

Unified boson sampling

Luca Bianchi^{1,*}, Carlo Marconi², Laura Ares³, Davide Bacco¹, and Jan Sperling³¹*Department of Physics and Astronomy, University of Florence, 50019 Firenze, Italy*²*Istituto Nazionale di Ottica del Consiglio Nazionale delle Ricerche (CNR-INO), 50125 Firenze, Italy*³*Theoretical Quantum Science, Institute for Photonic Quantum Systems (PhoQS), Paderborn University, Warburger Straße 100, 33098 Paderborn, Germany*

(Received 2 September 2025; accepted 13 November 2025; published 26 December 2025)

Boson sampling is a key candidate for demonstrating quantum advantage and has already yielded significant advances in quantum simulation, machine learning, and graph theory. In this work, a unification and extension of distinct forms of boson sampling is developed. The devised protocol merges discrete-variable scattershot boson sampling with continuous-variable Gaussian boson sampling. Therefore, it is rendered possible to harness the complexity of more interesting states, such as squeezed photons, in advanced sampling protocols. A generating function formalism is developed for the joint description of multiphoton and multimode light undergoing Gaussian transformations. The resulting analytical tools enable one to explore interfaces of different photonic quantum-information-processing platforms. A numerical simulation of unified sampling is carried out, benchmarking its performance, complexity, and scalability. Entanglement is characterized to exemplify the generation of quantum correlations from the nonlinear interactions of a unified sampler.

DOI: [10.1103/PhysRevResearch.7.L042068](https://doi.org/10.1103/PhysRevResearch.7.L042068)

Introduction. The development of quantum information processing [1] has been driven by the pursuit of physical platforms capable of efficiently encoding, manipulating, and transmitting quantum states while maintaining coherence and scalability [2,3]. Among the various candidate systems, quantum optical/photonic platforms [4,5] offer unique advantages due to the weak interaction of photons with their environment [6,7]. This intrinsic robustness against decoherence, coupled with well-established optical control techniques, has lifted photonics as a key architecture for quantum communication, quantum computing, quantum sensing, and quantum simulation [8,9].

Aside from single photons, which are the building blocks for discrete-variable optical quantum information [10], a particularly important class of states is squeezed states, which emerges from the family of Gaussian transformations that redistribute quantum uncertainty between conjugate quadratures of the optical field [11]. Such Gaussian states play a key role in continuous-variable quantum information [12–14]. The Gaussian nature of these states allows for deterministic state preparation and transformation via linear optics and homodyne detection [15].

Among all quantum technologies, boson sampling, introduced by Aaronson and Arkhipov [16], is a quantum optical implementation of a computational model that may surpass

classical algorithms, overcoming the extended Church-Turing thesis [17] and reaching quantum advantage. In its original version, the protocol consists of sending single-photon states through a randomly selected linear interferometer and performing measurements at the output. The sampled probability distribution, proportional to the modulus squared permanent of the scattering matrix, falls in the $\#P$ complexity class [18]. However, boson sampling experiments mostly rely on probabilistic photon sources [19], limiting the boson sampling scalability. To address this issue, many alternative models were introduced: Scattershot boson sampling (SBS) [20–22] improves the efficiency by using heralded photon pairs, while Gaussian boson sampling (GBS) [23–25] leverages squeezed states for the production of a large number of indistinguishable photons. In the GBS, the Hafnian of the scattering matrix [26,27] built by the measured modes provides the required $\#P$ complexity. Further applications of GBS can be found in molecular vibronic spectra simulation [28], graph-based problems [29,30], state preparation [31,32], and machine learning tasks [33,34].

More recently, however, many quantum information protocols were designed on hybrid platforms where both discrete and continuous properties are exploited [35]. Integrating squeezing transformations into large-scale photonic quantum networks presents an opportunity to take advantage of the best of both encodings by entering the domain of non-Gaussian quantum states [36].

In this work, we introduce a hybrid protocol for boson sampling. The protocol entails feeding single photons to optical parametric amplifiers, followed by a Haar random linear interferometer and photodetection. To describe this unified boson sampler, we first develop an exact formalism comprising both Gaussian and photonic resources that harnesses

*Contact author: luca.bianchi@unifi.it

the expectation values for momenta to arbitrary order. Our method represents photons via derivatives of a Gaussian generating function that carries all the information about the time evolution of the multiphoton system. We prove that the unified boson sampler encompasses early sampling models, and we validate this statement by determining the distance from SBS and GBS. Furthermore, we study the computational time as a function of the squeezing and the system size, providing a hint about the complexity of the here-devised sampler. We finally compute the entanglement negativity as a function of the mean total photon number, highlighting the dependency on the input photons as well as on the squeezing intensity.

