

General theory for nonzero subsymmetry-protected topological chiral boundary states

Yunlin Li , Yufu Liu, Xuezhi Wang, Haoran Zhang, and Xunya Jiang*

Department of Illuminating Engineering and Light Sources, College of Intelligent Robotics and Advanced Manufacturing, Fudan University, Shanghai 200433, China



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Symmetry plays an important role in the topological band theory. Recently, the concept of subsymmetry-protected topological (sub-SPT) phases [Nat. Phys. **19**, 992 (2023)] suggests that chiral edge states can exist in a system with broken symmetry. However, sub-SPT theory only focuses on the zero-energy edge states in the half-filling gap, which limits its application. In this work, we extend the scope of sub-SPT phases in both one-dimensional (1D) and two-dimensional (2D) multigap asymmetric systems through redefined sublattices. In three typical asymmetric models without subsymmetry, we find that all the nonzero edge (corner) states can be illustrated by sub-SPT chiral edge (corner) states after redefining sublattices. Moreover, these nonzero chiral boundary states can exhibit some exotic properties, such as the topological bound states in the continuum. As to the bulk-edge correspondence, a universal topological invariant $\bar{Z}$ is defined, which can precisely predict the appearance of chiral boundary states in both 1D and 2D systems. Finally, we propose an acoustic experiment where the nonzero chiral corner states are numerically observed. Our work exhibits a general approach to study the topological properties of asymmetric systems, opening up possibilities for topological phenomena.

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I. INTRODUCTION

In the last few decades, topological insulator has attracted much attention in both quantum and classical systems [1–4]. Since the discovery of spin-orbit-induced topological insulators [5,6], symmetry-protected topological (SPT) phases of matter play an important role in the study of topological band theory. In nontrivial SPT phases, symmetry-protected topological edge states can be found for the systems under open boundary condition (OBC) and the bulk-edge correspondence can be established. For instance, based on the tenfold classification [7,8], nontrivial topological phases protected by antiunitary symmetry can be found. Moreover, in topological crystalline insulators, topological edge states protected by crystalline point group symmetry [9,10] are extensively studied. As to the higher-order topological insulators [11,12], symmetry-protected topological corner states are found theoretically and experimentally [13–19].

Recently, studies have shown that nontrivial topology also exists in systems without certain symmetry. In the SSH3 model with neither chiral symmetry nor spatial inversion symmetry, the existence of topological edge states is reported extensively [20–26] while the bulk-edge correspondence is established through various approaches [22,27,28]. For one-dimensional (1D) biparticle asymmetric model, the

topological nature of its edge states is investigated through non-Bloch theory [29]. In these cases, exotic topological phenomena such as the topological phase transition without gap closing are observed, which go beyond the conventional understanding of the SPT phases.

Meanwhile, there has been a considerable interest to build the bridge between SPT phases and asymmetric systems. One intriguing example is the concept of subsymmetry-protected topological (sub-SPT) phases [30]. By combining the chiral operator with the sublattice projection operator, the chiral symmetry can be defined within certain subspace. As a result, robust chiral edge states can be observed in a system even when the overall chiral symmetry is broken. By introducing the SPT phases into asymmetric systems, the sub-SPT phase successfully provides a clear physical picture for the edge states in asymmetric systems, leading to topological phenomena in photonic systems [31] and superconducting systems [32].

However, in the framework of sub-SPT phases, only the zero-energy chiral edge states in the half-filling gap can be well illustrated. For multigap asymmetric systems with nonzero edge states, such as the SSH3 model mentioned above, the sub-SPT theory fails to work. This strict restriction on the edge state energy greatly hinders the application of sub-SPT theory since any asymmetric model with more than two sublattices is generally a multigap system with nonzero fractional-filling gaps. Therefore, developing a more general theory to explore the nontrivial topology in various asymmetric systems is an urgent need.

On the other hand, redefining sublattices through representation transformation has been utilized to explore topological phenomena, e.g., to construct topological bound states in the

*Contact author: jiangxunya@fudan.edu.cn

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continuum (TBICs) in mirror-stacked systems [33,34], to realize non-Abelian topological phases in acoustic systems [35] and to reconstruct the topology of the Rice-Mele ladder [36]. With redefined sublattices, the original system can be divided into effective subspaces where the nontrivial topological edge states might appear. More importantly, as the effective onsite energy of the redefined sublattices differs from the original ones, the Fermi level of the redefined subspace also changes, which results in an energy shift of the topological edge states. With these merits, redefining sublattices could also have great potential in expanding topological phases in asymmetric systems.

