

Transport and localization in driven disordered potentials with quantum and thermal fluctuations

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The transport and localization of quantum droplets in time-dependent periodic disorder potentials are systematically studied. Our numerical results indicate that such exquisite periodically driven structures may support quantum phases that are principally different compared to static systems. We observe an intriguing anomalous diffusion, with a diffusion exponent depending on the strength of quantum fluctuations and on noise parameters (amplitude and frequency). We show that the interplay of quantum fluctuations and noise tends to suppress localization especially in a large droplet, while the transport is subdiffusive in a small droplet only for a very weak noise. For reducing modulation strength and at fixed dephasing time, the system may exhibit Anderson localization of the Floquet states in the small droplet. We also elucidate the role of thermal effects in the transport properties of quantum droplets. The obtained anomalous diffusion is justified in the frame of a heuristic model.

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I. INTRODUCTION

Recently, transport of matter waves with ultracold atoms in disordered environment has attracted much attention due to their potential applications in quantum computing and sensing. In particular, the expansion of initially confined interacting Bose-Einstein condensates (BECs) in random potentials has been extensively explored (for review, see Refs. [1,2]). Such a dynamics is characterized by two distinct regimes, namely, localized (absence of transport) and diffusive (wave packet spreads freely) regimes. In the past two decades, plenty of works have dealt with transport of both noninteracting and weakly interacting BECs for different disorder configurations and geometries (see, e.g., Refs. [3–14]). There has been evidence that the long-time evolution of the condensate is characterized by (sub)diffusive dynamics suggesting that interactions may prohibit Anderson localization (AL).

However, to date few attempts are made to analyze the dynamics of BECs in disordered time-dependent potentials (i.e., random in space and time, also known as noise). It has been shown that the presence of time-dependent disorder destroys AL and thus diffusive or even superdiffusive transports are expected to emerge in such time-dependent media [15–17]. Experimentally, time-dependent disorder is generated using cross-correlations of density distributions

[18,19]. Nonequilibrium dynamics of BECs subjected in time-dependent random potentials has also been explored in Refs. [20–23].

Advances in the field of ultracold atoms have unlocked promising avenues for the realization of quantum droplets in Bose-Bose mixtures [24,25] and in dipolar BECs [26–28]. Quantum droplets, featuring their self-binding, ultradiluteness, precise controllability, and tunability, were first predicted by Petrov [29] and have been widely explored in the past few years with a variety of approaches (for review, see Refs. [30–32] and references therein). They originate due to the delicate balance between mean-field effects and quantum fluctuations arising from the Lee-Huang-Yang corrections [33]. Most recently, self-bound droplets in disordered potentials have attracted considerable attention [34–38] since they allow better understanding of the interplay between disorder, interactions, and quantum fluctuations. We have found that in the presence of relatively strong disorder, the droplet exhibits AL in the tail region, which is closer to a noninteracting gas, while the central (high-density) part of the distribution follows a power law [39].

So far, the transport of quantum droplets under the effect of a time-dependent disorder is yet to be explored. There must be definite answers to the questions: Does AL still survive in such a dynamics? to what extent the fluctuating force may induce a transition to delocalization?, and how the intriguing interplay of the drive, disorder, interactions, and quantum and thermal fluctuations affects the transport? The aim of this paper is then to investigate the transport of both large and small quantum droplets in time-dependent periodic disorder potentials. For this purpose, we solve numerically the underlying one-dimensional (1D) discrete generalized Gross-Pitaevskii equation (DGGPE) in the presence of two types of perturbation, namely, phase noise and Floquet-engineered noise. Such

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a Floquet drive that depends on the drive frequency gives rise to a rich dynamical phase structure and can be viewed as a potential route for creating exotic quantum phases.

In the static case, a crossover dynamics from superdiffusion to subdiffusion is observed depending on the disorder strength and on the droplet size. The observed transport regimes are compared with a heuristic model. We also show that deviations from normal diffusion for relatively strong disorder can be attributed to signatures of AL in the small droplet. To characterize the localized regime, we utilize the exponential decay profile of the 1D density of the droplets.

Going beyond static disorder, our numerics indicate that the presence of a periodic driving plays a crucial role in delocalizing particles giving rise to a destruction of AL in both droplets as the strength of modulation gets increased, henceforth allowing for an anomalous diffusion. We demonstrate, in particular, that beyond a certain critical modulation the small droplet undergoes a transition from AL to normal diffusion, while a transition from diffusive to superdiffusive regime is observed in a large droplet. The width of both structures grows with the drive time as t^α . Surprisingly, in the case of the Floquet drive, we show the occurrence of AL in the small droplet for small modulation amplitudes. Such an AL ceases to exist for sufficiently strong noise strength indicating that the motion of particles is governed by diffusion. However, the transport properties of a large droplet are dominated by superdiffusive behavior for which the diffusion coefficient is increased. Phase diagram summarizing different transport regimes of the droplet for both time-dependent disorders is provided. Here again, we explain the anomalous diffusion exponent with a heuristic model of interaction-Lee-Huang-Yang (LHY)-induced transitions.

Furthermore, we deeply analyze the impact of thermal effects on the transport properties of both droplets by numerically solving the finite-temperature DGGPE (FTDG-GPE) that incorporates an additional term stemming from thermal fluctuations. We show that even weak thermal corrections may substantially shrink or eventually destroy AL, and the motion of particles becomes either diffusive or superdiffusive in both droplets. Their transport exponents are related to the strength of thermal fluctuations, approximately as g_T^ν , where ν depends on the dephasing time and on disorder strength.

Although there are similarities between AL observed here and the Bose glass (BG) state [40–42] given the presence of disorder and interactions, they emerge through fundamentally distinct mechanisms. The BG phase generally involves different properties such as compressibility, absence of long-range phase coherence, and vanishing superfluid current, while the obtained AL occurs in quantum droplets that display liquid-like properties such as incompressibility and surface tension, via their stability aspects stemming from quantum fluctuations. Furthermore, while a BG state is a perfect insulator, our results show that periodic modulations may provide an energy injection that can suppress localization, especially in larger droplets. This creates a transition from localization to transport that does not exist in the standard static BG scenario. Therefore, our localized regimes should not be interpreted as a BG state.

II. MODEL

We consider a 1D quantum droplet in a symmetric Bose-Bose mixture confined in a harmonic potential $V(z) = m\omega_0^2 z^2/2$, with trap frequency ω_0 , and subjected to an external time-dependent disorder potential $U(z, t)$. The dynamics of the system is governed by the following dimensionless GGPE [38,39]:

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2}{\partial z^2} + V(z) + U(z) + |\psi|^2 - |\psi|\right)\psi. \quad (1)$$

The stationary solution of Eq. (1) is obtained through the transformation $\psi(z, t) = \psi(z)e^{-i\mu t}$, where μ is the chemical potential.