Methods. In order to describe the dynamics of an M -mode optical field, we introduce the vector of canonical variables in phase space [37], $\hat{\mathbf{a}}^T = [\hat{a}_1, \dots, \hat{a}_M, \hat{a}_1^\dagger, \dots, \hat{a}_M^\dagger]$, in such a way that canonical commutation relations can be recast as $[\hat{\mathbf{a}}, \hat{\mathbf{a}}^T] = i\mathbf{J}$ where $\mathbf{J} = -i\begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}$, with $\mathbb{0}$ and $\mathbb{1}$, respectively, denoting the null matrix and the identity. (Note that $\hat{I}$ is the identity operator acting on quantum states.) Throughout the work, bold quantities denote $2M$ -dimensional objects in the space of canonical variables.

All Gaussian transformations preserve canonical commutation relations [14], are closed under matrix multiplication, and form the complex symplectic group [38,39] $\text{Sp}_{\mathbb{C}}(2M) = \{\mathbf{T} \in \mathbb{C}^{2M \times 2M} \mid \mathbf{T}\mathbf{J}\mathbf{T}^{*T} = \mathbf{J}\}$, where we denote with $*T$ the Hermitian conjugation for phase-space matrices, while reserving the $\dagger$ symbol for operators acting on the Hilbert space of states. Therefore, any unitary operator $\hat{T}$ as a Gaussian transformation in Hilbert space can be expressed by the action of a complex symplectic matrix on vectors of canonical variables $\hat{T}\hat{\mathbf{a}}\hat{T}^\dagger = \mathbf{T}\hat{\mathbf{a}}$. Further, a generic $\mathbf{T} \in \text{Sp}_{\mathbb{C}}(2M)$ can always be written as four blocks made by $M \times M$ complex matrices [40] $\mathbf{T} = \begin{bmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{V}^* & \mathbf{U}^* \end{bmatrix}$ subject to the symplectic constraints $\mathbf{U}\mathbf{U}^{*T} - \mathbf{V}\mathbf{V}^{*T} = \mathbb{1}$, $\mathbf{U}\mathbf{V}^T = \mathbf{V}\mathbf{U}^T$. Linear interferometry ($\mathbf{V} = \mathbb{0}$) is now identified with the group of unitary transformations $\text{U}(M)$, which turns out also to be isomorphic to the maximal compact subgroup of $\text{Sp}_{\mathbb{C}}(2M)$ [41].

We may decompose Gaussian matrices with the so-called Bloch-Messiah decomposition [42] as $\mathbf{T} = \mathbf{L}\mathbf{S}\mathbf{R}$, with $\mathbf{L}, \mathbf{R}$ being M -mode unitaries and $\mathbf{S}$ taking the form $\mathbf{S} = \begin{bmatrix} \sigma_U & \sigma_V \\ \sigma_V^* & \sigma_U^* \end{bmatrix}$, with $\sigma_U = \text{diag}(\sigma_{u_1}, \dots, \sigma_{u_M})$ and $\sigma_V = \text{diag}(\sigma_{v_1}, \dots, \sigma_{v_M})$ containing the $2M$ singular values of $\mathbf{T}$, obeying $\sigma_{u_i} = \sqrt{1 + |\sigma_{v_i}|^2}$. This identity stems from the singular value decomposition [43] when applied to symplectic matrices. Thus, Bloch-Messiah states how any Gaussian operation can be decomposed into a linear interferometric transformation (unitary rotation), followed by single-mode squeezing (diagonal matrices) and another linear transformation [44].

To compactly describe input photons, we refer to the formalism of generating functions [45]. For M -mode photon states, it turns out that the matrix $\bigotimes_{k=1}^M |n_k\rangle\langle n_k|$ can be equivalently described by the operator $\hat{\Pi}_{\vec{n}} = \frac{1}{\vec{n}!} \prod_{k=1}^M \hat{E}_{x_k}^{n_k} \hat{E}(x_k)|_{\vec{x}=\vec{0}}$, with $\hat{E}(x_k) = x_k^{\hat{n}_k} =: e^{-(1-x_k)\hat{n}_k} :$, and “ $:\dots:$ ” denoting the normal ordering prescription [46]. Furthermore, we utilize a multi-index notation, namely, $\vec{x} = (x_1, \dots, x_M)$. For each mode k , $\hat{E}(x_k)$ simultaneously encodes the full information of all photon-number states. Such compact writing allows for efficient computations of statistical moments of the field

and was proven to be also useful for the description of thermal states, Fock states, and detector performance characterization. In particular, the generating function formalism can easily be adapted to describe photon losses by replacing $:\exp\{-(1-x)\}:\rightarrow:\exp\{-\eta(1-x)\}:$, with η denoting the quantum efficiency [47,48].

