

# No practical quantum broadcasting: General framework

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(Received 28 March 2025; accepted 22 July 2025; published 26 August 2025)

We introduce a comprehensive framework for no practical broadcasting, demonstrating that no linear map can simultaneously meet the conditions of sample efficiency, unitary covariance, permutation invariance, and classical consistency. This work establishes a rigorous theoretical foundation and provides an in-depth analysis of the employed techniques. In particular, we apply Schur-Weyl duality to provide a simplified alternative proof for the uniqueness of linear maps, referred to as the canonical virtual broadcasting map, that satisfy unitary covariance, permutation invariance, and classical consistency. The sample complexity of this canonical map is determined through semidefinite programming. Expanding beyond the 1-to-2 case, we derive the canonical form for 1-to- $N$  virtual broadcasting, analyze the sample complexity involved in its realization, and explore its connections with temporal quantum states and pseudodensity operators.

DOI: [10.1103/8g6j-w7ld](https://doi.org/10.1103/8g6j-w7ld)

## I. INTRODUCTION

In the rapidly advancing information era, the ability to clone unknown information is fundamental to the transmission, storage, and processing of data across communication networks. However, such a task is intrinsically prohibited by the linearity of quantum mechanics [1,2]. Even a relaxed version, known as broadcasting, is not realizable within the quantum theory [3]. More precisely, it is impossible to encode information into a quantum system and subsequently distribute it to multiple receivers in such a way that the marginal distributions of the output on each receiver's side align with the statistics of the original state. Importantly, the no-broadcasting theorem is not exclusive to quantum mechanics; it applies to any nonclassical finite-dimensional probabilistic model that adheres to the no-signaling principle [4].

This naturally raises the following question: Can we circumvent this fundamental no-go result in the quantum realm? In many protocols, the final output is classical in nature (for instance, when measuring observables of interest), making

the physical construction of the ideal quantum state less critical, provided the same statistical outcomes are faithfully reproduced. This insight drives the development of a hybrid framework that combines completely positive and trace-preserving (CPTP) operations with classical postprocessing, leading to the introduction of Hermitian-Perman trace-preserving (HPTP) operations. Referred to as virtual operations in the literature, this framework accelerates advancements in quantum simulation [5–7], error mitigation [8–12], property testing [13,14], and resource distillation [15–17]. These breakthroughs offer compelling avenues for overcoming the no-broadcasting theorem [18–20], paving the way for new possibilities in quantum information processing.

We explore this question by examining broadcasting within the general class of linear maps, assuming only the linearity of quantum mechanics. Four natural conditions are imposed on broadcasting, one of which, referred to as the “zeroth condition,” arises from fundamental considerations of sample complexity. The term “zeroth” underscores its foundational importance, setting it apart from the other conditions. Specifically, broadcasting can be viewed as an interaction between three agents (see Fig. 1): Alice, who aims to distribute an unknown quantum state to Bob and Claire in such a way that both parties obtain the same statistics from local measurements, as would be obtained from the original state. Without loss of generality, we assume that Bob and Claire wish to measure different observables  $\mathcal{O}_B$  and  $\mathcal{O}_C$ , respectively, requiring  $n_1$  and  $n_2$  copies of the original state to achieve a predefined confidence level. The broadcasting process must accomplish

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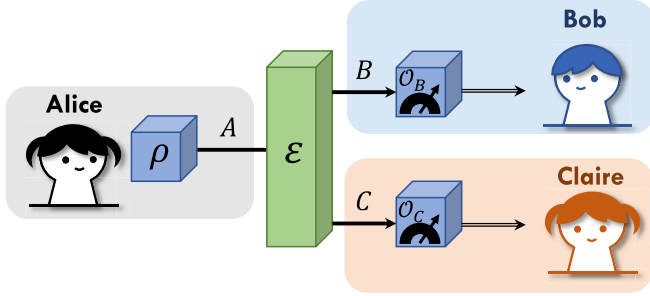


FIG. 1. Broadcasting through the linear map  $\mathcal{E} : A \rightarrow BC$ . Alice distributes independent and identically distributed copies of the quantum state  $\rho$  to agents Bob and Claire via a linear map  $\mathcal{E}$ . Bob and Claire then independently measure the observables  $\mathcal{O}_B$  and  $\mathcal{O}_C$  on their respective subsystems. To reach the desired confidence level, we assume that Bob requires  $n_1$  copies of  $\rho$ , while Claire needs  $n_2$  copies. A broadcasting map  $\mathcal{E}$  is said to satisfy the condition of sample efficiency (SE) if Bob and Claire can achieve the desired confidence using strictly fewer than  $n_1 + n_2$  copies of  $\rho$  through  $\mathcal{E}$ .

this using strictly fewer than  $n_1 + n_2$  copies; otherwise, simply sending  $n_1$  copies to Bob and  $n_2$  copies to Claire would trivialize the task. This essential requirement is referred to as the “zeroth condition” or *sample efficiency* (SE), ensuring that broadcasting remains a nontrivial operation distinct from direct state preparation and distribution. If this basic condition on sample complexity is not met, broadcasting loses its meaningful purpose.

In addition to SE, broadcasting maps should also satisfy three essential conditions (see Fig. 2). *Unitary covariance* (UC) requires that applying a unitary operation to Alice’s system before broadcasting is equivalent to performing the broadcasting operation first, followed by the same unitary applied to both Bob’s and Claire’s systems. *Permutation invariance* (PI) ensures that applying a SWAP operation between Bob and Claire after broadcasting leaves the overall output state unchanged, thereby preserving symmetry between the two recipients. *Classical consistency* (CC) stipulates that if all quantum systems undergo completely dephasing channel, the broadcasting process should reduce to classical broadcasting between the agents. A linear map that satisfies all four conditions, SE, UC, PI, and CC, is termed a *practical broadcasting map*.

We answer the previously posed question by showing that no practical broadcasting map exists within the class of all linear maps [21]. Extending the framework of virtual broadcasting under UC, PI, and CC [18], we introduce a new constraint: SE. This condition effectively distinguishes practical broadcasting from trivial operations, such as direct state preparation and distribution. Leveraging Schur-Weyl duality, we provide a significantly simpler alternative proof of the uniqueness of the canonical 1-to-2 virtual broadcasting map under UC, PI, and CC. Moreover, we show that achieving the desired confidence level for realizing the canonical 1-to-2 virtual broadcasting, as illustrated in Fig. 1, requires at least  $d^2 \max\{n_1, n_2\}$  state copies, where  $d$  represents the dimension of the system being broadcast. This value exceeds  $n_1 + n_2$ , violating the SE condition and conclusively showing that quantum information cannot be practically broadcast

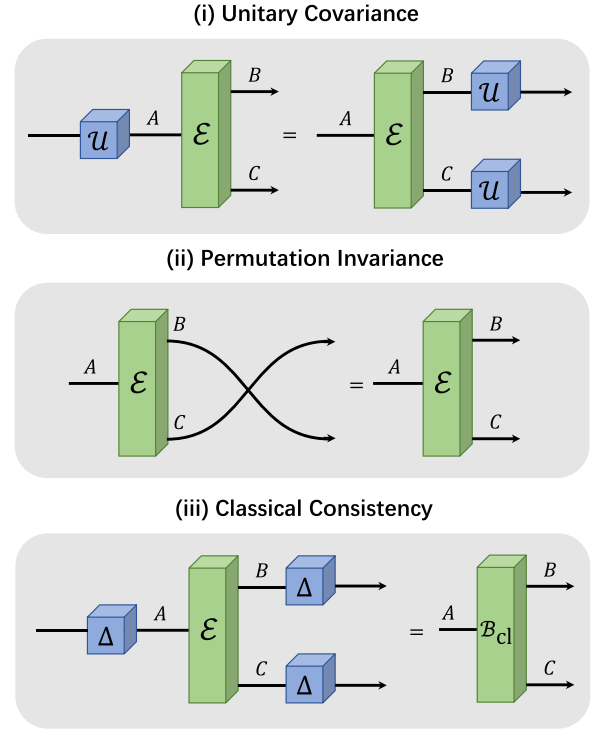


FIG. 2. Essential requirements for broadcasting maps. (i) Unitary covariance (UC): covariant under unitary transformations  $\mathcal{U}$  [see Eq. (48)]. (ii) Permutation invariance (PI): invariant under SWAP gate  $S$  [see Eq. (49)]. (iii) Classical consistency (CC): reduces to the classical broadcasting map  $\mathcal{B}_{cl}$  [see Eq. (51)] in the presence of completely dephasing channels  $\Delta$  [see Eq. (50)].

under the assumption of linearity. Finally, we derive the explicit form of the unique linear map satisfying UC, PI, and CC for the 1-to- $N$  scenario, alongside an analysis of its sample complexity. These results highlight a fundamental limitation of quantum broadcasting within the framework of quantum mechanics. Additionally, we explore the connection between temporal quantum states and pseudodensity operators, proposing protocols for constructing temporal quantum states across multiple time points using virtual broadcasting maps.

## II. OUTLINE

In this work, we investigate practical quantum broadcasting and establish the no-practical-broadcasting theorem, proving that no linear map can simultaneously satisfy sample efficiency (SE), unitary covariance (UC), permutation invariance (PI), and classical consistency (CC). Our results extend and provide a rigorous theoretical foundation for our companion paper [21], where the concept of no practical broadcasting was first introduced. In Sec. III, we introduce the tensor network diagram language to analyze key operators that determine the sample complexity of realizing the canonical 1-to-2 virtual broadcasting maps, uniquely defined under UC, PI, and CC. We also provide a brief overview of random unitary matrices and Schur-Weyl duality to support our analysis. In Sec. IV, we provide an alternative proof of the uniqueness of the canonical 1-to-2 virtual broadcasting map using Schur-Weyl

duality and tensor network formulations. Going beyond previous work, we demonstrate that any linear map satisfying UC and CC must also satisfy the broadcasting condition, showing that the broadcasting condition does not need to be assumed *a priori*. In Sec. V, we evaluate whether the canonical 1-to-2 virtual broadcasting map satisfies the SE condition by reformulating the problem as a semidefinite program and deriving a closed-form solution, which reveals its violation of the SE condition. This fundamental result establishes the no-practical-broadcasting theorem, highlighting intrinsic limitations imposed by quantum mechanics. Finally, in Sec. VI, we generalize our results to the 1-to- $N$  case and explore their connections with temporal quantum states and pseudodensity operators. We conclude that, as long as quantum mechanics remains linear, a practical quantum broadcasting protocol is fundamentally impossible.

### III. TENSOR NETWORKS: VISUALIZING QUANTUM DYNAMICS

The tensor network formalism (see Refs. [22–24]) provides a powerful framework for streamlining calculations. Combined with Schur-Weyl duality, it facilitates the uniqueness proof of the canonical virtual broadcasting map. In this section, we introduce the fundamentals of tensor networks, focusing on their foundational building blocks. We introduce our tensor network diagram conventions, construct key matrices for sample overhead calculations, and thoroughly investigate their algebraic structure and spectral properties, setting the stage for the subsequent analysis.

#### A. Preliminaries

Since our investigation focuses on a linear map  $\mathcal{E}$  from system  $A$  to systems  $B$  and  $C$ , e.g.,  $\mathcal{E} : A \rightarrow BC$ , we direct our attention to the matrix acting on the system  $ABC$ . For simplicity, the tensor notation between systems has been omitted, namely,  $AB := A \otimes B$ . Let the unnormalized maximally entangled state be denoted as  $\Gamma$ ; when it acts on the subsystem  $AB$ , we write it as  $\Gamma_{AB}$ . Additionally, we introduce the SWAP operator, denoted as  $S$ . Mathematically,  $\Gamma$  and  $S$  are given by

$$\Gamma_{AB} := \sum_{ij} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_B = \text{diagram}, \quad (1)$$

$$S_{AB} := \sum_{ij} |i\rangle\langle j|_A \otimes |j\rangle\langle i|_B = \text{diagram}. \quad (2)$$

The last column of the above equation represents the tensor network representations of  $\Gamma$  and  $S$ . In these diagrams, the first and second rows correspond to system  $A$  and  $B$ , respectively, and the open legs represent uncontracted indices. Building on the unnormalized maximally entangled state  $\Gamma$  and the SWAP operator  $S$ , we define the matrices  $M$  and  $N$  as follows

$$M := \Gamma_{AB} \otimes \mathbb{1}_C = \text{diagram}, \quad (3)$$

and

$$N := \mathbb{1}_A \otimes S_{BC} = \text{diagram}. \quad (4)$$

Here, the notation  $\mathbb{1}$  denotes the identity matrix, represented by a straight line. Throughout this work, we will frequently use tensor diagrams with three rows, where the top, middle, and bottom rows represent systems  $A$ ,  $B$ , and  $C$ , respectively. The squared matrix  $M^2$  satisfies the following relations:

$$M^2 = \text{diagram} = d \text{diagram} = dM. \quad (5)$$

In this context, the connection between different legs represents the index contraction and  $d$  represents the dimension of the system, specifically

$$d := \dim A = \dim B = \dim C. \quad (6)$$

Meanwhile, the matrix  $N^2$  is expressed as

$$N^2 = \text{diagram} = \text{diagram} = \mathbb{1}. \quad (7)$$

In the equation above [see Eq. (7)], the identity matrix  $\mathbb{1}_{ABC}$  operates on all systems,  $A$ ,  $B$ , and  $C$ . As is clear from the context, we will omit the subscript. Additionally, the matrix  $MNM$  can be written as

$$MNM = \text{diagram} = \text{diagram} = M. \quad (8)$$

The first building block  $x$ , which will be discussed and extensively used in this work, is defined as

$$x := M + NMN \quad (9)$$

$$= \text{diagram} + \text{diagram}. \quad (10)$$

Given that  $M$  and  $NMN$  are positive semidefinite,  $x$  is also positive semidefinite, namely  $x \geq 0$ . Consequently, all eigenvalues of  $x$  are non-negative.

Another important element is twice the Choi operator (see Refs. [25,26]) associated with the so-called *canonical 1-to-2 broadcasting map*  $\mathcal{B}_2$  (see Ref. [18]). For an input state  $\rho$ , the output of  $\mathcal{B}_2$  is given by

$$\mathcal{B}_2(\rho) := \frac{1}{2} \{\rho \otimes \mathbb{1}, S\}, \quad (11)$$

where  $\{\cdot, \cdot\}$  denotes the anticommutator. In the case of qubits,  $\mathcal{B}_2(\rho)$  can be understood as a pseudodensity operator (PDO) that describes the trivial evolution  $\mathcal{I}$  of a quantum system starting in state  $\rho$  (see Ref. [27]). When we move to qudits, the canonical 1-to-2 broadcasting map produces the real part of the two-point correlator (see Refs. [28,29]). The Choi operator  $J^{\mathcal{B}_2}$  for this map is expressed as

$$J^{\mathcal{B}_2} = \frac{1}{2}(MN + NM), \quad (12)$$

with  $M$  and  $N$  as defined in Eqs. (3) and (4). At this point, we formally introduce the matrix  $y := 2J^{\mathcal{B}_2}$ , namely,

$$y := MN + NM \quad (13)$$

$$= \text{diagram} + \text{diagram}. \quad (14)$$

TABLE I. Multiplication table of matrices  $x$  and  $y$ . In this table,  $x$  and  $y$  in the leftmost column represent matrices that multiply a given matrix from the left (premultiplication).  $x$  and  $y$  in the top row represent matrices that multiply from the right (postmultiplication). As an example, consider the entry in the bottom row and middle column. This entry represents the product  $yx$ , which is computed as  $x + dy$ .

| Multiplication table |          |          |
|----------------------|----------|----------|
|                      | $x$      | $y$      |
| $x$                  | $dx + y$ | $x + dy$ |
| $y$                  | $x + dy$ | $dx + y$ |

Using Eqs. (5), (7), and (8), we can easily derive the multiplication table for the matrices  $x$  [see Eq. (10)] and  $y$  [see Eq. (14)] shown below.

From Table I, we derive two useful facts: first, the matrices  $x$  and  $y$  commute, i.e.,

$$[x, y] = 0, \quad (15)$$

meaning they can be diagonalized simultaneously. Second, we have

$$x^2 = y^2. \quad (16)$$

Combining this with their commutativity, we conclude that the eigenvalues of  $x$ , when squared, are equal to those of  $y$ .