In the free-space case, with $V(z) = U(z) = 0$, Eq. (1) admits a ground-state solution in the form [43]

$$\psi(z, t) = -\frac{3\mu e^{-i\mu t}}{1 + \sqrt{1 + 9\mu/2} \cosh(\sqrt{-2\mu}z)}, \quad (2)$$

where $0 < -\mu < 2/9$ and the norm reads $N = (4/3) \ln[(\sqrt{-9\mu/2} + 1)/\sqrt{9\mu/2 + 1}] - 4\sqrt{-\mu/2}$.

To study the dynamics, we initially prepare the droplet near the ground state of the lattice, and in presence of a tight axial trap. We then switch off the trap and analyze the dynamics along the lattice through an additional radial confinement [17]. For a deep lattice, the wavefunction can be decomposed over localized Wannier functions, which form an orthonormal and complete set of functions, as $\psi(z) = \psi_j w_j(z)$. Projection of GGPE (1) onto the Wannier basis [44,45] yields the following DGGPE [46]:

$$\begin{aligned} i\frac{\partial\psi_j}{\partial t} &= -C(\psi_{j+1} + \psi_{j-1} - 2\psi_j) + U_j\psi_j + |\psi_j|^2\psi_j \\ &\quad - |\psi_j|\psi_j = 0, \end{aligned} \quad (3)$$

where $\psi_j(t)$ accounts for the mean-field wavefunction at site j , which is an integer, and C is an effective lattice coupling constant. The DGGPE (3) governs the dynamics of a quantum droplet loaded onto periodic optical lattices in the presence of an external potential. In the absence of the LHY corrections (last term), Eq. (3) reduces to the standard DGPE describing the dynamics of a disordered BEC.

Indeed, there are various distinctions between the DG-GPE (3) and the continuum GGPE (1). The discrete droplets described by Eq. (3) are characterized by a dynamical, self-maintained energy localization, due to competition between discreteness, mean-field nonlinearity, and the LHY term. They differ also from their continuum counterparts by the presence of the Peierls-Nabarro barrier arising from the broken translational invariance. While the continuum GGPE allows for continuous motion, the discrete model captures the lattice-induced localization effects more naturally. Our choice of the discrete model is motivated by its ability to accurately simulate long-time dynamics in disordered potentials while maintaining the essential stabilization mechanism of the LHY corrections.

III. STATIC DISORDER

Let us start by reporting the observation of density profiles and widths in the case of a static potential consisting of

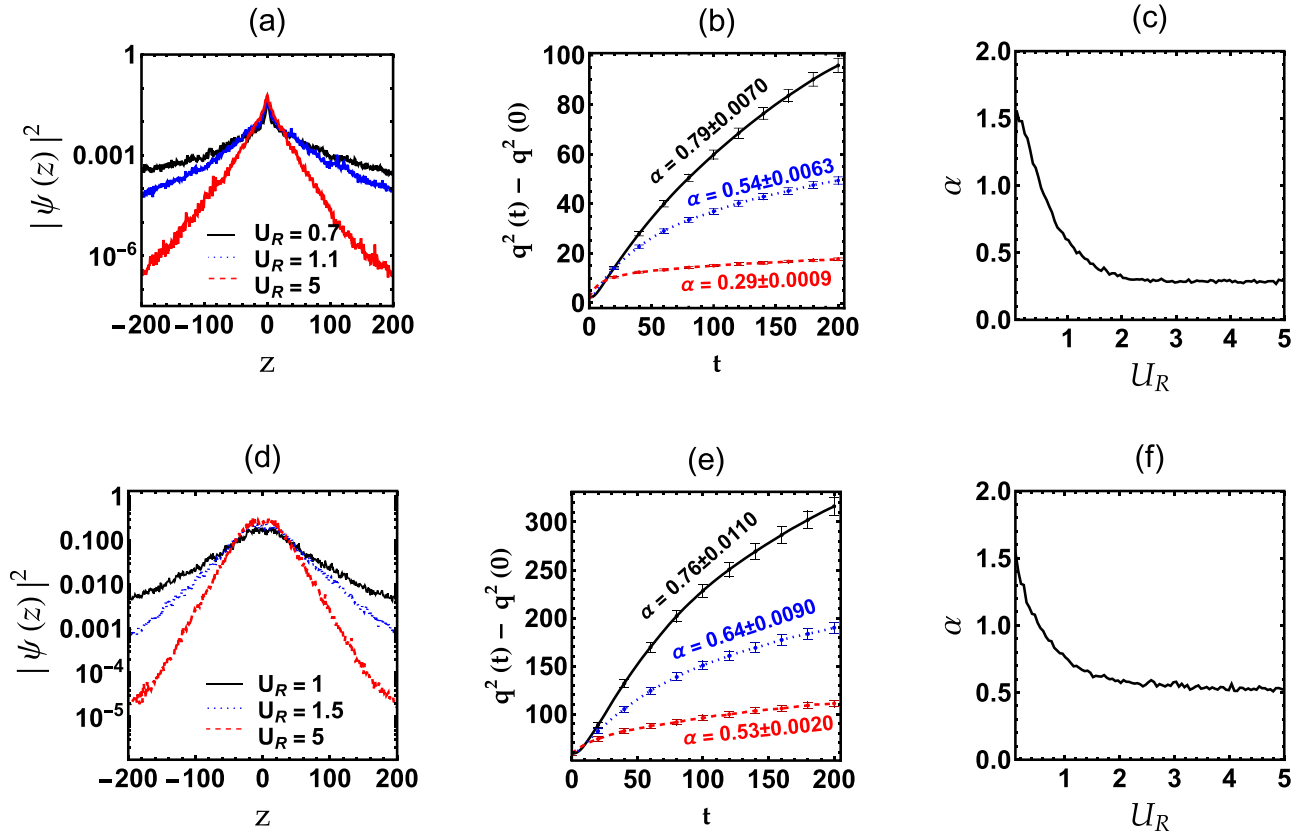


FIG. 1. (a) Stationary density profiles of a small droplet in an Anderson potential for different values of disorder strength, U_R . (b) Time evolution of the width of a small droplet for different values of U_R . (c) The diffusion exponent, α , as a function of U_R . Panels (d)–(f) are the same as panels (a)–(c) but for a large droplet. Here, the diffusion exponent, α , is extracted from fits of Eq. (5) to the data obtained from our numerical simulations in the time interval [10, 200] to exclude initial transient effects. The vertical lines represent the standard error bars of the width calculated across the 100 independent disorder realizations, while the statistical errors are very small to be seen for most points confirming the high precision of our numerical simulations.

an Anderson model characterized by a uniform random distribution. It is given by

$$U = U_R(s - 0.5), \quad (4)$$

where U_R is the disorder strength and s is a random number uniformly distributed in the interval $[-0.5, 0.5]$ to ensure that the disorder has vanishing ensemble. Therefore, the disorder values are uniformly distributed in the range $[-U_R/2, U_R/2]$ [47]. The disorder potential is also assumed to have a finite correlation. The numerical scheme used for the solution of the DGGPE (3) is based on a split-step Fourier method (SSFM) (see Appendix A for more details).