In this work, by redefining sublattices, we extend the sub-SPT phases to various asymmetric systems with nonzero topological chiral edge states. We start from 1D systems, investigating two typical asymmetric models without subsymmetry, namely the SSH m model and the asymmetric SSH ladder. We show that both models can be transformed into several coupled subspaces with redefined sublattices. After the transformation, each subspace is subsymmetric, which supports sub-SPT edge states. Moreover, due to the effective onsite energy, the energy of the chiral edge states shifts away from zero. As to the bulk-edge correspondence, we find that the chiral edge states originate from the Rice-Mele-like effective Hamiltonian. Based on the reduced wave function of the effective Hamiltonian, a quantized topological invariant $\bar{Z}$ can be defined through normalization. We show that the abrupt change in $\bar{Z}$ directly corresponds to the appearance of chiral edge states.

To take a step further, we then focus on 2D systems. To illustrate our points, we construct the 2D BBH3 model. Although the model itself is neither chiral symmetric nor subsymmetric, sub-SPT chiral corner states can still be found with redefined sublattices. These corner states exhibit fundamental differences from the conventional topological quadrupole corner states, e.g., they cannot be described by quantized quadrupole moment while their existence are independent of any spatial or chiral symmetry. The corner states can even move into the bulk continuous spectrum, exhibiting a new kind of TBICs. In terms of the topological invariant, we introduce $\bar{Z}$ into 2D systems and establish the edge-corner correspondence through the quantized normalized edge polarization $\bar{p}^{\text{edge}}$. At last, we propose a laboratory acoustic experiment to test topological phenomena of the asymmetric BBH3 model, where nonzero chiral corner states could be numerically observed.

Our work establishes a general theory to study the nonzero chiral characteristics of the boundary states in various asymmetric systems. By redefining sublattices, we extend the concept of sub-SPT phases to multigap systems with nonzero edge states. Moreover, the bulk-edge correspondence can be universally established with a general topological invariant.

II. SUB-SPT CHIRAL EDGE STATES IN 1D ASYMMETRIC MODELS

We would like to start with 1D asymmetric systems to illustrate our theory step by step. Here, we consider two typical asymmetric models, i.e., the SSH m model and the asymmetric SSH ladder.

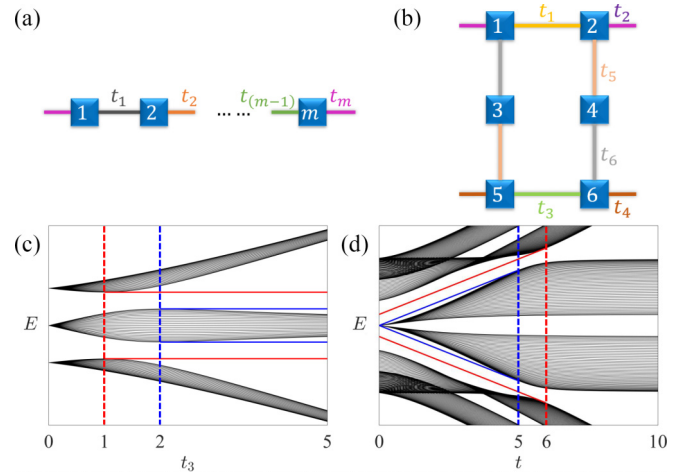


FIG. 1. (a) Schematic of the SSH m model. (b) Schematic of the asymmetric SSH ladder. (c) Band structure of the SSH3 model vs t_3 under OBC with $t_1 = 1$ and $t_2 = 2$. (d) Band structure of the asymmetric SSH ladder vs t under OBC with $t_1 = 5$, $t_2 = 1$, $t_3 = 2$, $t_4 = 4$, $t_5 = t + 1$, and $t_6 = t$. In panels (c) and (d), the in-gap edge states are marked by red and blue lines.

As shown in Fig. 1(a), the SSH m model is the 1D chain model whose unit cell contains m sublattices [37]. The Hamiltonian of the SSH m model can be written as

$$H_m = \sum_n \left[\sum_{j=1}^{m-1} t_j c_{n,(j+1)} c_{n,j}^\dagger + t_m c_{(n+1),1} c_{n,m}^\dagger + \text{H.c.} \right], \quad (1)$$

where n is the index of the unit cell and j is the index of the sublattice. Here, we only consider the nearest-neighbour hoppings. When $t_i \neq t_j$ and m is odd, both spatial inversion symmetry and chiral symmetry are broken.