### B. Partial traces in tensor networks

In conventional tensor network analysis, the partial trace is often depicted as a loop connecting the leftmost and rightmost parts of a system. However, when multiple systems are involved, these loops can clutter the diagram. To address this and maintain clarity, we propose an alternative method

for illustrating the partial trace. Instead of using loops, we introduce dashed vertical lines to represent boundaries. Dots aligned along the same row indicate that the boundaries at those points are connected, fulfilling the same function as a loop.

Let us consider  $\text{Tr}_B(\Gamma)$  [see Eq. (1)] as an example, where

$$\text{Tr}_B[\Gamma_{AB}] = \text{diagram} = \text{diagram} = \mathbb{1}_A. \quad (17)$$

For a tensor network with three systems, let us use  $MN$  as an example and examine the result of taking the partial trace over system  $B$ , which leads to

$$\text{Tr}_B[MN] = \text{diagram} = \text{diagram} = \Gamma_{AC}. \quad (18)$$

Having completed the necessary preparation, we present in Table II a summary of the relevant results for the partial traces of  $x$  [see Eq. (10)] and  $y$  [see Eq. (14)], including their constituent components  $M$ ,  $NMN$ ,  $MN$ , and  $NM$ . This table is instrumental in analyzing the structure of  $x$  [see Eq. (10)] and  $y$  [see Eq. (14)], particularly concerning the sampling overhead of realizing the canonical 1-to-2 broadcasting map  $\mathcal{B}_2$  [see Eq. (11)], which we will explore in the following sections.

### C. Random unitary matrices

Haar random unitaries represent the uniform distribution over the unitary group, uniquely defined by its invariance under both left and right translations, namely,

$$\int_{\text{Haar}} f(U) dU = \int_{\text{Haar}} f(UV) dU = \int_{\text{Haar}} f(VU) dU, \quad (19)$$

TABLE II. Partial trace results table. This table presents the partial traces ( $\text{Tr}_B$ ,  $\text{Tr}_C$ ,  $\text{Tr}_{BC}$ , and  $\text{Tr} = \text{Tr}_{ABC}$ ) of the components of matrices  $x$  [see Eq. (10)] and  $y$  [see Eq. (14)]. Each row corresponds to a specific partial trace operation. The first two columns of a row, when combined, give the result for the corresponding partial trace of  $x$ , and the second two columns give the result for  $y$ . For instance,  $\text{Tr}_B[x]$  can be directly determined from this table, namely,  $\text{Tr}_B[x] = \mathbb{1}_{AC} + d \Gamma_{AC}$ .

| Partial Trace Table |                                                                                                         |                                                                                                         |                                                                                                       |                                                                                                        |
|---------------------|---------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|
| Building Block      | $x$ (see Eq. (10))                                                                                      |                                                                                                         | $y$ (see Eq. (14))                                                                                    |                                                                                                        |
| Component           | $M$                                                                                                     | $NMN$                                                                                                   | $MN$                                                                                                  | $NM$                                                                                                   |
| $\text{Tr}_B$       |  $= \mathbb{1}_{AC}$ |  $= d \Gamma_{AC}$   |  $= \Gamma_{AC}$  |  $= \Gamma_{AC}$  |
| $\text{Tr}_C$       |  $= d \Gamma_{AB}$   |  $= \mathbb{1}_{AB}$ |  $= \Gamma_{AB}$  |  $= \Gamma_{AB}$  |
| $\text{Tr}_{BC}$    |  $= d \mathbb{1}_A$  |  $= d \mathbb{1}_A$  |  $= \mathbb{1}_A$ |  $= \mathbb{1}_A$ |
| $\text{Tr}_{ABC}$   |  $= d^2$             |  $= d^2$             |  $= d$            |  $= d$            |

holds for arbitrary function  $f(\cdot)$  and unitary  $V$ . According to *Schur-Weyl duality* [30], the integral over Haar random unitaries is closely linked to the representation theory of the symmetric group  $\mathfrak{S}_N$ ,

$$\int_{\text{Haar}} U^{\otimes N} X (U^\dagger)^{\otimes N} dU = \sum_{\pi, \sigma \in \mathfrak{S}_N} C_{\pi, \sigma} \text{Tr}(X W_\pi) W_\sigma. \quad (20)$$

Here,  $\pi$  and  $\sigma$  are elements of the symmetric group  $\mathfrak{S}_N$ , while  $W_\pi$  and  $W_\sigma$  denote the corresponding permutation operations. The term  $C_{\pi, \sigma}$  represents an element of the *Weingarten matrix*.

We can use the previously introduced tensor network diagram to develop a more intuitive understanding of this concept. As an example, consider the second-order integral, where  $W_\pi$  and  $W_\sigma$  simplify to the SWAP operator  $S$  and the identity operator  $\mathbb{1}$ . Applying Eq. (20), we obtain

$$\begin{aligned} & \int_{\text{Haar}} U^{\otimes 2} X U^{\dagger \otimes 2} dU \\ &= \frac{1}{d^2 - 1} \left[ \text{Tr}(X \mathbb{1}) \mathbb{1} - \frac{1}{d} \text{Tr}(XS) \mathbb{1} - \frac{1}{d} \text{Tr}(X \mathbb{1}) S + \text{Tr}(XS) S \right] \\ &= \frac{1}{d^2 - 1} \left[ \text{Diagram 1} - \frac{1}{d} \text{Diagram 2} - \frac{1}{d} \text{Diagram 3} + \text{Diagram 4} \right], \end{aligned} \quad (21)$$

where  $d$  denotes the dimension of the unitary operator  $U$ .

#### D. Eigenvalues and their algebraic multiplicities

The simultaneous diagonalizability of  $x$  [see Eq. (10)] and  $y$  [see Eq. (14)], established in Sec. III A, implies the existence of a shared eigenbasis. Here, we analyze the eigenvalues associated with this basis, along with their algebraic multiplicities. From Table I, we derive that

$$x^2 = dx + y, \quad (22)$$

which immediately implies that

$$x^2(x - d)^2 = y^2 = x^2. \quad (23)$$

The last equation also follows from Table I. Equation (23) yields three distinct eigenvalues  $\{\lambda_1, \lambda_2, \lambda_3\}$  for  $x$ , namely,

$$\lambda_1 = d + 1, \quad \lambda_2 = d - 1, \quad \lambda_3 = 0. \quad (24)$$

Returning to Eq. (22), we can rewrite it as

$$y = x^2 - dx, \quad (25)$$

which results in three distinct eigenvalues  $\{\mu_1, \mu_2, \mu_3\}$  for  $y$ , given by

$$\mu_1 = d + 1, \quad \mu_2 = -(d - 1), \quad \mu_3 = 0. \quad (26)$$

With the eigenvalues of  $x$  and  $y$  determined, we now examine the algebraic multiplicities of  $x$ 's eigenvalues. Let  $m_1$  and  $m_2$  denote the algebraic multiplicities of  $\lambda_1 = d + 1$  and  $\lambda_2 = d - 1$ , respectively. Table II provides

$$m_1(d + 1) + m_2(d - 1) \quad (27)$$

$$= \text{Tr}[x] \quad (28)$$

$$= \text{Diagram 1} + \text{Diagram 2} \quad (29)$$

$$= 2d^2. \quad (30)$$

Building on Table I, we employ  $x^2 = dx + y$  to further analyze  $m_1$  and  $m_2$ , resulting in the following equation:

$$m_1(d + 1)^2 + m_2(d - 1)^2 \quad (31)$$

$$= \text{Tr}[x^2] \quad (32)$$

$$= d \text{Tr}[x] + \text{Tr}[y] \quad (33)$$

$$= d \left( \text{Diagram 1} + \text{Diagram 2} \right) + \text{Diagram 3} + \text{Diagram 4} \quad (34)$$

$$= 2d^3 + 2d. \quad (35)$$

Solving Eqs. (27) and (31), we obtain

$$m_1 = m_2 = d. \quad (36)$$

In other words, the algebraic multiplicities of  $\lambda_1 = d + 1$  and  $\lambda_2 = d - 1$  are both  $d$ , while the multiplicity of  $\lambda_3 = 0$  is  $d^3 - 2d$ .

Similarly, let  $n_1$  and  $n_2$  represent the algebraic multiplicities of  $\mu_1 = d + 1$  and  $\mu_2 = -(d - 1)$  for  $y$ , respectively. The trace of  $y$  (see Table II) then gives us the following equation:

$$n_1(d + 1) - n_2(d - 1) \quad (37)$$

$$= \text{Tr}[y] \quad (38)$$

$$= \text{Diagram 1} + \text{Diagram 2} \quad (39)$$

$$= 2d. \quad (40)$$

Likewise, the trace of  $y^2$  reveals that

$$n_1(d + 1)^2 + n_2(d - 1)^2 \quad (41)$$

$$= \text{Tr}[y^2] \quad (42)$$

$$= d \text{Tr}[x] + \text{Tr}[y] \quad (43)$$

$$= d \left( \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} \right) \quad (44)$$

$$= 2d^3 + 2d. \quad (45)$$

From Eqs. (37) and (41), we arrive at

$$n_1 = n_2 = d. \quad (46)$$

That is, the eigenvalues  $\mu_1 = d + 1$  and  $\mu_2 = -(d - 1)$  are each  $d$ -fold degenerate, whereas the degeneracy of  $\mu_3 = 0$  is  $d^3 - 2d$ . In summary, Table III presents the eigenvalues and their corresponding algebraic multiplicities for both  $x$  [see Eq. (10)] and  $y$  [see Eq. (14)].

#### IV. 1-TO-2 QUANTUM BROADCASTING: VIRTUAL QUANTUM INFORMATION DUPLICATION

Recently, Ref. [18] established the existence of a unique Hermitian-preserving (HP) and trace-preserving (TP) 1-to-2 broadcasting map satisfying three fundamental constraints: (i) UC, (ii) PI, and (iii) CC. In this section, we prove a stronger result: any linear map  $\mathcal{E} : A \rightarrow BC$  satisfying these conditions is necessarily the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  [see Eq. (11)], from which both the HP and *broadcasting condition* (BC) follow, with BC further implying TP. Our simplified proof, which employs Schur-Weyl duality and tensor networks, offers a more direct path to this conclusion.

TABLE III. Spectrum table. Following the discussion in Sec. III D, we summarize the eigenvalues and their algebraic multiplicities for  $x$  [see Eq. (10)] and  $y$  [see Eq. (14)].

| Spectrum Table      |                |                    |                |
|---------------------|----------------|--------------------|----------------|
| $x$ (see Eq. (10))  |                | $y$ (see Eq. (14)) |                |
| Eigenvalues         | Multiplicities | Eigenvalues        | Multiplicities |
| $\lambda_1 = d + 1$ | $d$            | $\mu_1 = d + 1$    | $d$            |
| $\lambda_2 = d - 1$ | $d$            | $\mu_2 = -(d - 1)$ | $d$            |
| $\lambda_2 = 0$     | $d^3 - 2d$     | $\mu_3 = 0$        | $d^3 - 2d$     |

#### A. Fundamental conditions

Quantum state broadcasting, the process of generating a multipartite system whose marginals reproduce the state to be broadcast, is impossible using standard quantum operations, represented by CPTP linear maps. However, this becomes possible if we consider a broader class of operations called HPTP linear maps, also known as virtual operations. Specifically, there exists a unique HPTP linear map capable of broadcasting, termed the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  (see Ref. [18]).

In this section, we restrict our attention to the case of 1-to-2 broadcasting. A linear map  $\mathcal{E} : A \rightarrow BC$  satisfies the broadcasting condition if

$$\text{Tr}_B[\mathcal{E}(\rho)] = \text{Tr}_C[\mathcal{E}(\rho)] = \rho \quad (47)$$

holds for all quantum states  $\rho$ . It is worth noting that we do not need to assume  $\mathcal{E}$  is trace preserving *a priori*, as the broadcasting condition of Eq. (47) itself ensures that  $\mathcal{E}$  is trace preserving, namely,  $BC \Rightarrow TP$ . The canonical 1-to-2 virtual broadcasting map is uniquely defined by three fundamental constraints, the first of which is covariance under unitary evolution. This requires that for any unitary operator  $\mathcal{U}$ , the map  $\mathcal{E} : A \rightarrow BC$  satisfies the following relation:

$$(\mathcal{U} \otimes \mathcal{U}) \circ \mathcal{E} = \mathcal{E} \circ \mathcal{U}, \quad (48)$$

where  $\circ$  denotes the composition of maps. Figure 2(a) demonstrates the unitary covariance (UC) property of map  $\mathcal{E}$ , as expressed in Eq. (48). The second constraint is invariant under the SWAP channel  $\mathcal{S}(\cdot) = \mathcal{S} \cdot \mathcal{S}^\dagger$ , termed permutation invariance (PI), which requires the map  $\mathcal{E}$  to satisfy

$$\mathcal{S} \circ \mathcal{E} = \mathcal{E}, \quad (49)$$

as depicted in Fig. 2(b). Finally, completely dephasing channel  $\Delta(\cdot) := \sum_i \langle i || \cdot || i \rangle |i\rangle\langle i|$  with basis  $\{|i\rangle\}$  on all input and output systems must reduce  $\mathcal{E}$  to the classical broadcasting

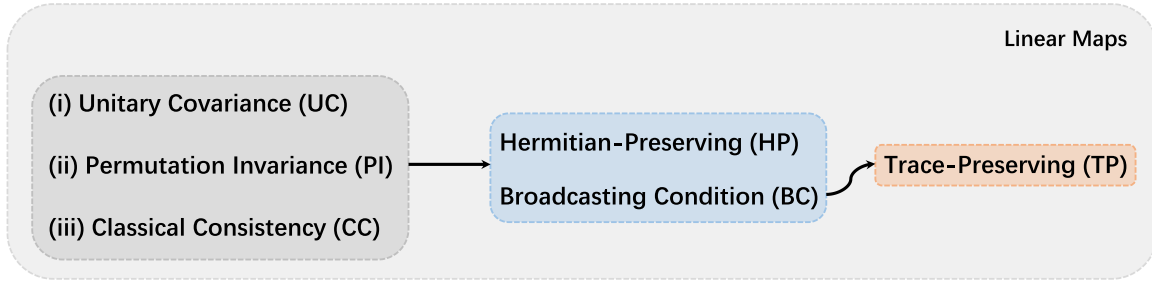


FIG. 3. Implication relationship diagram. As the diagram summarizes, for any linear map, unitary covariance (UC), permutation invariance (PI), and classical consistency (CC) imply both Hermitian preserving (HP) and broadcasting condition (BC), while BC implies trace preserving (TP).

map  $\mathcal{B}_{\text{cl}}$ , called classical consistency (CC), specifically:

$$(\Delta \otimes \Delta) \circ \mathcal{E} \circ \Delta = \mathcal{B}_{\text{cl}}, \quad (50)$$

where the classical broadcasting map  $\mathcal{B}_{\text{cl}}$  is defined as

$$\mathcal{B}_{\text{cl}}(|i\rangle\langle j|) = \delta_{ij} |i\rangle\langle i| \otimes |i\rangle\langle i|, \quad \forall i, j. \quad (51)$$

An illustration of Eq. (50) is shown in Fig. 2(c).

### B. Canonical virtual broadcasting

In this subsection, we present an alternative proof of the uniqueness of the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  [see Eq. (11)]. The central result of this subsection establishes that any linear map  $\mathcal{E} : A \rightarrow BC$  satisfying the conditions of UC [see Eq. (48)], PI [see Eq. (49)], and CC [see Eq. (50)] is necessarily equivalent to the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  (see Ref. [18]). We emphasize that this result is obtained without assuming  $\mathcal{E}$  is HP or satisfies BC. The theorem is stated formally below.

**Theorem 1 (1-to-2 broadcasting).** A linear map  $\mathcal{E} : A \rightarrow BC$  that is (a) unitarily covariant (UC) [see Eq. (48)], (b) permutation-invariant (PI) [see Eq. (49)], and (c) classically consistent (CC) [see Eq. (50)] must be the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  [see Eq. (11)]; that is,

$$\mathcal{E} = \mathcal{B}_2. \quad (52)$$

As a direct consequence of Theorem 1, if conditions (i) UC, (ii) PI, and (iii) CC are satisfied, then the linear map  $\mathcal{E}$  inherently fulfills both HP and BC. In other words, UC, PI & CC  $\Rightarrow$  HP & BC. Figure 3 shows how the discussed conditions relate to each other.