In our simulation, we consider a ^{39}K mixture droplet with effective 1D scattering lengths on the order of $0.1\text{--}1\text{ }\mu\text{m}$, which drastically reduce three-body losses [25]. The dimensional crossover from three-dimensional to a strict pure 1D regime is achieved experimentally through tightly confined geometries via deep optical lattices or microtraps, resulting in high transverse trapping frequencies. We then introduce the following characteristic units of length $z_0 = \pi \hbar^2 \sqrt{\delta g/2}/(m g^{3/2}) = 0.55\text{ }\mu\text{m}$, time $t_0 = \pi^2 \hbar^3 \delta g/(2m g^3) = 0.2\text{ ms}$, $\omega_0 = t_0^{-1}$, and energy $\mathcal{E} = \hbar/t_0 = 862.6\text{ Hz}$ [39,48]. The 1D coupling constants are given by $g = 2\hbar^2/(ma)$, $g_{12} = -2\hbar^2/(ma_{12})$, and $\delta g = g + g_{12}$ [43]. Furthermore, our total real-time evolution translates to a

window of 36 ms, covering the stable experimental lifetimes (30 ms) observed before atom decay takes over. Given the energy scaling unit $\mathcal{E} = 862.6\text{ Hz}$, the numerical values employed for the disorder strength, $U_R = 1\text{--}5$, map to $0.86\text{--}4.3\text{ kHz}$, which perfectly matches the experimental values [1,2], and the transverse trapping frequency is $\omega_\perp = 200\text{ kHz}$.

Figure 1(a) shows that for sufficiently strong disorder strength, $U_R = 5$, the density profile shows an exponential decay for a small droplet revealing the occurrence of AL. However, we see from Fig. 1(d) an absence of the exponential tail in a discrete large droplet even for strong disorder strength. Remarkably, the modulated density profile of the tail is almost symmetric in contrast to the random speckle potential [39].

One can expect that the droplet spreads normally or anomalously depending on the disorder whether it is weak or strong. To fit the obtained dynamics, we use the solution of a generalized diffusion equation:

$$q^2(t) = q^2(0) + 2D_\alpha t^\alpha, \quad (5)$$

where $q^2(t) = \int |\psi(z, t)|^2 z^2 dz$ is the time-dependent width that provides insights into the density spreading, $q(0)$ is the initial width, α is the diffusion exponent, and D_α is a generalized diffusion coefficient. For $\alpha = 0.5$, the discrete system undergoes normal diffusion, and localization for $\alpha < 0.5$ [2,49].

The dynamics of the droplet width for different values of disorder strength is exemplified in Figs. 1(b) and 1(e) for both small and large droplets. A large variety of regimes ranging from superdiffusion to subdiffusion are observed. For a large droplet, a transition from superdiffusive toward normal diffusive regime occurs with $\alpha \approx 0.76$ – 0.53 as the disorder strength increases, $U_R = 1$ – 5 . The interaction energy hardly transforms into kinetic energy during the expansion since the bulk is localized as a droplet (bound state). In the case of a small droplet, the regime is classified as subdiffusive, in which the extracted exponent quickly switches to $\alpha = 0.29$ for a strong disorder, $U_R = 5$, anticipating the localized state. This behavior is confirmed in Figs. 1(c) and 1(f) where we observe that α decreases with U_R and saturates to a constant value in the regime of strong disorder, $\alpha \sim 0.3$ for a small droplet and $\alpha \sim 0.5$ for a large droplet. This follows the expected physical behavior of nonlinear AL, where the disorder competes with the LHY and cubic nonlinearities preventing total localization with $\alpha = 0$.

The resulting dynamics can be described through a perturbation theory as interaction- and LHY-induced dephasing that prevents perfect destructive interference. In such a heuristic model, the transfer rate Γ is related to the variation of the width q via $\Gamma \propto |V'|^2$, where $V' \propto -N I_{ijkl} + \sqrt{N} J_{ijk}$ is essentially the total interaction energy times a spatial overlap coefficient that is of order unity for neighboring states [8], where $I_{ijkl} = \int \varphi_l^*(z) \varphi_k^*(z) \varphi_j(z) \varphi_i(z) dz$ and $J_{ijk} = \int \varphi_k^*(z) \varphi_j(z) \varphi_i(z) dz$ are, respectively, the mean-field and beyond-mean-field overlap tensors. This gives an exponent α in the range $1/3$ – $1/2$ depending on the droplet size (small or large), which is in qualitative agreement with the values of our numerical simulation (see Appendix B). These values are larger than those found for the subdiffusive spreading of a weakly interacting BEC characterized by $\alpha = 1/4$ – $1/6$ for both discrete and continuum cases [8,10,11].

IV. TIME-DEPENDENT PERIODIC DISORDER

In this section, we will see what happens if one introduces a random time-varying environment in addition to the underlying static disorder. The time-dependent periodic potential can be written as

$$U(t) = U[1 + U_s f(t)], \quad (6)$$

where U_s is the strength of modulation and $f(t)$ is the noise function. Experimentally, such periodic potentials can be achieved employing interference patterns of laser beams in geometries with tunable lattice spacings [50]. Here, we consider two types, namely, phase noise and Floquet-engineered noise:

$$f(t) = \begin{cases} \sin[\omega t + \phi(t)], & \text{phase noise,} \\ \sin(\omega t), & \text{Floquet-engineered noise,} \end{cases} \quad (7)$$

where the phase $\phi(t)$ and the frequency ω play a crucial role in the properties of the function $f(t)$ and in the noisy feature of the potential [17]. If the phase ϕ is constant, then the noisy part of the disorder can be represented by a Floquet state.

For numerical values, we will use $U_R = 5 \mathcal{E} = 4.3$ kHz, $U_s = 0.01 \mathcal{E} = 8.6$ Hz, and $\omega = 0.628 t_0^{-1} = 541.7$ Hz.

A. Phase noise

Let us start by investigating the effects of noise for a droplet in an optical lattice. We consider the situations reported above, keep constant the disorder parameters, and vary those defining the temporal part, i.e., the strength U_s and the frequency ω .

To produce a time-dependent random potential, we suddenly change the phase ϕ every fixed time step, known as dephasing time, T_d [17]. At long times, $t > T_d$, the resulting noise is decaying while at $t < T_d$, ϕ_j is randomly chosen within the interval $[-\pi, \pi]$. The effect of different values of the modulation strength on the dynamics of the droplet is analyzed for $T_d < T_0$, $T_d = T_0$, and $T_d > T_0$, where $T_0 = 2\pi/\omega$. The variation of the phase, $\phi(t)$, for $T_d = 5$ – 20 (or equivalently 0.9–3.7 ms) is displayed in Fig. 2. We see that a smaller T_d corresponds to faster fluctuations and more rapid phase changes, while larger T_d corresponds to slower fluctuations and more gradual phase changes.