Meanwhile, the asymmetric SSH ladder is illustrated in Fig. 1(b). Its Hamiltonian can be written as

$$\begin{aligned} \mathcal{H}_{\text{ladder}} = \sum_n & t_1 c_{n,1} c_{n,2}^\dagger + t_2 c_{(n+1),1} c_{n,2}^\dagger + t_3 c_{n,5} c_{n,6}^\dagger \\ & + t_4 c_{(n+1),5} c_{n,6}^\dagger + t_6 (c_{n,1} c_{n,3}^\dagger + c_{n,4} c_{n,6}^\dagger) \\ & + t_5 (c_{n,3} c_{n,5}^\dagger + c_{n,2} c_{n,4}^\dagger) + \text{H.c.} \end{aligned} \quad (2)$$

When $t_5 \neq t_6$, the spatial inversion symmetry is broken; when $t_1 \neq t_3$ and $t_2 \neq t_4$, the C_2 rotation symmetry is broken. On the other hand, $\mathcal{H}_{\text{ladder}}$ possesses chiral symmetry, which makes the band structure symmetric around $E = 0$.

The evolution of band structure for the asymmetric SSH3 model and the asymmetric SSH ladder under OBC is shown in Figs. 1(c) and 1(d), respectively. The hopping parameters are set to $t_1 = 2$ and $t_2 = 2$ for Fig. 1(c) and $t_1 = 5$, $t_2 = 1$, $t_3 = 2$, $t_4 = 4$, $t_5 = t + 1$, and $t_6 = t$ for Fig. 1(d). The in-gap edge states are marked by red and blue lines. Two critical phenomena can be observed. First, the emergence of the edge state does not coincide with gap closing, indicating that the conventional SPT phase transition cannot happen in these models. Second, all of the edge states exist in the fractional-filling gaps, indicating that their eigenenergies are nonzero. Therefore, the sub-SPT theory cannot be directly utilized to illustrate their topological properties.

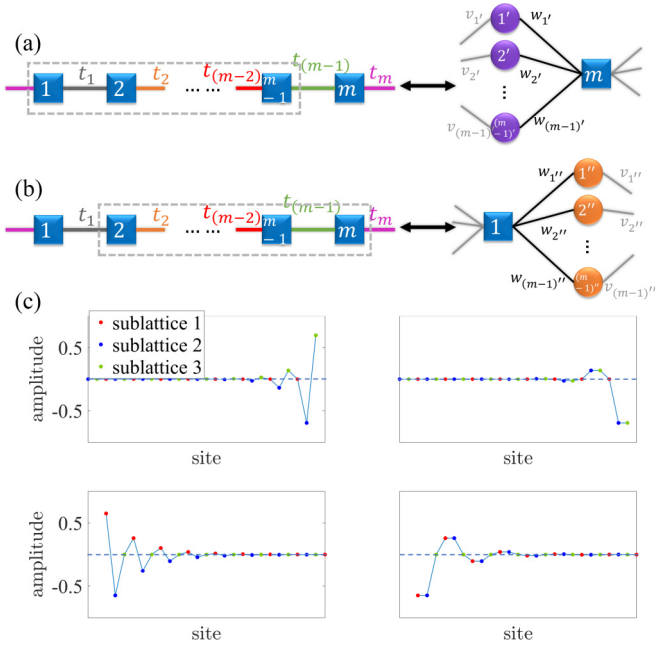


FIG. 2. Schematic of the SSH m model with redefined sublattices diagonalizing the subspace (a) $\mathcal{H}_{m-1}^L$ and (b) $\mathcal{H}_{m-1}^R$ marked by dashed box. (c) The spatial distribution of the edge states of the SSH3 model.

A. Sub-SPT phases in the SSH m model with redefined sublattices

Due to its simplicity, we first investigate the SSH m model. Here, we emphasize that we do not intend to repeat the results of previous works [22,24,28], but rather to exhibit the nontrivial topology of the edge states from the perspective of sub-SPT phases. The edge states of the SSH m model under OBC can be obtained analytically utilizing the non-Bloch theory [38,39]. After some calculation, one can see that the edge localized at the left edge can be written as (Appendix A)

$$|L\rangle = A_L \sum_{n=1}^N z_1^{n-1} \sum_{j=1}^m \phi_j^{(1)} |n, j\rangle, \quad (3)$$

where $|z_1| < 1$, $\phi_j^{(1)}$ is the eigenstate of $\mathcal{H}_{m-1}^L$, which is the Hamiltonian of the subspace composed of the sublattices 1 to $m-1$ marked by dashed lines in Fig. 2(a).

Similarly, the edge localized at the right edge can be written as

$$|R\rangle = A_R \sum_{n=1}^N z_2^{n-1} \sum_{j=1}^m \phi_j^{(2)} |n, j\rangle, \quad (4)$$

where $|z_2| > 1$, $\phi_j^{(2)}$ is the eigenstate of $\mathcal{H}_{m-1}^R$, which is the Hamiltonian for the subspace composed of the sublattices 2 to m marked by dashed lines in Fig. 2(b).

