

Hierarchy of collective ordering in swarmalator systems

Hyunsuk Hong^{1,*}, Junghyo Jo^{2,3,4} and Daniel D. Lee^{4,5}¹Department of Physics and Research Institute of Physics and Chemistry, Jeonbuk National University, Jeonju 54896, South Korea²Department of Physics Education, Seoul National University, Seoul 08826, South Korea³Center for Theoretical Physics and Artificial Intelligence Institute, Seoul National University, Seoul 08826, South Korea⁴School of Computational Sciences, Korea Institute for Advanced Study, Seoul 02455, South Korea⁵Department of Electrical and Computer Engineering, Cornell University, New York, New York 10044, USA

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We introduce an exactly solvable multispecies spherical spin model that unifies noisy Kuramoto dynamics and equilibrium swarmalator systems within a single mean-field framework. The hierarchy of collective ordering in swarmalator systems is elucidated by analytically mapping the full three-dimensional phase diagram, classifying the various collective phases, and determining their corresponding first- and second-order transition boundaries. Our results reveal that the diverse states of swarmalator systems are driven by the competition and cooperation between internal and cross-species couplings. This establishes the spherical spin model as a mathematically tractable foundation for understanding collective behavior in active phase systems and their higher-dimensional generalizations.

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Understanding how simple pairwise interactions give rise to a hierarchy of collective states is a central question in the statistical physics of oscillator and active matter systems. Swarmalators—agents that couple spatial motion to internal phase dynamics—have emerged as a striking example [1–12]. They exhibit a rich spectrum of collective patterns from disordered clouds to synchronized clusters and various non-stationary states [13–21]. Yet, despite intensive numerical and phenomenological exploration, a unified analytical account of *why* these diverse orderings arise, *how* they are connected, and *where* the transitions between them occur has remained out of reach. Existing analyses treat each state and each model—the Kuramoto model with higher-harmonic couplings, the swarmalator model, and their variants—as separate phenomena, obscuring the underlying organizing principles.

In this Letter, we address these questions within a single, exactly solvable framework: an m -species spherical spin model with general quadratic and quartic interactions. Exploiting the tractable nature of the spherical model [22–24], we derive the self-consistent mean field equations and free energy to obtain the full three-dimensional phase diagram in the space of intra- and interspecies couplings. Our analysis shows how a hierarchy of numerous collective phases follows from competition between intra- and interspecies interactions.

The model reveals that seemingly distinct synchronization phenomena are governed by the same underlying free-energy landscape. The one-species limit ($m = 1$) recovers the noisy Kuramoto model with a second-harmonic coupling [25–27], while switching off the intraspecies quartic couplings in the two-species case ($m = 2$) reduces the model to equilibrium swarmalator models [5–7]. Both appear here as special submanifolds of a broader phase diagram, and previously known transitions emerge as slices of our results. Competition between intra- and interspecies couplings organizes the entire hierarchy of collective states—a principle that can be extended to other systems, including ($m = 3$) Vicsek-type collective motion [28].

Consider an m -species swarmalator model consisting of two-dimensional spin vectors $\mathbf{s}_i^\alpha = (r_i^\alpha \cos \theta_i^\alpha, r_i^\alpha \sin \theta_i^\alpha)$, where each spin is associated with species $\alpha = 1, \dots, m$ and site $i = 1, \dots, N$. We study the following Hamiltonian system:

$$\mathcal{H} = -\frac{J_1}{2N} \sum_{\alpha=1}^m \sum_{i,j=1}^N \mathbf{s}_i^\alpha \cdot \mathbf{s}_j^\alpha - \frac{J_2^{AA}}{4N} \sum_{\alpha=1}^m \sum_{i,j=1}^N (\mathbf{s}_i^\alpha \cdot \mathbf{s}_j^\alpha)^2 - \frac{J_2^{AB}}{4N} \sum_{\substack{\alpha, \alpha'=1 \\ \alpha \neq \alpha'}}^m \sum_{i,j=1}^N (\mathbf{s}_i^\alpha \cdot \mathbf{s}_j^\alpha)(\mathbf{s}_i^{\alpha'} \cdot \mathbf{s}_j^{\alpha'}), \quad (1)$$

which is governed by three coupling parameters: J_1 , the quadratic coupling; J_2^{AA} , the intraspecies quartic coupling; and J_2^{AB} , the interspecies quartic coupling. When the spins are restricted to Ising variables, $s_i^\alpha \in \{-1, +1\}$, the Hamiltonian in Eq. (1) with $m = 2$ reduces to the mean-field Ashkin-Teller model [29]. Note that adopting an equilibrium Hamiltonian model implies that any nonreciprocal, time-irreversible behaviors are overlooked.