Once the phase $\phi(t)$ is determined, we can modify the degree of noisiness of the function $f(t)$ and hence examine the effect of the phase noise on the evolution of the droplet. In Fig. 3(a), the width of a small droplet is plotted as a function of time for different modulation strengths, U_s . One can clearly see a transition from localized to diffusive regime ($\alpha = 0.5$) as U_s increases regardless of the values of ω . However, a transition from diffusive to superdiffusive regime is observed in a large droplet ($\alpha > 0.5$) as seen in Fig. 3(d). After the evaporation of the large droplet, the repulsion force dominates the LHY term resulting in an accelerated movement of the particle inducing a superdiffusion. One can also explain the appearance of such a diffusive dynamics due to the fact that the noise may provide an additional energy to couple the interaction-shifted states to the lattice band causing a fast expansion. It is worth noticing that a similar superdiffusion effect with close values of the exponent α has been observed in the case of a BEC loaded in a quasiperiodic lattice with controlled noise and tunable interaction [17].

Now, we discuss the effect of the noise amplitude, U_s , on the behavior of the anomalous diffusion exponent and coefficient. These parameters are extracted using $\alpha(t) = d \log_{10} \Delta q(t) / (d \log_{10} t)$ and $D_\alpha(t) = (q^2(t) - q^2(0)) / (2t^\alpha)$, respectively. As shown in Figs. 3(b)–3(e), with increase of the noise strength the anomalous diffusion exponent, α , increases and then saturates after it reaches its critical value, U_s^c . The latter relies on the droplet size. The appearance of maximum and minimum in the curve of the diffusion coefficient D_α notably for mobile particles in a large droplet characterizes the anomalous diffusion as depicted in Figs. 3(c)–3(f). This reveals that at a critical modulation strength the particle motion in the droplet undergoes transitions between superdiffusion, normal diffusion, and subdiffusion.

B. Floquet-engineered noise

Here, we study the properties of the Floquet anomalous transport in quantum droplets. In such a case, noise is introduced via random modification of the Floquet timing (frequency). The numerical findings are sketched in Fig. 4.

Surprisingly, we observe that AL persists in the presence of the Floquet drive for small modulation strengths whatever T_d is chosen [see Fig. 4(a)]. When U_s is sufficiently strong, the

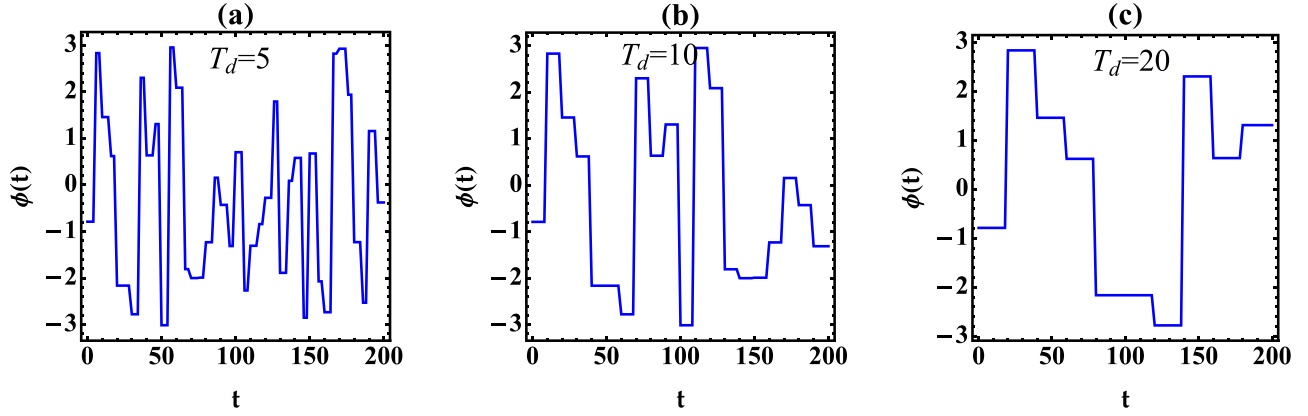


FIG. 2. The phase $\phi(t)$ for different values of T_d . Here, the phase is same for both small and large droplets since it is independent of N . Parameters are $U_R = 5$, $U_s = 0.01$, and $\omega = 0.628$.

system appears to exhibit diffusive dynamics [see Fig. 4(b)], suggesting that the correlation effects of the Floquet noise become unimportant. Such an anomalous diffusion is confirmed in the behavior of diffusion coefficient [see Fig. 4(c)]. However, the situation is quite different for a large droplet where a transition from diffusive to superdiffusive regime is observed by increasing U_s [see Figs. 4(d)–4(f)].

Remarkably, AL breaks down in the large droplet (in tail and central parts) for both types of noise. Why? In the tail region where the system can be considered as a noninteracting gas, AL is destroyed due to the interplay between disorder, phase, or Floquet noise and the LHY fluctuations. However, in the central part where most of the density resides and which therefore can be still described as a quantum