Remark that Theorem 1 was previously derived in Ref. [18]; here, we present an alternative proof based on Schur-Weyl duality and tensor network techniques. This new approach not only streamlines the derivation by clarifying the role of each assumption, but also reveals structural insights, for example, it becomes evident that the UC and CC conditions jointly imply BC. Moreover, our construction offers a natural intuition for generalizing to 1-to- $N$  scenarios. From a practical standpoint, the tensor network formalism employed here also provides a unified framework for analyzing sample complexity, enabling both the derivation of the canonical virtual broadcasting map and an estimation of the sample cost required for its physical implementation. For these reasons, we include the alternative proof in full detail.

*Proof.* We begin by considering the condition of UC and rewrite Eq. (48) as

$$(\mathcal{U} \otimes \mathcal{U}) \circ \mathcal{E} \circ \mathcal{U}^{-1} = \mathcal{E}, \quad (53)$$

which allows us to express the Choi operator (see Refs. [25,26]) of the linear map  $\mathcal{E}$  as

$$J^{\mathcal{E}} = (U^* \otimes U \otimes U) \cdot J^{\mathcal{E}} \cdot (U^{\text{T}} \otimes U^{\dagger} \otimes U^{\dagger}). \quad (54)$$

Here,  $*$  denotes the complex conjugate of a matrix. Applying the partial transpose with respect to system  $A$  yields

$$(J^{\mathcal{E}})^{\text{T}_A} = U \otimes U \otimes U \cdot (J^{\mathcal{E}})^{\text{T}_A} \cdot U^{\dagger} \otimes U^{\dagger} \otimes U^{\dagger}. \quad (55)$$

By averaging over all unitaries with respect to the Haar measure, we obtain

$$(J^{\mathcal{E}})^{\text{T}_A} = \int_{\text{Haar}} dU [U \otimes U \otimes U \cdot (J^{\mathcal{E}})^{\text{T}_A} \cdot U^{\dagger} \otimes U^{\dagger} \otimes U^{\dagger}] \quad (56)$$

$$= a_1 \text{---} + a_2 \text{---} + a_3 \text{---} + a_4 \text{---} + a_5 \text{---} + a_6 \text{---}, \quad (57)$$

holds for some coefficients  $a_i$  with  $i \in [6]$ . These terms correspond to the permutation matrices in the symmetric group  $\mathfrak{S}_3$ . It follows that

$$\mathcal{E}(\rho) = J^{\mathcal{E}} \star \rho = \text{Tr}_A[(J^{\mathcal{E}})^{\text{T}_A} \cdot (\rho \otimes \mathbb{1}_B \otimes \mathbb{1}_C)] \quad (58)$$

$$= a_1 \text{---} + a_2 \text{---} + a_3 \text{---} + a_4 \text{---} + a_5 \text{---} + a_6 \text{---}. \quad (59)$$

Here,  $\star$  denotes the link product between the Choi operators of quantum dynamics [31,32], which is crucial for investigating complex quantum behaviors, including non-Markovian quantum dynamics [33]. Specifically, given two matrices  $M$  and  $N$  that share a common subsystem  $A$ , the link product  $\star$  between them, denoted as  $M \star N$ , is defined as

$$M \star N := \text{Tr}_A[M^{\text{T}_A} \cdot N]. \quad (60)$$

This concept finds applications in various fields, such as non-Markovian metrology [34–36], adaptive quantum

communication [37], and causal uncertainty relations [38], among others.

The second step involves enforcing CC. Equation (50) dictates that if the input state is incoherent, specifically  $|i\rangle\langle i|$ , then broadcasting and subsequently applying a complete dephasing channel  $\Delta$  to all output systems should reproduce a cloned version of  $|i\rangle\langle i|$ :

$$(\Delta \otimes \Delta) \circ \mathcal{E}(|i\rangle\langle i|) = |i\rangle\langle i| \otimes |i\rangle\langle i|. \quad (61)$$

Using Eq. (58), the left-hand side of Eq. (61) becomes

$$(\Delta \otimes \Delta) \circ \mathcal{E}(|i\rangle\langle i|) = a_1 \text{---} + a_2 \text{---} + a_3 \text{---} + a_4 \text{---} + a_5 \text{---} + a_6 \text{---}. \quad (62)$$

In the fourth term shown above, we use blue to highlight the summation over the repeated index  $j$ , indicating that it corresponds to the operator  $\sum_j a_4 |j\rangle\langle j| \otimes |j\rangle\langle j|$ . The output being  $|i\rangle\langle i| \otimes |i\rangle\langle i|$ , we therefore conclude

$$a_1 = a_2 = a_3 = a_4 = 0 \quad (63)$$

and

$$a_5 + a_6 = 1. \quad (64)$$

Up to this point, we have established that under the conditions of UC [see Eq. (48)] and CC [see Eq. (50)], there are only two nonvanishing terms in  $\mathcal{E}(\rho)$ , which can be written in full as

$$\mathcal{E}(\rho) = a_5 \text{---} + a_6 \text{---}. \quad (65)$$

According to PI, we have

$$\mathcal{S} \circ \mathcal{E}(\rho) = a_6 \text{---} + a_5 \text{---}, \quad (66)$$

which implies

$$a_5 = a_6. \quad (67)$$

Substituting this into Eq. (64) gives

$$a_5 = a_6 = \frac{1}{2}. \quad (68)$$

Hence,

$$\mathcal{E}(\rho) = \frac{1}{2} \left( \text{---} + \text{---} \right) = \mathcal{B}_2(\rho), \quad (69)$$

holds for all quantum states  $\rho$ , thereby concluding our proof. ■

Our proof clearly shows that when a linear map  $\mathcal{E} : A \rightarrow BC$  satisfies both UC [see Eq. (48)] and CC [see Eq. (50)], it can be expressed as a linear combination of

$$\text{---}, \quad (70)$$

and

$$\text{---}. \quad (71)$$

Since both terms satisfy BC, which can be visually interpreted as follows:

$$\text{---} = \text{---} = \text{---}, \quad (72)$$

and

$$\text{---} = \text{---} = \text{---}, \quad (73)$$

the right-hand side of Eq. (69) does as well. Unlike previous studies that assume BC, we derive it directly from UC and CC. This is particularly counterintuitive, as one might expect the broadcasting condition to be a necessary assumption when analyzing a broadcasting map; however, we demonstrate that it naturally emerges from these fundamental constraints.

### C. Venn diagram analysis of fundamental conditions

Section IV B established the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  [see Eq. (11)] as the unique linear map satisfying the fundamental conditions of UC, PI, and CC. However, the relationships between the linear maps satisfying these conditions individually remain unexplored. In this subsection, we address this gap, offering deeper understanding of virtual broadcasting and the relationships among these fundamental constraints.

To complete the task, we provide several examples of maps belonging to the subsets shown in Fig. 4. These examples are defined as follows:

$$\mathcal{E}_1(\rho) := \frac{1}{d} \text{---}, \quad (74)$$

$$\mathcal{E}_2(\rho) := \frac{1}{d^2} \text{---}, \quad (75)$$

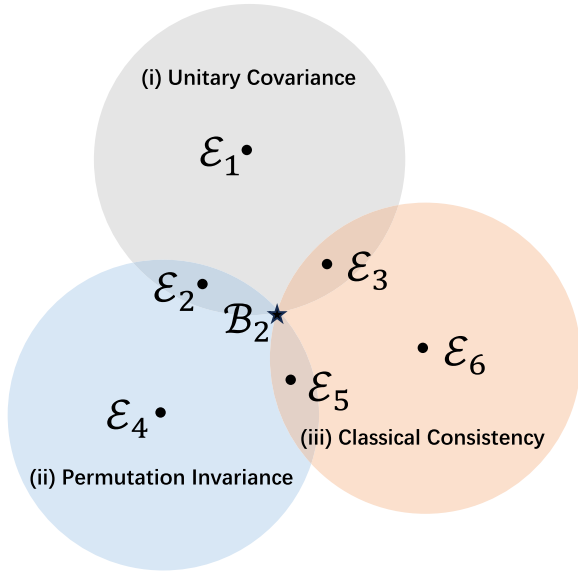


FIG. 4. Visualizing UC, PI, and CC with a Venn diagram. The intersections in the Venn diagram highlight linear maps that possess multiple properties. Quantum channel  $\mathcal{E}_2$ , for instance, satisfies both UC and PI, underscoring the connection between these important conditions. Notably, the canonical virtual broadcasting map  $\mathcal{B}_2$  uniquely satisfies all three (see Theorem 1), placing it at the single point where all sets intersect.

$$\mathcal{E}_3(\rho) := \frac{2}{3} \begin{array}{c} \text{---} \rho \text{---} \\ \text{---} \text{---} \end{array} + \frac{1}{3} \begin{array}{c} \text{---} \text{---} \\ \text{---} \rho \text{---} \end{array}, \quad (76)$$

$$\mathcal{E}_4(\rho) := \frac{\text{Tr}[\rho]}{d} \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array}, \quad (77)$$

$$\mathcal{E}_5(\rho) := \sum_i \text{Tr}[|i\rangle\langle i| \rho] \begin{array}{c} \text{---} i \text{---} \\ \text{---} i \text{---} \end{array}, \quad (78)$$

$$\begin{aligned} \epsilon_6(\rho) := & \sum_i \text{Tr}[|i\rangle\langle i| \rho] \left( |i\rangle\langle i| + \frac{1}{2}(|i\rangle\langle i+1| + |i+1\rangle\langle i|) \right) \\ & \otimes \left( |i\rangle\langle i| + \frac{1}{3}(|i\rangle\langle i+1| + |i+1\rangle\langle i|) \right). \end{aligned} \quad (79)$$

In Eq. (79), the summation is evaluated modulo  $d$ . All examples provided above, with the exception of  $\mathcal{E}_3$ , are quantum channels, characterized as CPTP maps. In contrast,  $\mathcal{E}_3$  is merely a linear map that fails to satisfy even the HP condition. It is worth noting that while  $\mathcal{E}_3$  in Eq. (76) is not HP, there exist other HP maps that satisfy both UC and CC, as demonstrated by the following example:

$$\frac{1+i}{2} \begin{array}{c} \text{---} \rho \text{---} \\ \text{---} \text{---} \end{array} + \frac{1-i}{2} \begin{array}{c} \text{---} \text{---} \\ \text{---} \rho \text{---} \end{array}. \quad (80)$$

To conclude this subsection, it is clear that the relationships between the constraints of UC, PI, and CC with CP and HP are intricate and not fully understood. This complexity suggests that a deeper investigation is required to unravel these connections and gain a more comprehensive understanding

of their interplay. Further exploration in this area could yield valuable insights into the underlying structure and behavior of these linear maps in broadcasting tasks.

## V. ZEROth CONDITION FOR BROADCASTING: THE PRIMACY OF SAMPLE COMPLEXITY

Prior work on broadcasting has established three fundamental conditions (see Sec. IV), culminating in a canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  [see Eq. (11)]. However, these conditions overlook a critical prerequisite: a “zeroth” condition concerning sample complexity. Effective broadcasting must reduce, not merely replicate, the sample resources needed for distributed communication. Simple replication and transmission fail this basic requirement, offering no advantage in sample complexity. We prove that even this minimal sample complexity condition (the zeroth condition), SE, together with UC, PI, and CC, precludes the existence of any broadcasting map within the set of linear maps. This no-practical-broadcasting theorem reveals a fundamental limitation that applies not only to quantum operations but also to the broader class of all linear maps.

### A. Gamification of broadcasting

We introduce the concept of *practical broadcasting* by formulating it as a strategic game. In this game, Alice possesses a source that generates independent and identically distributed (i.i.d.) copies of an unknown quantum state  $\rho$ . She then distributes these states to two agents, Bob and Claire, via a linear map  $\mathcal{E} : A \rightarrow BC$ , where  $A$  represents Alice’s system and  $B$  and  $C$  represent Bob’s and Claire’s systems, respectively. Bob and Claire each perform measurements on their respective subsystems to infer properties of  $\rho$ . Specifically, Bob aims to estimate the expectation value of an observable  $\mathcal{O}_B$ , while Claire aims to estimate the expectation value of an observable  $\mathcal{O}_C$ . Assume, without loss of generality, that Bob requires at least  $n_1$  copies of his reduced state to estimate  $\mathcal{O}_B$  with a desired confidence level, and Claire requires at least  $n_2$  copies of her reduced state to estimate  $\mathcal{O}_C$  with a similar confidence level. Alice’s objective is to distribute the quantum states such that both Bob and Claire can successfully estimate their respective observables. She wins the game if both players achieve the desired confidence level. Conversely, Alice loses if either Bob or Claire fails to do so. The central challenge for Alice is to minimize the total number of copies distributed, potentially achieving a savings compared to the naive approach of distributing

$$n_C := n_1 + n_2 \quad (81)$$

copies by exploiting quantum correlations between Bob’s and Claire’s subsystems. A schematic illustration of this game is provided in Fig. 1.

A quantum broadcasting map, if it existed, would allow Alice to win the game with

$$n_Q := \max\{n_1, n_2\} \quad (82)$$

copies [see Fig. 5(a)], a clear improvement over the naive  $n_1 + n_2$  [see Fig. 5(b)]. Unfortunately, the no-broadcasting theorem demonstrates that this scenario is impossible [3]; a

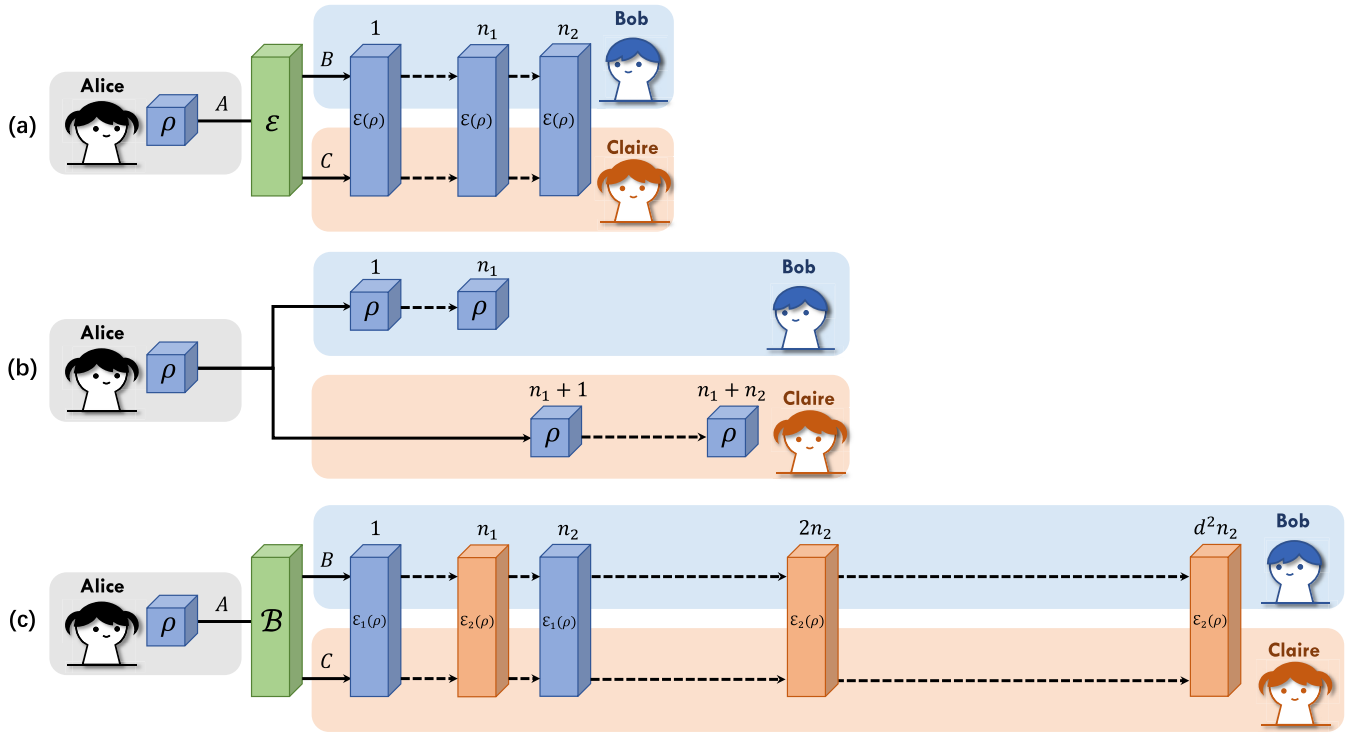


FIG. 5. Broadcasting strategies. In the game shown in Fig. 1, Bob and Claire need  $n_1$  and  $n_2$  copies of the quantum state, respectively, to estimate observables  $\mathcal{O}_B$  and  $\mathcal{O}_C$  with a given confidence level. Assuming  $n_2 > n_1$  without loss of generality, we have  $n_Q = n_2$  [see Eq. (82)]. (a) While  $n_2$  copies of the state would guarantee Alice's victory if quantum broadcasting were allowed, the no-broadcasting theorem rules out this approach. (b) A straightforward strategy involves distributing  $n_1$  copies of  $\rho$  to Bob and  $n_2$  copies to Claire. (c) If the canonical 1-to-2 virtual broadcasting map  $B_2$  can be decomposed as  $aC_1 - bC_2$ , then Alice can win the game by implementing the quantum channels  $\mathcal{E}_1$  and  $\mathcal{E}_2$  and using the measurement data to simulate  $B_2$ .

universal quantum broadcasting map cannot be realized. This raises a key question: Can Alice still secure a win with fewer than  $n_1 + n_2$  copies by utilizing the canonical 1-to-2 virtual broadcasting map  $B_2$  [see Eq. (11)]? If not [see Fig. 5(c)], then the straightforward strategy of sending  $n_1$  copies of the state  $\rho$  to Bob and  $n_2$  copies to Claire becomes a more efficient method of “broadcasting” for their observable measurements. This direct approach achieves the same level of confidence while requiring fewer total copies than the protocol involving  $B_2$ . Furthermore, this observation establishes a baseline condition for practical broadcasting in terms of sample complexity.

