

Generation of squeezed optical states via stored classical pulses in a Bose gas

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We propose and analyze a scheme to generate squeezed light by storing a classical probe pulse in a Bose-Einstein condensate (BEC) and exploiting the nonlinear evolution caused by atom-atom collisions during the storage time. A Λ -type optical memory interface maps a chosen temporal probe mode onto a single phase-matched collective spin wave; for a coherent input, this prepares a tunable coherent spin state of a two-component BEC, with its initial spin orientation set by the stored mean excitation number and the phase relation between the probe and control fields. Collisional interactions during storage then implement one-axis-twisting dynamics and generate spin squeezing in the atomic ensemble. We account for realistic loss and finite memory and retrieval efficiencies, and model readout as a single-mode beam-splitter mapping that transfers the atomic quadrature squeezing onto a propagating optical mode. The model identifies optimal storage times and predicts that, under realistic conditions, several decibels of squeezing can be transferred to the retrieved light.

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I. INTRODUCTION

The generation of squeezed states is a cornerstone of modern quantum technologies: It enables measurement sensitivities beyond the standard quantum limit (SQL) and provides a key resource for quantum information processing. In collective spin systems, spin squeezing [1–3] denotes the redistribution of quantum fluctuations such that the variance of one collective spin component is reduced below the SQL, at the expense of increased fluctuations in a noncommuting component. Spin-squeezed states have been realized in a variety of atomic platforms through both measurement-induced and interaction-driven protocols [4–7]. In quantum nondemolition schemes, dispersive probing yields information about a collective spin projection (e.g., $\hat{J}_z$) while ideally preserving its mean value; the associated measurement backaction conditionally reduces its fluctuations, and with feedback and/or cavity enhancement can produce strongly metrologically useful squeezing [5,7–9]. Complementarily, collisional interactions in a two-component Bose-Einstein condensate (BEC) realize an effective one-axis-twisting (OAT) nonlinearity, $\hat{H}_{\text{OAT}} = \hbar\chi\hat{J}_z^2$, where χ is set by the relevant scattering

lengths and the trapped-mode geometry. In this unitary picture, population-dependent mean-field shifts shear the collective Bloch vector, squeezing one spin quadrature while antisqueezing its conjugate. Interaction-driven squeezing and entanglement generation have been demonstrated in condensates using controlled collisional dynamics and internal-state interferometry [4,10–15]. Related spin-mixing processes in spinor condensates provide additional routes to nonclassical states, enabling twin-beam interferometry and spin-nematic squeezing [16–18]. These BEC-based platforms also allow spatially resolved access to many-body correlations, including Bell correlations and Einstein-Podolsky-Rosen steering [19,20].

In parallel, electromagnetically induced transparency (EIT) [21,22] provides a coherent and mode-selective interface between propagating optical fields and long-lived ground-state coherences [23,24]. Implemented in many different atomic systems, EIT and related light storage techniques have enabled ultraslow and halted light as well as long-lived optical storage [23,25–30], offering a route to coherently map and transport collective excitations between light and matter, enabling the development of quantum memory devices [31,32] that have significant importance in quantum information science.

In this work, we combine these two ingredients and analyze how a Λ -type EIT interface can prepare a two-component BEC in a controllable collective spin state using a classical probe pulse, while collisional interactions subsequently generate OAT-driven spin squeezing within the ensemble. Crucially, the same phase-matched atom-light coupling also defines a

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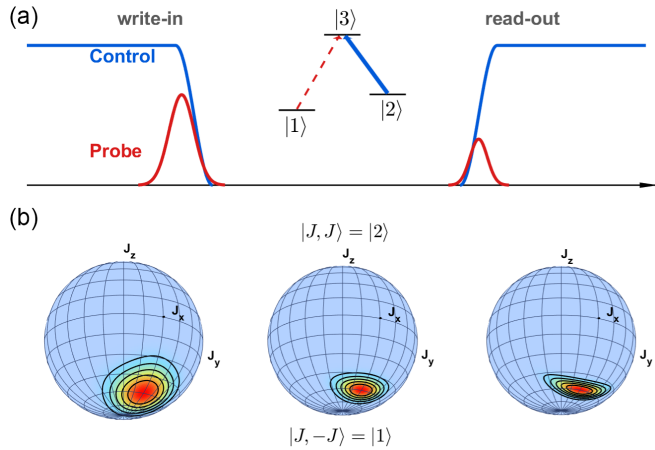


FIG. 1. (a) Timing diagram of the Λ -EIT sequence. A short probe pulse (red) is incident while the control field (blue) is on (write-in). Turning the control off maps the optical excitation onto a collective $|1\rangle - |2\rangle$ ground-state coherence. After a storage interval with the light fields off, the control is turned back on (read-out) to convert the spin wave back into an emitted probe pulse. Inset: Λ -level scheme with the probe coupling $|1\rangle \leftrightarrow |3\rangle$ and the control coupling $|2\rangle \leftrightarrow |3\rangle$. (b) Illustrative Bloch-sphere representations of the collective pseudospin formed by the two ground states $|1\rangle$ and $|2\rangle$ (south/north poles $|J, -J\rangle \equiv |1\rangle$ and $|J, J\rangle \equiv |2\rangle$). The colored patches schematically indicate the spin-noise distribution on the sphere: starting from a near-isotropic coherent-state spot and evolving into an elliptical distribution during the interaction-only evolution, illustrating the build up of spin squeezing and the rotation of the principal noise axes.

specific optical output mode, enabling the squeezing generated in the collective spin to be mapped onto the retrieved light field.

II. PHYSICAL SYSTEM

We consider a BEC of N_0 identical atoms with two long-lived internal states $\{|1\rangle, |2\rangle\}$, e.g., two hyperfine (or Zeeman) sublevels, forming an effective collective pseudospin $J = N_0/2$. The internal dynamics is described by the collective spin operator $\hat{\mathbf{J}} = \sum_{j=1}^{N_0} \hat{\mathbf{j}}^{(j)}$, with $\hat{\mathbf{j}}^{(j)}$ the spin-1/2 operator of atom j in the $\{|1\rangle, |2\rangle\}$ manifold. The state with all atoms in $|1\rangle$ corresponds to the fully polarized Dicke state $|J, -J\rangle$, i.e., the south pole of the collective Bloch sphere [Fig. 1(b)].

The two ground states $|1\rangle$ and $|2\rangle$ are coupled to a common excited state $|3\rangle$ in a Λ configuration by a weak probe field with mean photon number of μ_{in} , and a strong classical control field, with corresponding optical Rabi frequencies Ω_p and Ω_c , respectively. In the adiabatic EIT regime [27,33] (or far-detuned Raman limit), eliminating $|3\rangle$ yields an effective interaction between one selected probe temporal mode $\hat{a}$ and one collective ground-state spin-wave mode $\hat{S}$ of the form [34,35]

$$\hat{H}_{\text{int}}(t) = \hbar \kappa(t) (\hat{a} \hat{S}^\dagger + \hat{a}^\dagger \hat{S}), \quad (1)$$

where $\kappa(t)$ is controlled by the control-field envelope and detuning. For an ensemble of atoms at positions $\mathbf{r}_j$,

$$\hat{S}^\dagger \equiv \frac{1}{\sqrt{N_0}} \sum_{j=1}^{N_0} e^{i\Delta\mathbf{k}\cdot\mathbf{r}_j} |2\rangle_j \langle 1|, \quad (2)$$

meaning that a single excitation is delocalized over N_0 atoms, with $\Delta\mathbf{k} = \mathbf{k}_p - \mathbf{k}_c$, where $\mathbf{k}_p$ and $\mathbf{k}_c$ are the probe and control wave vectors, respectively; thus, copropagating probe/control beams ($|\Delta\mathbf{k}| \simeq 0$) address a spatially uniform collective mode.

Since $\kappa(t)$ is a scalar prefactor, the interaction Hamiltonian is proportional to a time-independent operator and therefore satisfies $[\hat{H}_{\text{int}}(t), \hat{H}_{\text{int}}(t')] = 0$. The write process thus generates the beam-splitter unitary

$$\hat{U}_w = \exp[-i\vartheta (\hat{a} \hat{S}^\dagger + \hat{a}^\dagger \hat{S})], \quad (3)$$

with $\vartheta = \int_{\text{write}} dt \kappa(t)$, which mixes the modes according to $\hat{U}_w^\dagger \hat{a} \hat{U}_w = \hat{a} \cos \vartheta - i \hat{S} \sin \vartheta$ and $\hat{U}_w^\dagger \hat{S} \hat{U}_w = \hat{S} \cos \vartheta - i \hat{a} \sin \vartheta$. For a coherent probe input $|\alpha\rangle$ with $|\alpha|^2 = \mu_{\text{in}}$ and spin-wave vacuum $|0\rangle_S$, the ideal state-level mapping is (beam splitters map coherent states to coherent states [36])

$$\hat{U}_w |\alpha\rangle |0\rangle_S = |\alpha \cos \vartheta\rangle \otimes |\tilde{\beta}\rangle_S, \quad \tilde{\beta} = -i\alpha \sin \vartheta, \quad (4)$$

so that the ideally generated spin wave is a bosonic coherent state with mean excitation $|\tilde{\beta}|^2$.

In practice, finite optical depth, spontaneous-emission loss, and imperfect mode matching reduce the amplitude stored in the desired collective mode. We capture these effects by defining an effective stored spin-wave coherent state $|\beta\rangle_S$ with the same phase as the ideal mapping, $\arg \beta = \arg \tilde{\beta}$, but reduced magnitude. Specifically, we introduce a net write-in efficiency η_{write} (defined for the chosen write sequence and thus including any incomplete transfer implicit in ϑ) and a spatial mode overlap ζ_{spatial} , such that

$$\mu_{\text{stored}} \equiv \langle \hat{S}^\dagger \hat{S} \rangle = |\beta|^2 = \eta_{\text{write}} \zeta_{\text{spatial}} |\alpha|^2. \quad (5)$$

In the following, $|\beta\rangle_S$ is the atomic state we use as the initial condition.