To show the topological origin of the left-boundary edge states $|L\rangle$, we perform a proper representation transformation

U to diagonalize $\mathcal{H}_{m-1}^L$:

$$\mathcal{H}_{m-1}^L = U \mathcal{H}_{m-1}^L U^\dagger = \begin{bmatrix} V_1 & & & 0 \\ & V_2 & & \\ & & \ddots & \\ 0 & & & V_{(m-1)'} \end{bmatrix}. \quad (5)$$

In this representation, the sublattices are redefined as $\{|j'\rangle\}$, $j = 1, \dots, m-1$. Then, the Hamiltonian of the SSH m model can be written as

$$\mathcal{H}_m'(k) = \begin{bmatrix} V_1 & & & 0 & t_1'(k) \\ & V_2 & & & t_2'(k) \\ & & \ddots & & \vdots \\ 0 & & & V_{(m-1)'} & t_{(m-1)'}'(k) \\ t_1^\dagger(k) & t_2^\dagger(k) & \cdots & t_{(m-1)'}^\dagger(k) & 0 \end{bmatrix}, \quad (6)$$

where $t_j'(k) = w_j + v_j e^{ik}$, and w_j and v_j can be calculated by

$$\begin{aligned} w_j &= \langle n, j' | H_m | n, m \rangle = t_{m-1} \phi_{m-1}^{j'}, \\ v_j &= \langle n, m | H_m | n+1, j' \rangle = t_m \phi_1^{j'}. \end{aligned} \quad (7)$$

Here, $\phi_j^{j'}$ is the component of the redefined $|j'\rangle$ on the original sublattice $|i\rangle$ and H_m stands for the Hamiltonian of SSH m model in Eq. (1). As shown in Fig. 2(a), $\mathcal{H}_m'(k)$ is equivalent to $m-1$ SSH models:

$$H_m' = \sum_{j'=1'}^{(m-1)'} H_j^{\text{SSH}} + V_j P_{j'}, \quad (8)$$

where

$$H_j^{\text{SSH}} = \sum_n w_j c_{n,m} c_{n,j'}^\dagger + v_j c_{(n+1),j'} c_{n,m}^\dagger + \text{H.c.} \quad (9)$$

is the SSH model composed of $|j'\rangle$ and $|m\rangle$, V_j is the effective onsite energy of $|j'\rangle$, and $P_{j'} = \sum_n |n, j'\rangle \langle n, j'|$ is the projection operator over sublattice $|j'\rangle$. These $m-1$ SSH models are not independent. As shown in Fig. 2(a), they share the same sublattice $|m\rangle$. The coupling terms break the chiral symmetry of each SSH model:

$$\Gamma_{i'}^\dagger \sum_{j'=1'}^{(m-1)'} H_j^{\text{SSH}} \Gamma_{i'} \neq - \sum_{j'=1'}^{(m-1)'} H_j^{\text{SSH}} P_{j'}, \quad (10)$$

where $\Gamma_{i'} = \sum_n |n, i'\rangle \langle n, i'| - |n, m\rangle \langle n, m|$.

However, the subsymmetry [30] on the redefined sublattice $|j'\rangle$ is still satisfied:

$$\Gamma_{i'}^\dagger \sum_{j'=1'}^{(m-1)'} H_j^{\text{SSH}} \Gamma_{i'} P_{i'} = - \sum_{j'=1'}^{(m-1)'} H_j^{\text{SSH}} P_{j'}. \quad (11)$$

The subsymmetry protects the zero-energy chiral edge states $|L_{i'}\rangle$ occupying sublattice $|i'\rangle$:

$$\sum_{j'=1'}^{(m-1)'} H_j^{\text{SSH}} |L_{i'}\rangle = 0, \quad P_{i'} |L_{i'}\rangle = |L_{i'}\rangle. \quad (12)$$

Together with Eq. (8), we can obtain

$$H_m' |L_{i'}\rangle = V_{i'} |L_{i'}\rangle. \quad (13)$$

Similar to the case of Rice-Mele model [36,40], the effective onsite terms V_i only shift the energy of the chiral edge state without affecting its wave function and topological property. Therefore, when $|v_i| > |w_i|$, i.e., $|t_{m-1}\phi_{m-1}^{j'}/t_m\phi_1^{j'}| < 1$, the subsymmetry-protected chiral edge state with the energy of V_i can be found.