*Contact author: hhong@jbnu.ac.kr

To clarify the role of each interaction term, we project the Langevin spin dynamics onto the tangential (angular) directions:

$$\begin{aligned} r_i^\alpha \dot{\theta}_i^\alpha = & \frac{1}{N} \sum_{j=1}^N \left[J_1 r_j^\alpha \sin(\theta_j^\alpha - \theta_i^\alpha) + \frac{J_2^{AA}}{2} r_i^\alpha r_j^{\alpha 2} \sin 2(\theta_j^\alpha - \theta_i^\alpha) \right. \\ & \left. + J_2^{AB} \sum_{\alpha' \neq \alpha} r_i^{\alpha'} r_j^{\alpha'} \cos(\theta_j^{\alpha'} - \theta_i^{\alpha'}) \sin(\theta_j^\alpha - \theta_i^\alpha) \right] \\ & + \xi_i^\alpha(t), \end{aligned} \quad (2)$$

where $\xi_i^\alpha(t)$ is Brownian noise with zero mean and $\langle \xi_i^\alpha(t) \xi_j^{\alpha'}(t') \rangle = 2k_B T \delta_{ij} \delta_{\alpha\alpha'} \delta(t - t')$ at temperature T .

Under the XY model with constraints $r_i^\alpha = 1$, the J_1 term in Eq. (2) is identical to that of the Kuramoto model, the paradigmatic model of synchronization in coupled oscillator systems [13–21]. For $J_1 > 0$, this interaction drives each species toward phase synchronization, $R_1^\alpha \equiv |\langle e^{i\theta^\alpha} \rangle| > 0$.

The quartic interactions are characterized by two independent coupling parameters governing the intra- and interspecies interactions. The intraspecies coupling, J_2^{AA} , is a second-harmonic coupling of the Kuramoto model that naturally favors nematic ordering, characterized by $R_2^\alpha \equiv |\langle e^{2i\theta^\alpha} \rangle| > 0$, with $R_1^\alpha = 0$ [25–27]. The system thus supports two distinct synchronization transitions: polar ordering induced by the J_1 term and nematic ordering induced by the J_2^{AA} term.

The other quartic interaction is modulated by the interspecies coupling J_2^{AB} in Eq. (2). For $J_1 = 0$, this term coincides with the equation of motion of the *equilibrium* one-dimensional swarmalator model by coupling the phase and spatial angular variables [6,7]. The sine term promotes phase synchronization in species α , while the cosine term modulates the effective coupling strength with the phase alignment in species α' . Consequently, swarmalators synchronize more strongly when the companion spins are also closely aligned.

The mean-field order parameters for Eq. (1) are given by the mean spin polarizations $S_\alpha \in \mathbb{R}^2$ and second-order spin moments $Q_{\alpha\alpha'} \in \mathbb{R}^{2 \times 2}$:

$$S_\alpha = \frac{1}{N} \sum_{i=1}^N s_i^\alpha, \quad Q_{\alpha\alpha'} = Q_{\alpha'\alpha}^\top = \frac{1}{N} \sum_{i=1}^N s_i^\alpha s_i^{\alpha'\top}. \quad (3)$$

In the present work, we focus on $m = 2$ spin species ($\alpha = \{a, b\}$) with order parameters $S = [S_a; S_b] \in \mathbb{R}^4$ and $Q = [Q_{aa} \ Q_{ab}; Q_{ba} \ Q_{bb}] \in \mathbb{R}^{4 \times 4}$. Rather than fixing the length of each individual spin, $r_i^\alpha = 1$, we instead impose the spherical constraint $\sum_{i=1}^N \|s_i^\alpha\|^2 = \sum_{i=1}^N (r_i^\alpha)^2 = N$ for each species α , allowing the individual spin amplitudes to fluctuate around their mean squared average. The spherical model is a long established soft-spin generalization of fixed-length spin models that admits exact solutions [22–24].