To achieve the normally ordered form of matrices describing propagated input states, we rely on the group theoretical properties of Gaussian Hamiltonians. In particular, for M -mode Gaussian transformations ladder operators turn out to be an infinite-dimensional representation of an $\mathfrak{sp}(2M+2)$ subalgebra [49,50]. Thus, tacking operations on group elements that only involve products is representation independent [51], meaning that the reordering of operators can be performed into a simpler, finite-dimensional representation. Combining the aforementioned technique with the Bloch-Messiah decomposition leads to a normally ordered form for

$$\hat{T} = e^{-\frac{1}{2}\hat{\mathbf{a}}^\dagger \mathbf{U}^{-*T} \mathbf{V}^T \hat{\mathbf{a}}^\dagger} e^{-\log \mathbf{U}^{*T} (\hat{\mathbf{a}}^\dagger \hat{\mathbf{a}} + \hat{I}/2)} e^{\frac{1}{2}\hat{\mathbf{a}} \mathbf{V}^{*T} \mathbf{U}^{-*T} \hat{\mathbf{a}}}, \quad (1)$$

(see the Supplemental Material [52]). Previously, similar techniques were shown to be useful, e.g., in Refs. [53–55]. From the above result, we can compute the normally ordered time evolution of photon-number states:

$$\hat{\Pi}'_{\vec{n}} = \partial_{\vec{x}}^{\vec{n}} : \frac{\exp\left\{\frac{1}{2}\hat{\mathbf{a}}^T \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{A}^* \end{bmatrix} \hat{\mathbf{a}}\right\}}{\vec{n}! |\det[\mathbf{U}]| \sqrt{\prod_{i=1}^M D_i}} : \Big|_{\vec{x}=\vec{0}}, \quad (2)$$

with

$$\begin{aligned} \mathbf{D} &= \text{diag}(\mathbb{1} - \mathbf{X}\mathbf{U}^{-*} \mathbf{V}^* \mathbf{X} \mathbf{V}^T \mathbf{U}^{-T}), \\ \mathbf{A} &= -\mathbf{V}^* \mathbf{U}^- + \mathbf{U}^{-T} \mathbf{X} \mathbf{D}^- \mathbf{V}^{*T} \mathbf{U}^{-*T} \mathbf{X} \mathbf{U}^-, \\ \mathbf{B} &= -(\mathbb{1} - \mathbf{U}^{-*T} \mathbf{X} \mathbf{D}^- \mathbf{U}^-), \end{aligned} \quad (3)$$

where $\mathbf{X} = \text{diag}(x_1, \dots, x_M)$ and a superscript “ $-$ ” denotes the inverse (cf. Supplemental Material [52]). Next, we show how this formalism unifies the description of any process involving generation, transformation, and detection of any photon-number state combined with Gaussian resources, thus subsuming previous results, e.g., in Refs. [56,57]. Notice that a complementary analysis of the above scenario has been developed in terms of loop Hafnians in Refs. [58–60].

Unified boson sampling. Having developed a formalism to analytically describe Gaussian transformations of photon states, we can incorporate both the SBS and the GBS in our model and describe the physics that lies in between; i.e., we can give a joint description of these boson sampling problems, a protocol that we dub unified boson sampling (UBS). For simplicity, let $N_{\text{in}} = N_{\text{out}} = N$ single input and output photons. By calling $\mathbf{G}$ the matrix in the exponent of Eq. (2), it can be shown (see the Supplemental Material [52]) that probability outcomes are described by the following expression:

$$\begin{aligned} P_{N \rightarrow N} &= \frac{1}{|\det[\mathbf{U}]|} \sum_{\mathbf{s}} \text{Haf}(|\mathbf{U}^{-1} \mathbf{V}|_{(\vec{\beta}, \vec{\beta})}^2) \\ &\quad \times \frac{\partial^{N-|\vec{\beta}|}}{\prod_{i \notin \mathbf{s}} \partial x_i} \text{Haf}(\mathbf{G}^{\mathbf{s}}) \Big|_{\vec{x}=\vec{0}}, \end{aligned} \quad (4)$$

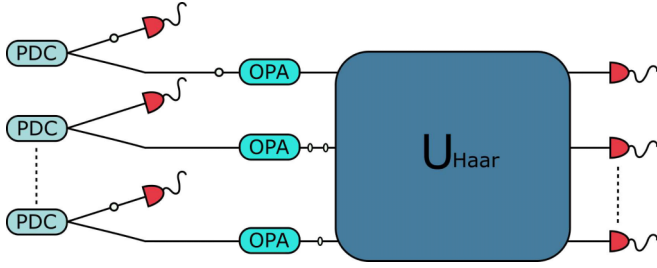


FIG. 1. Implementation of a unified boson sampler. Input photons, generated via parametric down-conversion processes (PDCs) and conditional measurements, are fed into optical parametric amplifiers (OPAs) to generate squeezed photons. The latter are mixed by a Haar-random unitary (U_{Haar}) before an array of photodetectors projects the state onto one element of the photon-number basis.

where S runs over the sets $\{1, \dots, N\}$ and $|\dots|^2$ stands for the elementwise modulus square of a matrix. Given $\tilde{\beta}$ the subset of derivation variables, the notation $(\tilde{\beta}, \tilde{\beta})$ stems from the subtraction of the $\tilde{\beta}$ rows and columns. G^S indicates the submatrix selected by the modes in which photons are detected at the output.

