \rangle} \int_{tu} \phi_{jt} M_{jk}^{-1} F_{tu}^{-1} \phi_{ku},$$

and resulting second moment

$$\langle \phi_{jt} \phi_{ku} \rangle = -4\Gamma M_{jk} F_{tu}. \quad (\text{G1})$$

A shift $\phi_{jt} \rightarrow \phi_{jt} + \frac{4\Gamma D}{\lambda} \int_u \sum_k F_{tu} M_{jk} (BB^\dagger)_{ku}$ removes the quartic interaction but adds a linear $\phi(BB^\dagger)$ coupling, so that

We substitute the representation $G = T \bar{G} T^{-1}$ with the mean-field solution [Eq. (61)] and the T matrices given by Eq. (12) into this action. The resulting expressions can be organized according to the number of B -matrices. To zeroth and second order, the action vanishes as a consequence of causality. These terms contain the saddle point Green's function in combinations $(\bar{G}_{t't}^s)^2 (\bar{G}_{t't}^s)^2$, which evaluate to zero because $\Theta(s(t - t'))\Theta(s(t' - t)) = 0$. Terms of first and third order vanish because of the off-diagonal Keldysh block structure of the matrix T . The expansion thus starts at fourth order, as in the discrete time model:

the quadratic action is now given by

$$S^{(2)}[B, \Phi] = D \sum_j \int_t B_{jt}^\dagger \left(\frac{2}{\lambda} \partial_t - \phi_{jt} \right) B_{jt},$$

where the time derivative acts on the center coordinate: $\partial_t B_t = (\partial_{t_+} + \partial_{t_-}) B_{t_+, t_-}$. We remove the "potential" ϕ via the gauge transformation

$$B_t \rightarrow e^{\Theta_t} B_t, \quad B_t^\dagger \rightarrow B_t^\dagger e^{-\Theta_t},$$

with

$$\Theta_t = \Theta_{t_+, t_-} = \frac{\lambda}{2} \int_{-\infty}^0 ds \phi_{t_+, s, t_-, s+}.$$

The continuum time analog of the gauged form factor (C6) then reads

$$\begin{aligned} K(t) &\rightarrow \frac{D^{2L}}{\lambda^{3L}} \int_\tau \prod_j \langle (B_{t\tau} B_{\tau 0}^\dagger B_{0(t-\tau)}^\dagger B_{(t-\tau)t})_j \rangle_B \\ &\quad \times \langle e^{\sum_j (\Theta_{t\tau} - \Theta_{0\tau} - \Theta_{(t-\tau)0} + \Theta_{(t-\tau)t})_j} \rangle_\phi, \end{aligned} \quad (\text{G2})$$

where $\tau = \{\tau_j\}$ defines the set of qudit time integration variables (cf. Fig. 7). We simplify the exponent in analogy to Eq. (C7):

$$\begin{aligned} \Theta_{t\tau} - \Theta_{0\tau} - \Theta_{(t-\tau)0} + \Theta_{(t-\tau)t} \\ = \frac{\lambda}{2} \int_{-t}^0 ds \phi_{t+s, \tau+s+t\Theta(s+\tau)} \equiv X[\tau, \phi], \end{aligned}$$

and hence obtain for the averaged exponent,

$$\begin{aligned} \langle e^{\sum_j X[\tau_j, \phi_j]} \rangle_\phi &= e^{\frac{1}{2} \sum_{j,k} \langle X[\tau_j, \phi_j] X[\tau_k, \phi_k] \rangle_\phi} \\ &= e^{-\frac{\Gamma\lambda}{2} \sum_{\langle j,k \rangle} \int_{-t}^t ds du F_{(t+s)(\tau_j+s+t\Theta(s+\tau_j))} (t+u)(\tau_k+u+t\Theta(u+\tau_k))} \\ &= e^{-\Gamma t \sum_{\langle j,k \rangle} (1 - \frac{8}{\lambda} \delta_{\tau_j - \tau_k})}. \end{aligned}$$

We finally recall that all δ functions are smeared over intervals $\sim \lambda^{-1}$. We also know that for a synchronized configuration $\theta_j = \text{const}$, interactions will not dephase the form factor. Using this condition to fix the normalization of the last remaining

δ function as $1 = 8\lambda^{-1}\delta_0$, we identify the exponent with the function specified below [Eq. (66)].

In a final step, we compute the average $\langle \cdots \rangle_B$ in Eq. (G2). Each of the contractions of fields B_j over the action [Eq. (63)]

brings down a factor $\sim \lambda^2/D$. Absorbing a factor λ^L in the measure of integration over the time arguments τ_j , we arrive at Eq. (66).

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