We compute the partition function for the spherical model, $Z = \int \prod_{\alpha,i} ds_i^\alpha e^{-\beta \mathcal{H}}$, with $\beta \equiv 1/(k_B T)$, in the thermodynamic limit $N \rightarrow \infty$. This gives the self-consistent mean-field equations (see the Supplemental Material [30] for derivations):

$$S = \beta J_1 C S, \quad (4)$$

$$\begin{aligned} [C^{-1}]_{\alpha\alpha'} &= \delta_{\alpha\alpha'} [k_\alpha I - \beta J_2^{AA} Q_{\alpha\alpha}] \\ &\quad - (1 - \delta_{\alpha\alpha'}) \beta J_2^{AB} Q_{\alpha\alpha'}, \end{aligned} \quad (5)$$

where $C \equiv Q - S S^\top$ denotes the connected correlation matrix and $[C^{-1}]_{\alpha\alpha'}$ is the (α, α') block of the matrix inverse of C . k_α are Lagrange multipliers enforcing the spherical constraints, $\text{Tr } Q_{\alpha\alpha} = 1$. The corresponding free energy per site is given by

$$\begin{aligned} \beta f \equiv -\frac{1}{N} \ln Z = & \frac{\beta J_2^{AA}}{4} (\text{Tr } Q_{aa}^2 + \text{Tr } Q_{bb}^2 - 2) \\ & + \frac{\beta J_2^{AB}}{2} \text{Tr } Q_{ab}^\top Q_{ab} - \frac{1}{4} \text{Tr } C^{-1} \\ & - \frac{1}{2} \ln \det C - 2 \ln 2\pi. \end{aligned} \quad (6)$$

The mean-field equations admit multiple solutions and each distinct phase is depicted in Fig. 1, showing the most stable phase with lowest free energy as a function of $(\beta J_1, \beta J_2^{AA}, \beta J_2^{AB})$. For small values of βJ_1 , several phases emerge, all characterized by the absence of polar ordering, $S_a = S_b = 0$. The *disordered* (D) phase has an isotropic covariance, $Q_{aa} = Q_{bb} = \frac{1}{2}I$, and no interspecies correlations, $Q_{ab} = 0$. The D phase corresponds to the incoherent (desynchronized) state in swarmalator systems, where neither the spatial nor the internal-phase variables exhibit collective ordering.

The other two phases without polar synchronization exhibit nematic ordering. Their second-order moments are anisotropic, $Q_{aa} \neq \frac{1}{2}I$ and $Q_{bb} \neq \frac{1}{2}I$, and characterized by splitting of the eigenvalues of Q_{aa} and Q_{bb} . We denote the nematic phase without interspecies correlations, $Q_{ab} = 0$, as the *nematic-independent* (NI) phase. In swarmalator systems, this corresponds to a state in which spatial and phase nematic ordering emerge independently while remaining statistically uncorrelated. In contrast, when interspecies correlations develop so that $Q_{ab} \neq 0$, the system enters the *nematic-correlated* (NC) phase, where the nematic ordering of the spatial and internal-phase variables becomes correlated through the interspecies interaction. The corresponding swarmalator state exhibits correlated spatial and phase organizations, reflecting the coupling between the two dynamical phase variables.

There are also several other phases that exhibit some degree of polarization, $S_\alpha \neq 0$. In the symmetric *polar-polar* (PP) phase, the phases of the spin species may be rotated so that the polarizations are equal, $S_a = S_b$. This phase in swarmalators corresponds to a synchronized state in which both the spatial distribution and the internal phases are collectively ordered, reflecting simultaneous spatial aggregation and phase synchronization. There are also two interesting phases of broken symmetry that emerge when one species becomes polarized, $S_a \neq 0$, while the other remains unpolarized, $S_b = 0$. We refer to these phases as the *polar-disordered* (PD) phase when $Q_{bb} = \frac{1}{2}I$, and the *polar-nematic* (PN) phase when $Q_{bb} \neq \frac{1}{2}I$. There also exists a broken symmetry *polar-polar'* (PP') phase as a solution of the mean-field equations with unequal polarizations, $S_a \neq S_b \neq 0$. However, for the present spherical Hamiltonian, the PP' phase is only metastable. In swarmalator systems, the PD and PN phases correspond to partially synchronized states in which only one of the two dynamical variables develops polar ordering, such as when