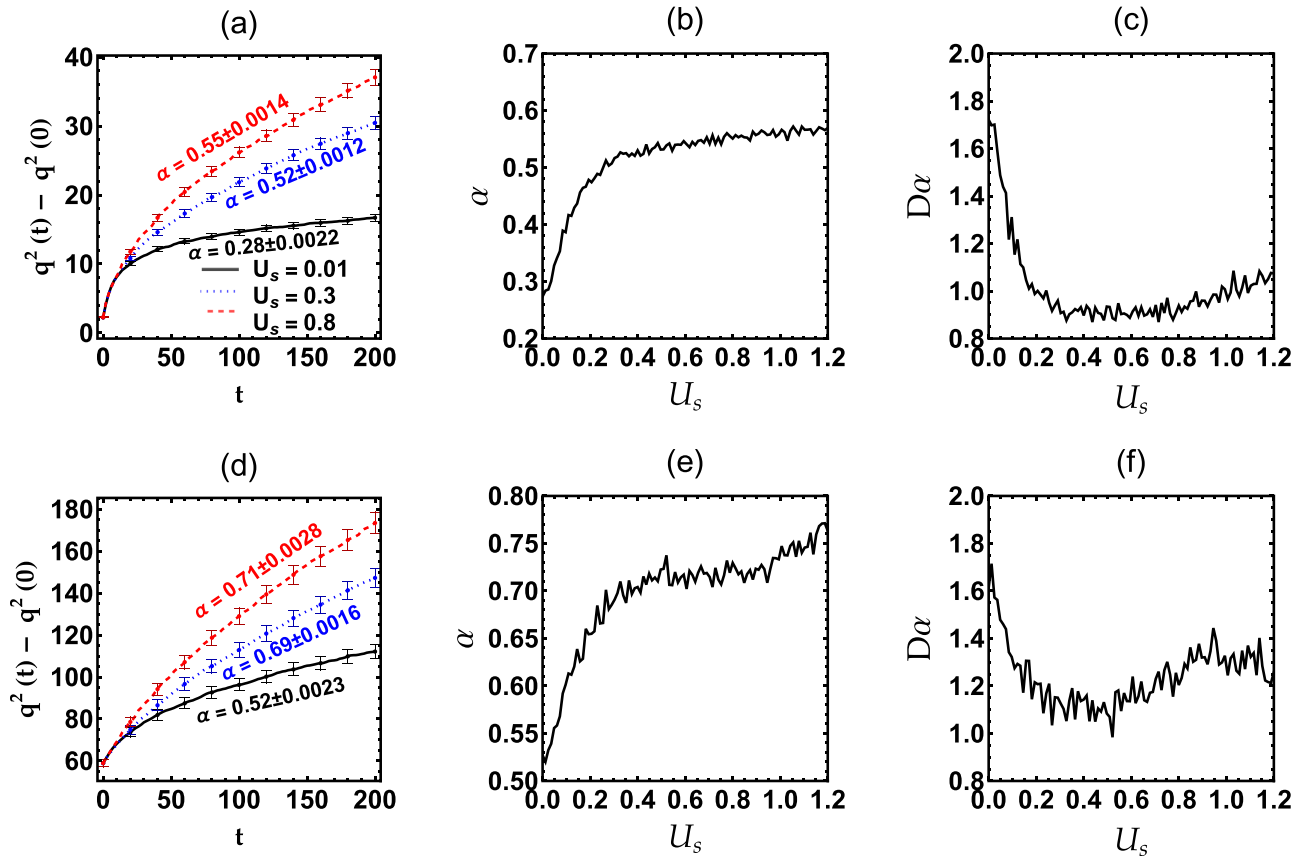


FIG. 3. (a) Time evolution of the width of a small droplet for different values of U_s . (b) The diffusion exponent, α , as a function of U_s . (c) The diffusive coefficient, $D\alpha$, as a function of U_s . Panels (d)–(f) are the same as panels (a)–(c) but for a large droplet. Parameters are $U_R = 5$, $\omega = 0.628$, and $T_d = 10$. As in Fig. 1, the diffusion exponent, α , is extracted from fits of Eq. (5) to the data obtained from our numerical simulations in the time interval [10, 200]. The vertical lines represent the standard error bars of the width calculated across the 100 independent disorder realizations, while the statistical errors are very small to be seen for most points confirming the high precision of our numerical simulations.

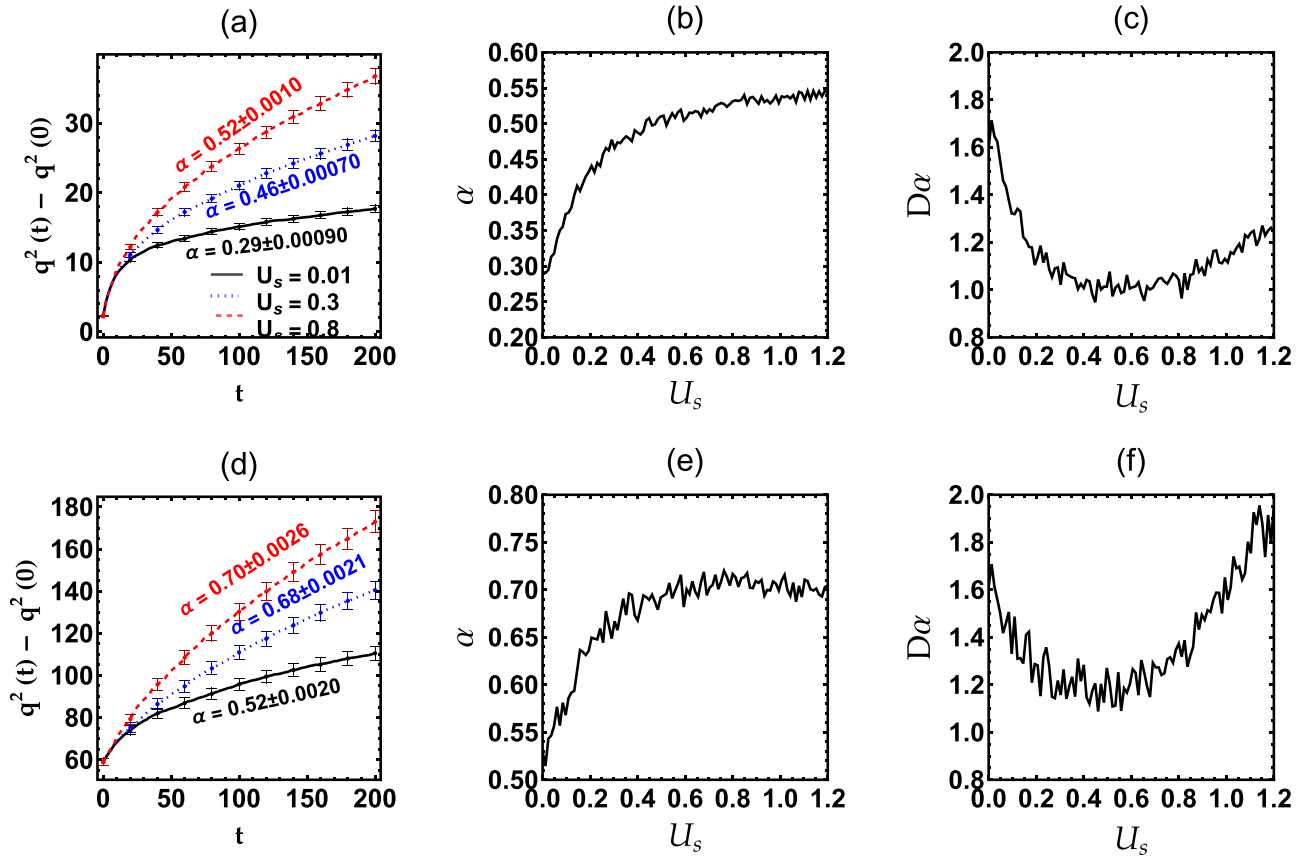


FIG. 4. The same as in Fig. 3 but for Floquet-engineered noise with $U_R = 5$ and $\omega = 0.5$.

droplet, nonlinear effects and noise become stronger, which enhance the transport and the coupling between localized states leading to significantly weakened disorder-induced localization. Therefore, the strong self-interaction (including the LHY correction) acts as an effective dephasing mechanism that breaks the phase coherence required for AL. One can expect that for sufficiently strong disorder and for a high-frequency periodic drive, the quantum droplet (central part of the distribution) is undergoing many-body localization provided that interacting particles avoid thermalization.

Note that the values of the exponent α are also confirmed by the extended heuristic model that incorporates, in addition to interaction and LHY contributions, a noise (phase or Floquet) term (see Appendix B).