**Definition 1 (Sample efficiency (SE)).** To achieve the desired confidence level for all receivers, the number of quantum state copies required for successful broadcasting should be no greater than that needed by a naive distribution strategy, which involves sending independent sets of copies to each remote receiver.

For 1-to-2 broadcasting, achieving sample efficiency (SE) requires Alice to win the game with a maximum of  $n_C$  [see Eq. (81)] copies of the state  $\rho$ . We are now ready to formally define *practical broadcasting*.

**Definition 2 (Practical broadcasting).** A linear map that satisfies sample efficiency (SE), unitary covariance (UC), permutation invariance (PI), and classical consistency (CC) is called a practical broadcasting map.

Below, we state one of our main results. Proof, requiring a more rigorous treatment of sample complexity, is given in subsequent Sec. VB.

**Theorem 2 (No practical broadcasting).** A practical broadcasting map does not exist.

Figure 5 illustrates the main idea of Theorem 2: using canonical 1-to-2 virtual broadcasting  $B_2$  to help Bob and Claire measure their observables  $\mathcal{O}_B$  and  $\mathcal{O}_C$  with the desired confidence level requires  $d^2 n_Q$  copies of the quantum state, where  $d := \dim A = \dim B = \dim C$ . In contrast, the same accuracy can be achieved by the naive approach of providing  $n_1$  copies of  $\rho$  to Bob and  $n_2$  copies to Claire. This violates the condition of sample efficiency (SE) (see Definition 1), indicating that no practical broadcasting map can exist. The sample overhead factor  $d^2$  is nontrivial and is derived in Sec. VB. The analysis relies on semidefinite programming (SDP), with supporting results detailed in Sec. III.

## B. Sample complexity analysis

This subsection analyzes the number of state copies needed to achieve a desired confidence level in experimental data. We use concentration inequalities and introduce the  $\epsilon - \delta$  test to determine the sample complexity of implementing the canonical 1-to-2 virtual broadcasting map  $B_2$ . Specifically, we utilize *Hoeffding's inequality* [39], a powerful concentration inequality, to analyze the sample complexity. This inequality offers a probabilistic upper bound on the deviation between the observed average of bounded independent random variables and their expected value. Formally, it states the following.

*Lemma 1 (Hoeffding's inequality [39]).* Let  $X_1, \dots, X_n$  be independent random variables with  $X_i \in [a_i, b_i]$ , where  $-\infty < a_i \leq b_i < \infty$ ,  $\bar{X} := (1/n) \sum_i X_i$ , and  $e := \mathbb{E}[X_i]$ . Then for all error  $\epsilon > 0$ ,

$$\Pr(|\bar{X} - e| \geq \epsilon) \leq 2 \exp - \frac{2n^2 \epsilon^2}{\sum_i (b_i - a_i)^2}. \quad (83)$$

In the quantum regime, suppose we are given a state  $\rho$  and wish to explore the observable  $\mathcal{O}$  through measurement. By averaging the measurement outcomes, we approximate  $\text{Tr}[\mathcal{O}\rho]$ . If the difference between the approximation and the true value is smaller than  $\epsilon$ , the result is considered a success; otherwise, it is a failure. To ensure the probability of success is at least  $1 - \delta$ , we can establish a lower bound on the number of samples needed for the experiment. Before proceeding, let us formally define the  $\epsilon - \delta$  test in experiments.

*Definition 3 ( $\epsilon - \delta$  test).* In an experiment where we investigate the observable  $\mathcal{O}$  with respect to the quantum state  $\rho$ , we perform measurements and denote the outcome of each round as  $X_i$ , with the average outcome denoted as  $\bar{X}$ . If we can ensure that the probability of the error, measured by the difference between  $\bar{X}$  and the true value  $\text{Tr}[\mathcal{O}\rho]$ , exceeding  $\epsilon$  is at most  $\delta$ , the experiment is said to pass the  $\epsilon - \delta$  test.

If an experiment passes the  $\epsilon - \delta$  test, it is said to achieve a confidence level of  $1 - \delta$  with an accuracy of  $1 - \epsilon$ . Thanks to Lemma 1, we can now establish a lower bound on the number of independent and identically distributed samples of  $\rho$  needed to successfully pass the  $\epsilon - \delta$  test, which we will refer to as the *sample cost*.

*Corollary 1 (Sample cost).* Given  $n$  copies of  $\rho$  for the observer to measure  $\mathcal{O}$ ,  $n$  must satisfy the following condition in order to pass the  $\epsilon - \delta$  test (see Definition 3):

$$n \geq \frac{c^2}{2\epsilon^2} \ln \frac{2}{\delta}, \quad (84)$$

where  $c := \max_i (b_i - a_i)$  for all  $i$  (see Lemma 1), representing the maximum difference between the upper and lower bounds of all possible measurement outcomes.

We will continue to explore the sample complexity of implementing virtual broadcasting. In contrast to simulating the effects of quantum broadcasting, which involves obtaining a physical state, the virtual protocol centers on simulating the measurement statistics of a target. This approach relaxes the operations from CPTP maps (i.e., quantum channels) to HPTP maps (i.e., virtual operations), enabling realization through the combination of measurement outcomes across different quantum channels. Virtual operations expand the range of achievable tasks compared to quantum channels, but at the expense of requiring more samples.

Returning to the game described in Fig. 1, we can now use the  $\epsilon - \delta$  test (see Definition 3) to rigorously quantify the desired confidence level. Upon receiving the state, Bob and Claire perform local measurements on the observables  $\mathcal{O}_B$  and  $\mathcal{O}_C$ , respectively. Bob's measurement is subjected to an  $\epsilon_1 - \delta_1$  test, while Claire's is evaluated using an  $\epsilon_2 - \delta_2$  test, with the parameters potentially differing between them. Alice wins the game if both Bob and Claire pass their respective tests. This leads to the following crucial question: How many state copies must Alice prepare to guarantee a win? The

answer lies in the sample complexity required to implement the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$ .

Let  $n_1$  and  $n_2$  represent the number of copies required for Bob and Claire to pass their respective tests, namely,  $\epsilon_1 - \delta_1$  test for Bob and  $\epsilon_2 - \delta_2$  test for Claire. Throughout this work, we assume that the difference between the upper and lower bounds of all possible measurement outcomes is bounded by a constant  $c$ . Then, from Corollary 1, we know that these quantities must satisfy the following inequality:

$$n_i \geq \frac{c^2}{2\epsilon_i^2} \ln \frac{2}{\delta_i}, \quad i \in \{1, 2\}. \quad (85)$$

The total quantity  $n_C = n_1 + n_2$  [see Eq. (81)] serves as a benchmark for practical broadcasting. Specifically, SE refers to the scenario where Alice can win the game using fewer than  $n_C$  copies of  $\rho$  (see Definition 1).

In canonical 1-to-2 virtual broadcasting, two quantum channels,  $\mathcal{E}_1$  and  $\mathcal{E}_2$ , are used to simulate the target virtual operation  $\mathcal{B}_2$  [see Eq. (11)] with weighting coefficients  $a$  and  $b$ :

$$a \mathcal{E}_1 - b \mathcal{E}_2 = \mathcal{B}_2. \quad (86)$$

When measuring  $\mathcal{O}_B$  on Bob's side, the equation becomes

$$\text{Tr}[\mathcal{O}_B \text{Tr}_C[(a \mathcal{E}_1 - b \mathcal{E}_2)(\rho)]] \quad (87)$$

$$= a \text{Tr}[\mathcal{O}_B \text{Tr}_C[\mathcal{E}_1(\rho)]] - b \text{Tr}[\mathcal{O}_B \text{Tr}_C[\mathcal{E}_2(\rho)]] \quad (88)$$

$$= (a + b) \left( \frac{a}{a + b} \text{Tr}[\mathcal{O}_B \text{Tr}_C[\mathcal{E}_1(\rho)]] - \frac{b}{a + b} \text{Tr}[\mathcal{O}_B \text{Tr}_C[\mathcal{E}_2(\rho)]] \right). \quad (89)$$

To implement the process physically, we apply  $\mathcal{E}_1$  with probability  $a/(a + b)$  and  $\mathcal{E}_2$  with probability  $b/(a + b)$ , then scale their average by a factor of  $a + b$ . Since this holds for all possible values of  $a$  and  $b$ , minimizing their sum yields the lowest sample overhead. Let  $u_2$  denote the minimum value of  $a + b$ . As a result of Corollary 1, the number of copies required for Bob to pass the  $\epsilon_1 - \delta_1$  test increases to  $u_2^2 n_1$ . Similarly,  $u_2^2 n_2$  copies of  $\rho$  will enable Claire to pass her  $\epsilon_2 - \delta_2$  test. In other words, for Alice to win the game, the number of copies, or the sample overhead required for the canonical 1-to-2 virtual broadcasting  $\mathcal{B}_2$ , must be no less than

$$n_V := u_2^2 n_Q, \quad (90)$$

where  $n_Q$  is defined in Eq. (82).

Since the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  is the only map that satisfies the conditions of UC, PI, and CC, the existence of a practical broadcasting map depends on whether  $n_V$  is strictly smaller than  $n_C$ . If  $n_V < n_C$ , then  $\mathcal{B}_2$  qualifies as a practical broadcasting map. Otherwise, no practical broadcasting map exists. The key challenge is finding  $u_2$ , which we will address in the next subsection.

### C. Estimating sample overhead with semidefinite programming

This subsection addresses the problem of minimizing sample overhead for the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  [see Eq. (11)] expressed as a linear combination of two quantum channels. We formulate and solve this optimization

problem using semidefinite programming (SDP), leveraging insights from the previous section's algebraic structure (see Sec. III).

We begin by formally introducing the minimal sample overhead, which is characterized by the following optimization problem:

$$u_2 := \min \quad a + b \quad (91)$$

$$\text{s.t.} \quad J_1 - J_2 = \frac{1}{2}y, \quad (92)$$

$$\text{Tr}_{BC}[J_1] = a \mathbb{1}_A, \quad (93)$$

$$\text{Tr}_{BC}[J_2] = b \mathbb{1}_A, \quad (94)$$

$$J_1 \geq 0, J_2 \geq 0. \quad (95)$$

Here,  $J_1$  denotes  $a$  times the Choi operator of quantum channel  $\mathcal{E}_1$ , while  $J_2$  denotes  $b$  times the Choi operator of quantum channel  $\mathcal{E}_2$ . Thus,  $J_1 - J_2$  represents an arbitrary linear combination of these quantum channels. The optimization is performed over all possible combinations, resulting in a SDP (see Ref. [40]). Solving this SDP is critical, as it directly governs the sample overhead required for practical simulation of the canonical 1-to-2 broadcasting map  $\mathcal{B}_2$ . Beyond simply providing a solution, we aim to establish a systematic approach applicable to a broader class of problems. We accomplish this by outlining our methodology for estimating the optimal value of such optimization problems, followed by demonstrating its achievability.

Our first step is to derive a lower bound for the primal problem in Eq. (91) by constructing its Lagrangian and corresponding dual problem. We begin by formulating the Lagrangian, written as

$$\begin{aligned} \mathcal{L} := & a + b - \text{Tr}[X(J_1 - J_2 - \frac{1}{2}y)] \\ & - \text{Tr}_A[Y_1(\text{Tr}_{BC}[J_1] - a \mathbb{1}_A)] - \text{Tr}_A[Y_2(\text{Tr}_{BC}[J_2] - b \mathbb{1}_A)] \\ & - \text{Tr}[Z_1 J_1] - \text{Tr}[Z_2 J_2] \end{aligned} \quad (96)$$

$$\begin{aligned} = & a(1 + \text{Tr}_A[Y_1]) + b(1 + \text{Tr}_A[Y_2]) \\ & + \text{Tr}[J_1(-X - Y_1 \otimes \mathbb{1}_{BC} - Z_1)] \\ & + \text{Tr}[J_2(X - Y_2 \otimes \mathbb{1}_{BC} - Z_2)] + \frac{1}{2}\text{Tr}[Xy]. \end{aligned} \quad (97)$$

Here, the operators  $X$ ,  $Y_1$ , and  $Y_2$  are Hermitian, while  $Z_1$  and  $Z_2$  are positive semidefinite. To guarantee that the Lagrangian  $\mathcal{L}$  is bounded, we obtain the equations

$$\text{Tr}_A[Y_1] = \text{Tr}_A[Y_2] = -1, \quad (98)$$

$$-X - Y_1 \otimes \mathbb{1}_{BC} = Z_1, \quad (99)$$

$$X - Y_2 \otimes \mathbb{1}_{BC} = Z_2. \quad (100)$$

Consequently, the dual problem of Eq. (91) takes the form

$$v_2 := \frac{1}{2} \max \quad \text{Tr}[Xy] \quad (101)$$

$$\text{s.t.} \quad \text{Tr}[Y_1] = -\text{Tr}[Y_2] = 1, \quad (102)$$

$$Y_2 \otimes \mathbb{1}_{BC} \leq X \leq Y_1 \otimes \mathbb{1}_{BC}. \quad (103)$$

The key lies in identifying a feasible solution to the dual problem. However, this is not a process of arbitrary guessing; it should begin with the careful selection of appropriate  $Y_1$  and  $Y_2$ . Specifically, we should choose  $Y_1$  and  $Y_2$  such that  $Y_1 \otimes \mathbb{1}_{BC}$  and  $Y_2 \otimes \mathbb{1}_{BC}$  commute with  $y$  [see Eq. (14)], thereby simplifying our estimation and reducing the complexity of calculating  $\text{Tr}[Xy]$ . The most natural and effective choice should be

$$Y_1 = -Y_2 = \frac{1}{d} \mathbb{1}_A, \quad (104)$$

ensuring that Eq. (102) is satisfied. Without loss of generality, we assume that  $y$  can be written as

$$y = U_y \text{diag}(\mu) U_y^\dagger \quad (105)$$

for some unitary matrix  $U_y$  and diagonal matrix  $\text{diag}(\mu)$  with  $\mu$  defined as

$$\mu := (\underbrace{d+1, \dots, d+1}_d, \underbrace{-(d-1), \dots, -(d-1)}_d, \underbrace{0, \dots, 0}_{d^3-2d}). \quad (106)$$

To maximize  $\text{Tr}[Xy]$  while satisfying Eq. (103), we choose  $X$  as

$$X := U_y \text{diag}(\underbrace{\frac{1}{d}, \dots, \frac{1}{d}}_d, \underbrace{-\frac{1}{d}, \dots, -\frac{1}{d}}_d, \underbrace{0, \dots, 0}_{d^3-2d}) U_y^\dagger, \quad (107)$$

which gives the following feasible solution to the dual problem in Eq. (101):

$$\left\{ X = U_y \text{diag} \left( \underbrace{\frac{1}{d}, \dots, \frac{1}{d}}_d, \underbrace{-\frac{1}{d}, \dots, -\frac{1}{d}}_d, \underbrace{0, \dots, 0}_{d^3-2d} \right) U_y^\dagger, Y_1 = \frac{1}{d} \mathbb{1}_A, Y_2 = -\frac{1}{d} \mathbb{1}_A \right\}. \quad (108)$$

In this case, Eq. (101) simplifies to

$$\frac{1}{2} \text{Tr}[Xy] = \frac{1}{2} \text{Tr}[U_y \text{diag} \left( \underbrace{\frac{1}{d}, \dots, \frac{1}{d}}_d, \underbrace{-\frac{1}{d}, \dots, -\frac{1}{d}}_d, \underbrace{0, \dots, 0}_{d^3-2d} \right) \text{diag}(\mu) U_y^\dagger] = \frac{1}{2} \left( d \frac{1}{d} (d+1) + d \frac{1}{d} (d-1) \right) = d, \quad (109)$$

which leads to the following chain of inequalities:

$$d \leq v_2 \leq u_2, \quad (110)$$

where the final inequality of Eq. (110) follows from the weak duality of SDP. To summarize our findings so far, we formalize our results in the following lemma.