We fix the total atom number N_0 in two internal modes with bosonic operators $\hat{b}_{1,2}$. The collective spin and number operators are $\hat{J}_z = \frac{1}{2}(\hat{n}_2 - \hat{n}_1)$, $\hat{J}_+ = \hat{b}_2^\dagger \hat{b}_1$, $\hat{J}_- = \hat{b}_1^\dagger \hat{b}_2$, and $\hat{n}_1 = N_0/2 - \hat{J}_z$ and $\hat{n}_2 = N_0/2 + \hat{J}_z$. In the Holstein-Primakoff regime, which is equivalent to weak-probe EIT in our case, $\langle \hat{S}^\dagger \hat{S} \rangle \ll N_0$, the spin-wave creation operator can be embedded into the spin algebra as [37]

$$\hat{S}^\dagger \simeq \frac{1}{\sqrt{N_0}} \hat{J}_+ = \frac{1}{\sqrt{N_0}} \hat{b}_2^\dagger \hat{b}_1. \quad (6)$$

Within the single-mode assumption, i.e., atoms in states $|1\rangle$ and $|2\rangle$ occupy the same spatial mode, the commutator satisfies $[\hat{S}, \hat{S}^\dagger] = (\hat{n}_1 - \hat{n}_2)/N_0 = 1 - 2\hat{n}_2/N_0 \simeq 1$ for $\langle \hat{n}_2 \rangle \ll N_0$, which reduces to $[\hat{S}, \hat{S}^\dagger] \simeq 1$.

We take the spin-wave vacuum as $|0\rangle_S \equiv |N_0, 0\rangle$ (all atoms in $|1\rangle$). We describe the input probe field as an optical coherent state with complex amplitude α , $|\alpha\rangle_{\text{in}}$, normalized such that $|\alpha|^2 = \mu_{\text{in}}$. Assuming a phase-preserving linear write process with overall (mode-matched) storage factor $\eta_{\text{write}} \zeta_{\text{spatial}}$, the stored collective amplitude is $\beta = \sqrt{\eta_{\text{write}} \zeta_{\text{spatial}}} \alpha$, so

that the mean stored excitation number is $\mu_{\text{stored}} = |\beta|^2 = \eta_{\text{write}} \zeta_{\text{spatial}} \mu_{\text{in}}$.

A stored spin-wave coherent state is obtained by displacing the spin-wave vacuum, $|\beta\rangle_S \equiv \hat{D}_S(\beta) |0\rangle_S$, with the standard displacement operator $\hat{D}_S(\beta) = \exp(\beta \hat{S}^\dagger - \beta^* \hat{S})$. Using the normally ordered identity $\hat{D}_S(\beta) = e^{-|\beta|^2/2} e^{\beta \hat{S}^\dagger} e^{-\beta^* \hat{S}}$ and the vacuum property $\hat{S} |0\rangle_S = 0$, the rightmost factor acts trivially on $|0\rangle_S$, which is why the $-\beta^* \hat{S}$ term does not appear explicitly once the operator is applied to the vacuum. In the Holstein-Primakoff (weak-excitation) regime $\mu_{\text{stored}} = |\beta|^2 \ll N_0$, we further use $\hat{S}^\dagger \simeq (\hat{b}_2^\dagger \hat{b}_1)/\sqrt{N_0}$, yielding the fixed- N_0 ladder form

$$|\beta\rangle_S = \hat{D}_S(\beta) |0\rangle_S = e^{-|\beta|^2/2} \exp(\beta \hat{S}^\dagger) |0\rangle_S \simeq e^{-|\beta|^2/2} \exp\left(\frac{\beta}{\sqrt{N_0}} \hat{b}_2^\dagger \hat{b}_1\right) |N_0, 0\rangle. \quad (7)$$

Atomic coherent states, or coherent spin states (CSS), may be defined as rotations $|\theta, \varphi_0\rangle = e^{-i\varphi_0 \hat{J}_z} e^{-i\theta \hat{J}_y} |J, -J\rangle$ and admit an equivalent ladder form [37] $(1 + |\tau|^2)^{-J} \exp(\tau \hat{J}_+) |J, -J\rangle$ with $\tau = e^{i\varphi_0} \tan(\theta/2)$; the derivation and conventions are summarized in Appendix B. Comparison with Eq. (7) identifies $\tau = \beta/\sqrt{N_0} = \sqrt{\eta_{\text{write}} \zeta_{\text{spatial}}/N_0} \alpha$ and $\varphi_0 = \arg \beta$ (set by the probe/control phase reference). In the same weak-excitation regime $|\tau|^2 \ll 1$, the CSS normalization satisfies $(1 + |\tau|^2)^{-J} \simeq \exp(-J|\tau|^2) = \exp(-|\beta|^2/2)$, which matches the bosonic prefactor in $|\beta\rangle_S$ (Appendix B). For a coherent spin state, the mean transferred population is $\mu_{\text{stored}} = \langle \hat{n}_2 \rangle = N_0 \sin^2(\theta/2) \simeq N_0(\theta/2)^2$ for $\mu_{\text{stored}} \ll N_0$, hence $\theta \simeq 2\sqrt{\mu_{\text{stored}}/N_0} = 2\sqrt{\eta_{\text{write}} \zeta_{\text{spatial}} \mu_{\text{in}}/N_0}$. Thus, the mean input excitation μ_{in} together with η_{write} and ζ_{spatial} sets the initial, small tilt θ , while the probe/control phase sets the azimuth φ_0 . Furthermore, we limit μ_{in} so that we are operating in the weak probe regime.

During the storage stage, the optical fields are off and the dynamics of the two-component condensate is governed by single-particle energies and S -wave collisional interactions. In a two-mode description, we introduce bosonic mode operators $\hat{b}_i$ for the two internal states ($i = 1, 2$) and number operators $\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i$. The effective Hamiltonian reads

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}, \quad (8)$$

with the single-particle part

$$\hat{H}_0 \simeq E_\phi(\hat{n}_1 + \hat{n}_2) + \epsilon_1 \hat{n}_1 + \epsilon_2 \hat{n}_2 \quad (9)$$

and the collisional interaction

$$\hat{H}_{\text{int}} = \frac{U_{11}}{2} \hat{b}_1^{\dagger 2} \hat{b}_1^2 + \frac{U_{22}}{2} \hat{b}_2^{\dagger 2} \hat{b}_2^2 + U_{12} \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_1 \hat{b}_2, \quad (10)$$

where E_ϕ is the common single-particle mode energy (kinetic plus trap energy) of the shared spatial mode, ϵ_i are the internal-state energy offsets in the rotating frame (only the difference $\epsilon_1 - \epsilon_2$ affects the relative phase evolution), and U_{ij} are the effective interaction matrix elements (e.g., $U_{ij} = g_{ij} \int d^3r |\phi(\mathbf{r})|^4$ in a single-spatial-mode picture with $g_{ij} = 4\pi \hbar^2 a_{ij}/m$, where a_{ij} is the S -wave scattering length between the states $|1\rangle$ and $|2\rangle$).

For fixed total atom number $\hat{n}_1 + \hat{n}_2 = N_0$, the term $E_\phi(\hat{n}_1 + \hat{n}_2) = E_\phi N_0$ is an additive constant that contributes

only a global phase and will be dropped. Substituting $\hat{J}_z$ and $\hat{n}_{1,2}$ into $\hat{H}_0 + \hat{H}_{\text{int}}$ (discarding additive constants) yields the OAT Hamiltonian

$$\hat{H}_{\text{OAT}} = \hbar \chi \hat{J}_z^2 + \hbar \Omega \hat{J}_z + \text{const}, \quad (11)$$

with $\chi = (U_{11} + U_{22} - 2U_{12})/(2\hbar)$ and $\Omega = (\epsilon_2 - \epsilon_1 + (N_0 - 1)(U_{22} - U_{11})/2)/\hbar$. The nonlinear term $\propto \hat{J}_z^2$ is the one-axis-twisting interaction that generates spin squeezing, while the linear term $\propto \hat{J}_z$ corresponds to a collective rotation about the z axis (differential phase accumulation) and can be removed by working in a rotating frame with respect to $\hbar \Omega \hat{J}_z$ or by appropriate phase referencing.

III. SQUEEZING DYNAMICS, LOSS, AND OPTICAL READOUT

After the write process prepares an initial coherent spin state $|\theta, \varphi_0\rangle$ (Sec. II), the two-component condensate evolves under the effective OAT Hamiltonian [Eq. (11)]. We work throughout in the rotating frame of the linear term $\hbar \Omega \hat{J}_z$ (equivalently, we absorb this deterministic z rotation into the phase reference) and thus retain only the nonlinear term in Eq. (11). The interaction-induced phase shift then depends on the population difference, which shears the collective noise distribution on the Bloch sphere and generates squeezing in an optimally chosen quadrature. Because $[\hat{H}_{\text{OAT}}, \hat{J}_z] = 0$, unitary OAT conserves $\hat{J}_z$ and hence the instantaneous populations; the nontrivial dynamics is therefore encoded in the transverse spin components and their correlations with $\hat{J}_z$. As elaborated in Sec. II, we focus on the weak-excitation regime $\mu_{\text{stored}} \ll N_0$, where the initial tilt is small and the relevant squeezing dynamics is well captured by transverse fluctuations around the mean spin.

A. Exact moments from Heisenberg correlators

To quantify squeezing, we need the first moments $\langle \hat{\mathbf{J}} \rangle$ and the symmetrized second moments that form the covariance matrix (Sec. III B). We work directly in the fixed- J Dicke manifold and compute these moments exactly without propagating the full many-body state. For one-axis twisting (OAT), $\hat{J}_z$ is conserved and the ladder operators acquire phases depending only on the $\hat{J}_z$ eigenvalue m . This reduces the problem to evaluating a small set of Heisenberg-picture correlators as finite Dicke-basis sums.

We introduce the following four expectation values:

$$E_1(t) \equiv \langle \hat{J}_+(t) \rangle, \quad E_2(t) \equiv \langle \hat{J}_-(t) \rangle, \quad E_3(t) \equiv \langle \hat{J}_+^2(t) \rangle, \quad (12)$$

$$E_4(t) \equiv \langle \hat{J}_-^2(t) \rangle.$$

They determine the transverse mean spin and the second moments within the $x - y$ plane. Mixed moments that involve $\hat{J}_z$ are expressed through additional number-weighted correlators $E_5, \dots, E_{12}$. Their explicit forms are collected in Appendix A 1. From E_1 and E_2 , we obtain the transverse mean spin components

$$\langle \hat{J}_x \rangle = \frac{1}{2}(E_1 + E_2), \quad \langle \hat{J}_y \rangle = \frac{1}{2i}(E_1 - E_2). \quad (13)$$

For the transverse second moments, we use the fixed- J relations $\hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_z^2 + \hat{J}_z$ and $\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z^2 - \hat{J}_z$. This gives

$$\langle \hat{J}_x^2 \rangle = \frac{E_4 + E_3}{4} + \frac{1}{2}(J(J+1) - \langle \hat{J}_z^2 \rangle), \quad (14)$$

$$\langle \hat{J}_y^2 \rangle = -\frac{E_4 + E_3}{4} + \frac{1}{2}(J(J+1) - \langle \hat{J}_z^2 \rangle), \quad (15)$$

$$\langle \frac{1}{2} \{ \hat{J}_x, \hat{J}_y \} \rangle = \frac{E_3 - E_4}{4i}. \quad (16)$$

The two-quantum coherences E_3 and E_4 set the anisotropy and orientation of the transverse noise ellipse. The remaining term $J(J+1) - \langle \hat{J}_z^2 \rangle$ provides the isotropic offset fixed by the spin length and the $\hat{J}_z$ statistics.