C. Phase diagram

To gain deeper insights into the dynamic regimes in terms of droplet size (number of particles) and noise amplitude, a phase diagram is presented in Fig. 5. The numerical outcomes show several transport regimes for both small and large droplets. For $N \lesssim 4$ and $U_s < 0.05$, one finds $\alpha \lesssim 0.07$ indicating the occurrence of AL, while for $N < 16$ and $0.05 < U_s < 0.12$, the dynamics of the droplet is subdiffusive with $\alpha \lesssim 0.2$. Thereafter, the exponent takes the value of either normal diffusion or superdiffusion with $\alpha \gtrsim 0.5$. The deviation of the observed exponent might be attributed to the competition between quantum fluctuations, nonlinearity, and noise. The same behavior holds for a Floquet noise but with

slight modifications of the diffusion coefficient. Importantly, for large N and U_s , AL is destroyed by the quantum fluctuations and replaced by a subdiffusive behavior. One can deduce that the addition of somehow large temporal noise (Floquet or phase) acts as a delocalization mechanism. By fixing U_R and increasing U_s , the exponent α increases marking a transition from a highly subdiffusive regime to a more transport-active regime. This systematic dependence demonstrates that the observed exponents represent the underlying scaling physics of the interplay between quantum fluctuations, nonlinearity, and stochastic driving.

V. FINITE-TEMPERATURE EFFECTS

Since finite-temperature effects cannot be fully ignored in cold atom experiments, it is therefore crucial to include the thermal corrections to the DGGPE (3) and study their influence on the transport of disordered quantum droplets. At finite temperature, the dynamics of the system is governed by the following dimensionless FTDGPE [38]:

$$i \frac{\partial \psi_j}{\partial t} = -C(\psi_{j+1} + \psi_{j-1} - 2\psi_j) + U_j \psi_j + |\psi_j|^2 \psi_j - |\psi_j| \psi_j + g_T |\psi_j|^{-3} \psi_j = 0, \quad (8)$$

where $g_T = \pi^3 \hbar T^2 (\delta g/g)^2 / (48 m g^{3/2})$ accounts for the strength of thermal fluctuations. The FTDGPE (8) is valid in the weak-coupling limit in which the ratio between the global interaction energy including the LHY term

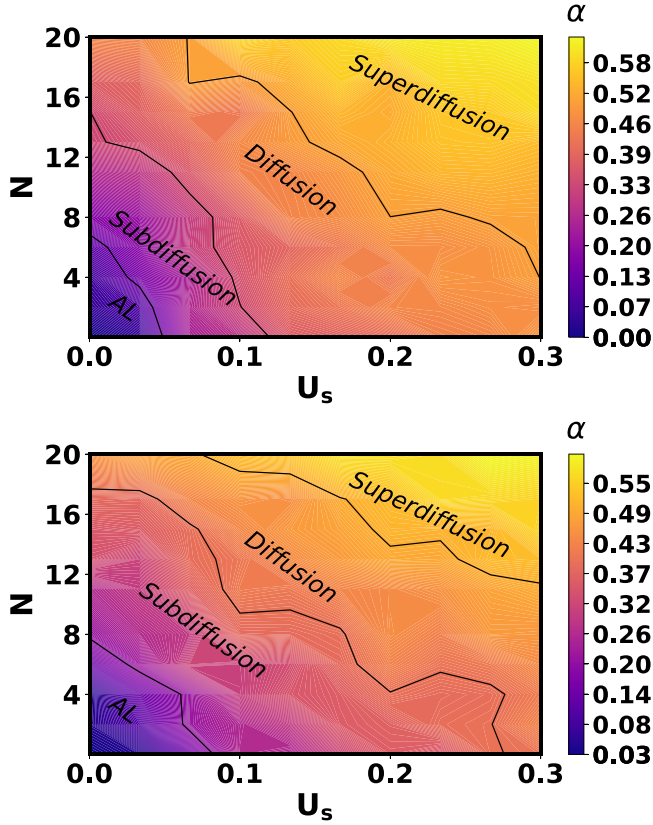


FIG. 5. (a) Phase diagram summarizing different transport regimes of the droplet for the phase noise. (b) The same but for Floquet noise. Parameters are $U_R = 5$, $\omega = 0.628$, and $T_d = 10$.

and the kinetic energy is much smaller than unity [51]. In such a regime, a dominant coherent fraction is still preserved (i.e., the thermal depletion is negligible), which typically corresponds to temperatures less than the threshold temperature for thermally destabilizing Bose mixture droplets [52–54]. The presence of disorder may invoke an additional assumption that the disorder potential must vary smoothly in space on a length scale comparable to the healing length [36,38]. Therefore, the coherent-wavefunction description still captures the essential transport dynamics of disordered droplets at small but nonzero temperatures.

We can then proceed to solve the FTDGGPE (8) in the presence of phase noise disorder from Eq. (7) using the same numerical scheme presented in Sec. IV. As shown in Fig. 6, the diffusion exponent of both large and small droplets grows with the strength of thermal fluctuations, approximately as g_T^ν , where $\nu \approx 0.01$ – 0.1 depending on the dephasing time, the disorder strength, and the droplet size. It increases with thermal fluctuations' strength whatever the values of U_s and T_d , and then it saturates when thermal fluctuations reach a certain critical value ($g_T \gtrsim 0.5 \times 10^{-4}$). The figure also depicts that the effect of the noise is pronounced only for weak thermal fluctuations ($g_T \leq 0.2 \times 10^{-4}$). Importantly, as the temperature increases, a continuous transition from subdiffusion to other type of diffusion such as normal diffusion and superdiffusion ($\alpha \approx 0.28$ – 0.56) even for a very weak noise, $U_s = 0.1$ (or equivalently $U_s = 86.2$ Hz), is observed regardless of the

value of T_d [see Figs. 6(a)–6(c)]. This is presumably due to the fact that as the thermal effects increase, particles gain more kinetic energy and hence can diffuse more quickly giving rise to a complete delocalization of the localized states at larger g_T . However, the motion of particles in the large droplet is purely superdiffusive with $\alpha \gtrsim 0.53$ at any temperature and for any dephasing time [see Figs. 6(d)–6(f)].