*Lemma 2 (Lower bound).* The objective function of the optimization problem in Eq. (91) is bounded below by  $d$  [see Eq. (110)], the dimension of the state to be broadcast.

With this, we have completed the first step, establishing a lower bound for the primal problem in Eq. (91).

Our second step is to establish an upper bound for the primal problem in Eq. (91) by identifying a feasible solution. Ideally, we also want  $d$  to be the optimal solution of the primal problem, meaning that we must find a feasible solution to Eq. (91) that attains  $d$ . To achieve this, we employ a few strategic techniques. From Eq. (92), we know that if feasible solutions  $a$  and  $b$  exist, they must satisfy

$$a - b = 1. \quad (111)$$

Subsequently, the objective function becomes

$$a + b = 2a - 1. \quad (112)$$

For the rest, we only need to determine a suitable channel  $\mathcal{E}_2$ , such that

$$bJ^{\mathcal{E}_2} + \frac{1}{2}y \geq 0, \quad (113)$$

with  $J^{\mathcal{E}_2}$  representing the Choi operator of quantum channel  $\mathcal{E}_2$ . When  $d$  is attainable, we have

$$a = \frac{d+1}{2}, \quad (114)$$

$$b = \frac{d-1}{2}. \quad (115)$$

For the choice of  $J^{\mathcal{E}_2}$ , we can construct it using the combination of  $x$  [see Eq. (10)] and  $y$  [see Eq. (14)]. From Table II, it is clear that both  $x + y$  and  $x - y$  are positive semidefinite. We begin with  $x - y$ . To guarantee that the corresponding channel is TP, it must satisfy

$$\text{Tr}_{BC}[J^{\mathcal{E}_2}] = \mathbb{1}_A, \quad (116)$$

leading to the following form of  $J^{\mathcal{E}_2}$ ,

$$J^{\mathcal{E}_2} = \frac{x - y}{2(d-1)}, \quad (117)$$

which guarantees that  $\mathcal{E}_2$  is CPTP. It is now straightforward to verify that

$$bJ^{\mathcal{E}_2} + \frac{1}{2}y = \frac{d-1}{2} \frac{x-y}{2(d-1)} + \frac{1}{2}y = \frac{1}{4}(x+y) \geq 0. \quad (118)$$

By defining the Choi operator  $J^{\mathcal{E}_1}$  of quantum channel  $\mathcal{E}_1$  as

$$J^{\mathcal{E}_1} = \frac{1}{a} \left( bJ^{\mathcal{E}_2} + \frac{1}{2}y \right) = \frac{x+y}{2(d+1)}, \quad (119)$$

we obtain the following feasible solution to the primal problem in Eq. (91):

$$\left\{ a = \frac{d+1}{2}, b = \frac{d-1}{2}, J_1 := aJ^{\mathcal{E}_1} = \frac{x+y}{4}, \right. \\ \left. J_2 := bJ^{\mathcal{E}_2} = \frac{x-y}{4} \right\}, \quad (120)$$

accompanied by an upper bound  $d$ , i.e.,

$$u_2 \leq d, \quad (121)$$

which completes our second step. To encapsulate our findings in this step, we formalize them in the following lemma.

*Lemma 3 (Upper bound).* As shown in Eq. (121), the objective function of the optimization problem [Eq. (91)] is upper bounded by  $d$ , which represents the dimension of the state being broadcast.

Application of Eqs. (110) and (121) yields the result

$$d \leq v_2 \leq u_2 \leq d. \quad (122)$$

Hence, the optimal value of the SDP in Eq. (91), which expresses the minimal overhead in virtual broadcasting, is  $d$ :

$$u_2 = d. \quad (123)$$

The results presented in this subsection are summarized by the following theorem.

*Theorem 3 (Exact solution).* The objective function of the optimization problem [Eq. (91)] is exactly  $d$  [Eq. (123)], where  $d$  is the dimension of the broadcast state.

Thanks to Theorem 3, we now have the following result regarding the sample complexity required to realize the canonical 1-to-2 virtual broadcasting  $\mathcal{B}_2$  [see Eq. (11)].

*Theorem 4 (Sample complexity of  $\mathcal{B}_2$ ).* In the game shown in Fig. 1, where Bob and Claire need  $n_1$  and  $n_2$  copies of  $\rho$  to pass their respective  $\epsilon - \delta$  tests (see Definition 3), and Alice employs the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  to distribute states to both Bob and Claire, the number of copies of  $\rho$  that Alice must prepare to win the game must be no less than

$$n_V = d^2 n_Q. \quad (124)$$

In quantum theory,  $d$  should be at least 2, and hence we have (see Fig. 5)

$$n_V = d^2 n_Q > 2n_Q \geq n_C = n_1 + n_2. \quad (125)$$

Consequently, the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  does not fulfill the zeroth condition of SE (see Definition 1), thereby giving rise to the no-practical-broadcasting theorem (see Theorem 2). This fundamental limitation is visually illustrated in Fig. 6.

## VI. 1-TO- $N$ QUANTUM BROADCASTING: TOWARDS MULTIRECIPIENT INFORMATION DISTRIBUTION

1-to-2 quantum broadcasting is well studied, but the generalization to 1-to- $N$  broadcasting is essential for realizing the full potential of quantum communication and computation. Distributing quantum information among multiple parties is fundamental to numerous applications, including quantum

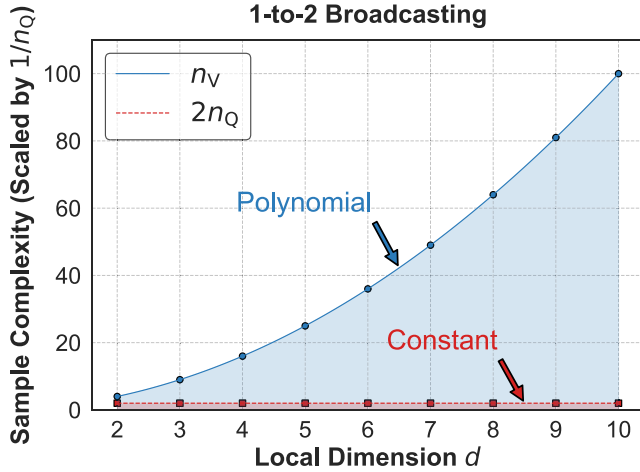


FIG. 6. Sample complexity of realizing  $\mathcal{B}_2$ . The horizontal axis represents the dimension  $d$  of the quantum state to be broadcast, while the vertical axis shows the sample complexity, scaled by a factor of  $1/n_Q$ . The blue line indicates the sample complexity required for implementing the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  [see Eq. (11)], expressed as  $d^2 n_Q$ . Meanwhile, the red dashed line denotes an upper bound,  $2n_Q$ , on the number of copies required for the naive approach [see Fig. 5(b)], corresponding to  $n_C$ . The figure illustrates that the canonical 1-to-2 virtual broadcasting map (blue line) exhibits polynomial sample complexity as the dimension  $d$  increases. In contrast, the naive approach (red dashed line) shows constant sample complexity, suggesting that the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  fails to satisfy the zeroth condition of sample efficiency (SE) (see Definition 1). This implies that the canonical 1-to-2 virtual broadcasting map is not a realistic option for broadcasting quantum information.

networks, multiparty cryptography, and distributed quantum computing and sensing. Scaling these applications hinges on the ability to efficiently share quantum information while preserving quantum properties like correlations and coherence. Furthermore, analyzing the sample complexity and resource requirements of 1-to- $N$  broadcasting provides critical insights into the fundamental limits imposed by no-broadcasting theorem. In this section, we investigate the 1-to- $N$  virtual broadcasting map, addressing two fundamental questions: Does such a map exist? And, if so, is it unique? The answers to these questions are not only crucial for advancing our theoretical understanding of quantum broadcasting, but also have implications for the practical scalability of quantum technologies.

### A. Canonical forms

To investigate the 1-to- $N$  broadcasting map, we consider a linear map  $\mathcal{E}$  from system  $A$  to the composite system  $B_1 \dots B_N$ , where  $d := \dim A = \dots = \dim B_N$ . Analogous to the 1-to-2 case, we impose three fundamental constraints on  $\mathcal{E}$ : generalized unitary covariance (GUC), generalized permutation invariance (GPI), and generalized classical consistency (GCC). Specifically, the GUC, GPI and GCC conditions for  $\mathcal{E}$

are formulated as follows:

$$\text{GUC: } \underbrace{\mathcal{U} \otimes \dots \otimes \mathcal{U}}_N \circ \mathcal{E} = \mathcal{E} \circ \mathcal{U}, \quad (126)$$

$$\text{GPI: } P_\pi \circ \mathcal{E} = \mathcal{E}, \quad \forall \pi \in \mathfrak{S}_N, \quad (127)$$

$$\text{GCC: } \underbrace{\Delta \otimes \dots \otimes \Delta}_N \circ \mathcal{E} \circ \Delta = \mathcal{B}_{N\text{-cl}}. \quad (128)$$

In Eq. (127),  $\pi$  denotes a permutation in the symmetry group  $\mathfrak{S}_N$ , with its matrix representation given by  $V_\pi$  acting on systems  $B_1 \dots B_N$ . Using this notation, we define the linear map  $P_\pi$  as

$$P_\pi(X) := V_\pi X V_\pi^T. \quad (129)$$

The generalized classical broadcasting map,  $\mathcal{B}_{N\text{-cl}}$ , is defined as follows:

$$\mathcal{B}_{N\text{-cl}}(|i\rangle\langle j|) = \delta_{ij} \underbrace{|i\rangle\langle i| \otimes \dots \otimes |i\rangle\langle i|}_N, \quad \forall i, j. \quad (130)$$

This formulation ensures that  $\mathcal{B}_{N\text{-cl}}$  perfectly replicates any incoherent basis state  $|i\rangle$  into  $N$  identical copies.

Following a similar approach in Sec. IV B, we analyze the structure of  $\mathcal{E}$  under GUC, GPI, and GCC conditions by rewriting Eq. (126) as

$$\underbrace{\mathcal{U} \otimes \dots \otimes \mathcal{U}}_N \circ \mathcal{E} \circ \mathcal{U}^{-1} = \mathcal{E}. \quad (131)$$

Expressing Eq. (131) in terms of the Choi operator and then performing a partial transpose on system  $A$ , we arrive at

$$(J^\mathcal{E})^{\text{T}_A} = \underbrace{U \otimes \dots \otimes U}_{N+1} \cdot (J^\mathcal{E})^{\text{T}_A} \cdot \underbrace{U^\dagger \otimes \dots \otimes U^\dagger}_{N+1} \quad (132)$$

$$= \sum_{\eta \in \mathfrak{S}_{N+1}} a_\eta V_\eta, \quad (133)$$

where  $a_\eta$  are coefficients related with the permutation  $\eta$ , with its matrix representation  $V_\eta$  acting on systems  $AB_1 \dots B_N$ . Applying  $\mathcal{E}$  to the input state  $\rho$  yields the following output state:

$$\mathcal{E}(\rho) = J^\mathcal{E} \star \rho \quad (134)$$

$$= \sum_{\pi_0 \in \mathfrak{S}_N} b_{(0, \pi_0)} V_{\pi_0} + \sum_{i=1}^N \sum_{\pi_i \in \mathfrak{S}_N} b_{(i, \pi_i)} \rho_i V_{\pi_i}. \quad (135)$$

Here, the coefficients are related to those in Eq. (132). For instance,  $b_{\pi_0}$  corresponds to the coefficient  $a_\eta$  in Eq. (132), where  $\eta$  fixes system  $A$ . We adopt the convention that  $\rho_i$  represents the operator  $\rho$  acting on the  $B_i$  subsystem, while the identity operator acts on all remaining subsystems, that is,

$$\rho_i := \underbrace{\mathbb{1}_{B_1} \otimes \dots \otimes \mathbb{1}_{B_{i-1}}}_{i-1} \otimes \rho \otimes \underbrace{\mathbb{1}_{B_{i+1}} \otimes \dots \otimes \mathbb{1}_{B_N}}_{N-i}. \quad (136)$$

The GCC condition [see Eq. (128)] requires that all  $b_{(0, \pi_0)}$  be zero. Furthermore,  $b_{(i, \pi_i)}$  must also be zero for any  $\pi_i$  which is not an  $N$ -cycle in  $\mathfrak{S}_N$ . Consequently, under the GCC condition, we conclude that

$$\mathcal{E}(\rho) = \sum_{i=1}^N \sum_{\substack{\pi_i: N\text{-cycle} \\ \pi_i \in \mathfrak{S}_N}} b_{(i, \pi_i)} \rho_i V_{\pi_i}, \quad (137)$$

where the coefficients satisfy

$$\sum_{i=1}^N \sum_{\substack{\pi_i: N\text{-cycle} \\ \pi_i \in \mathfrak{S}_N}} b_{(i, \pi_i)} = 1. \quad (138)$$

For any two  $N$ -cycles  $\alpha$  and  $\beta \in \mathfrak{S}_N$ , there exists a permutation  $\gamma$

$$\gamma = \begin{pmatrix} \alpha(1) & \dots & \alpha(N) \\ \beta(1) & \dots & \beta(N) \end{pmatrix} \quad (139)$$

such that they are conjugate. Thus, for a given index  $i$ , all  $N$ -cycles share the same coefficient, which we denote as  $b_i$ . Therefore,  $b_{(i, \pi_i)} = b_i$  for any  $\pi_i \in \mathfrak{S}_N$ . By further examining the result under  $P_\pi \circ (i, j) \circ P_\pi^{-1}$ , where  $(i, j)$  denotes the transposition between the  $i$ th and  $j$ th systems, we obtain  $b_i = b_j$ . This result confirms that, under GPI condition, we have

$$b_i = \frac{1}{N!}, \quad \forall i \in \{1, \dots, N\}. \quad (140)$$

Having established the formalism, it is straightforward to verify that it satisfies both HP and BC. The broadcasting condition, in particular, requires that the partial trace over all output systems except  $B_i$  results in a noiseless channel from  $A$  to  $B_i$ . This is captured by the equation

$$\text{Tr}_{B_1 \dots B_{i-1} B_{i+1} \dots B_N} \circ \mathcal{E}_{A \rightarrow B_1 \dots B_N} = \mathcal{I}_{A \rightarrow B_i}, \quad \forall i. \quad (141)$$

$$\mathcal{B}_3(\rho) = \frac{1}{6} \left( \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right). \quad (146)$$

In the 1-to-2 broadcasting case, the canonical virtual broadcasting map can be formulated using the anti-commutator  $\{\cdot, \cdot\}$ . Similarly, the 1-to-3 canonical virtual broadcasting map can also be represented in terms of the anticommutator. To develop some intuition, let us start by analyzing  $\rho_1 V_{(123)}$  in Eq. (157), where  $(123)$  represents the cyclic permutation operation. A straightforward observation reveals

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}. \quad (147)$$

In algebraic form, this is equivalently written as

$$\rho_1 V_{(123)} = V_{(123)} \rho_2. \quad (148)$$

By applying the same technique,  $\mathcal{B}_3(\rho)$  can be reformulated in the form

$$\mathcal{B}_3(\rho) = \frac{1}{6} \left( \frac{1}{2} \rho_1 V_{(123)} + \frac{1}{2} V_{(123)} \rho_2 + \frac{1}{2} \rho_2 V_{(123)} + \frac{1}{2} V_{(123)} \rho_3 \right. \quad (149)$$

$$\left. + \frac{1}{2} \rho_3 V_{(123)} + \frac{1}{2} V_{(123)} \rho_1 + \frac{1}{2} \rho_1 V_{(132)} + \frac{1}{2} V_{(132)} \rho_3 + \frac{1}{2} \rho_2 V_{(132)} + \frac{1}{2} V_{(132)} \rho_1 \right. \quad (150)$$

$$\left. + \frac{1}{2} \rho_3 V_{(132)} + \frac{1}{2} V_{(132)} \rho_2 \right)$$

The following theorem summarizes the results derived thus far.