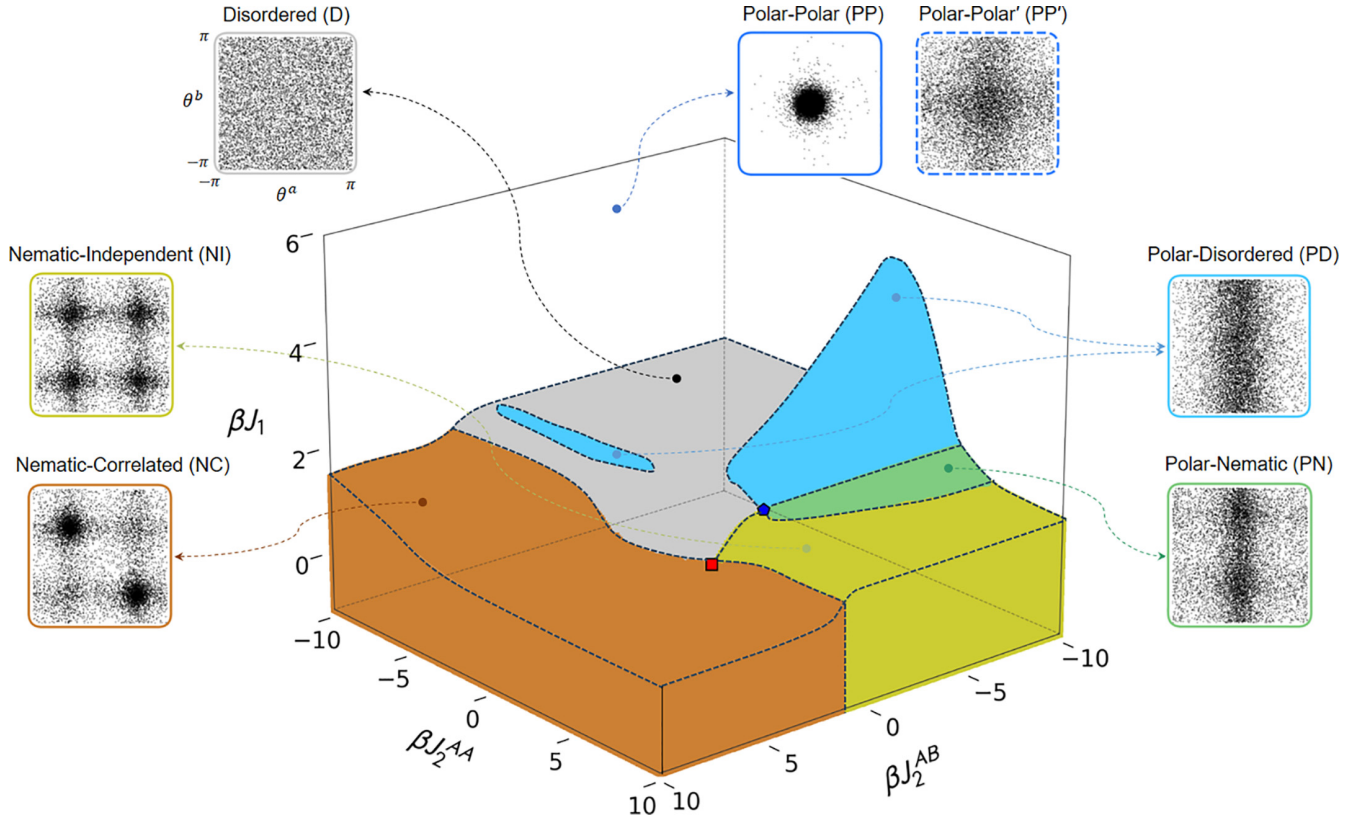


FIG. 1. Three-dimensional phase diagram in $(\beta J_1, \beta J_2^{AA}, \beta J_2^{AB})$ space for $m = 2$, with representative scatterplots of the spin angle configurations (θ^a, θ^b) for each phase. The PP region is left uncolored to reveal the internal structure of the diagram; the symmetry-broken PP' phase with unequal polarizations is only metastable. The blue pentagon marks the quintuple point where five phases (D, NI, PD, PN, and PP) coexist, and the red square marks where four phases (D, NI, NC, and PP) coexist.

fireflies or Japanese tree frogs exhibit synchronized flashing (or calling) while remaining spatially disordered.

The phase diagram contains two multicritical points, marked by a blue pentagon and a red square in Fig. 1: Five phases (D, NI, PD, PN, and PP) coexist at $(\beta J_1, \beta J_2^{AA}, \beta J_2^{AB}) \approx (1.665, 4, -0.05)$ (blue pentagon), while four phases (D, NI, NC, and PP) coexist at $(\beta J_1, \beta J_2^{AA}, \beta J_2^{AB}) \approx (0.95, 4, 2.5)$ (red square) (see the Supplemental Material [30] for details). At these points, the thermodynamic driving forces for polar phase synchronization and nematic ordering become degenerate and the energy contribution from the cross-species coupling (βJ_2^{AB}) is necessary to compensate for any entropy lost in breaking the rotational symmetry between the spin species.