To determine the orientation of the squeezing ellipse (i.e., the principal axes of the transverse fluctuations) with respect to the mean-spin direction, we also require the mixed symmetrized moments with $\hat{J}_z$. Using the number-weighted correlators from Appendix A, we define

$$C_{xz}(t) \equiv (E_{12} - E_{11}) + (E_{10} - E_9) + (E_7 - E_5) + (E_8 - E_6), \quad (17)$$

$$C_{yz}(t) \equiv (E_{11} - E_{12}) + (E_{10} - E_9) + (E_7 - E_5) + (E_6 - E_8). \quad (18)$$

The corresponding anticommutators are

$$\langle \frac{1}{2} \{ \hat{J}_x, \hat{J}_z \} \rangle = \frac{C_{xz}(t)}{8}, \quad (19)$$

$$\langle \frac{1}{2} \{ \hat{J}_y, \hat{J}_z \} \rangle = \frac{C_{yz}(t)}{8i}. \quad (20)$$

Together with the longitudinal moments $\langle \hat{J}_z \rangle$ and $\langle \hat{J}_z^2 \rangle$, Eqs. (13)–(20) fully specify the covariance matrix $\Gamma(t)$ and hence the minimal transverse variance $\lambda_{\min}(t)$ (Sec. III B). In the following subsections, we extend this unitary description to include particle loss and to map the resulting atomic squeezing onto the retrieved optical field.

B. Covariance matrix and transverse squeezing

Quantum fluctuations of the collective spin are summarized by the symmetrized covariance matrix

$$\Gamma_{ij}(t) \equiv \frac{1}{2} \langle \Delta \hat{J}_i \Delta \hat{J}_j + \Delta \hat{J}_j \Delta \hat{J}_i \rangle, \quad (21)$$

where $\Delta \hat{J}_i = \hat{J}_i - \langle \hat{J}_i \rangle$, with $i, j \in \{x, y, z\}$. For any real unit vector $\mathbf{u}$, the variance of the spin projection $\hat{J}_{\mathbf{u}} \equiv \mathbf{u} \cdot \hat{\mathbf{J}}$ is given by $\text{Var}(\hat{J}_{\mathbf{u}}) = \mathbf{u}^T \Gamma \mathbf{u}$.

The metrological signal is carried by the mean spin vector $\mathbf{J}(t) = \langle \hat{\mathbf{J}}(t) \rangle$. Noise relevant for phase estimation is the noise in directions transverse to $\mathbf{J}(t)$, i.e., in the plane orthogonal to the unit vector $\mathbf{n}(t) = \mathbf{J}(t)/|\mathbf{J}(t)|$. To isolate these fluctuations, we introduce the projector $\mathbf{P}(t) = \mathbb{I}_3 - \mathbf{n}(t)\mathbf{n}(t)^T$ and define the projected covariance $\Gamma_{\perp}(t) = \mathbf{P}(t) \Gamma(t) \mathbf{P}(t)$. This projection removes the component parallel to $\mathbf{n}(t)$ and retains fluctuations within the transverse plane. Accordingly, $\Gamma_{\perp}(t)$ has one null direction along $\mathbf{n}(t)$ and two nonzero eigenvalues, which are the variances along the principal axes of the transverse noise ellipse. We denote by $\lambda_{\min}(t)$ the smaller of these two eigenvalues. Equivalently, $\lambda_{\min}(t)$ is the minimum

variance of a transverse spin component $\hat{J}_{\mathbf{u}} = \mathbf{u} \cdot \hat{\mathbf{J}}$ over all unit vectors $\mathbf{u}$ satisfying $\mathbf{u} \cdot \mathbf{n}(t) = 0$.

We characterize atomic squeezing by the smallest fluctuation of a collective-spin component orthogonal to the instantaneous mean spin $\mathbf{J}(t) \equiv \langle \hat{\mathbf{J}}(t) \rangle$ [1]. Let $\lambda_{\min}(t)$ denote the smaller eigenvalue of the covariance matrix projected onto the plane perpendicular to $\mathbf{J}(t)$, i.e., the minimal transverse spin variance $\Delta J_{\perp, \min}^2(t)$. For a coherent spin state with the same mean-spin length, the transverse variance is $|\mathbf{J}(t)|/2$. We therefore define the dimensionless squeezing parameter

$$v_{A, \min}(t) \equiv \frac{\lambda_{\min}(t)}{|\mathbf{J}(t)|/2}, \quad (22)$$

so that $v_{A, \min} = 1$ corresponds to the CSS, i.e., shot-noise, level. In the unitary, high-contrast limit $|\mathbf{J}(t)| \simeq J = N/2$, this reduces to the Kitagawa-Ueda definition $\xi_S^2 = 4 \Delta J_{\perp, \min}^2/N$ [1]. This is the quantity entering our optical readout model (details in Sec. III D): When the local-oscillator phase is aligned with the minimal-noise axis, the retrieved optical quadrature variance is obtained by mixing $v_{A, \min}(t)$ with vacuum according to the readout efficiency.

C. Population loss and injected transverse noise

We include irreversible losses at the level of mean populations by evolving $N_i(t) = \langle \hat{n}_i \rangle$ with rate equations that include one-, two-, and three-body processes. These deterministic equations set the longitudinal first moment $\langle \hat{J}_z \rangle = [N_2(t) - N_1(t)]/2$ and define an instantaneous effective spin length $J_{\text{stored}}(t) = N_e(t)/2$, with $N_e(t) = N_1(t) + N_2(t)$. We model the population dynamics as

$$\dot{N}_i(t) = -K_{1,i} N_i - \sum_j K_{2,ij} N_i N_j - \sum_{j,k} K_{3,ijk} N_i N_j N_k, \quad (23)$$

where $K_{1,i}$ are one-body loss rates (units of s^{-1}), $K_{2,ij}$ are two-body loss coefficients (units of m^3/s), and $K_{3,ijk}$ are three-body loss coefficients (units of m^6/s). The indices indicate the internal states participating in the event. See Appendix A 2 for detailed explanation and numerical values used in our simulations.

Loss also reduces collective $|1\rangle - |2\rangle$ coherence. We model this by a minimal, number-consistent renormalization of the unitary OAT correlators $E_k^{(u)}(t)$ obtained from unitary evolution under $\hat{H}_{\text{OAT}}$ (i.e., with all loss channels set to zero) on the fixed- J manifold (Appendix A 2). Concretely, we write $E_k(t) = S_k(t) E_k^{(u)}(t)$, where the prefactor $S_k(t)$ is constructed from the instantaneous mean populations $N_{1,2}(t)$ and depends only on whether the correlator contains one or two ladder operators $\hat{J}_{\pm}$ and whether it is weighted by $\hat{n}_1$ or $\hat{n}_2$. The explicit mapping $k \mapsto S_k(t)$ is given in Appendix A 2. A useful by-product of this construction is a simple, “memory-efficiency” factor. The stored spin wave is proportional to the collective coherence operator $\hat{J}_+ \equiv \hat{b}_2^\dagger \hat{b}_1$. For a fully phase-coherent two-component state with populations N_1 and N_2 , the collective coherence reaches $|\langle \hat{J}_+ \rangle| = \sqrt{N_1 N_2}$ [38]. Since the retrieved probe field amplitude in EIT readout is linear in the stored Raman coherence [23,39], the retrieved intensity (and hence an effective quadrature-mapping strength) scales as $|\langle \hat{J}_+ \rangle|^2 \propto N_1 N_2$. This motivates the intrinsic,

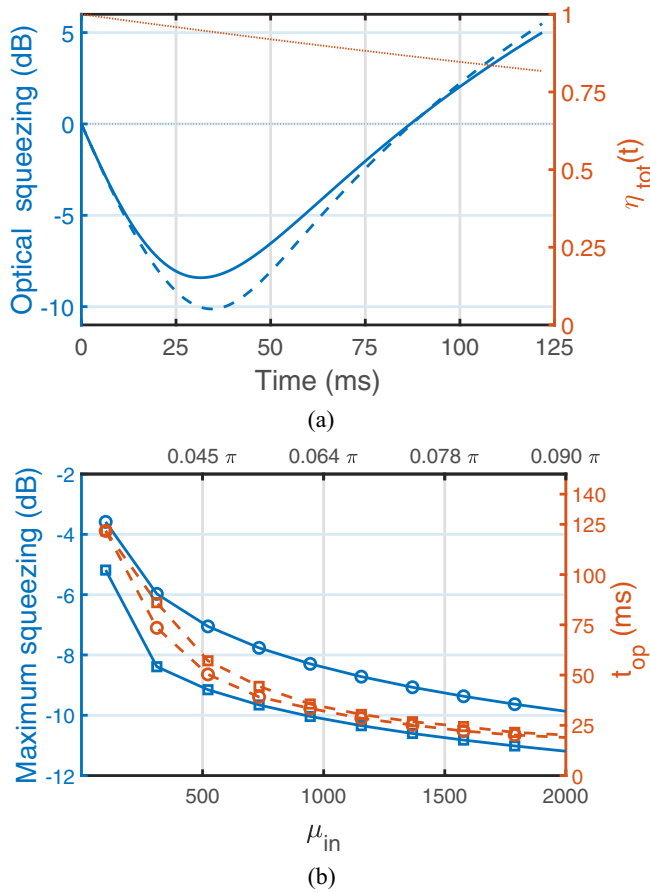


FIG. 2. (a) Time evolution of the retrieved optical variance $10 \log_{10} V_{\text{opt}}(t)$ (blue, left axis) for $\mu_{\text{in}} = 10^3$; solid (lossy) and dashed (lossless) curves are shown. The dotted red curve (right axis) shows the total memory efficiency $\eta_{\text{tot}}(t)$ for the lossy case. (b) Best (minimum) retrieved squeezing $\min_t[10 \log_{10} V_{\text{opt}}(t)]$ (blue, left axis) and the corresponding optimal time t_{op} (red, right axis) vs μ_{in} ; the upper axis indicates the equivalent initial spin-tilt angle θ , corresponding to the respective μ_{in} . Circles: lossy case, squares: lossless case.

coherence-limited factor

$$\eta_{\text{coh}}(t) = \frac{N_1(t)N_2(t)}{N_{1,0}N_{2,0}}. \quad (24)$$

In other words, the same population-loss model that determines $N_{1,2}(t)$ also determines the natural “memory decay” through loss of coherence, $\eta_{\text{coh}}(t)$. We will use this quantity as time-dependent memory efficiency, $\eta_{\text{tot}}(t) = \eta_{\text{coh}}(t) \times \eta_{\text{write}} \times \eta_{\text{read}}$, as shown by the orange curve in Fig. 2(a).