Following the same strategy, we can also illustrate the topological origin of the edge states at the right boundary $|R\rangle$. To be precise, here we perform another representation transformation to diagonalize $\mathcal{H}_{m-1}^R$ with the redefined sublattices $|j''\rangle$. The Hamiltonian can then be written as

$$\mathcal{H}_m''(k) = \begin{bmatrix} 0 & t_1''(k) & t_2''(k) & \cdots & t_{(m-1)}'' \\ t_1^\dagger(k) & V_1'' & & & 0 \\ t_2^\dagger(k) & & V_2'' & & \\ \vdots & & & \ddots & \\ t_{(m-1)}^\dagger(k) & 0 & & & V_{(m-1)}'' \end{bmatrix}, \quad (14)$$

where $t_{j''}(k) = w_{j''} + v_{j''}e^{ik}$, and $w_{j''}$ and $v_{j''}$ can be calculated by

$$\begin{aligned} w_{j''} &= \langle n, 1 | H_m | n, j'' \rangle, \\ v_{j''} &= \langle n, j'' | H_m | n+1, 1 \rangle. \end{aligned} \quad (15)$$

As shown in Fig. 2(b), $\mathcal{H}_m''(k)$ also contains $m-1$ coupled SSH models sharing sublattice $|1\rangle$. As a result, the chiral edge states occupying $|j''\rangle$ are subsymmetry-protected.

Taking the SSH3 model in Fig. 1(a) as an example, the redefined sublattices can be written as

$$\begin{aligned} |1'\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \quad V_{1'} = t_1, \\ |2'\rangle &= \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle), \quad V_{2'} = -t_1, \\ |1''\rangle &= \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle), \quad V_{1''} = t_2, \\ |2''\rangle &= \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle), \quad V_{2''} = -t_2. \end{aligned} \quad (16)$$

Therefore, chiral edge states occupying $|1'\rangle$ and $|2'\rangle$ ($|1''\rangle$ and $|2''\rangle$) can be expected at $E = \pm t_1$ ($E = \pm t_2$), which are exactly the edge states plotted in Fig. 2(c).

We note that apart from the simplest SSH m model with the nearest-neighbor hoppings only, the method introduced in this section may also bring insights into the topology of the extended SSH m model with long-range hoppings. In Appendix B, we identify the differences between the intracell and intercell long-range hoppings from the perspective of subsymmetry-protected topology.

B. Sub-SPT phases in the asymmetric SSH ladder with redefined sublattices

Next, we move on to the more complicated asymmetric SSH ladder. While non-Bloch theory can be quite difficult to analyze the topological nature of the asymmetric ladder model [29], our method still works. As shown in Fig. 3(a), we perform the representation transformation to diagonalize

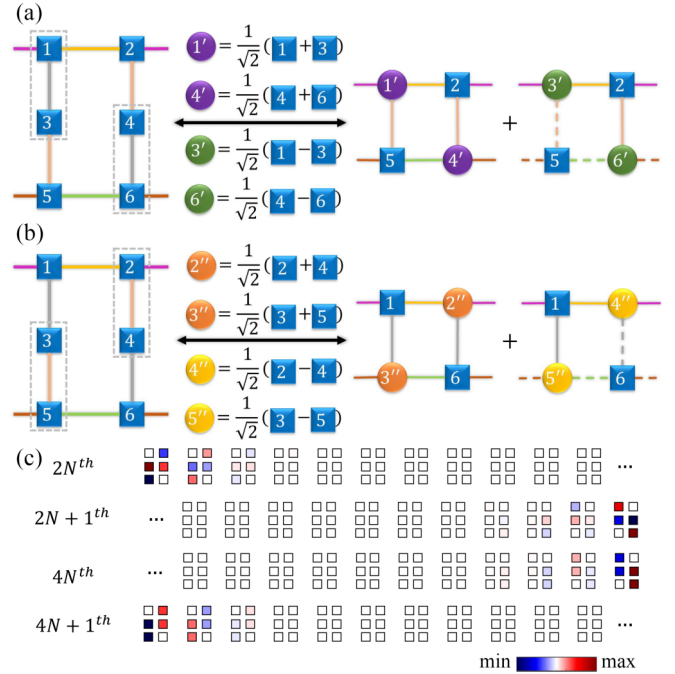


FIG. 3. (a), (b) Schematic of the asymmetric SSH ladder with redefined sublattices diagonalizing the subspace composed of (a) $\{|1\rangle, |3\rangle, |4\rangle, |6\rangle\}$ and (b) $\{|2\rangle, |3\rangle, |4\rangle, |5\rangle\}$. In the redefined sublattices, the onsite energies are distinguished by different colors, while the hopping terms are $1/\sqrt{2}$ of the ones sharing the same color in the original sublattices. The dashed lines indicate negative hoppings. (c) The spatial distribution of the edge states of the asymmetric SSH ladder.