**Theorem 5 (1-to- $N$  broadcasting).** If a linear map from system  $A$  to systems  $B_1 \dots B_N$  satisfies the conditions of generalized unitary covariance (GUC) [Eq. (126)], generalized permutation invariance (GPI) [Eq. (127)], and generalized classical consistency (GCC) [Eq. (128)], then, for any input state  $\rho$ , it must have the form

$$\mathcal{B}_N(\rho) := \sum_{i=1}^N \sum_{\substack{\pi: N\text{-cycle} \\ \pi \in \mathfrak{S}_N}} \frac{1}{N!} \rho_i V_\pi, \quad (142)$$

where  $\rho_i$  is defined as in Eq. (136), and  $V_\pi$  is the matrix representation of the  $N$ -cycle  $\pi \in \mathfrak{S}_N$ . Since  $\mathcal{B}_N$  is the unique 1-to- $N$  virtual broadcasting map satisfying the aforementioned conditions, we call it the *canonical 1-to- $N$  virtual broadcasting map*.

In the case of  $N = 2$ , corresponding to 1-to-2 virtual broadcasting, Eq. (142) simplifies to

$$\mathcal{B}_2(\rho) = \frac{1}{2} (\rho \otimes \mathbb{1} \cdot S + \mathbb{1} \otimes \rho \cdot S) \quad (143)$$

$$= \frac{1}{2} (\rho \otimes \mathbb{1} \cdot S + S \cdot \rho \otimes \mathbb{1}) \quad (144)$$

$$= \frac{1}{2} \{\rho \otimes \mathbb{1}, S\}, \quad (145)$$

thus recovering the previously established canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  (see Ref. [18]). Building upon this, we investigate the 1-to-3 canonical virtual broadcasting scenario by evaluating Eq. (142) at  $N = 3$ , resulting in

$$= \frac{1}{12} (\{\rho_1, V_{(123)}\} + \{\rho_2, V_{(123)}\} + \{\rho_3, V_{(123)}\} \\ + \{\rho_1, V_{(132)}\} + \{\rho_2, V_{(132)}\} + \{\rho_3, V_{(132)}\}) \quad (151)$$

$$= \frac{1}{2} \{\rho_3, V_3\}. \quad (152)$$

The average state, denoted  $\rho_{\mathbb{N}}$ , is defined as

$$\rho_{\mathbb{N}} := \frac{1}{N} \sum_{i=1}^N \rho_i. \quad (153)$$

Specifically, for  $N = 3$ , we have

$$\rho_3 = \frac{1}{3} (\rho_1 + \rho_2 + \rho_3). \quad (154)$$

We define  $V_{\mathbb{N}}$  as the average over all permutation matrices of  $N$ -cycles within the symmetric group  $\mathfrak{S}_N$ , namely,

$$V_{\mathbb{N}} := \frac{1}{(N-1)!} \sum_{\substack{\pi: N\text{-cycle} \\ \pi \in \mathfrak{S}_N}} V_\pi. \quad (155)$$

When  $N = 3$ , we obtain

$$V_3 = \frac{1}{2} (V_{(123)} + V_{(132)}). \quad (156)$$

Therefore, the corollary below characterizes the canonical 1-to-3 virtual broadcasting map  $\mathcal{B}_3$ .

*Corollary 2 (1-to-3 broadcasting).* A linear map from system  $A$  to systems  $B_1$ ,  $B_2$ , and  $B_3$  that satisfies generalized unitary covariance (GUC) [Eq. (126)], generalized permutation invariance (GPI) [Eq. (127)], and generalized classical consistency (GCC) [Eq. (128)] must, for any input state  $\rho$ , take the form

$$\mathcal{B}_3(\rho) = \frac{1}{2}\{\rho_3, V_3\}, \quad (157)$$

where  $\rho_3$  and  $V_3$  denote the average state and the average permutation matrix of 3-cycles in  $\mathfrak{S}_3$ , as defined in Eqs. (154) and (156), respectively. Given that  $\mathcal{B}_3$  uniquely satisfies all the aforementioned conditions for a 1-to-3 virtual broadcasting map, we refer to it as the *canonical 1-to-3 virtual broadcasting map*.

Following a similar methodology, the canonical 1-to- $N$  virtual broadcasting map  $\mathcal{B}_N$  [see Eq. (142)] is simplified by using the average state  $\rho_N$  [see Eq. (153)], the average permutation matrix  $V_N$  over  $N$ -cycles in  $\mathfrak{S}_N$  [see Eq. (155)], and the anticommutator

$$\sum_{i=1}^N \sum_{\substack{\pi: N\text{-cycle} \\ \pi \in \mathfrak{S}_N}} \frac{1}{N!} \rho_i V_\pi \quad (158)$$

$$= \frac{1}{N!} \sum_{i=1}^N \sum_{\substack{\pi: N\text{-cycle} \\ \pi \in \mathfrak{S}_N}} \frac{1}{2} (\rho_i V_\pi + V_\pi \rho_{\pi(i)}) \quad (159)$$

$$= \frac{1}{2(N-1)!} \sum_{\substack{\pi: N\text{-cycle} \\ \pi \in \mathfrak{S}_N}} \{\rho_N, V_\pi\} \quad (160)$$

$$= \frac{1}{2} \{\rho_N, V_N\}. \quad (161)$$

Consequently, the canonical 1-to- $N$  virtual broadcasting map  $\mathcal{B}_N$  is reformulated in terms of the anticommutator, as stated by the following corollary.

*Corollary 3 (1-to- $N$  broadcasting).* The canonical 1-to- $N$  virtual broadcasting map  $\mathcal{B}_N$  [see Eq. (142)] from system  $A$  to systems  $B_1 \dots B_N$  can be expressed as the anticommutator of the averaged state  $\rho_N$  and the average permutation matrix  $V_N$  of  $N$ -cycles in  $\mathfrak{S}_N$ , scaled by  $\frac{1}{2}$ :

$$\mathcal{B}_N(\rho) = \frac{1}{2} \{\rho_N, V_N\}. \quad (162)$$

The definitions of  $\rho_N$  and  $V_N$  are given in Eqs. (153) and (155), respectively.

The canonical virtual broadcasting map admits a feasible experimental implementation. For the 1-to-2 case, the two terms appearing in  $\mathcal{B}_2$  [see Eq. (11)] can be simultaneously realized using a single quantum circuit based on the SWAP test. This approach has been demonstrated in Ref. [20], where virtual broadcasting was implemented on a nuclear magnetic resonance (NMR) quantum processor. Similar circuit architectures can also be realized on other platforms such as nitrogen-vacancy (NV) centers and photonic systems. For canonical 1-to- $N$  virtual broadcasting map  $\mathcal{B}_N$ ,  $N$ -cycle permutations can be implemented by extending the SWAP test,

replacing pairwise SWAPs with generalized cyclic permutation operations.

## B. Fundamental limits of sample complexity

To determine whether a practical 1-to- $N$  broadcasting map exists, we first examine the fundamental condition of SE for 1-to- $N$  cases. Using the game framework introduced in Sec. V A, we analyze the required resources. Suppose Alice aims to broadcast her state  $\rho$  to  $N$  distinct and spatially separated agents,  $B_1, B_2, \dots, B_N$ , each performing an  $\epsilon - \delta$  test to measure observables  $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_N$  (see Definition 3). Without loss of generality, let agents  $B_1, B_2, \dots, B_N$  require  $n_1, n_2, \dots, n_N$  copies of  $\rho$  to pass their respective tests. A naive estimate of the total copies Alice needs to win the game is simply the sum of these individual requirements, denoted as

$$n_C(N) := \sum_{i=1}^N n_i. \quad (163)$$

Any strategy requiring more resources than this naive approach would be inefficient, as Alice could simply send  $n_i$  copies to each agent  $B_i$  individually. This sets a fundamental benchmark for the sample complexity of practical broadcasting. Specifically, in the case of 1-to- $N$  broadcasting, the SE requirement (see Definition 1) is equivalent to stating that the sample size needed should be no greater than  $n_C(N)$ . We assume that the maximum value of  $n_i$  across all agents is given by  $n_Q(N)$ , that is,

$$n_Q(N) := \max_i n_i. \quad (164)$$

From the analysis in Sec. V B, we determine that the sample complexity required for implementing the canonical 1-to- $N$  virtual broadcasting  $\mathcal{B}_N$  [see Eq. (142)] is given by

$$n_V(N) := u_N^2 n_Q(N). \quad (165)$$

The sample overhead, i.e.,  $u_N$ , is determined by a variant of SDP outlined in Eq. (91), where  $y/2$  in Eq. (92) is replaced with  $J^{\mathcal{B}_N}$ , the Choi operator of canonical 1-to- $N$  virtual broadcasting map  $\mathcal{B}_N$  [see Eq. (142)]. Specifically, we define  $\Gamma_N$  as

$$\Gamma_N := \frac{1}{N} \sum_{i=1}^N \Gamma_{AB_i}, \quad (166)$$

where  $\Gamma_{AB_i}$  represents the unnormalized maximally entangled state on the systems  $A$  and  $B_i$  with identity operators in all the other systems in  $B_j$  with  $j \neq i$ , and  $y_N$  is defined as

$$y_N := \Gamma_N(\mathbb{1}_A \otimes V_N) + (\mathbb{1}_A \otimes V_N)\Gamma_N. \quad (167)$$

Using this,  $u_N$  can be written as

$$u_N := \min a + b \quad (168)$$

$$\text{s.t. } J_1 - J_2 = \frac{1}{2}y_{\mathbb{N}}, \quad (169)$$

$$\text{Tr}_{B_1 \dots B_N}[J_1] = a \mathbb{1}_A, \quad (170)$$

$$\text{Tr}_{B_1 \dots B_N}[J_2] = b \mathbb{1}_A, \quad (171)$$

$$J_1 \geq 0, J_2 \geq 0. \quad (172)$$

Generalizing practical broadcasting from the 1-to-2 case (see Definition 2) to the 1-to- $N$  case, we formally define *1-to- $N$  practical broadcasting* as follows.

**Definition 4 (Practical broadcasting).** A linear map from system  $A$  to systems  $B_1 \dots B_N$  is considered a 1-to- $N$  practical broadcasting map if it satisfies the properties of sample efficiency (SE), generalized unitary covariance (GUC), generalized permutation invariance (GPI), and generalized classical consistency (GCC).

The uniqueness of the canonical 1-to- $N$  virtual broadcasting map  $\mathcal{B}_N$  [see Eq. (142)], established by the conditions of generalized unitary covariance [GUC, see Eq. (126)], generalized permutation invariance [GPI, see Eq. (127)], and generalized classical consistency [GCC, see Eq. (128)], directly determines the feasibility of practical 1-to- $N$  broadcasting. The existence of a practical broadcast implementation depends on  $n_V(N)$  [see Eq. (165)] being less than or equal to the number of samples required in the naive protocol, namely  $n_C(N)$  [see Eq. (163)]. Therefore,  $n_V(N) < n_C(N)$  guarantees practical broadcasting, while  $n_V(N) \geq n_C(N)$  precludes it. However, the comparison between  $n_V(N) = u_N^2 n_Q(N)$ , and  $N n_Q(N)$  in Fig. 7 highlights the impossibility of practical broadcasting.

Virtual cooling marked a turning point in leveraging classical postprocessing alongside quantum operations to overcome limitations that are otherwise insurmountable within quantum theory [5]. This hybrid paradigm has since enabled meaningful gains in tasks such as quantum error mitigation [8–12] and virtual resource distillation [15–17]. However, these advantages come at a cost: significant sample overhead inherent to classical inference. Crucially, such benefits are not universal. Our no-practical-broadcasting theorem delineates fundamental constraints on protocols that seek to distribute quantum information across multiple recipients using limited quantum resources. In quantum networking, while virtual broadcasting may appear to offer an efficient route for replicating quantum states over distributed nodes, our results show that this cannot be accomplished in a sample-efficient or operationally consistent manner within standard quantum mechanics. These findings highlight a key message of virtual operations: their practical utility must be weighed against the often-prohibitive sampling demands they impose.

To conclude our discussion on no practical broadcasting (see Theorem 2), it is instructive to place our analysis in the broader context of fundamental constraints in quantum information theory, particularly the class of so-called no-go theorems, and to ask when such limitations might be overcome by extending the operational framework. The no-cloning theorem [1,2] establishes that quantum information

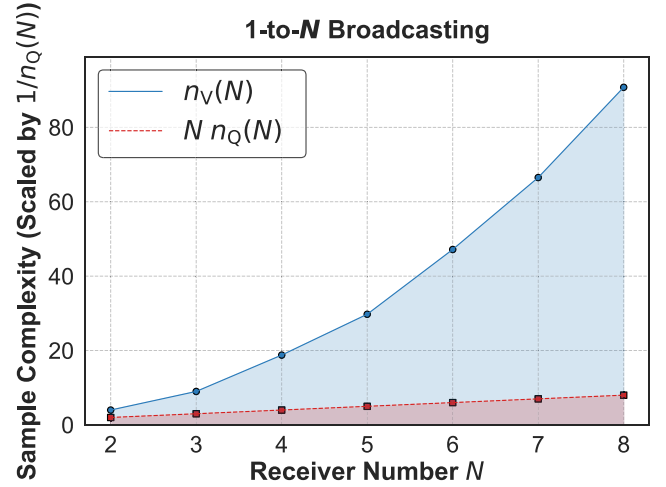


FIG. 7. Sample complexity of realizing  $\mathcal{B}_N$ . The horizontal axis represents the number of receivers  $N$  in the broadcasting task, while the vertical axis shows the sample complexity, scaled by a factor of  $1/n_Q(N)$ , with the system to be broadcast taken as a qubit system, i.e.,  $d = 2$ . The blue line depicts the sample complexity required for implementing the canonical 1-to- $N$  virtual broadcasting map  $\mathcal{B}_N$  [see Eq. (142)], expressed as  $u_N^2 n_Q(N)$ , whereas the red dashed line represents an upper bound  $N n_Q(N)$  on the number of copies needed for the naive approach, given by  $n_C(N)$  [see Eq. (163)]. The figure reveals that the canonical 1-to- $N$  virtual broadcasting map requires significantly more state copies than the naive approach, indicating that  $\mathcal{B}_N$  fails to meet the zeroth condition of sample efficiency (SE) (see Definition 1). This suggests that the canonical 1-to- $N$  virtual broadcasting map is impractical for broadcasting quantum information.

cannot be perfectly duplicated: there exists no CPTP map  $\mathcal{E} : A \rightarrow BC$  satisfying

$$\mathcal{E}(\psi) = \psi \otimes \psi \quad (173)$$

for all pure states  $\psi$ . Broadcasting represents a natural relaxation [3], requiring only that the marginals of the output reproduce the input state, i.e.,

$$\text{Tr}_B[\mathcal{E}(\rho)] = \text{Tr}_C[\mathcal{E}(\rho)] = \rho. \quad (174)$$

Cloning is thus a special case of broadcasting, and any obstruction to broadcasting immediately implies the impossibility of cloning. Our result rules out the possibility of practical broadcasting under all linear maps that satisfy sample efficiency (SE), unitary covariance (UC), permutation invariance (PI), and classical consistency (CC). As a corollary, no such map can implement cloning either, within these constraints. This highlights the rigidity of quantum information: even when augmented with classical postprocessing, virtual extensions of quantum operations remain bound by fundamental limitations.