The UBS can be compared to the aforementioned standard boson sampling algorithms. It is proven in the Supplemental Material [52] how the probability distribution of the UBS reduces to the SBS one when sending $V \rightarrow 0$; i.e., one obtains the squared permanent of a submatrix of the (now unitary) U block of T . The GBS limiting case, i.e., the Hafnian of G^S , is more easily recovered by sending $N_{\text{in}} \rightarrow 0$. Furthermore, the result in Eq. (4) can also be rewritten using the Faà di Bruno's formula as

$$P_{N \rightarrow N} = \frac{1}{|\det[U]|} \sum_S \text{Haf}(|U^{-1}V|^2_{(\tilde{\beta}, \tilde{\beta})}) \times \sum_{\pi \in \Pi} \text{Haf}(G^S_{[i, j]}) \prod_{\substack{B \in \pi \\ (k, l) \in (i, j)}} \left. \frac{\partial^{|B|} G^S_{kl}(\bar{x})}{\prod_{r \in B} \partial x_r} \right|_{\bar{x}=\tilde{0}}, \quad (5)$$

with Π being the set of partitions of the indices $\{1, \dots, N - |\tilde{\beta}|\}$, B the size of each partition, $i = i_1, \dots, i_{|\pi|}$, $j = j_1, \dots, j_{|\pi|}$ indices belonging to each partition π , and $[i, j]$ standing for the removal of all i, j rows and columns. Equation (5) points out how the number of Hafnians that should be computed for each probability grows as the number of partitions of $N - \tilde{\beta}$ elements. This quantity scales with the Bell number of $N - \tilde{\beta}$, showing a superexponential behavior in the number of Hafnians and thus an increased complexity.

Our results are generalized to multiple photons per mode by means of MacMahon's theorem [61], by removing/adding multiple rows and columns conditioned to the number of initial as well as the number of measured photons. However, since the complexity of the Hafnian increases with the rank of the matrix [62], any boson sampling implementation usually tries to avoid multiple clicks in the same detector. For realizing UBS, we outline an experimental setup in Fig. 1, in which multiple seeded-generated single photons undergo a Gaussian transformation performed by K independent single-mode optical parametric amplifiers, followed by a

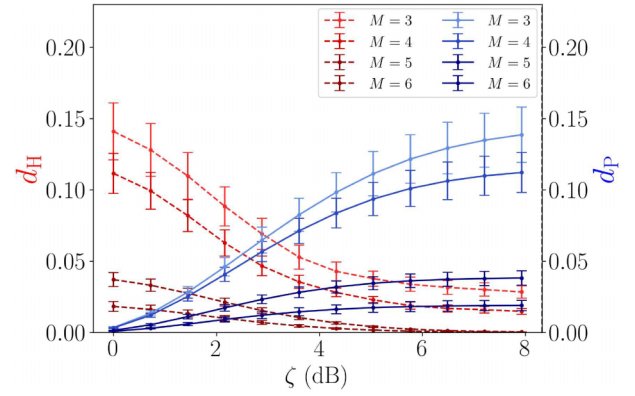


FIG. 2. Average L_1 distances between UBS and SBS (d_P) and GBS (d_H) for different numbers M of modes in the so-called saturated scenario as a function of the squeezing ζ . On the left vertical axis, we display the average distance between UBS and GBS (dashed, red curves), while on the right one between UBS probabilities and the permanents in SBS (solid, blue curves). We have a minimum distance for d_P and a maximum for d_H at $\zeta = 0$ dB (no squeezing, pure SBS regime), while we have a maximum for d_P and a minimum for d_H at $\zeta = 8$ dB (high value of squeezing intensity, approaching the GBS regime). Statistical fluctuations from the Haar-random selection are displayed via bars.

Haar-random unitary matrix [63] and a postselection onto N -photon events.

Numerical simulations. Our UBS renders possible to continuously transition from SBS to GBS, exploring the full intermediate regime. To this end, we simulate multiple rounds of the UBS. We study the L_1 distance between the UBS probabilities and the Hafnians, $d_H = |P_{\text{UBS}} - P_{\text{GBS}}|$, and similarly to the permanents, $d_P = |P_{\text{UBS}} - P_{\text{SBS}}|$, both as a function of the squeezing in the region $\zeta \in [0, 8]$ dB. Such distances show the behavior of UBS probability distributions, displaying their deviations from those associated with SBS and GBS. For derivatives, we harness automatic differentiation techniques widely popular for neural network training tasks, as they are proven to perform with high precision [64,65]. We compute the distances for 12 values of ζ in the aforementioned interval, in a range from two to six modes for a saturated scenario, defined as $N = \lceil M/2 \rceil$ and $K = M$. The same was done in a diluted scenario, where $K = N = \lceil M/2 \rceil$ [52].

We run the simulation multiple times to avoid biases in the choice of particular unitaries or photon patterns, which are randomly generated. The sample variance over Haar-random unitaries is given by the average estimator, i.e., $(\sigma/N_{\text{sample}})^{1/2}$ with σ the standard deviation for the average over the runs, and $N_{\text{run}} = 50$. Furthermore, in Fig. 2, we can observe an intersection point around $\zeta \approx 3$ dB, representing a midpoint of UBS between GBS and SBS.