The self-consistent equation for S in Eq. (4) implies that only the nonpolarized phases D, NI, and NC can exist for $\beta J_1 < \frac{1}{2}$; their transitions are shown in Fig. 2(a). The dashed lines denote the second-order phase boundaries: D-NI, $\beta J_2^{AA} = 4$; D-NC, $\beta J_2^{AB} = 4$; and the NI-NC boundary $\sqrt{\beta J_2^{AB}} = 2/[1 + \sqrt{1 - 4/(\beta J_2^{AA})}]$, respectively (see the Supplemental Material [30] for derivations). At intermediate coupling strengths, however, these continuous transitions are preempted by a first-order line (solid).

Figure 2(b) shows the phase diagram for $J_2^{AB} = 0$, where the two species decouple and the model reduces to the spherical counterpart of the noisy Kuramoto model with

second-harmonic coupling. The noisy Kuramoto model has previously been studied via a truncated Fourier-mode expansion of the Fokker-Planck equation (see the Supplemental Material [30] for a detailed comparison) [31–35]. Along the line $\beta J_2^{AA} = 0$, the model further reduces to the conventional noisy Kuramoto model, whose synchronization transition between D and PP at $\beta J_1 = 2$ (horizontal dashed line) has been extensively studied [36–38]. We also see a nematic transition that occurs at $\beta J_2^{AA} = 4$ (vertical dashed line separating D-NI), which can be related to the $\beta J_1 = 2$ synchronization transition by a scale transformation of the angular variables in the model [6]. However, in the XY model where individual spin amplitudes are fixed ($r_i = 1$), the critical coupling doubles to $\beta J_2^{AA} = 8$. The difference in the nematic transition between the spherical and XY constraints arises from spin amplitude fluctuations in the spherical model. The spins can stretch along the nematic axis, strengthening the response to the nematic field so that ordering occurs at weaker second-harmonic coupling compared to the XY model. The spherical constraint halves the nematic threshold, reflecting the difference in spin amplitude statistics between the two models (see the Supplemental Material for further details [30]).

Finally, there is a second-order transition (dashed line) separating NI and PP phases, $\beta J_1 = 2/(1 + \sqrt{1 - 4/(\beta J_2^{AA})})$. This line approaches $\beta J_1 \simeq 1 + 1/\beta J_2^{AA}$ for large βJ_2^{AA} and