All first and second moments are then reconstructed from Eqs. (13)–(20), using the rescaled correlators $E_k(t)$ and replacing $J \rightarrow J_{\text{stored}}(t)$ in the static $J(J+1)$ contributions. Importantly, because the coherence reduction enters already at the level of the rescaled correlators (through R_1 and R_2), the resulting $v_{A,\text{min}}(t)$ refers directly to the retrievable collective mode at time t ; we therefore do not apply an additional $\eta_{\text{coh}}(t)$ factor again at the optical beam-splitter stage (Sec. III D).

The rescaling above accounts for average damping of mean values but not for the extra fluctuations generated by stochastic loss events (quantum jumps). Following Refs. [40,41],

we include their leading-order effect as an approximately isotropic diffusion in the plane transverse to the instantaneous mean spin. We implement this by the covariance update

$$\mathbf{\Gamma}(t + dt) = \mathbf{\Gamma}(t) + \frac{|\langle \hat{\mathbf{J}}(t) \rangle|^2}{N_e(t)} \frac{s_q(t)}{3} dt \mathbf{P}(t), \quad (25)$$

where $s_q(t) = \gamma_1(t) + 2\gamma_2(t) + 3\gamma_3(t)$ is the weighted sum of effective one-, two-, and three-body loss rates evaluated on the instantaneous mean state (explicit forms of γ_i are given in Appendix A 2). In our numerics, $\mathbf{\Gamma}(t)$ in Eq. (25) denotes only the accumulated loss-induced noise contribution to the covariance (initialized to zero); it is added to the covariance reconstructed from the rescaled OAT correlators to form the total covariance used to compute $\lambda_{\text{min}}(t)$ and hence $v_{A,\text{min}}(t)$. This makes explicit that the rescaling captures average coherence damping, whereas Eq. (25) accounts for the additional transverse fluctuations injected by random loss events.

D. Optical readout: Mapping atomic to optical squeezing

The read process is the time reverse of the write-in beam splitter and maps the collective Raman coherence (spin-wave mode) back onto a single optical output mode. In the adiabatic EIT picture, this corresponds to rotating the dark-state polariton from matterlike to lightlike by turning on the control field, thereby coherently transferring the stored atomic excitation back to the probe field mode [23,39]. We therefore model the external readout and detection as a linear loss channel of efficiency η_{read} acting on the relevant atomic quadrature, with the unused port fed by vacuum.

The retrieval of nonclassical optical states from atomic ensembles has been demonstrated in a number of experiments, including the storage and recall of squeezed and entangled light fields (and related nonclassical photonic inputs) [42–46]. In those works, however, the nonclassicality is primarily injected into the memory via the input light and then preserved during storage. In contrast, our proposal uses the memory interface as part of the state-preparation mechanism: The write-store-read sequence prepares a two-component BEC whose intrinsic interactions generate spin squeezing during storage, which is subsequently mapped onto the retrieved optical mode.

When the local-oscillator (LO) phase is aligned with the instantaneous minimal-noise axis of the atomic state, the detected optical quadrature variance is

$$V_{\text{opt}}(t) = [1 - \eta_{\text{read}}] + \eta_{\text{read}} v_{A,\text{min}}(t). \quad (26)$$

Equation (26) is the usual beam-splitter relation for variances: A retrieval channel of efficiency η_{read} transmits the squeezed atomic quadrature with weight η_{read} and admixes vacuum with weight $1 - \eta_{\text{read}}$, thereby reducing the observable squeezing. In our model, the atomic variance $v_{A,\text{min}}(t)$ already includes the time-dependent in-medium degradation of the spin wave due to loss (including the coherence-limited reduction absorbed into the correlator rescaling in Sec. III C) and any write-in imperfections; the conversion to the detected optical mode is therefore described solely by the additional external efficiency η_{read} .

IV. RESULTS

To benchmark our model with experimentally grounded parameters, we simulate a two-component ^{23}Na BEC in an optical dipole trap. We adopt the Λ -scheme EIT and interaction parameters at the working bias field from the long-lived EIT memory experiment of Ref. [28], but use a more compact, symmetrical cloud (Thomas-Fermi diameter $d_{\text{TF}} \simeq 10 \mu\text{m}$) with atom number ($N_0 = 1 \times 10^5$) and size comparable to the ^{87}Rb BEC optical memory experiment [47]. All results in this section are obtained from numerical simulations of the effective two-component collective-spin model introduced above: We evaluate the unitary OAT dynamics and combine it with numerical integration of the mean population-loss equations and the associated injected transverse noise. Additional technical decoherence is treated phenomenologically in Appendix D. There, we quantify magnetic-field noise explicitly and discuss how differential light shifts, density-dependent phase noise, and residual motional dephasing enter a phenomenological Gaussian decay of the effective Raman coherence.

We model a Λ -type EIT interface on the sodium D_1 line, $3S_{1/2} \rightarrow 3P_{1/2}$, with ground states $|1\rangle \equiv |3S_{1/2}, F=1, M=0\rangle$ and $|2\rangle \equiv |3S_{1/2}, F=2, M=-2\rangle$, coupled via the excited state $|3\rangle \equiv |3P_{1/2}, F'=1, M'=-1\rangle$; the coupling field is resonant with $|2\rangle \rightarrow |3\rangle$ and the probe field addresses $|1\rangle \rightarrow |3\rangle$ [28]. The excitation is then transferred to $|4\rangle \equiv |3S_{1/2}, F=1, M=-1\rangle$ for storage and back to $|2\rangle$ for readout [28]. The subsequent internal-state evolution is modeled as OAT with χ computed from the elastic S -wave scattering lengths at the working bias field $B \simeq 132.4$ G, namely, $a_{11} = 2.8$ nm, $a_{22} = 3.4$ nm, and $a_{12} = 3.4$ nm [28]. At this field, inelastic losses in the $|1\rangle$ – $|2\rangle$ channel are strongly suppressed: $\text{Im}(a_{12})$ approaches zero and it is assumed to be $-1 \times 10^{-3} \times a_0$ in our simulations, and $\text{Im}(a_{11})$ and $\text{Im}(a_{22})$ are negligible [28]. Numerical values of K parameters of Eq. (23) are detailed in Appendix A 2. At these elastic parameters, one has $a_{12}^2 > a_{11}a_{22}$, i.e., the mixture is formally immiscible, consistent with the phase-separation dynamics reported in Ref. [28] for strong stored-pulse imprints. In contrast, our simulations focus on the weak-excitation (Holstein-Primakoff) regime $n_2 \ll n_1$, for which the spin-demixing instability rate is parametrically suppressed by the minority density n_2 and, for our parameters, the most unstable demixing wavelength exceeds the cloud size; see Appendix C for a quantitative estimate based on the Bogoliubov spectrum of a binary condensate [48,49]. This separation-of-timescales requirement is essential for spin squeezing: Once significant demixing occurs, the spatial overlap $\int d^3r |\phi_1(\mathbf{r})|^2 |\phi_2(\mathbf{r})|^2$ drops rapidly, which quenches the effective intercomponent nonlinearity and coherence on which OAT squeezing relies; thus, we require the demixing time τ_{MI} to exceed the squeezing-development time t_{sq} [approximately the time to reach the minimum of $v_{A,\min}(t)$] so that appreciable squeezing can build up before overlap is lost. Optical readout is modeled by a constant read-out efficiency η_{read} , and unless stated otherwise, we set $\zeta_{\text{spatial}} = \eta_{\text{write}} = \eta_{\text{read}} = 1$.

Figure 2(a) summarizes the central prediction of our model: An optical pulse stored by EIT in a BEC

prepares a CSS that subsequently develops interaction-driven spin squeezing, and this atomic squeezing can be mapped back onto a single, well-defined optical output mode upon readout. We quantify squeezing by the minimal transverse atomic variance $v_{A,\min}(t)$ (normalized to the CSS noise, and shown in decibels as $10 \log_{10} v_{A,\min}$) and convert it to the detected optical variance via Eq. (26), $V_{\text{det},\min}(t) = (1 - \eta_{\text{read}}) + \eta_{\text{read}} v_{A,\min}(t)$. The dashed blue curve shows a lossless reference evolution with $\text{Im}(a_{12}) = 0$ and all inelastic channels disabled ($K_1 = K_2 = K_3 = 0$), while the solid blue curve includes inelastic processes and enables one-, two-, and three-body loss channels (see Appendix A 2 for details and sources). For an input pulse with mean photon number $\mu_{\text{in}} = 10^3$ (corresponding to an initial tilt $\theta \simeq 0.064\pi$ for the parameters used here), the lossless dynamics reaches a minimum of $\simeq -10.1$ dB at $t \simeq 35$ ms before over-twisting causes the variance to rise, whereas including loss both injects additional transverse noise and reduces the collective coherence, yielding a shallower optimum of $\simeq -8.4$ dB at $t \simeq 31$ ms. The orange curve shows the corresponding efficiency proxy $\eta_{\text{tot}}(t)$ in the lossy case, which decays steadily during storage (reaching $\eta_{\text{tot}} \simeq 0.82$ by $t \simeq 125$ ms), reflecting the progressive reduction of retrievable collective coherence.