To further benchmark the model, we report the computational time as a function of system size in Fig. 3. We fix the squeezing intensity to the midpoint value $\zeta \approx 3$ dB, as we observe no changes in the computational time as a function of the squeezing parameter (cf. Supplemental Material [52]). In Fig. 3, we plot the averaged computational time as a function of the input and output photon numbers N_{in} and N_{out} . We can see that, even for low dimensions, the computational

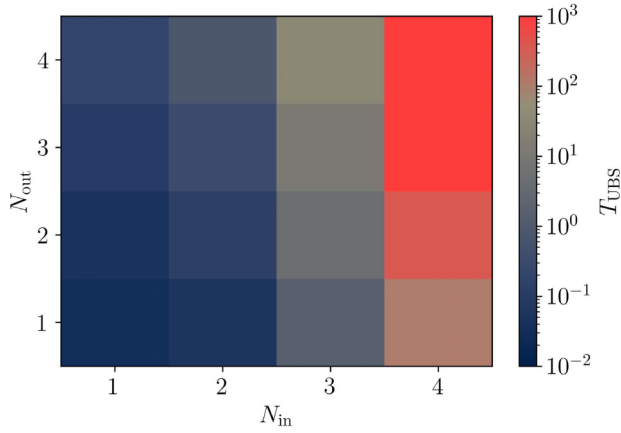


FIG. 3. Average computational time for the UBS at $\zeta \approx 3$ dB. The average was taken over ten rounds of simulation for each $(N_{\text{in}}, N_{\text{out}})$ pair. The computational time increases more significantly with the input when compared to the output, pointing out the super-exponential behavior, captured through Eq. (5).

time scales more rapidly with the number of input photons with respect to the output photon number, capturing the increased complexity given by the presence of non-Gaussian input states.

Finally, we study quantum correlations within the UBS. More specifically, we apply the Shchukin-Vogel criterium [66] to a matrix of second-order moments: The presence of negative eigenvalues on a matrix of moments certifies entanglement. We quantify the bipartite entanglement by computing the logarithmic negativity $\log_2(2\mathcal{N} + 1)$ of the quantum states, with $\mathcal{N}$ being the absolute value of the sum over negative eigenvalues of the partially transposed matrix of moments of the quantum field, i.e., $\mathcal{N} = |\sum_i \lambda_i^{(<0)}|$. We focus on an $M = 2d$ -mode scenario for $d = 6$ and different number of input photons $N_{\text{in}} = \{2, 4, 6, 8\}$ equally split between the subsystems, a case of particular interest for qudit technologies [56]. To this end, we too fix $U_{\text{Haar}} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbb{1} & i\mathbb{1} \\ i\mathbb{1} & \mathbb{1} \end{bmatrix}$, where $\mathbb{1}$ represents a $d \times d$ identity matrix. The computation of moments is given by the following formula (see the Supplemental Material [52]):

$$\begin{aligned} & \langle \hat{a}_1^{k_1} \dots \hat{a}_M^{k_M} \hat{a}_1^{\dagger l_1} \dots \hat{a}_M^{\dagger l_M} \rangle \\ &= \frac{(-1)^{\sum_{i=1}^M k_i} \partial_{\vec{x}}^{\vec{n}} \partial_{\vec{y}^*}^{\vec{k}} \partial_{\vec{y}}^{\vec{l}}}{\bar{n}! |\det[U]|} \frac{e^{\frac{1}{2} \mathbf{y}^T \mathbf{G}^{-1} \mathbf{y}}}{\sqrt{\det[-\mathbf{G}] \prod_{i=1}^M D_i}} \Big|_{\vec{y}=\vec{0}, \vec{y}^*=\vec{0}, \vec{x}=\vec{0}} \end{aligned} \quad (6)$$

with $\mathbf{y}^T = [-y_1^*, \dots, -y_M^*, y_1, \dots, y_M]$.

To perform the numerical computations, we rely on a second-order Richardson interpolation [67] of the central finite-difference technique for derivation. (For a comprehensive analysis on the error estimation of the simulation, see the Supplemental Material [52]). In Fig. 4, we show the behavior of the logarithmic negativity as a function of the mean total photon number. Each curve represents a different number of input photons, and each point on the curves stems from a particular squeezing, highlighting the dependency of the entanglement on the two different quantities. From this analysis,