the subspace composed of the $\{|1\rangle, |3\rangle, |4\rangle, |6\rangle\}$:

$$\begin{aligned} |1'\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle), \quad V_{1'} = t_6, \\ |4'\rangle &= \frac{1}{\sqrt{2}}(|4\rangle + |6\rangle), \quad V_{4'} = t_6, \\ |3'\rangle &= \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle), \quad V_{3'} = -t_6, \\ |6'\rangle &= \frac{1}{\sqrt{2}}(|4\rangle - |6\rangle), \quad V_{6'} = -t_6. \end{aligned} \quad (17)$$

The Hamiltonian after transformation can be divided into two coupled Rice-Mele ladders [36] sharing sublattices $|2\rangle$ and $|5\rangle$:

$$H'_{\text{ladder}} = H_{\{1',2,5,4'\}}^{\text{RM}} + H_{\{3',2,5,6'\}}^{\text{RM}}, \quad (18)$$

where $H_{\{\alpha,\beta,\gamma,\delta\}}^{\text{RM}}$ stands for the Rice-Mele ladder defined on sublattices $|\alpha\rangle, |\beta\rangle, |\gamma\rangle$, and $|\delta\rangle$. It is known that Rice-Mele ladder supports chiral edge states occupying the opposite sublattices. Based on the analysis above, the chiral edge states occupying sublattices $|1'\rangle, |4'\rangle$ and $|3'\rangle, |6'\rangle$ are subsymmetry protected. The detailed derivation can be found in Appendix C. These are exactly the $4N$ th and $(2N+1)$ th states shown in Fig. 3(c).

Similarly, we can also diagonalize the subspace composed of the complement set $\{|2\rangle, |3\rangle, |4\rangle, |5\rangle\}$:

$$\begin{aligned} |2''\rangle &= \frac{1}{\sqrt{2}}(|2\rangle + |4\rangle), \quad V_{2''} = t_5, \\ |3''\rangle &= \frac{1}{\sqrt{2}}(|3\rangle + |5\rangle), \quad V_{3''} = t_5, \\ |4''\rangle &= \frac{1}{\sqrt{2}}(|2\rangle - |4\rangle), \quad V_{4''} = -t_5, \\ |5''\rangle &= \frac{1}{\sqrt{2}}(|3\rangle - |5\rangle), \quad V_{5''} = -t_5, \end{aligned} \quad (19)$$

which also results in two coupled Rice-Mele ladders sharing sublattices $|1\rangle, |6\rangle$ shown in Fig. 3(b):

$$H''_{\text{ladder}} = H_{\{1,2'',3'',6\}}^{RM} + H_{\{1,4'',5'',6\}}^{RM}. \quad (20)$$

In this case, the chiral edge states occupying sublattice $|2''\rangle, |3''\rangle$ and $|4''\rangle, |5''\rangle$ are subsymmetry protected, which correspond to the $(4N + 1)$ th and $2N$ th states in Fig. 3(c).

Moreover, from the perspective of wave function, the edge states in both the SSH m model and the asymmetric SSH ladder also exhibit clear evidence of sub-SPT chiral property. From Figs. 2(c) and 3(c), it is obvious that these edge states always have zero amplitude over certain sublattice, which meets the criterion of sub-SPT edge states [30,31]. We will see in the next section that this chiral property helps us construct the bulk-edge correspondence.

To sum up, by redefining sublattices, all of the nonzero edge states in SSH m model and the asymmetric SSH ladder can be understood as sub-SPT chiral edge states.

III. THE BULK-EDGE CORRESPONDENCE FOR ASYMMETRIC SYSTEMS WITH NONZERO CHIRAL EDGE STATES

In Sec. II, we show the hidden sub-SPT chiral edge states in two 1D asymmetric models by redefining sublattices. However, as stated in Ref. [30], defining a topological invariant in a system with subsymmetry is not straightforward. In this section, based on isospectral reduction, we propose a proper topological invariant to establish the bulk-edge correspondence. For the asymmetric model with chiral edge states, we show that its effective Hamiltonian might have a Rice-Mele-like form. Then, the mirror symmetry of its reduced wave function can be reconstructed through normalization. Therefore, a quantized $\bar{Z}$ can be calculated by the normalized reduced wave function. Based on the chiral property of the nonzero edge states, the relationship between $\bar{Z}$ and the appearance of chiral edge states is exhibited.