Figure 2(b) summarizes the best (minimum) retrievable optical squeezing and the readout time t_{op} at which it occurs as the stored pulse strength μ_{in} is varied. As μ_{in} increases from 10^2 to 2×10^3 (equivalently $\theta \simeq 0.02\pi$ to 0.09π on the top axis), the optimal squeezing improves monotonically (reaching ~ -10 dB) while the optimal readout time shifts earlier, from $\sim 10^{-1}$ s down to a few $\times 10^{-2}$ s. Including the ^{23}Na loss channels in the simulation (circles) slightly degrades the best achievable squeezing and favors earlier retrieval compared with the lossless reference (squares). The improvement with increasing μ_{in} is expected because a larger μ_{in} prepares a larger initial tilt θ of the coherent spin state, i.e., a state further from a J_z eigenstate; this increases the available transverse spin length and the spread in J_z that the χJ_z^2 nonlinearity can shear, so stronger squeezing builds up faster before loss and decoherence take over. In Appendix D 3, we provide more detailed analysis on the dependence of achievable squeezing on readout efficiency and on the operating point (N_0, μ_{in}).

Figure 3 shows the predicted outcome statistics of a balanced-homodyne readout of the retrieved optical mode as the local-oscillator phase ϕ_{LO} is scanned. For each ϕ_{LO} , we generate $N_{\text{sample}} = 10^6$ single-shot quadrature outcomes ϵ by modeling the retrieved field as a displaced Gaussian state and sampling

$$\epsilon = \mu(\phi_{\text{LO}}) + \sqrt{V_{\text{det}}(\phi_{\text{LO}})} \xi, \quad \xi \sim \mathcal{N}(0, 1), \quad (27)$$

where the coherent fringe is $\mu(\phi_{\text{LO}}) = A_{\text{coh}} \cos(\phi_{\text{LO}} - \phi_{\text{coh}})$, with $A_{\text{coh}} = 2\sqrt{\eta_{\text{read}} \eta_{\text{coh}}(t) \mu_{\text{stored}}}$ and $\mu_{\text{stored}} = \eta_{\text{write}} \zeta_{\text{spatial}} \mu_{\text{in}}$ (clipped to $\leq N_0$). Thus, the vertical excursion reflects the coherent displacement, while the band thickness at fixed ϕ_{LO} directly encodes the quadrature noise. The phase-dependent variance is obtained from the transverse collective-spin covariance ellipse at the chosen storage time through Eq. (26). We overlay the lossless reference (red) and the lossy evolution (blue), evaluating each at the storage time where that case achieves its best squeezing. For visual clarity,

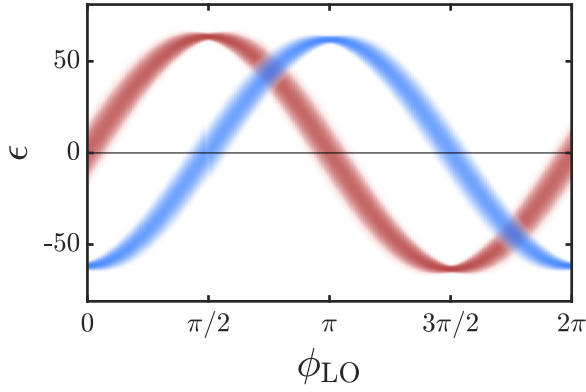


FIG. 3. Simulated balanced-homodyne LO-phase scan of the retrieved optical mode at the optimal storage time t^* . For each ϕ_{LO} , we generate N_{sample} single-shot quadrature outcomes ϵ by sampling $\epsilon = \mu(\phi_{LO}) + \sqrt{V_{\text{det}}(\phi_{LO})}\xi$, with $\xi \sim \mathcal{N}(0, 1)$, where $\mu(\phi_{LO}) = A_{\text{coh}} \cos(\phi_{LO} - \phi_{\text{coh}})$ and $V_{\text{det}}(\phi_{LO}) = (1 - \eta_{\text{read}}) + \eta_{\text{read}} v_A(\phi_{LO})$. The heatmaps show $\log_{10}(\text{counts} + 1)$ from binned samples: red denotes the lossless reference [$\text{Im}(a_{12}) = 0$, all loss channels off] and blue (shifted by $\pi/2$ to avoid overlap with the red curve) denotes the lossy ^{23}Na case (loss channels on; representative $\text{Im}(a_{12}) = -0.001 a_0$ used to set the interstate two-body loss in the model).

the lossy (blue) distribution is displayed with a horizontal offset $\phi_{LO} \mapsto \phi_{LO} + \pi/2$ so that the two bands do not obscure one another. In each case, the narrowest portion of the band identifies the squeezed quadrature, and its angular offset from the extrema of the coherent fringe reflects the rotation of the noise ellipse familiar from one-axis twisting and optical homodyne detection [1, 50, 51]. Comparing the two overlays highlights how inelastic processes both suppress the coherent modulation (via the reduced effective retrieval amplitude) and broaden/reshape the quadrature-noise band through the modified covariance ellipse.

V. CONCLUSIONS

We have developed a quantitative theory for generating squeezed optical states using a Bose-Einstein-condensate EIT memory as an interaction-enabled nonlinear element during the storage interval. A weak coherent probe pulse is mapped onto a single, phase-matched collective spin-wave mode of a two-component condensate, preparing a coherent spin state whose initial Bloch-sphere tilt is set by the stored mean excitation number. Collisional interactions then implement one-axis twisting during the hold time, generating transverse spin squeezing that can be exported back into a single, well-defined propagating optical mode on readout.

A key outcome is that the squeezing generation is intrinsically memory-synchronized: The EIT interface fixes the retrieved spatiotemporal mode, while the storage time controls the accumulated nonlinear evolution, enabling on-demand emission of a squeezed optical pulse. Our loss-aware simulations identify an optimal readout time t_{op} set by the competition between interaction-driven correlations and dissipative processes. In particular, inelastic two-body loss both reduces retrievable coherence and injects transverse noise,

shifting the best attainable squeezing to earlier times than in the lossless case; nevertheless, several decibels of optical squeezing remain accessible in realistic parameter regimes. This positions the protocol as a hybrid route to mode-matched squeezed light compatible with quantum networking and precision measurement, where controlled release and mode selectivity are often as important as the squeezing level itself.

More broadly, our results connect early proposals that stopped light in condensates could enable nonlinear and quantum optical effects [39, 52] with the capabilities of modern high-efficiency, long-lived BEC memories. Looking ahead, the approach should benefit directly from schemes that extend coherence times and improve the ratio of coherent interaction strength to inelastic loss, for example, by optimizing internal-state choices and densities at fixed optical depth, or by leveraging extended-storage platforms such as microgravity BEC memories [53]. In this way, EIT memories offer a unified platform for programmable squeezed-state generation and retrieval, with potential impact for quantum-enhanced optical interferometry and hybrid atom-light metrology.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: HEISENBERG CORRELATORS USED IN THE OAT SIMULATION

We work with two bosonic internal modes $\hat{b}_{1,2}$ and number operators $\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i$. The collective spin operators are

$$\hat{J}_+ = \hat{b}_2^\dagger \hat{b}_1, \quad \hat{J}_- = \hat{b}_1^\dagger \hat{b}_2, \quad \hat{J}_z = \frac{1}{2}(\hat{n}_2 - \hat{n}_1). \quad (\text{A1})$$

In the Heisenberg picture, all operators are time dependent; for readability, we often omit explicit “(t)” on $\hat{b}_i(t)$ inside the expectation values below.

1. Operator definitions (with bosonic/number-operator forms)

The twelve basic correlators are defined as

$$E_1(t) \equiv \langle \hat{J}_+(t) \rangle = \langle \hat{b}_2^\dagger \hat{b}_1 \rangle, \quad (\text{A2})$$

$$E_2(t) \equiv \langle \hat{J}_-(t) \rangle = \langle \hat{b}_1^\dagger \hat{b}_2 \rangle, \quad (\text{A3})$$

$$E_3(t) \equiv \langle \hat{J}_+^2(t) \rangle = \langle (\hat{b}_2^\dagger)^2 \hat{b}_1^2 \rangle, \quad (\text{A4})$$

$$E_4(t) \equiv \langle \hat{J}_-^2(t) \rangle = \langle (\hat{b}_1^\dagger)^2 \hat{b}_2^2 \rangle, \quad (\text{A5})$$

and the number-weighted correlators

$$E_5(t) \equiv \langle \hat{n}_1 \hat{J}_+(t) \rangle = \langle (\hat{b}_1^\dagger \hat{b}_1)(\hat{b}_2^\dagger \hat{b}_1) \rangle, \quad (\text{A6})$$

$$E_6(t) \equiv \langle \hat{n}_1 \hat{J}_-(t) \rangle = \langle (\hat{b}_1^\dagger \hat{b}_1)(\hat{b}_1^\dagger \hat{b}_2) \rangle, \quad (\text{A7})$$

$$E_7(t) \equiv \langle \hat{n}_2 \hat{J}_+(t) \rangle = \langle (\hat{b}_2^\dagger \hat{b}_2)(\hat{b}_2^\dagger \hat{b}_1) \rangle, \quad (\text{A8})$$

$$E_8(t) \equiv \langle \hat{n}_2 \hat{J}_-(t) \rangle = \langle (\hat{b}_2^\dagger \hat{b}_2)(\hat{b}_1^\dagger \hat{b}_2) \rangle, \quad (\text{A9})$$

$$E_9(t) \equiv \langle \hat{J}_+(t) \hat{n}_1 \rangle = \langle (\hat{b}_2^\dagger \hat{b}_1)(\hat{b}_1^\dagger \hat{b}_1) \rangle, \quad (\text{A10})$$

$$E_{10}(t) \equiv \langle \hat{J}_+(t) \hat{n}_2 \rangle = \langle (\hat{b}_2^\dagger \hat{b}_1)(\hat{b}_2^\dagger \hat{b}_2) \rangle, \quad (\text{A11})$$

$$E_{11}(t) \equiv \langle \hat{J}_-(t) \hat{n}_1 \rangle = \langle (\hat{b}_1^\dagger \hat{b}_2)(\hat{b}_1^\dagger \hat{b}_1) \rangle, \quad (\text{A12})$$

$$E_{12}(t) \equiv \langle \hat{J}_-(t) \hat{n}_2 \rangle = \langle (\hat{b}_1^\dagger \hat{b}_2)(\hat{b}_2^\dagger \hat{b}_2) \rangle. \quad (\text{A13})$$

For $\hat{H}_{\text{OAT}} = \hbar \chi \hat{J}_z^2$, the relevant matrix elements acquire simple m -dependent phases. Let the initial state be $|\theta, \varphi_0\rangle = \sum_{m=-J}^J c_m |J, m\rangle$, with

$$c_m = \sqrt{\binom{2J}{J+m}} \left(\cos \frac{\theta}{2} \right)^{J-m} \left(\sin \frac{\theta}{2} \right)^{J+m} e^{-im\varphi_0}. \quad (\text{A14})$$