The landscape becomes more intricate when we consider other no-go theorems. The no-deleting theorem [41], for instance, forbids any unitary transformation that erases one copy of a quantum state from the input  $|\psi\rangle \otimes |\psi\rangle \otimes |A\rangle$ , yielding  $|\psi\rangle \otimes |0\rangle \otimes |A_\psi\rangle$ . Yet in contrast to cloning and broadcasting, deletion becomes possible if the set of operations is extended from unitaries to general quantum channels. This contrast raises a fundamental question: Which quantum no-go

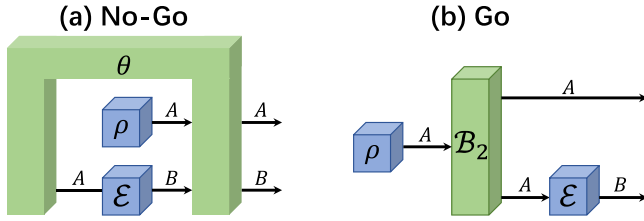


FIG. 8. Absence of a quantum state over time. (a) Given a copy of the initial state  $\rho$  and its evolution  $\mathcal{E}$ , no physical operation, described by the superchannel  $\theta$ , can generate a bipartite quantum state with marginals  $\rho$  and  $\mathcal{E}(\rho)$ , respectively. (b) However, when virtual operations are permitted,  $(\mathcal{I} \otimes \mathcal{E}) \circ \mathcal{B}_2(\rho)$  represents a quantum state over time with marginals  $\rho$  and  $\mathcal{E}(\rho)$ . In general,  $(\mathcal{I} \otimes \mathcal{E}) \circ \mathcal{B}_2(\rho)$  is not a positive-semidefinite operator.

theorems are preserved under operational extensions, such as the inclusion of ancillary systems, measurement-based feedback, or classical postprocessing, and which remain inviolable regardless of such generalizations? The answer is far from settled. As we have shown, broadcasting and cloning resist such extensions. Deletion, by contrast, yields. Identifying the operational boundaries that separate mutable constraints from immutable ones is more than a matter of theoretical elegance, it could offer a road map for reengineering quantum protocols at the edge of what is physically realizable. Clarifying this distinction remains an open challenge, but one whose resolution may illuminate new pathways for both foundational insight and technological innovation in quantum information science.

### C. Temporal quantum states

The temporal evolution of a quantum state  $\rho$  under a quantum channel  $\mathcal{E}$  inherently encodes spatiotemporal correlations, acting as the quantum counterpart to a classical conditional probability distribution  $P(X_t|X_{t-1})$  [42,43]. Crucially, a single instance of the initial state and the channel's action are insufficient to physically reconstruct the full time-dependent quantum state, a constraint imposed by the inherent limitations of quantum operations, represented here by superchannels, as illustrated in Fig. 8. This highlights a fundamental asymmetry between space and time in quantum theory. However, the barrier can be overcome by relaxing the operational constraints from CPTP maps to HPTP maps, facilitating a virtual reconstruction of the state's evolution. Building upon this insight, we generalize this approach to encompass arbitrary multitime quantum processes, establishing a formal analogy of stochastic processes  $P(X_t|X_{t-1}, X_{t-2}, \dots)$  in classical probability theory.

To capture and analyze the evolving spatiotemporal correlations of a quantum system across a finite time series, we introduce the temporal quantum state, a mathematical framework formally defined as follows.

**Definition 5 (Temporal quantum state).** Consider an initial state  $\rho$  acting on system  $A$ , which encodes the history of evolution, along with its future quantum dynamics  $\mathcal{E}_1 : B_1 \rightarrow B_2, \dots, \mathcal{E}_{N-1} : B_{N-1} \rightarrow B_N$ . The collection of these operations is denoted by  $\mathfrak{N} := \{\mathcal{E}_i\}_{i=0}^{N-1}$ , with the convention that  $\mathcal{E}_0 := \mathcal{I}$ , where  $\mathcal{I}$  represents the noiseless quantum channel

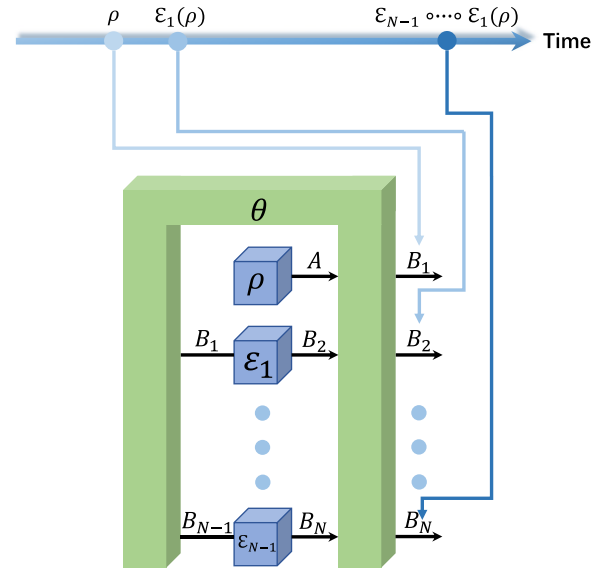


FIG. 9. Temporal quantum state. Creating a multipartite quantum state  $\theta(\rho, \mathfrak{N})$  (see Definition 5) using the (higher-order) quantum operation  $\theta$ , such that the reduced state on each subsystem  $B_i$  corresponds to  $\mathcal{E}_i \circ \dots \circ \mathcal{E}_0(\rho)$  for all  $i \in \{0, \dots, N-1\}$ .

from system  $A$  to system  $B_1$ . If there exists a physical operation  $\theta$  such that applying  $\theta$  to  $\rho$  and  $\mathfrak{N}$  generates a multipartite quantum state  $\theta(\rho, \mathfrak{N})$  on systems  $B_1 \dots B_N$ , whose marginal is given by

$$\text{Tr}_{B_1 \dots B_{i-1} B_{i+1} \dots B_N} [\theta(\rho, \mathfrak{N})] = \mathcal{E}_{i-1} \circ \dots \circ \mathcal{E}_0(\rho), \quad (175)$$

for any  $i \in \{1, \dots, N\}$ , then we refer to  $\theta(\rho, \mathfrak{N})$  as the temporal quantum state associated with  $\rho$  and  $\mathfrak{N}$ . An illustration of temporal quantum state is provided in Fig. 9.

The above Definition 5 naturally extends the concept of a quantum state over time, originally considered for two time points, to the most general case involving a finite sequence of time points. It is well established that a quantum state over time satisfying Eq. (175) for just two time points does not exist [42]. Consequently, a general temporal quantum state for an arbitrary number ( $>1$ ) of time points is also fundamentally unattainable under physical operations.

However, if we relax these constraints and allow for virtual operations, a key question emerges: How can we systematically construct temporal quantum states under these broader conditions? The answer depends on how virtual operations are utilized. Different protocols can be designed based on the consumption of channel copies. Here, we introduce two distinct constructions: one leveraging our previously established canonical 1-to- $N$  virtual broadcasting map  $\mathcal{B}_N$  [see Eq. (142)], and the other employing the sequential application of the canonical 1-to-2 virtual broadcasting map  $\mathcal{B}_2$  [see Eq. (11)].

We begin with the first approach, where each quantum channel  $\mathcal{E}_i$  is applied  $N-i$  times. By combining this with the canonical 1-to- $N$  virtual broadcasting map  $\mathcal{B}_N$  (see

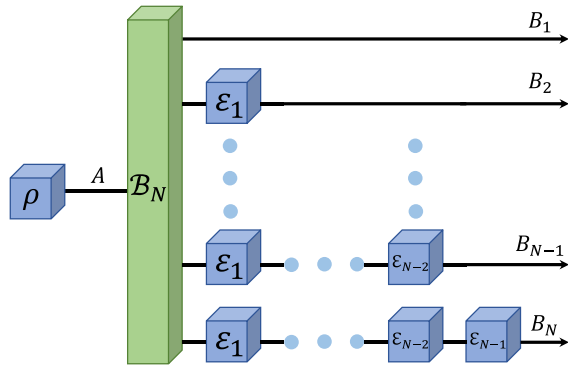


FIG. 10. Generating temporal quantum states: Protocol I. We construct the temporal quantum state by first applying the canonical 1-to- $N$  virtual broadcasting map  $\mathcal{B}_N$  [see Eq. (142)] to the initial state  $\rho$ , followed by a sequence of channels  $\mathcal{E}_{i-1}, \dots, \mathcal{E}_1$  acting on the  $i$ th output system. This process yields the state given in Eq. (176), whose marginal precisely corresponds to  $\mathcal{E}_{i-1} \circ \dots \circ \mathcal{E}_0(\rho)$ , with  $\mathcal{E}_0 = \mathcal{I}$ .

Eq. (142)], we construct an  $N$ -partite virtual state

$$\left( \mathcal{E}_0 \otimes \mathcal{E}_1 \otimes \underbrace{(\mathcal{E}_2 \circ \mathcal{E}_1)}_2 \otimes \dots \otimes \underbrace{(\mathcal{E}_{N-1} \circ \dots \circ \mathcal{E}_1)}_{N-1} \right) \circ \mathcal{B}_N(\rho), \quad (176)$$

whose marginal on system  $B_i$  precisely reproduces the target state  $\mathcal{E}_{i-1} \circ \dots \circ \mathcal{E}_0(\rho)$  at the corresponding time point. This construction is visualized in Fig. 10.

Our second construction involves a sequential application of canonical 1-to-2 virtual broadcasting maps  $\mathcal{B}_2$  [see Eq. (11)]. This design ensures that each channel  $\mathcal{E}_i$  ( $i \in \{0, \dots, N-1\}$ ) is employed precisely once, preventing redundancy and optimizing resource allocation. The virtual state

generated by this method is

$$\begin{aligned} & \underbrace{\mathcal{I} \otimes \dots \otimes \mathcal{I}}_{N-1} \otimes \mathcal{E}_{N-1} \circ \underbrace{\mathcal{I} \otimes \dots \otimes \mathcal{I}}_{N-2} \otimes \mathcal{B}_2 \\ & \times \left[ \underbrace{\mathcal{I} \otimes \dots \otimes \mathcal{I}}_{N-2} \otimes \mathcal{E}_{N-2} \circ \right. \\ & \left. \dots \left( \underbrace{\mathcal{I} \otimes \mathcal{I}}_2 \otimes \mathcal{E}_2 \circ (\mathcal{I} \otimes \mathcal{B}_2[\mathcal{I} \otimes \mathcal{E}_1 \circ \mathcal{B}_2(\rho)]) \right) \right]. \end{aligned} \quad (177)$$

The marginal state on subsystem  $B_i$  recovers the target state  $\mathcal{E}_{i-1} \circ \dots \circ \mathcal{E}_0(\rho)$  at the corresponding temporal step. Figure 11 demonstrates the recursive nature of this construction.

The recursive mechanism in our second construction [see Eq. (177) and Fig. 11] shares similarities with collision models in nonequilibrium quantum thermodynamics [44] and agent models in quantum stochastic processes [45]. Importantly, these models can be viewed as specific instances of quantum circuit fragments [37,38], also known as quantum combs in Refs. [31,32] and process tensors in Ref. [33]. This perspective allows us to interpret our construction as a temporal broadcasting protocol. This interpretation raises several fundamental questions: How does temporal broadcasting diverge from conventional spatial broadcasting? Can a practical temporal broadcasting protocol be implemented in physical systems? Furthermore, are there inherent constraints, similar to unitary covariance (UC), permutation invariance (PI), and classical consistency (CC), that uniquely dictate the form of a temporal broadcasting protocol? Addressing these questions is essential for understanding the nature of quantum information flow over time, but requires a dedicated investigation beyond the current scope. Therefore, we leave them for future works.

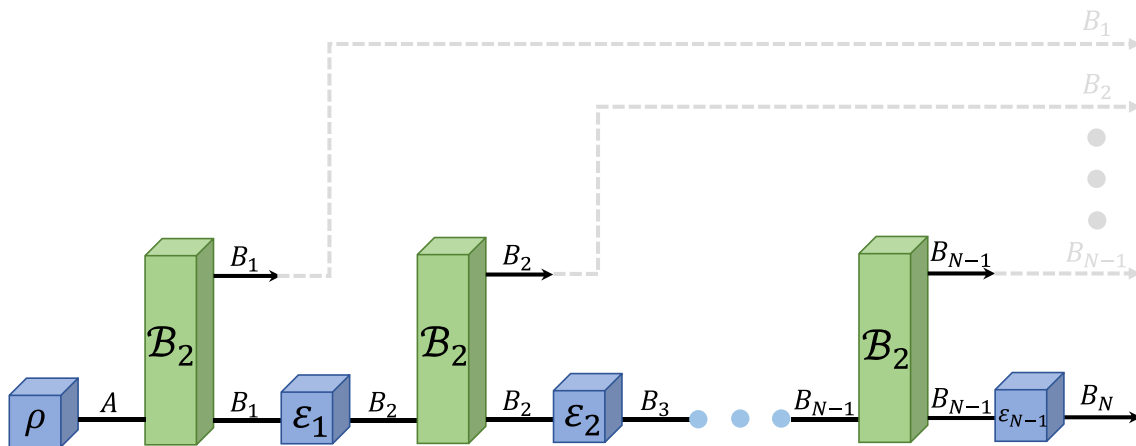


FIG. 11. Generating temporal quantum states: Protocol II. In the first step of the process, we apply the canonical 1-to-2 virtual broadcasting maps  $\mathcal{B}_2$  [see Eq. (11)], keeping one of its output systems, denoted as system  $B_1$ , and applying channel  $\mathcal{E}_1$  to the other. Using this as a building block, we iterate the process. In the  $i$ th step, we apply the broadcasting map  $\mathcal{B}_2$ , followed by the channel  $\mathcal{E}_i$  on one of its output systems. The final state across all output systems constitutes the temporal quantum state (see Definition 5).

### D. Pseudodensity operator

As discussed earlier, this work establishes the canonical form of 1-to- $N$  virtual broadcasting (see Corollary 3). We now explore its connection to the pseudodensity operator (PDO) [27], which extends the conventional density matrix formalism by associating distinct Hilbert spaces with different time points, thereby generalizing quantum states across multiple time points. In Ref. [18], it was shown that for the qubit case, the canonical 1-to-2 virtual broadcasting coincides with the PDO of the state  $\rho$  under identity evolution  $\mathcal{I}$ . This coincidence can be generalized to general quantum channel using Fig. 8(b). Here, we extend this analysis and demonstrate that this equivalence does not hold in general. Specifically, we compare the canonical 1-to-3 virtual broadcasting  $\mathcal{B}_3$  with the PDO across three time points [46], showing that the former contains additional terms beyond those present in the PDO.