A. Chiral edge states in Rice-Mele-like effective Hamiltonian

To begin with, we would like to briefly introduce the concept of isospectral reduction proposed in latent symmetry [41–43]. For a system with a Hamiltonian $\mathcal{H}(k)$, the sublattices in each unit cell can be divided into two sets: S and its complement $\bar{S}$. Here, set S is composed of all the sublattices related to the intercell hopping terms. Then the Hamiltonian

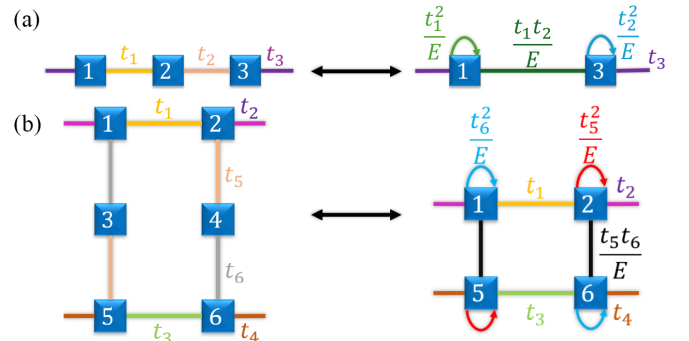


FIG. 4. (a) Schematic of the effective Hamiltonian of SSH3 model with $S = \{|1\rangle, |3\rangle\}$. (b) Schematic of the effective Hamiltonian of the asymmetric SSH ladder with $S = \{|1\rangle, |2\rangle, |5\rangle, |6\rangle\}$.

can be represented as

$$\mathcal{H}(k) = \begin{bmatrix} \mathcal{H}_{SS}(k) & \mathcal{H}_{S\bar{S}} \\ \mathcal{H}_{\bar{S}S} & \mathcal{H}_{\bar{S}\bar{S}} \end{bmatrix}. \quad (21)$$

The eigenequation of $\mathcal{H}(k)$ can be reduced into the generalized eigenequation of set S :

$$\tilde{\mathcal{H}}_S(k, E)\phi_S(k) = E\phi_S(k), \quad (22)$$

with

$$\tilde{\mathcal{H}}_S(k, E) = \mathcal{H}_{SS} - \mathcal{H}_{S\bar{S}}(\mathcal{H}_{\bar{S}\bar{S}} - EI)^{-1}\mathcal{H}_{\bar{S}S}. \quad (23)$$

If $\tilde{\mathcal{H}}_S(k, E)$ possesses mirror symmetry, $\mathcal{H}(k)$ is referred to as being latent symmetric [41].

The merit of isospectral reduction is to describe a given system with minimum sublattices while the intracell structure is neglected. Comparing with the original Hamiltonian, the effective Hamiltonian $\tilde{\mathcal{H}}_S$ might reveal the properties of the system in a more fundamental way. For the SSH m model discussed in Sec. II A, we can obtain $\tilde{\mathcal{H}}_S$ with $S = \{|1\rangle, |m\rangle\}$. In this case, $\tilde{\mathcal{H}}_S(k)$ takes the form of Rice-Mele model:

$$\tilde{\mathcal{H}}_S(k, E) = \begin{bmatrix} v_1(E) & h(k, E) \\ h^\dagger(k, E) & v_2(E) \end{bmatrix}. \quad (24)$$

In particular, for the SSH3 model shown in Fig. 1(a), we have $S = \{|1\rangle, |3\rangle\}$. As shown in Fig. 4(a), $\tilde{\mathcal{H}}_S(k, E)$ reads as [28,44]

$$\tilde{\mathcal{H}}_S(k, E) = \begin{bmatrix} \frac{t_1^2}{E} & \frac{t_1 t_2}{E} + t_3 e^{ik} \\ \frac{t_1 t_2}{E} + t_3 e^{-ik} & \frac{t_2^2}{E} \end{bmatrix}. \quad (25)$$

Meanwhile, for the more complicated asymmetric SSH ladder discussed in Sec. II B, we have $S = \{|1\rangle, |2\rangle, |5\rangle, |6\rangle\}$. As shown in Fig. 4(b), its effective Hamiltonian $\tilde{\mathcal{H}}_S(k, E)$ also exhibits a Rice-Mele form:

$$\tilde{\mathcal{H}}_S(k, E) = \begin{bmatrix} \frac{t_6^2}{E} I_2 & h(k, E) \\ h^\dagger(k, E) & \frac{t_5^2}{E} I_2 \end{bmatrix}, \quad (26)$$

with

$$h(k, E) = \begin{bmatrix} t_1 + t_2 e^{ik} & \frac{t_5 t_6}{E} \\ \frac{t_5 t_6}{E} & t_3 + t_4 e^{-ik} \end{bmatrix}. \quad (27)$$