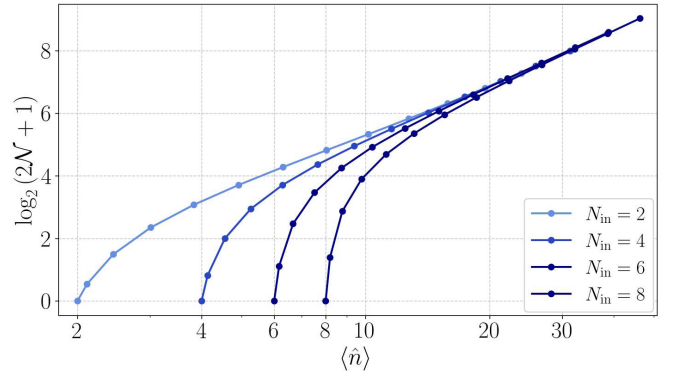


FIG. 4. Quantum correlations of the UBS. The logarithmic negativity of the unified boson sampler for the bipartite scenario is plotted as a function of the mean photon number. Different curves correspond to different total numbers of input photons in a 12-mode scenario, while points on each curve represent different squeezing intensities ζ . Deviations from the purely Gaussian regime are more marked for low squeezing values and high number of initial photons.

we can see how in the high-squeezing region, the negativity converges to a purely GBS regime, as the number of input photons becomes more negligible for increasing values of squeezing. Conversely, in the low-squeezing region, for any fixed ζ , we observe significant deviations between different N_{in} .

Conclusions. In this work, we propose an advanced model of boson sampling, called unified boson sampling, encompassing Gaussian and scattershot samplers as special cases. We develop the formalism to exactly characterize our unified sampler, harnessing the theory of generating functions and representations of the symplectic group. This allows us to compute quantum-physical quantities by means of derivatives of a normally ordered operator containing the Gaussian nature of the time evolution together with photon-number states. Then, we present a theoretical proposal to implement the devised protocol using state-of-the-art components. In numerical simulations, we quantify the distinctiveness of our UBS and the Hafnian and permanents from common samplers in terms of distances. This is done for different numbers of modes and squeezing values. We further analyze the computational time as a function of the squeezing and the system size. This quantity might serve as a benchmark of the complexity of the UBS. Finally, we certify the amount of entanglement in the UBS apparatus via the logarithmic negativity, highlighting the different contributions of photon numbers and squeezing. Future research is needed in order to thoroughly prove the potential application of the unified sampler, in particular, to test its performances and its complexity under photon losses. However, the generating function formalism straightforwardly generalizes to account for photon losses that always affect single photons and squeezed states, providing analytical and numerical tools for the description of experiments within reach.

Acknowledgments. The authors thank Nicolás Quesada for useful comments. L.B. is grateful to Paderborn University for the support. L.B. and D.B. acknowledge support from the European Union ERC StG, QOMUNE, 101077917. C.M. acknowledges support from the European Union—NextGeneration EU, “Integrated infrastructure initiative in

Photonic and Quantum Sciences”—I-PHOQS [IR0000016, ID D2B8D520, CUP B53C22001750006]. L.A. and J.S. acknowledge funding through the Ministry of Culture and Science of the State of North Rhine-Westphalia (PhoQC initiative) and the QuantERA project QuCABOoSE.

Data availability. The data that support the findings of this article are not publicly available because they contain commercially sensitive information. The data are available from the authors upon reasonable request.