VI. CONCLUSIONS

We studied different kinds of dynamics of quantum droplets loaded in a quasiperiodic lattice with controlled disorder and noise in the presence of competing mean-field and beyond-mean-field nonlinearities. Our numerical simulations reveal that by varying the control disorder strength, a transition between superdiffusion, normal diffusion, and subdiffusion is observed in a small droplet, while the motion of atoms in the large droplet is either diffusive or superdiffusive characterized by a diverging diffusion coefficient. However, in the presence of phase and Floquet noises, we observed instead a complex anomalous diffusion in both structures due to the intricate interplay of noise, interaction, and quantum fluctuations. Importantly, we showed that AL can occur for wave packets obeying Floquet drive in small droplets. A phase diagram characterizing different transport regimes in terms of droplet size and noise amplitude has been mapped out. Moreover, we demonstrated that due to thermal fluctuations, the particles in the droplets gain more kinetic energy and hence can diffuse more quickly. In such a case, the diffusion exponent depends on the strength of thermal fluctuations and on the details of the noise. Our findings offer an ideal platform for simulations of quantum many-body problems and for quantum computing.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: NUMERICAL SIMULATION

1. Numerical method

The dynamics is investigated by suddenly releasing the droplet initially prepared in its ground state of Eq. (3) to a disorder potential. To simulate this evolution, we employ the SSFM by time evolving the initial equilibrium solution (B1) dependent on a chemical potential μ , and it is initialized as a complex-valued wave packet. The basic idea is to split the evolution over a small time step Δt into two parts. (1) Potential and nonlinear part: In this stage, the wavefunction is updated in real space using an exponential factor that depends on both the potential and the nonlinear terms $|\psi|^2\psi - |\psi|\psi$. This simulates the effect of internal and external forces on the wave packet over half a time step. (2) Kinetic part: Here, the wavefunction is Fourier transformed to momentum space, where the kinetic term $k^2/2$ is applied as a phase factor. After that, the wavefunction is transformed back to real space to complete the time step.

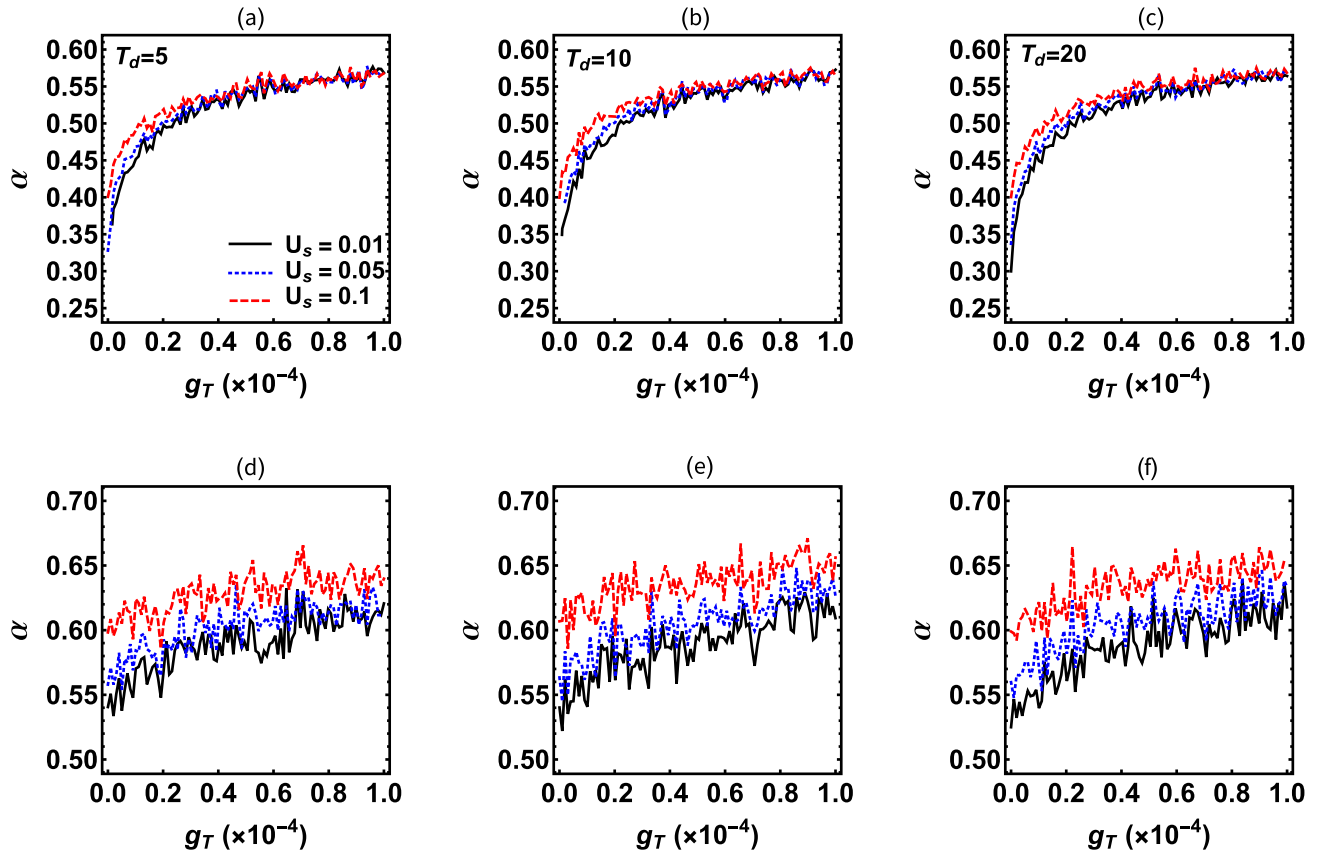


FIG. 6. (a)–(c) The diffusion exponent, α , associated with a small droplet as a function of thermal fluctuations’ strength, g_T , for different values of U_s and dephasing time T_d . (d)–(f) Same as panels (a)–(c) but for a large droplet. Parameters are $U_R = 5$ and $\omega = 0.628$.

The numerical approach involves a spatial grid with $L = 800$ sites with spatial and time steps $\Delta z = 1$ and $\Delta t = 0.5$, respectively, and the simulation is averaged over 100 different realizations of disorder to obtain statistically meaningful results. Time evolution is implemented using the SSFM, which alternates between applying the nonlinear interaction including the LHY correction, disorder potential in real space, and the kinetic term in Fourier space. The SSFM presents the same accuracy for predicting effects of thermal fluctuations described by the FTDGPE (8).

2. Analysis of the diffusion exponent

All fitting procedures have been performed in the late-time regime, specifically in the interval $t \in [10, 200]$, which is chosen to ensure accuracy and exclude any initial transient effects and to focus on the stabilized transport dynamics. We find that excluding the initial 5%–10% of the simulation time results in diffusion exponent, α , values consistent with those obtained from the full-time range. Note that the extracted diffusion exponent remains unchanged when increasing the lattice size L and varying the number of disorder realizations, ensuring the robustness of the asymptotic transport behavior, especially near crossover regimes.

To evaluate the statistical reliability of our results, error bars for both the cloud width and the diffusion exponent α have been calculated across 100 independent disorder realizations. For the dynamic evolution, the true statistical

uncertainties of the width are plotted at regular time intervals (every ten time steps for better visualization). Meanwhile, the error bars for the final exponent α have been determined using the covariance matrix. For both static and dynamic disorders, all statistical and relative errors have been found to be remarkably small. This high precision not only confirms the stability of our numerical simulations but also demonstrates that the 100 disorder realizations are entirely sufficient to ensure converged and accurate results.