Define $L_\pm(m) = \sqrt{J(J+1) - m(m \mp 1)}$ and $\tau = \chi t$. Then,

$$E_1(t) = \sum_m c_m^* c_{m-1} L_+(m) e^{+i(2m-1)\tau}, \quad (\text{A15})$$

$$E_2(t) = \sum_m c_m^* c_{m+1} L_-(m) e^{-i(2m+1)\tau}, \quad (\text{A16})$$

$$E_3(t) = \sum_m c_m^* c_{m-2} L_+(m) L_+(m-1) e^{+i(4m-4)\tau}, \quad (\text{A17})$$

$$E_4(t) = \sum_m c_m^* c_{m+2} L_-(m) L_-(m+1) e^{-i(4m+4)\tau}, \quad (\text{A18})$$

and the number-weighted correlators become

$$E_5(t) = \sum_m (J-m) c_m^* c_{m-1} L_+(m) e^{+i(2m-1)\tau}, \quad (\text{A19})$$

$$E_6(t) = \sum_m (J-m) c_m^* c_{m+1} L_-(m) e^{-i(2m+1)\tau}, \quad (\text{A20})$$

$$E_7(t) = \sum_m (J+m) c_m^* c_{m-1} L_+(m) e^{+i(2m-1)\tau}, \quad (\text{A21})$$

$$E_8(t) = \sum_m (J+m) c_m^* c_{m+1} L_-(m) e^{-i(2m+1)\tau}, \quad (\text{A22})$$

$$E_9(t) = \sum_m (J-m+1) c_m^* c_{m-1} L_+(m) e^{+i(2m-1)\tau}, \quad (\text{A23})$$

$$E_{10}(t) = \sum_m (J+m-1) c_m^* c_{m-1} L_+(m) e^{+i(2m-1)\tau}, \quad (\text{A24})$$

$$E_{11}(t) = \sum_m (J-m-1) c_m^* c_{m+1} L_-(m) e^{-i(2m+1)\tau}, \quad (\text{A25})$$

$$E_{12}(t) = \sum_m (J+m+1) c_m^* c_{m+1} L_-(m) e^{-i(2m+1)\tau}. \quad (\text{A26})$$

All sums run over those m values for which the shifted coefficients exist (e.g., c_{m-2} requires $m \geq -J+2$, etc.).

2. Loss-induced rescaling of Heisenberg correlators

Let $E_k^{(u)}(t)$ denote the unitary (lossless) OAT correlators on the fixed- J manifold. To incorporate the reduction of coherence caused by population loss while keeping the OAT phases intact, we use a minimal, number-consistent renormalization based on the mean populations $N_{1,2}(t)$ from Eq. (23). We define the population ratios $f_1(t) = N_1(t)/N_{1,0}$ and $f_2(t) = N_2(t)/N_{2,0}$, and a coherence ratio motivated by $|\langle \hat{J}_\pm \rangle| \propto \sqrt{N_1 N_2}$ for a phase-coherent two-mode state [38], namely, $R_1(t) = \sqrt{N_1(t)N_2(t)/(N_{1,0}N_{2,0})}$ and $R_2(t) = R_1^2(t) \equiv \eta_{\text{coh}}$. Here, $N_{i,0} = N_i(0)$.

We then rescale the correlators according to their operator content: $E_{1,2}(t) = R_1(t) E_{1,2}^{(u)}(t)$, $E_{3,4}(t) = R_2(t) E_{3,4}^{(u)}(t)$, $E_{5,6,9,11}(t) = f_1(t) R_1(t) E_{5,6,9,11}^{(u)}(t)$, and $E_{7,8,10,12}(t) = f_2(t) R_1(t) E_{7,8,10,12}^{(u)}(t)$. This prescription reduces to the unitary limit when losses vanish. It also ensures that correlators with one (two) ladder operators carry one (two) powers of the coherence factor, while explicit number weighting tracks the corresponding population decay.

To evaluate the diffusion strength $s_q(t)$ used in Eq. (25), we introduce effective one-, two-, and three-body loss rates $\gamma_{1,2,3}(t)$ obtained by evaluating the corresponding mean-field loss terms on the instantaneous mean state. Writing $N(t) = N_1(t) + N_2(t)$ and defining the mode-overlap integrals of the normalized spatial mode $\phi(\mathbf{r})$ as $I_2 = \int d^3r |\phi(\mathbf{r})|^4$ and $I_3 = \int d^3r |\phi(\mathbf{r})|^6$, and following Eqs. (14) and (20) of Ref. [40], we write $\gamma_1(t) = [K_{1,1}N_1(t) + K_{1,2}N_2(t)]/N(t)$, $\gamma_2(t) = I_2 [K_{2,11}N_1(t) + K_{2,22}N_2(t) + \frac{1}{2}K_{2,12}N(t)]$, and $\gamma_3(t) = I_3 [K_{3,111}N_1^2(t) + K_{3,222}N_2^2(t) + (K_{3,112} + K_{3,122})N_1(t)N_2(t)]$. I_2 and I_3 are evaluated by the experimental value of Thomas-Fermi diameter, d_{TF} . The one-, two-, and three-body loss coefficients K are used to model the population dynamics [see Eq. (23)]. The weighted combination entering the Li-Castin-Sinatra diffusion update is then $s_q(t) = \gamma_1(t) + 2\gamma_2(t) + 3\gamma_3(t)$.

For the simulations used in the article, i.e., for ^{23}Na in the stretched states $|1\rangle \equiv |F=1, m_F=-1\rangle$ and $|4\rangle \equiv |F=2, m_F=-2\rangle$, we take the one-body loss rate $K_{1,1} = K_{1,2} = 2.9 \times 10^{-2} \text{ s}^{-1}$ and the three-body coefficients

$$K_{3,111} = 1.57 \times 10^{-42} \text{ m}^6 \text{ s}^{-1}, \quad (\text{A27})$$

$$K_{3,222} = 1.53 \times 10^{-41} \text{ m}^6 \text{ s}^{-1} \text{ (upper bound)}, \quad (\text{A28})$$

from condensate lifetime measurements in a large-volume optical dipole trap [54], and $K_{2,11} = K_{2,22} = 0$; and $K_{2,12} = \frac{8\pi\hbar}{m_{\text{Na}}} \text{Im}(a_{12})$, with $\text{Im}(a_{12}) = -1 \times 10^{-3} a_0$ at $B \simeq 132.4 \text{ G}$ [28]. Mixed-channel three-body coefficients are not available; therefore, we approximate them by the geometric mean, $K_{3,112} = K_{3,122} = \sqrt{K_{3,111}K_{3,222}}$.

APPENDIX B: MAPPING BETWEEN OPTICAL COHERENT AMPLITUDE α AND ATOMIC COHERENT SPIN STATES

This Appendix clarifies the relation between the optical input amplitude α (with $|\alpha|^2 = \mu_{\text{in}}$), the stored spin-wave coherent amplitude β , and the rotation/ladder parametrizations of atomic (Bloch) coherent spin states. The key point is that in the weak-excitation (Holstein-Primakoff) regime $\mu_{\text{stored}} \ll N_0$, the bosonic spin-wave coherent state $|\beta\rangle_S$ and the atomic coherent spin state $|\theta, \varphi_0\rangle$ describe the same physical state, expressed in two equivalent languages.

1. From optical input to stored spin-wave amplitude

We take the input probe mode to be in a coherent state $|\alpha\rangle_{\text{in}}$ with $|\alpha|^2 = \mu_{\text{in}}$. For a phase-preserving linear write process with overall mode-matched storage factor $\eta_{\text{write}}\zeta_{\text{spatial}}$, coherent amplitudes scale with the square root of the efficiency, giving $\beta = \sqrt{\eta_{\text{write}}\zeta_{\text{spatial}}} \alpha$. Consequently, the mean stored excitation number is $\mu_{\text{stored}} = |\beta|^2 = \eta_{\text{write}}\zeta_{\text{spatial}}\mu_{\text{in}}$, and the stored azimuthal phase is $\varphi_0 = \arg \beta$.

2. Spin-wave coherent state and Holstein-Primakoff reduction

Define the spin-wave vacuum as $|0\rangle_S \equiv |N_0, 0\rangle$ and the spin-wave coherent state $|\beta\rangle_S = \hat{D}_S(\beta)|0\rangle_S = e^{-|\beta|^2/2} \exp(\beta \hat{S}^\dagger)|0\rangle_S$. In the Holstein-Primakoff regime $\mu_{\text{stored}} = \langle \hat{S}^\dagger \hat{S} \rangle \ll N_0$, the collective excitation operator may be embedded as $\hat{S}^\dagger \simeq (\hat{b}_2^\dagger \hat{b}_1)/\sqrt{N_0}$. Substituting yields the ladder representation on the fixed- N_0 manifold,

$$|\beta\rangle_S \simeq e^{-|\beta|^2/2} \exp\left[\frac{\beta}{\sqrt{N_0}} \hat{b}_2^\dagger \hat{b}_1\right] |N_0, 0\rangle, \quad (\text{B1})$$

which is the form used in the main text after inserting $\beta = \sqrt{\eta_{\text{write}}\zeta_{\text{spatial}}} \alpha$.

3. Atomic coherent spin states: Rotation and ladder forms

Introduce collective spin operators in the Schwinger representation $\hat{J}_+ = \hat{b}_2^\dagger \hat{b}_1$, $\hat{J}_- = \hat{b}_1^\dagger \hat{b}_2$, and $\hat{J}_z = (\hat{b}_2^\dagger \hat{b}_2 - \hat{b}_1^\dagger \hat{b}_1)/2$, with total spin $J = N_0/2$ and $|N_0, 0\rangle \equiv |J, -J\rangle$. Atomic coherent spin states are defined as rotations of an extremal Dicke state,

$$|\theta, \varphi\rangle \equiv e^{-i\varphi \hat{J}_z} e^{-i\theta \hat{J}_y} |J, -J\rangle. \quad (\text{B2})$$

Using the standard disentangling identity for SU(2) rotations, the same state can be written in ladder form as $|\theta, \varphi\rangle = (1 + |\tau|^2)^{-J} \exp(\tau \hat{J}_+) |J, -J\rangle$, where $\tau = e^{i\varphi} \tan(\theta/2)$. Comparing with Eq. (B1) identifies $\tau = \beta/\sqrt{N_0}$, i.e., $\tau = \sqrt{\eta_{\text{write}}\zeta_{\text{spatial}}/N_0} \alpha$ and $\varphi_0 = \arg \beta$.