Consider a quantum system with density matrix  $\rho \in \mathcal{H}^{\otimes M}$ , where  $\mathcal{H}$  is the Hilbert space of a single qubit, i.e.,  $\dim \mathcal{H} = 2$ , evolving over  $N$  discrete time points

$$t_1 \xrightarrow{\mathcal{E}_1} t_2 \xrightarrow{\mathcal{E}_2} t_3 \xrightarrow{\mathcal{E}_3} \cdots \xrightarrow{\mathcal{E}_{N-1}} t_N. \quad (178)$$

The corresponding  $N$ -time  $M$ -qubit PDO is then defined as

$$R_N = \frac{1}{2^{MN}} \sum_{i_N=0}^{4^M-1} \cdots \sum_{i_1=0}^{4^M-1} \langle \{\sigma_{\vec{i}_j}\}_{j=1}^N \rangle \bigotimes_{j=1}^N \sigma_{\vec{i}_j}. \quad (179)$$

Here,  $\vec{i}_j \in \{0, 1, 2, 3\}^{\otimes M}$  represents a quaternary string, and  $\sigma_{\vec{i}_j}$  denotes the corresponding  $M$ -qubit Pauli matrix at time  $j$ . In this case, the operator  $\bigotimes_{j=1}^N \sigma_{\vec{i}_j}$  serves as an observable across  $N$  time points, with its expectation value given by  $\langle \{\sigma_{\vec{i}_j}\}_{j=1}^N \rangle$ . The conventional quantum density matrix at a single time point is recovered by tracing out the Hilbert spaces for all but one time, say  $t_j$ , such that

$$\rho_j := \mathcal{E}_{j-1} \cdots \mathcal{E}_1(\rho) \quad (180)$$

$$= \text{Tr}_{12\dots(j-1)(j+1)\dots N}[R_N]. \quad (181)$$

PDO possesses several key properties. It is Hermitian and has a unit trace; however, unlike density matrices, it can have negative eigenvalues. While negativity is not a mandatory feature, it provides a sufficient condition for detecting quantum temporal correlations. Furthermore, taking the partial trace

over the PDO yields a valid PDO. A particularly useful scenario arises when the measurement basis consists of projectors onto the  $\pm 1$  eigenspaces of the Pauli operators  $\sigma_{\vec{i}_j}$ , enabling the calculation of expectation values  $\langle \{\sigma_{\vec{i}_j}\}_{\alpha=1}^m \rangle$ . Under this choice of measurement scheme, the  $N$ -time PDO is given by the following iterative expression [46]:

$$R_N = \frac{1}{2} \{R_{N-1}, J_{\mathbf{T}}^{\mathcal{E}_{N-1}}\}, \quad (182)$$

with the initial condition

$$R_2 = \frac{1}{2} \{\rho, J_{\mathbf{T}}^{\mathcal{E}_1}\}, \quad (183)$$

where  $\{\cdot, \cdot\}$  denotes the anticommutator, and  $J_{\mathbf{T}}^{\mathcal{E}_{N-1}}$  refers to the Jamiołkowski operator of the  $(N-1)$ th quantum channel  $\mathcal{E}_{N-1}$  [26]:

$$J_{\mathbf{T}}^{\mathcal{E}_{N-1}} := (\mathcal{I} \otimes \mathcal{E}_{N-1}) \circ (\mathcal{I} \otimes \mathbf{T})(\Gamma) \quad (184)$$

$$= \sum_{ij} |i\rangle\langle j| \otimes \mathcal{E}_{N-1}(|j\rangle\langle i|). \quad (185)$$

Here,  $\mathbf{T}$  denotes the transpose operation, and  $\Gamma$  represents the unnormalized maximally entangled state [see Eq. (1)]. Since the systems are clear from the context, we have omitted the identity matrix  $\mathbb{1}$  for simplicity. Remark that employing Jamiołkowski's formalism [26] for quantum channels, rather than the conventional Choi formalism [25], is a key factor contributing to the negativity of the PDO. This is because, even for the identity channel  $\mathcal{I}$ , its Choi operator  $J^{\mathcal{I}} := (\mathcal{I} \otimes \mathcal{I})(\Gamma) = \Gamma$  is positive semidefinite, namely  $J^{\mathcal{I}} \geq 0$ , whereas its Jamiołkowski operator  $J_{\mathbf{T}}^{\mathcal{I}} = S$  [see Eq. (2)] is not. Although initially formulated for qubit systems, the expression of PDO can be extended to general  $d$ -dimensional systems [47]. Therefore, we will refer to the PDO as describing a quantum system of arbitrary dimension across time.

Let us now consider two examples of the PDO. As the first example, consider the two time points PDO with trivial dynamics  $\mathcal{I}$ , demonstrated in Fig. 12(a), which takes the form

$$R_2 = \frac{1}{2} \{\rho, S\} = \mathcal{B}_2(\rho). \quad (186)$$

The second example involves the three time points PDO with two trivial evolutions  $\mathcal{I}$ , as illustrated in Fig. 12(b), and can be expressed as follows:

$$R_3 = \frac{1}{4} \{R_2 \otimes \mathbb{1}_3, \mathbb{1}_1 \otimes S_{23}\} \quad (187)$$

$$= \frac{1}{4} \left( \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right). \quad (188)$$

This coincides with the result of applying the canonical 1-to-2 virtual broadcasting  $\mathcal{B}_2$  twice consecutively to the input system  $\rho$  (see Fig. 11). However, it does not match the canonical 1-to-3 virtual broadcasting  $\mathcal{B}_3$  [see Eq. (157)], leaving room to explore their differences and the reasons for these discrepancies. Specifically, compared to  $\mathcal{B}_3$ , the following two elements

are missing in  $R_3$  [see Eq. (188)]:

$$\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \quad \text{and} \quad \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array}. \quad (189)$$

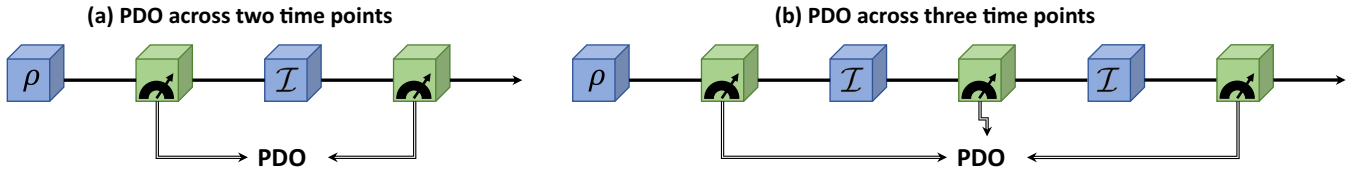


FIG. 12. Pseudodensity operators under identity evolution. (a) Shows pseudodensity operators for two time points under identity evolution  $\mathcal{I}$ , and (b) for three time points. The measuring devices are quantum instruments that output both classical outcomes and quantum states, with the quantum output serving as the input for the next step. All quantum instruments presented here are represented by Pauli matrices.

Building on the core physical idea behind the original PDO [27], we introduce a generalization that captures spatiotemporal correlations across multiple time points. For simplicity, we consider two time points, each associated with observables  $\mathcal{O}_{i_1}$  and  $\mathcal{O}_{i_2}$ . Here,  $\mathcal{O}_{i_1}$  is measured first, and its quantum output is fed back into the quantum dynamics  $\theta$  before  $\mathcal{O}_{i_2}$  is measured. We assume that the quantum dynamics  $\theta$  follows a definite causal order, as illustrated in Fig. 13. Denoting the expectation values with respect to  $\mathcal{O}_{i_1}$  and  $\mathcal{O}_{i_2}$  as  $e_{i_1 i_2}$ , we define the corresponding PDO as

$$\sum_{i_1, i_2} e_{i_1 i_2} \mathcal{O}_{i_1} \otimes \mathcal{O}_{i_2}. \quad (190)$$

This definition imposes no restrictions on the dimensions of  $\mathcal{O}_{i_1}$  or  $\mathcal{O}_{i_2}$ , and the systems on which they act may overlap. While our formulation differs mathematically from the original PDO [27], being more general, we retain the term “PDO” for conceptual clarity and consistency.

Now, let us address the final undefined quantity, which could potentially undermine the rigor of Eq. (190): the expectation value  $e_{i_1 i_2}$ . This value is primarily determined by the method of measurement. For clarity, consider two time points. Suppose the first observable is the Pauli Z operator,  $\sigma_Z$ . In this case, the measurement process involves projecting the input state onto the computational basis and then preparing the corresponding state based on the measurement outcome. For example, if the measurement yields the outcome 0, the state is

prepared as  $|0\rangle\langle 0|$  and passed on to the subsequent quantum dynamics. The Choi operator associated with this process is expressed by the formula

$$(|0\rangle\langle 0|)^T \otimes |0\rangle\langle 0|. \quad (191)$$

In general, we assume that in order to measure observable  $\mathcal{O}_{i_1}$  (as shown in Fig. 13), a positive operator-valued measure (POVM)  $\{M_{i_1}^x\}_x$  has been implemented. When outcome  $x$  is observed, the state  $\rho_{i_1}^x$  is prepared and sent to the subsequent quantum dynamics. For the second observable  $\mathcal{O}_{i_2}$ , we assume its measurement is governed by the POVM  $\{M_{i_2}^y\}_y$ . Here, measuring  $\mathcal{O}_{i_1}$  forms a quantum instrument, as it includes both a classical outcome  $x$  and a quantum output  $\rho_{i_1}^x$ , while measuring  $\mathcal{O}_{i_2}$  only requires a POVM. Mathematically, the expectation value  $e_{i_1 i_2}$  is given by

$$e_{i_1 i_2} := \sum_{x, y} xy [((M_{i_1}^x)^T \otimes \rho_{i_1}^x \otimes (M_{i_2}^y)^T) \star J^\theta]. \quad (192)$$

Here,  $\star$  denotes the operation of the link product [31,32]. To conclude, the formal definition of the PDO with definite causal order is provided by the following.

**Definition 6 (PDO with DCO).** Consider the quantum dynamics  $\theta$  and two observables  $\mathcal{O}_{i_1}$  and  $\mathcal{O}_{i_2}$ , as illustrated in Fig. 13. The measurement process for  $\mathcal{O}_{i_1}$  involves a POVM  $\{M_{i_1}^x\}_x$ , followed by the preparation of the state  $\rho_{i_1}^x$ . Specifically, upon measuring  $M_{i_1}^x$ , the state  $\rho_{i_1}^x$  is prepared. In contrast, the measurement process for  $\mathcal{O}_{i_2}$  consists solely of the POVM  $\{M_{i_2}^y\}_y$ . The resulting PDO with respect to  $\mathcal{O}_{i_1}$  and  $\mathcal{O}_{i_2}$  for the quantum dynamics  $\theta$  is given by Eq. (190), where  $e_{i_1 i_2}$  is defined in Eq. (192).

This framework [see Eq. (190)] naturally extends to more than two time points, so we omit a detailed discussion for brevity. Furthermore, a similar idea can be extended to cases with indefinite causal order, as illustrated in Fig. 14. It is mathematically defined as follows.

**Definition 7 (PDO with ICO).** Given the quantum dynamics  $\vartheta$  and two observables  $\mathcal{O}_{i_1}$  and  $\mathcal{O}_{i_2}$ , as depicted in Fig. 14, the measurement process for  $\mathcal{O}_{i_1}$  involves a POVM  $\{M_{i_1}^x\}_x$ , followed by the preparation of the state  $\rho_{i_1}^x$ . Specifically, when  $M_{i_1}^x$  is measured, the state  $\rho_{i_1}^x$  is prepared. Meanwhile, the measurement process for  $\mathcal{O}_{i_2}$  involves the POVM  $\{M_{i_2}^y\}_y$  and the preparation of the state  $\rho_{i_2}^y$ . The PDO corresponding to  $\mathcal{O}_{i_1}$  and  $\mathcal{O}_{i_2}$  for  $\vartheta$  is then given by Eq. (190), with  $e_{i_1 i_2}$  defined as

$$\sum_{x, y} xy [((M_{i_1}^x)^T \otimes \rho_{i_1}^x \otimes (M_{i_2}^y)^T \otimes \rho_{i_2}^y) \star J^\vartheta]. \quad (193)$$

### PDO with definite causal order

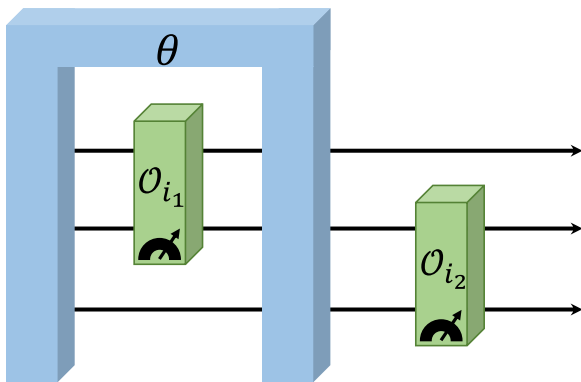


FIG. 13. Pseudodensity operators with definite causal order (DCO). Measuring the observables  $\mathcal{O}_{i_1}$  and  $\mathcal{O}_{i_2}$  sequentially, with a definite causal order in the quantum dynamics  $\theta$ , the resulting expectation values will yield a PDO with definite causal order.

### PDO with indefinite causal order

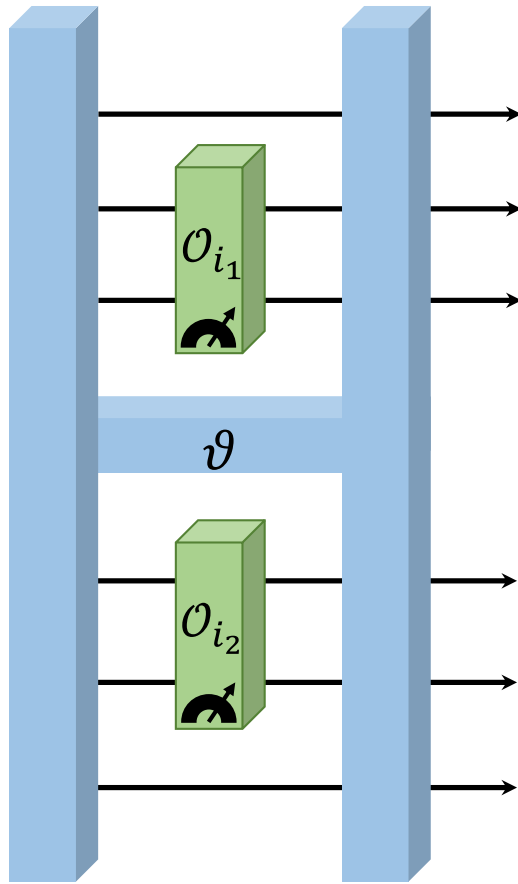


FIG. 14. Pseudodensity operators with indefinite causal order (ICO). When measuring the observables  $\mathcal{O}_{i_1}$  and  $\mathcal{O}_{i_2}$  in space and time, where the underlying quantum dynamics  $\vartheta$  lacks a definite causal order (such as in the case of quantum switch [48]), the resulting expectation values will give rise to a PDO with indefinite causal order.

Finally, recall that PDO has been applied in quantum causal inference [49–51], channel capacity [52], temporal teleportation [53], the arrow of time [54], and indefinite causal order [55], among others [56,57]. A natural question arises: Can the canonical virtual broadcasting map also find applications in these areas, or could the differences between the PDO and canonical virtual broadcasting yield deeper insights? Further investigation into these questions could provide valuable perspectives.

## VII. DISCUSSIONS

In this work, we have established the no-practical-broadcasting theorem by proving that no linear map can simultaneously satisfy sample efficiency, unitary covariance, permutation invariance, and classical consistency. Our approach combines Schur-Weyl duality, tensor networks, and semidefinite programming. As a by-product, we have derived the canonical 1-to- $N$  virtual broadcasting map and explored its connections to temporal quantum states and pseudodensity operators.

Quantum cloning defines the ultimate limit of eavesdropping: the best an adversary can do is produce an imperfect copy of the sender’s state while relaying the original to the receiver [58–61]. At a more fundamental level, quantum broadcasting represents a stronger paradigm, one that, in principle, allows for maximal extraction of information about sender’s key. If virtual broadcasting could be implemented with sample efficiency, such a strategy might compromise the foundational security of quantum cryptography. Our no-practical-broadcasting theorem, however, establishes that such a threat is intrinsically ruled out by the laws of quantum mechanics.

Looking ahead, a promising direction is the extension of our framework to channel broadcasting, where linear supermaps are used to broadcast quantum channels, with potential applications in quantum cryptography and distributed quantum computing. Another intriguing question is whether relaxing some of the constraints considered here allows for a linear map that still satisfies sample efficiency. Addressing these open problems could provide deeper insights into the feasibility of practical broadcasting and help delineate the fundamental limits of quantum information distribution.

## ACKNOWLEDGMENTS

Y.X. gratefully acknowledges S. Popescu, G. Chiribella, F. Buscemi, and M. Murao for stimulating discussions during QMQI 2024. Appreciation is also extended to Y. Yang, H. Fu, P. Yao, and N. Jing for insightful conversations at QIP 2025. Furthermore, he thanks J. Zhang, R. Takagi, V. Narasimhachar, D. Aghamalyan, X. Wang, M. S. Tan, B. Zhao, J. Thompson, K. Bharti, S. Hyung Lie, and J. Ye for valuable discussions. This research is supported by A\*STAR under Grant No. C23091703 and Q.InC Strategic Research and Translational Thrust. Y.X. acknowledges support from A\*STAR under its Central Research Funds and Career Development Fund (Grant No. C243512002). X.L. is funded by the National Research Foundation, Prime Minister’s Office, Singapore, under its Campus for Research Excellence and Technological Enterprise (CREATE) programme, the National Research Foundation through the NRF Investigatorship on Quantum-Enhanced Agents (Grant No. NRF-NRFI09-0010) and the National Quantum Office, hosted in A\*STAR, under its Centre for Quantum Technologies Funding Initiative (S24Q2d0009), the Singapore Ministry of Education Tier 1 Grants No. RT4/23 and No. RG77/22 (S), the FQXi R-710-000-146-720 Grant “Are quantum agents more energetically efficient at making predictions?” from the Foundational Questions Institute, Fetzer Franklin Fund (a donor-advised fund of Silicon Valley Community Foundation). Z.L. acknowledges the support of the National Natural Science Foundation of China Grant No. 12174216 and the Innovation Program for Quantum Science and Technology Grants No. 2021ZD0300804 and No. 2021ZD0300702.

## DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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