To verify the robustness of our results, we test different fitting windows (e.g., comparing $t \in [5, 200]$ with $[10, 200]$). The extracted values of α remained consistent and unchanged within the reported error bars, demonstrating that our results do not depend on the specific choice of the interval.

In order to check the inferred transport regimes, we also investigate finite-time effects by monitoring the local slope of the mean squared displacement (i.e., droplet width squared). We find that for $t = 200$, the exponent α reaches a plateau (saturation), indicating that the system is in the asymptotic subdiffusive regime. Therefore, increasing the simulation time further does not alter the physical conclusions.

APPENDIX B: HEURISTIC SCALING MODEL

For reasons of completeness, we report in this Appendix a heuristic microscopic model explaining that the observed dynamics relies onto a perturbation theory [7,8]. The presence of the interaction and the LHY term may break the

orthogonality between localized states giving rise to a transfer of population between neighboring states [8]. Thus, the expansion of the wave packet in Anderson modes in terms of time-dependent expansion coefficients, $a_j(t)$, at site j reads $\psi(z, t) = \sum_j a_j(t) e^{-iE_j t} \varphi_j(z)$. In the absence of the LHY term, the transfer rate Γ is related to the variation of the width q via [7,8]

$$\Gamma = \frac{2\pi}{\hbar} \frac{|V'_{ijkl}|^2}{\Delta E}, \quad (\text{B1})$$

where $V'_{ijkl} = \frac{1}{2} n d \int \varphi_l^*(z) \varphi_k^*(z) \varphi_j(z) \varphi_i(z) dz$ is the interaction term of this process, with d being the average distance that is of order unity for neighboring states [8], and $\Delta E = |E_i + E_j - E_k - E_l|$. In disordered systems, the conservation energy requires the following condition: $\Delta E < V'_{ijkl}$ [7]. This leads to an exponent $\alpha = 1/4$ in the strong-coupling regime and $\alpha = 1/6$ when the number of possible couplings decreases [8].

Now, in the presence of the LHY term, the interaction-assisted term is given as $V' = V'_{\text{MF}} + V'_{\text{BMF}}$, where

$$V'_{\text{MF}} \propto -N \int \varphi_l^*(z) \varphi_k^*(z) \varphi_j(z) \varphi_i(z) dz = -N I_{ijkl} \quad (\text{B2})$$

and

$$V'_{\text{BMF}} \propto \sqrt{N} \int \varphi_k^*(z) \varphi_j(z) \varphi_i(z) dz = \sqrt{N} J_{ijk}, \quad (\text{B3})$$

where I_{ijkl} and J_{ijk} are, respectively, the mean-field and beyond-mean-field overlap tensors. Since every Anderson eigenstate decays as $\varphi(z) \sim e^{-|z-z_i|/L_{\text{loc}}}$, where L_{loc} is the localization length, we have then $I_{ijkl} \approx J_{ijk} = e^{-d/L_{\text{loc}}}$. Therefore, the transfer rate becomes

$$\Gamma = \frac{2\pi}{\hbar} (N^2 I_{ijkl}^2 - 2N^{3/2} I_{ijkl} J_{ijk} + N J_{ijk}^2). \quad (\text{B4})$$

Here, $N = nL_{\text{loc}}$.

For a small droplet where the density is low and the droplet is compressible, one can set $n \propto 1/q$. This gives

$$\Gamma = \frac{I^2}{q^2} - \frac{2IJ}{q^{3/2}} + \frac{J^2}{q}. \quad (\text{B5})$$

This equation contains three transport channels: mean-field channel q^{-2} , mean-field and beyond-mean-field interference $q^{-3/2}$, and pure beyond-mean-field term q^{-1} . Given that the small droplet size is comparable to only few localization lengths, and the density is not saturated, the mean-field force dominates, $V'_{\text{BMF}} \ll V'_{\text{MF}}$, and hence $V' \simeq V'_{\text{MF}}$. This implies that each interaction-assisted transition between neighboring Anderson modes occurs with a well-defined phase. The transfer rate interfere constructively, $\Gamma \propto q^{-2}$, and the expansion is

described by coherent interaction-assisted population transfer, $d[q(t)]/dt \propto \Gamma$. This immediately gives

$$\Gamma = \frac{d[q(t)]}{dt} \propto q^{-2}. \quad (\text{B6})$$

Equation (B6) can be integrated with the initial condition $q(t=0) = q(0)$, yielding a diffusion exponent $\alpha = 1/3$.

For a large droplet where $V'_{\text{BMF}} \approx V'_{\text{MF}}$, the bulk density remains approximately constant $n = n_0$, with $n_0 = 8m g^3 / (9\pi^2 \hbar^2 \delta g^2)$ being the equilibrium density. So, the interaction no longer provides an efficient mechanism for coherent mode-to-mode transfer due to balance between the mean-field force and the LHY term. Instead, transport is governed by the accumulation of many weak, statistically independent scattering events arising from disorder and residual fluctuations. Therefore, the expansion obeys

$$\Gamma = \frac{d[q^2(t)]}{dt} = \text{constant}, \quad (\text{B7})$$

giving a larger exponent, $\alpha = 1/2$. The expansion is then much slower than $t^{1/3}$ and can approach pinning.

The crossover from $\alpha = 1/3$ to $\alpha = 1/2$ is not caused by a loss of condensate coherence, but by the progressive cancellation of the repulsive mean-field and attractive beyond-mean-field interaction matrix elements. This suppresses coherent interaction-assisted transport, allowing disorder-induced diffusion to become the dominant expansion mechanism. Such a crossover of the diffusion exponent, which is absent in the case of a BEC in quasiperiodic potentials [8], can be probed experimentally by independently tuning the ratio of mean-field and the LHY terms through the variation of interaction parameters in a Bose mixture.

Importantly, both values of α measured by such a simplified heuristic model are in qualitative agreement with our numerical results.

As a result, the obtained anomalous diffusion can be modeled with the following extended diffusion equation:

$$\frac{d[q^2(t)]}{dt} = D^{\text{noise}} + D_{\alpha}^{\text{int}}(t) + D_{\alpha}^{\text{LHY}}(t), \quad (\text{B8})$$

where $D^{\text{noise}} \propto A^2(d + L_{\text{loc}})^2 / (1 + e^{d/L_{\text{loc}}})$ is the time-independent diffusion coefficient with a constant relying on tunneling energy, $D_{\alpha}^{\text{int}}(t) = 2\alpha(q_0^{\alpha-1}/t_0)q(t)^{2-\alpha-1}$ is the time-dependent diffusion coefficient from Eq. (B1) for the interaction alone [8], and $D_{\alpha}^{\text{LHY}}(t)$ is the time-dependent diffusion coefficient for the LHY correction alone. It can be easily deduced from Eqs. (B6) and (B7). One should stress that in the case of a static disorder, the diffusion equation (B8) reduces to $d[q^2(t)]/dt = D_{\alpha}^{\text{int}}(t) + D_{\alpha}^{\text{LHY}}(t)$ [17].

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