4. Normalization matching and small-angle limit

The ladder form includes the CSS normalization $(1 + |\tau|^2)^{-J}$. With $J = N_0/2$ and $|\tau|^2 = |\beta|^2/N_0 = \mu_{\text{stored}}/N_0 \ll 1$, one has $(1 + |\tau|^2)^{-J} \simeq \exp(-J|\tau|^2) = \exp(-|\beta|^2/2)$, which coincides with the bosonic coherent-state prefactor in $|\beta\rangle_S$. Finally, for an atomic coherent state, the mean population in state $|2\rangle$ is $\mu_{\text{stored}} = \langle \hat{n}_2 \rangle = N_0 \sin^2(\theta/2)$, so in the weak-excitation regime $\mu_{\text{stored}} \ll N_0$ one finds $\theta \simeq 2\sqrt{\mu_{\text{stored}}/N_0} = 2\sqrt{\eta_{\text{write}}\zeta_{\text{spatial}}\mu_{\text{in}}/N_0}$.

APPENDIX C: SUPPRESSION OF SPIN DEMIXING IN THE WEAK-EXCITATION REGIME

Here, we justify *a posteriori* why spatial phase separation (spin demixing) does not invalidate the collective-spin, single-mode simulations reported in the main text, despite the fact that the sodium interaction parameters adopted from Ref. [28] satisfy the immiscibility condition $a_{12}^2 > a_{11}a_{22}$. The key point is that in the weak-excitation (Holstein-Primakoff) regime $N_2 \ll N_1$, the demixing instability rate scales linearly with the minority density n_2 , and the most unstable demixing wavelength grows as $n_2^{-1/2}$, so demixing becomes parametrically slow and can be further suppressed by finite system size.

We summarize the standard linear-stability (Bogoliubov) theory for a homogeneous two-component condensate with equal atomic mass m and repulsive contact interactions $g_{ij} = 4\pi\hbar^2 a_{ij}/m$. Linearizing the coupled Gross-Pitaevskii equations around uniform background densities n_1 and n_2 yields two excitation branches $\Omega_{\pm}(k)$ with [48,49]

$$\Omega_{\pm}^2(k) = \varepsilon_k \left[\varepsilon_k + g_{11}n_1 + g_{22}n_2 \pm \sqrt{(g_{11}n_1 - g_{22}n_2)^2 + 4g_{12}^2 n_1 n_2} \right], \quad (\text{C1})$$

with $\varepsilon_k = \frac{\hbar^2 k^2}{2m}$. For repulsive interactions, the long-wavelength miscibility criterion is $g_{12}^2 < g_{11}g_{22}$ (equivalently $a_{12}^2 < a_{11}a_{22}$), whereas, in the immiscible regime $g_{12}^2 > g_{11}g_{22}$, the lower branch $\Omega_-(k)$ becomes imaginary for sufficiently small k , implying exponential growth of spin-density modulations (modulational instability) [48,49]. For the sodium parameters quoted by Ref. [28], $a_{11} = 2.8$ nm, $a_{22} = 3.4$ nm, $a_{12} = 3.4$ nm, one finds $a_{12}^2/(a_{11}a_{22}) \simeq 3.4/2.8 \simeq 1.21 > 1$, consistent with the phase-separating behavior reported in Ref. [28] for strong stored pulses.

To extract a demixing timescale, define $A \equiv g_{11}n_1 + g_{22}n_2$ and $D \equiv \sqrt{(g_{11}n_1 - g_{22}n_2)^2 + 4g_{12}^2 n_1 n_2}$, and introduce $\bar{\Delta} \equiv D - A$. In the immiscible regime, one has $\bar{\Delta} > 0$, and the unstable band corresponds to $0 < \varepsilon_k < \bar{\Delta}$, for which $\Omega_-(k) = i\Gamma(k)$ with growth rate $\Gamma(k) = \hbar^{-1} \sqrt{\varepsilon_k(\bar{\Delta} - \varepsilon_k)}$ [48]. The maximum growth rate occurs at $\varepsilon_k = \bar{\Delta}/2$, yielding $\Gamma_{\text{max}} = \bar{\Delta}/(2\hbar)$ and an e -folding time $\tau_{\text{MI}} \equiv \Gamma_{\text{max}}^{-1} = 2\hbar/\bar{\Delta}$. The corresponding most unstable wave number satisfies $\varepsilon_{k_*} = \bar{\Delta}/2$, i.e., $k_* = \sqrt{m\bar{\Delta}}/\hbar$ and $\lambda_* = 2\pi/k_* = 2\pi\hbar/\sqrt{m\bar{\Delta}}$.

In the weak-excitation limit $n_2 \ll n_1$ (relevant for $N_2 \ll N_1$), one can expand $\bar{\Delta}$ to first order in n_2 and obtain

$$\bar{\Delta} \simeq 2n_2 \left(\frac{g_{12}^2}{g_{11}} - g_{22} \right), \quad \Gamma_{\text{max}} \simeq \frac{n_2}{\hbar} \left(\frac{g_{12}^2}{g_{11}} - g_{22} \right), \quad \tau_{\text{MI}} \propto \frac{1}{n_2}, \quad (\text{C2})$$

so the demixing instability rate is expected to be parametrically suppressed by the minority density.

It is worth emphasizing what the weak-excitation limit does (and does not) imply. The immiscibility condition $g_{12}^2 > g_{11}g_{22}$ still determines whether the uniform mixture is

unstable at long wavelengths. However, in the strongly imbalanced case $n_2 \ll n_1$ the dynamically relevant instability is carried by the out-of-phase (spin-density) branch, and its maximum growth rate is proportional to the minority density, $\Gamma_{\max} \propto n_2$ (equivalently $\tau_{\text{MI}} \propto 1/n_2$), while the most unstable wavelength increases as $\lambda_* \propto n_2^{-1/2}$. In contrast, the in-phase density branch remains essentially the majority-component phonon and can scale with n_1 without setting the demixing timescale.

In our collective-spin setting, the minority population is set by the stored excitation number, with $N_2(0) = N_0 \sin^2(\theta/2) \simeq \mu_{\text{eff}}$ for $\mu_{\text{eff}} \ll N_0$, and therefore $n_2 \propto N_2 \propto \mu_{\text{eff}}$. As a result, τ_{MI} grows as μ_{eff}^{-1} and the most unstable wavelength grows as $\lambda_* \propto \bar{\Delta}^{-1/2} \propto n_2^{-1/2} \propto \mu_{\text{eff}}^{-1/2}$; i.e., the demixing pattern becomes longer-wavelength and harder to realize as the stored excitation is reduced.

A further stabilizing effect is finite system size. In a trapped cloud of characteristic radius R , the smallest accessible wave number is $k_{\min} \sim \pi/R$ and the corresponding kinetic energy is $\varepsilon_{\min} = \hbar^2 k_{\min}^2 / (2m)$. Since the instability requires $\varepsilon_k < \bar{\Delta}$, a necessary condition for any unstable mode to fit is $\varepsilon_{\min} < \bar{\Delta}$, or equivalently $\lambda_* \lesssim 2R$, up to factors of order unity. For the baseline parameters used in our simulations ($N_0 = 1 \times 10^5$, $d_{\text{TF}} \simeq 10 \mu\text{m}$ so $R \simeq 5 \mu\text{m}$, and $\mu_{\text{eff}} \sim 10^3$), evaluating the above expressions with effective peak densities gives $\Gamma_{\max}^{-1} \sim 15 \text{ ms}$ and $\lambda_* \sim 30 \mu\text{m}$, i.e., λ_* significantly exceeds the cloud diameter and the condition $\varepsilon_{\min} < \bar{\Delta}$ is not met. Therefore, the modulational instability is strongly suppressed on the ~ 10 – 50 ms timescales relevant to the squeezing dynamics discussed in the main text. For much larger excitations, e.g., $\mu_{\text{eff}} \gtrsim 10^4$, one expects λ_* to decrease toward the system size and demixing to become dynamically relevant, consistent with the strong-pulse regime and larger condensate sizes explored in Ref. [28].

Finally, our stability estimate based on coupled mean-field Gross-Pitaevskii theory is conservative in that it neglects beyond-mean-field quantum-fluctuation effects, which become especially relevant when a collective (spin-density) mode softens near an instability. In Bose-Bose mixtures, the leading Lee-Huang-Yang correction provides an additional positive “quantum pressure” term in the energy functional that stiffens long-wavelength density and spin-density modulations; in particular, it can arrest mean-field instabilities and favor a stable overlapped state with a well-defined composition. This mechanism underlies the formation of ultradilute quantum droplets predicted in Ref. [55] and observed experimentally in the ^{39}K mixture [56]. While our squeezing simulations do not rely on droplet formation, the same fluctuation-induced stiffness would further reduce the propensity for spatial spin demixing on the millisecond timescales of interest here, complementing the suppression already implied by the small minority density and finite system size discussed above.

APPENDIX D: ADDITIONAL ROBUSTNESS CHECKS

1. Magnetic-field noise, gradients, and effective dephasing

For the sodium levels used in Ref. [28], a Breit-Rabi estimate at $B_0 \simeq 132.4 \text{ G}$ gives the differential magnetic

sensitivity

$$\beta_B \equiv \left| \frac{\partial v_{21}}{\partial B} \right| \simeq 1.1 \text{ MHz/G}. \quad (\text{D1})$$

Residual bias-field fluctuations therefore produce a two-photon detuning $\delta v_B(t) = \beta_B \delta B(t)$. In the collective-spin description, this gives

$$H/\hbar = \chi \hat{J}_z^2 + 2\pi \delta v_B(t) \hat{J}_z. \quad (\text{D2})$$

The second term is a technical phase-noise term, not an additional atom-loss channel. As a scale, the $18.5 \mu\text{G}$ rms noise and $<7 \mu\text{G}$ long-time instability demonstrated in an ultracold atom apparatus with active feedback [57] correspond to $\sigma_v \simeq 20 \text{ Hz}$ and $\sigma_v \simeq 8 \text{ Hz}$ for this Na transition. The associated inverse angular times, $1/(2\pi\sigma_v)$, are about 8 and 20 ms. These are untracked-detuning scales; slow common-mode components can be reduced by phase referencing, calibration, active stabilization, or echo-type refocusing.