-
- [1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2001), Vol. 2.
 - [2] C. H. Bennett and D. P. DiVincenzo, Quantum information: An introduction, *Nature (London)* **404**, 247 (2000).
 - [3] M. A. Khan, M. N. Aman, and B. Sikdar, Beyond bits: A review of quantum embedding techniques for efficient information processing, *IEEE Access* **12**, 46118 (2024).
 - [4] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, 1995).
 - [5] W. Vogel and D.-G. Welsch, *Quantum Optics* (John Wiley & Sons, 2006).
 - [6] S. Slussarenko and G. J. Pryde, Photonic quantum information processing: A concise review, *Appl. Phys. Rev.* **6**, 041303 (2019).
 - [7] F. Flamini, N. Spagnolo, and F. Sciarrino, Photonic quantum information processing: A review, *Rep. Prog. Phys.* **82**, 016001 (2018).
 - [8] J. W. Silverstone, *et al.*, Silicon quantum photonics, *IEEE J. Sel. Top. Quantum Electron.* **22**, 390 (2016).
 - [9] J. L. O’Brien, A. Furusawa, and J. Vučković, Photonic quantum technologies, *Nat. Photonics* **3**, 687 (2009).
 - [10] P. Kok, W. J. Munro, K. Nemoto, T. C. Ralph, J. P. Dowling, and G. J. Milburn, Linear optical quantum computing with photonic qubits, *Rev. Mod. Phys.* **79**, 135 (2007).
 - [11] R. Schnabel, Squeezed states of light and their applications in laser interferometers, *Phys. Rep.* **684**, 1 (2017).
 - [12] S. Lloyd and S. L. Braunstein, Quantum computation over continuous variables, *Phys. Rev. Lett.* **82**, 1784 (1999).
 - [13] S. L. Braunstein and P. van Loock, Quantum information with continuous variables, *Rev. Mod. Phys.* **77**, 513 (2005).
 - [14] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, *Rev. Mod. Phys.* **84**, 621 (2012).
 - [15] F. Lenzini, *et al.*, Integrated photonic platform for quantum information with continuous variables, *Sci. Adv.* **4**, eaat9331 (2018).
 - [16] S. Aaronson and A. Arkhipov, The computational complexity of linear optics, in *Proceedings of the 43rd Annual ACM Symposium on Theory of Computing* (2011), pp. 333–342.
 - [17] A. M. Ben-Amram, The Church-Turing thesis and its look-alikes, *ACM SIGACT News* **36**, 113 (2005).
 - [18] L. G. Valiant, The complexity of computing the permanent, *Theor. Comput. Sci.* **8**, 189 (1979).
 - [19] M. Tillmann, B. Dakić, R. Heilmann, S. Nolte, A. Szameit, and P. Walther, Experimental boson sampling, *Nat. Photon.* **7**, 540 (2013).
 - [20] A. P. Lund, A. Laing, S. Rahimi-Keshari, T. Rudolph, J. L. O’Brien, and T. C. Ralph, Boson sampling from a Gaussian state, *Phys. Rev. Lett.* **113**, 100502 (2014).
 - [21] M. Bentivegna, *et al.*, Experimental scattershot boson sampling, *Sci. Adv.* **1**, e1400255 (2015).
 - [22] S. Barkhofen, T. J. Bartley, L. Sansoni, R. Kruse, C. S. Hamilton, I. Jex, and C. Silberhorn, Driven boson sampling, *Phys. Rev. Lett.* **118**, 020502 (2017).
 - [23] C. S. Hamilton, R. Kruse, L. Sansoni, S. Barkhofen, C. Silberhorn, and I. Jex, Gaussian boson sampling, *Phys. Rev. Lett.* **119**, 170501 (2017).
 - [24] R. Kruse, C. S. Hamilton, L. Sansoni, S. Barkhofen, C. Silberhorn, and I. Jex, Detailed study of Gaussian boson sampling, *Phys. Rev. A* **100**, 032326 (2019).
 - [25] H.-S. Zhong, *et al.*, Experimental Gaussian boson sampling, *Sci. Bull.* **64**, 511 (2019).
 - [26] D. Callan, A combinatorial survey of identities for the double factorial, *arXiv:0906.1317*.
 - [27] M. Rudelson, A. Samorodnitsky, and O. Zeitouni, Hafnians, perfect matchings and Gaussian matrices, *Ann. Probab.* **44**, 2858 (2016).
 - [28] J. Huh, G. G. Guerreschi, B. Peropadre, J. R. McClean, and A. Aspuru-Guzik, Boson sampling for molecular vibronic spectra, *Nat. Photon.* **9**, 615 (2015).
 - [29] Y.-H. Deng, *et al.*, Solving graph problems using Gaussian boson sampling, *Phys. Rev. Lett.* **130**, 190601 (2023).
 - [30] J. M. Arrazola and T. R. Bromley, Using Gaussian boson sampling to find dense subgraphs, *Phys. Rev. Lett.* **121**, 030503 (2018).
 - [31] K. K. Sabapathy, H. Qi, J. Izaac, and C. Weedbrook, Production of photonic universal quantum gates enhanced by machine learning, *Phys. Rev. A* **100**, 012326 (2019).
 - [32] J. E. Bourassa, *et al.*, Blueprint for a scalable photonic fault-tolerant quantum computer, *Quantum* **5**, 392 (2021).
 - [33] F. Hoch, *et al.*, Quantum machine learning with adaptive boson sampling via post-selection, *Nat. Commun.* **16**, 902 (2025).
 - [34] I. Agresti, N. Viggianiello, F. Flamini, N. Spagnolo, A. Crespi, R. Osellame, N. Wiebe, and F. Sciarrino, Pattern recognition techniques for boson sampling validation, *Phys. Rev. X* **9**, 011013 (2019).
 - [35] U. L. Andersen, J. S. Neergaard-Nielsen, P. Van Loock, and A. Furusawa, Hybrid discrete- and continuous-variable quantum information, *Nat. Phys.* **11**, 713 (2015).
 - [36] M. Walschaers, Non-Gaussian quantum states and where to find them, *PRX Quantum* **2**, 030204 (2021).
 - [37] A. Serafini, *Quantum Continuous Variables: A Primer of Theoretical Methods* (CRC Press, 2023).
 - [38] Arvind, B. Dutta, N. Mukunda, and R. Simon, The real symplectic groups in quantum mechanics and optics, *Pramana* **45**, 471 (1995).
 - [39] B. C. Hall, *Lie Groups, Lie Algebras, and Representations* (Springer, 2013), pp. 333–366.
 - [40] J. B. Brask, Gaussian states and operations—A quick reference, *arXiv:2102.05748*.