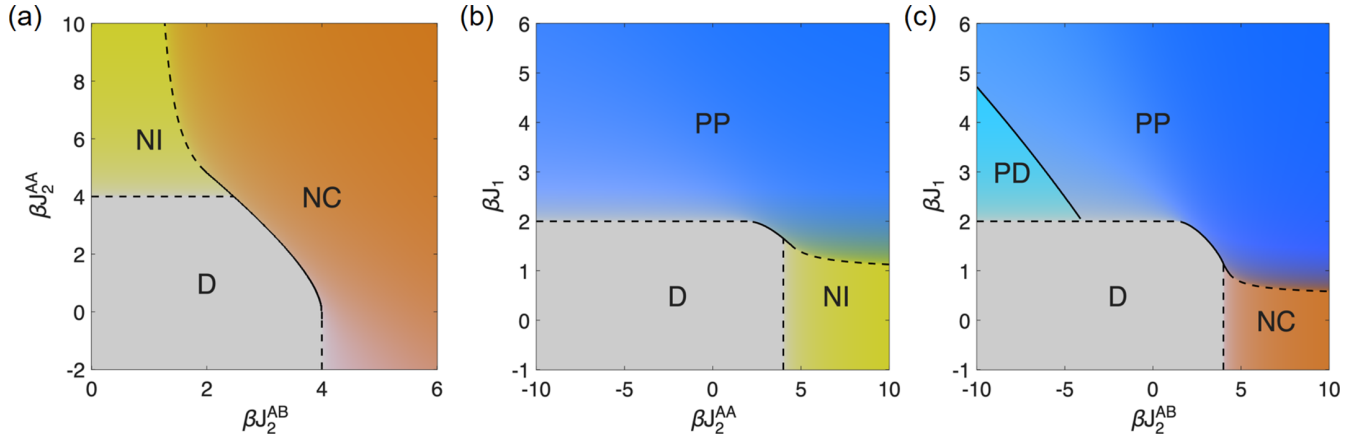


FIG. 2. Two-dimensional slices of the phase diagram at (a) $\beta J_1 = 0$, (b) $\beta J_2^{AA} = 0$, and (c) $\beta J_2^{AA} = 0$. Dashed and solid lines denote second- and first-order transitions, respectively, determined by free-energy minimization. First-order transitions occur for (a) $2.02 < \beta J_2^{AB} < 3.99$, (b) $2.26 < \beta J_2^{AB} < 4.40$, and (c) $\beta J_2^{AB} < -4.0$ and $1.57 < \beta J_2^{AB} < 4.32$; the analytic equations for the second-order transitions are described in the text. The RGB colors of the phases linearly encode the relevant order parameters.

matches a linear stability analysis of the NI phase against polar perturbations.

Figure 2(c) shows the complementary limit $J_2^{AA} = 0$, where the interspecies interaction without the second-harmonic term causes four phases to appear: D, NC, PD, and PP. This limit connects directly to the equilibrium swarmalator model: At $\beta J_1 = 0$, the D-NC transition occurs at $\beta J_2^{AB} = 4$ (vertical dashed line), in agreement with Ref. [6]. For smaller values βJ_2^{AB} , including at $\beta J_2^{AB} = 0$, the Kuramoto synchronization transition at $\beta J_1 = 2$ is recovered. There is also a second-order transition between NC and PP for $\beta J_2^{AB} > 4$ when $C_{11} = \frac{1}{2}[1 + \sqrt{1 - 4/(\beta J_2^{AB})}]$ and $(\beta J_1)^{-1} = C_{11} + \sqrt{C_{11}^2 - 1/(\beta J_2^{AB})}$ [30]. Notably, the symmetry-broken PD phase—in which one species develops polar order while the other remains disordered—arises in the region of negative βJ_2^{AB} . In the swarmalator context, this is a partially synchronized state in which only one of the two dynamical variables (spatial or internal-phase) orders. The symmetry-broken PP' phase also emerges as a metastable phase in the spherical model near the PD-PP boundary. Under the XY constraint, the phase diagram remains qualitatively unchanged, except that the PD-PP boundary shifts toward larger βJ_1 . With the spin amplitudes frozen, a stronger quadratic coupling is required to stabilize the fully polarized PP phase and allows the PP' phase to become stabilized.

We also note a connection between our m -species model to a spherical spin glass model $\mathcal{H} = -\frac{1}{2} \sum_{ij} J_{ij} s_i s_j$ with random Gaussian interactions, $J_{ij} = J_{ji} \sim \mathcal{N}(\frac{\bar{J}}{N}, \frac{\Delta^2}{N})$ [39]. In the replica calculation of Z^m , integrating over the bond randomness generates quartic interactions between the m replicas, analogous to interactions between the m species in our swarmalator model with parameters $J_1 = \bar{J}$ and $J_2^{AA} = J_2^{AB} = \beta \Delta^2$ (see the Supplemental Material [30] for further details).

In summary, the present work establishes an analytically solvable reference theory for collective synchronization with multiple interacting order parameters, placing Kuramoto, higher-harmonic synchronization, and equilibrium swarmalator models within a single unified framework. The

free-energy analysis of our spherical multispecies spin model yields a complete three-dimensional phase diagram, comprising numerous collective phases—disordered, nematic, cross-correlated nematic, and polar phases with full, partial, or species-selective ordering—together with closed-form expressions for the phase boundaries. The comparison with the XY-constrained model yields qualitative similarities but also some quantitative differences in the location of the transitions due to spin amplitude fluctuations. The spherical formulation thus provides an exactly solvable reference point for the hierarchy of collective ordering in swarmalator systems, just as it can provide a tractable model for randomly coupled spin systems [40]. A natural next step is to combine the two frameworks by introducing quenched randomness into the couplings of the present model, thereby opening a solvable route to better understanding glassy swarmalator systems in the future.

Beyond the present Hamiltonian formulation, several extensions would introduce time-irreversible behaviors that break detailed balance, requiring further nonequilibrium analysis techniques and numerical simulations. These include nonreciprocal couplings ($J_2^{AB} \neq J_2^{BA}$), quenched heterogeneity in natural frequencies and self-propulsion velocities [41], and pulse-driven interactions mediated by discrete signals [42,43]. Exploring these extensions could reveal complex dynamical phases such as traveling waves, active vortices, and chimera-like states that expand upon the equilibrium phases mapped in this study.

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Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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