Because the condensate is short, bias-field fluctuations from the coils or ambient laboratory fields are expected to be nearly common-mode across the sample. A residual gradient $G = \partial_z B$ across a length L produces an inhomogeneous frequency spread

$$\Delta v_{\text{grad}} \simeq \beta_B G L \simeq 5.5 \text{ Hz} \left(\frac{G}{1 \text{ mG/cm}} \right) \left(\frac{L}{50 \mu\text{m}} \right). \quad (\text{D3})$$

For $L = 50 \mu\text{m}$, this gives 0.55 and 5.5 Hz for gradients of 0.1 and 1 mG/cm, respectively, which are already within experimental reach [58–60].

For the specific detuning term in Eq. (D2), a transverse π pulse changes $\hat{J}_z \rightarrow -\hat{J}_z$ while leaving $\hat{J}_z^2$ invariant. Slow detuning noise that is linear in $\hat{J}_z$ can therefore be refocused without reversing the OAT interaction. For magnetic-noise suppression in cold atom memories, dynamical decoupling (DD) has been used to protect stored spin-wave superpositions against inhomogeneous Zeeman broadening and magnetic-field-induced phase noise [61]. In a BEC, DD has also been used to cancel strong magnetic noise while preserving interaction-induced nonlinear spin dynamics and squeezing [62]. These examples support echo-type mitigation when the dominant residual noise is slow and enters linearly in $\hat{J}_z$.

We include the residual loss of phase-matched Raman coherence through the phenomenological Gaussian envelope

$$\eta_\phi(t) = \exp[-(t/T_\phi)^2], \quad (\text{D4})$$

where T_ϕ is the $1/e$ decay time of the mode-matched retrieval efficiency associated with dephasing mechanisms not included in the population-loss model. The Gaussian form is a phenomenological effective model for slow residual detuning noise and related technical dephasing. The detected variance becomes

$$V_{\text{opt}}^{(\phi)}(t) = 1 - \eta_{\text{read}} \eta_\phi(t) + \eta_{\text{read}} \eta_\phi(t) v_{A,\min}(t). \quad (\text{D5})$$

This form treats dephasing as an additional reduction of the phase-matched retrieval efficiency. A purely common-mode

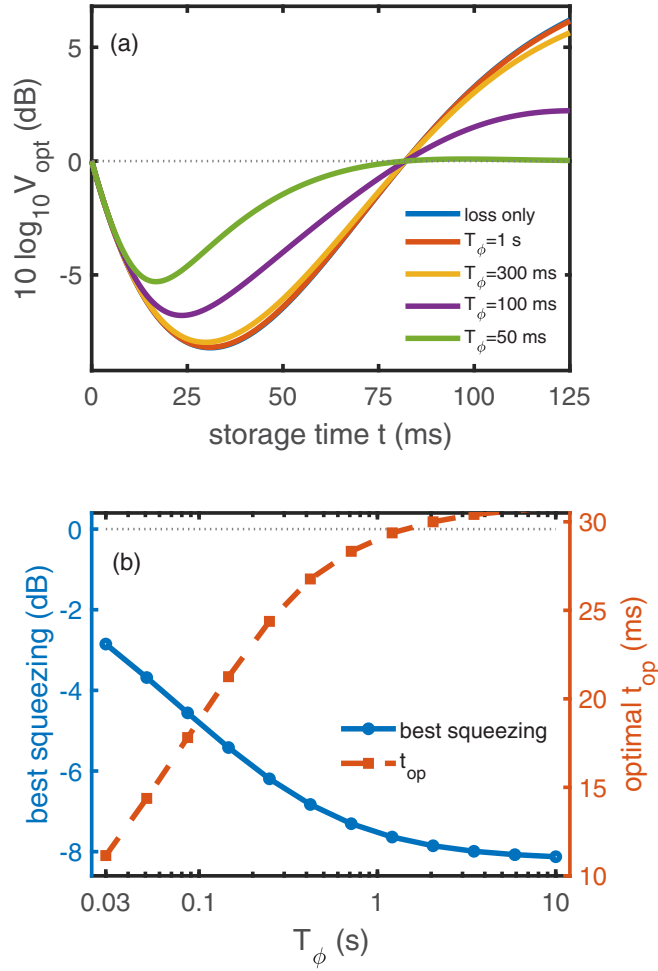


FIG. 4. Phenomenological Gaussian dephasing during storage. The calculation uses the same baseline parameters as Fig. 2(a), with $\mu_{\text{in}} = 10^3$ and $\eta_{\text{read}} = 1$. (a) Retrieved optical variance for several dephasing times T_ϕ , where $T_\phi = \infty$ corresponds to the loss-only model. (b) Best retrieved squeezing and the corresponding optimal storage time as functions of T_ϕ .

phase drift of the recalled field relative to the homodyne local oscillator is a separate phase-reference problem and must be stabilized or tracked experimentally.

For the present Na states, T_ϕ is set by the residual coherence after magnetic-field stabilization [57], light-shift control, phase referencing, and any refocusing pulses. We do not take the long storage time of Ref. [28] as a value of T_ϕ , since that experiment used phase separation to protect the stored imprint, whereas the OAT dynamics considered here requires the two components to remain overlapped, as discussed in Appendix C. Figure 4(a) shows the sensitivity of the predicted optical squeezing to T_ϕ . In Fig. 4(b), we show how t_{op} and the best achievable squeezing change with different dephasing timescales. If T_ϕ is long compared with the OAT optimum, the degradation is small. If T_ϕ becomes comparable to the tens-of-millisecond squeezing time, the best squeezing decreases and the optimum shifts earlier.

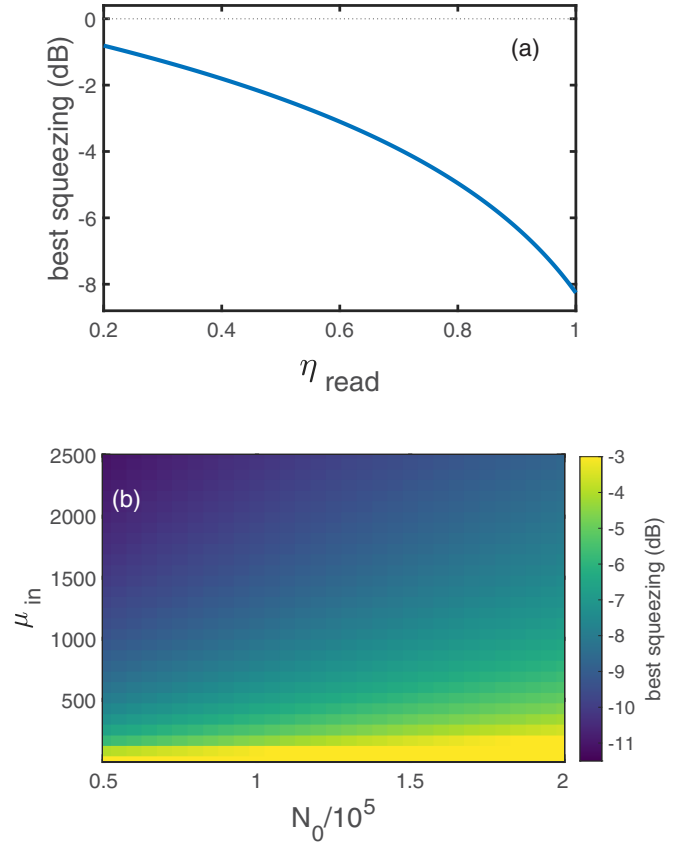


FIG. 5. Additional parameter dependences. (a) Best retrieved squeezing vs readout efficiency for the baseline $N_0 = 10^5$, $\mu_{\text{in}} = 10^3$ case. (b) Best retrieved squeezing as a function of atom number and input photon number, with the spatial mode, loss parameters, and $\eta_{\text{read}} = 1$ kept fixed.

2. Differential light shifts, motion, and density fluctuations

Differential light shifts enter in the same way as a two-photon detuning. During the ideal storage interval, the EIT probe and control fields are off, but the trapping light remains on and imperfect extinction or switching transients can add residual AC-Stark shifts. A spatially uniform, slowly varying differential shift contributes a term $2\pi\delta\nu_{\text{ac}}(t)\hat{J}_z$ and can be absorbed into the phase reference or refocused by the same echo/DD argument used above. Static inhomogeneous differential shifts can also be reduced by echo sequences if the atoms sample the same shift before and after the pulse; fluctuating intensity noise above the DD bandwidth and spatial mode changes are instead included in the effective T_ϕ . This is consistent with cold atom light-storage work, where differential AC-Stark shifts and motional effects are important technical limits and are mitigated using state-insensitive trapping conditions, magic-field operation, and dynamical decoupling [63,64].

Motional dephasing is governed by the spin-wave wave vector $\Delta\mathbf{k} = \mathbf{k}_p - \mathbf{k}_c$. The retrieval contrast contains the factor $C_{\text{mot}}(t) = \langle e^{i\Delta\mathbf{k} \cdot [\mathbf{r}(t) - \mathbf{r}(0)]} \rangle$. For the near copropagating geometry assumed here, $|\Delta\mathbf{k}|$ is set mainly by the ground-state splitting rather than by the optical wave number. Using the sodium hyperfine scale $\nu_{\text{hf}} \simeq 1.77$ GHz gives

$|\Delta k| \simeq 2\pi v_{\text{hf}}/c \simeq 37 \text{ m}^{-1}$, corresponding to a spin-wave period of about 17 cm. Even a 50 μm displacement would then produce a phase $|\Delta k|L < 2 \times 10^{-3} \text{ rad}$. Thus, motional dephasing from spin-wave phase winding is negligible for the copropagating BEC geometry, whereas it can become a dominant limitation for noncopropagating geometries or thermal ensembles [47,64,65].

3. Readout efficiency and operating point

The absolute write/read efficiency is separate from the storage-time decay. It is set by EIT-interface engineering, including optical depth, control-field shaping, probe bandwidth,

spatial mode matching, and retrieval geometry. In the present notation, these effects are represented by η_{write} , ζ_{spatial} , and η_{read} , and should be optimized independently of the internal collisional dynamics.

Figure 5(a) shows the dependence of the best retrieved squeezing on η_{read} for the baseline case. The beam-splitter relation gives a direct reduction of observable squeezing: For the same atomic evolution, the best values are about -2.4 , -3.9 , -6.3 , and -8.3 dB for $\eta_{\text{read}} = 0.5, 0.7, 0.9$, and 1 , respectively. Figure 5(b) shows a scan over N_0 and μ_{in} with the other parameters fixed. The plot is intended as an operating-range guide; large stored excitation numbers must still satisfy the weak-excitation and no-demixing conditions discussed in Appendix C.

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