- [41] M. Houde, W. McCutcheon, and N. Quesada, Matrix decompositions in quantum optics: Takagi/Autonne, Bloch–Messiah/Euler, Iwasawa, and Williamson, *Can. J. Phys.* **102**, 497 (2024).
- [42] R. Simon, S. Chaturvedi, and V. Srinivasan, Congruences and canonical forms for a positive matrix: Application to the Schweinler–Wigner extremum principle, *J. Math. Phys.* **40**, 3632 (1999).
- [43] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, 2012).
- [44] S. L. Braunstein, Squeezing as an irreducible resource, *Phys. Rev. A* **71**, 055801 (2005).
- [45] E. B. McBride, *Obtaining Generating Functions* (Springer Science & Business Media, 2012), Vol. 21.
- [46] P. Blasiak, A. Horzela, K. A. Penson, A. I. Solomon, and G. H. Duchamp, Combinatorics and boson normal ordering: A gentle introduction, *Am. J. Phys.* **75**, 639 (2007).
- [47] J. Sperling, W. Vogel, and G. S. Agarwal, True photocounting statistics of multiple on-off detectors, *Phys. Rev. A* **85**, 023820 (2012).
- [48] J. Sperling, W. Vogel, and G. S. Agarwal, Quantum state engineering by click counting, *Phys. Rev. A* **89**, 043829 (2014).
- [49] L. Frappat, A. Sciarrino, and P. Sorba, Dictionary on Lie superalgebras, [arXiv:hep-th/9607161](https://arxiv.org/abs/hep-th/9607161).
- [50] W.-M. Zhang, *et al.*, Coherent states: Theory and some applications, *Rev. Mod. Phys.* **62**, 867 (1990).
- [51] A. W. Knap and A. W. Knap, *Lie Groups Beyond an Introduction* (Springer, 1996), Vol. 140.
- [52] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/8hy1-m5gg> for additional considerations pertaining to the method of Lie algebras and generating functions, as well as simulation details, which included Refs. [42,68–71].
- [53] J. Schwinger, *A Quantum Legacy: Seminal Papers of Julian Schwinger*, edited by A. M. Kimball (World Scientific, Singapore, 2000), Vol. 26.
- [54] C. M. Caves, Reframing SU(1,1) interferometry, *Adv. Quantum Technol.* **3**, 1900138 (2020).
- [55] A. DasGupta, Disentanglement formulas: An alternative derivation and some applications to squeezed coherent states, *Am. J. Phys.* **64**, 1422 (1996).
- [56] L. Bianchi, C. Marconi, J. Sperling, and D. Bacco, Predetection squeezing as a resource for high-dimensional Bell-state measurements, *Phys. Rev. Res.* **7**, 023038 (2025).
- [57] M. Engelke, L. Lorz, S. De, B. Brecht, I. Dhand, M. B. Plenio, C. Silberhorn, and J. Sperling, Quantum photonics with active feedback loops, *Phys. Rev. A* **102**, 023712 (2020).
- [58] N. Quesada, Franck-Condon factors by counting perfect matchings of graphs with loops, *J. Chem. Phys.* **150**, 164113 (2019).
- [59] F. M. Miatto and N. Quesada, Fast optimization of parametrized quantum optical circuits, *Quantum* **4**, 366 (2020).
- [60] Y. Cardin and N. Quesada, Photon-number moments and cumulants of Gaussian states, *Quantum* **8**, 1521 (2024).
- [61] V. V. Kocharovsky, V. V. Kocharovsky, and S. V. Tarasov, The Hafnian master theorem, *Linear Algebra Appl.* **651**, 144 (2022).
- [62] A. Björklund, B. Gupt, and N. Quesada, A faster Hafnian formula for complex matrices and its benchmarking on a supercomputer, *ACM J. Exp. Algorithmics* **24**, 1 (2019).
- [63] E. S. Meckes, *The Random Matrix Theory of the Classical Compact Groups* (Cambridge University Press, 2019), Vol. 218.
- [64] C. C. Margossian, A review of automatic differentiation and its efficient implementation, *WIREs Data Min. Knowl. Discovery* **9**, e1305 (2019).
- [65] A. Griewank, A mathematical view of automatic differentiation, *Acta Numer.* **12**, 321 (2003).
- [66] E. Shchukin and W. Vogel, Inseparability criteria for continuous bipartite quantum states, *Phys. Rev. Lett.* **95**, 230502 (2005).
- [67] L. F. Richardson, IX. The approximate arithmetical solution by finite differences of physical problems involving differential equations, with an application to the stresses in a masonry dam, *Philos. Trans. R. Soc. London, Ser. A* **210**, 307 (1911).
- [68] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory* (Springer Science & Business Media, 2012), Vol. 9.
- [69] U. Leonhardt, Quantum physics of simple optical instruments, *Rep. Prog. Phys.* **66**, 1207 (2003).
- [70] N. Quesada, J. M. Arrazola, and N. Killoran, Gaussian boson sampling using threshold detectors, *Phys. Rev. A* **98**, 062322 (2018).
- [71] J. F. Bulmer, S. Paesani, R. S. Chadwick, and N. Quesada, Threshold detection statistics of bosonic states, *Phys. Rev. A* **106**, 043712 (2022).