The connection between nonzero sub-SPT chiral edge state and Rice-Mele-like effective Hamiltonian can be illustrated

as follows. Recalling the fact that chiral edge states in the latent SSH model with the effective Hamiltonian $\tilde{\mathcal{H}}_S(k, E) = \begin{bmatrix} v(E) & h(k, E) \\ h^\dagger(k, E) & v(E) \end{bmatrix}$ can be found by solving $E = v(E)$ [42], similar strategy can be utilized to investigate the chiral edge states in systems with Rice-Mele-like effective Hamiltonian. That is to say, for any system whose effective Hamiltonian takes the following form:

$$\tilde{\mathcal{H}}_S(k, E) = \begin{bmatrix} v_1(E)I & h(k, E) \\ h^\dagger(k, E) & v_2(E)I \end{bmatrix}. \quad (28)$$

Here, $h(k, E)$ can be a matrix of any dimension, $v_1(E)$, $v_2(E)$ are the effective onsite terms, and I is the identity matrix. Its generalized eigenequation reads as

$$\begin{bmatrix} v_1(E)I & h(k, E) \\ h^\dagger(k, E) & v_2(E)I \end{bmatrix} \begin{bmatrix} \phi_A(k) \\ \phi_B(k) \end{bmatrix} = E \begin{bmatrix} \phi_A(k) \\ \phi_B(k) \end{bmatrix}, \quad (29)$$

where the sublattice set S is further divided into two sets A and B . In this case, $\tilde{\mathcal{H}}_S(k, E)$ also supports two chiral edge states when $E = v_1(E)$ and $E = v_2(E)$. These chiral edge states have zero amplitude over sublattices B and A , respectively.

This conclusion can be checked by examining the two sample models. For the SSHm model with Eq. (25), we have $A = \{|1\rangle\}$ and $B = \{|3\rangle\}$. Therefore, chiral edge states can be found at $\frac{t_1}{E} = E$ or $\frac{t_2}{E} = E$, i.e., $E = \pm t_1$ or $E = \pm t_2$. These edge states have zero amplitude on sublattices $\{|3\rangle\}$ or $\{|1\rangle\}$, which coincide with the topological properties of the SSH3 model shown in Figs. 1(c) and 2(c). On the other hand, for the asymmetric SSH ladder with Eq. (28), we have $A = \{|1\rangle, |6\rangle\}$ and $B = \{|2\rangle, |5\rangle\}$. Therefore, chiral edge states with zero amplitude over sublattices $\{|2\rangle, |5\rangle\}$ ($\{|1\rangle, |6\rangle\}$) can be found with $E = \frac{t_2}{E}$ ($E = \frac{t_1}{E}$), which are exactly the edge states shown in Figs. 1(d) and 3(c).

B. The normalized Zak phase $\bar{Z}$

Having shown that the nonzero sub-SPT chiral edge states originate from a Rice-Mele-like effective Hamiltonian, we would like to establish the bulk-edge correspondence by defining a proper topological invariant. For a given system with Rice-Mele-like effective Hamiltonian $\tilde{\mathcal{H}}_S(k, E)$, we define $\tilde{\mathcal{H}}_S^0(k, E)$ as its counterpart without onsite terms:

$$\tilde{\mathcal{H}}_S^0(k, E) = \begin{bmatrix} 0 & h(k, E) \\ h^\dagger(k, E) & 0 \end{bmatrix}. \quad (30)$$

If $\tilde{\mathcal{H}}_S^0(k, E)$ further possesses mirror symmetry, i.e.,

$$M_x^{-1} \tilde{\mathcal{H}}_S^0(k, E) M_x = \tilde{\mathcal{H}}_S^0(-k, E), \quad (31)$$

where $M_x = \sigma_x \otimes I$, we can define the normalized reduced wave function over set A and B , respectively,

$$\tilde{\psi}_S(k) = \begin{bmatrix} \tilde{\phi}_A(k) \\ \tilde{\phi}_B(k) \end{bmatrix}, \quad \tilde{\phi}_\sigma(k) = \frac{1}{\sqrt{2\langle \phi_\sigma | \phi_\sigma \rangle}} \phi_\sigma(k), \quad (32)$$

with $\sigma = A, B$. It can be proved that $\tilde{\psi}_S(k)$ is mirror symmetric, i.e., the normalized reduced wave function at $\pm k$ is related by M_x up to a phase factor $\theta(k)$ (Appendix D):

$$M_x \tilde{\psi}_S(k) = e^{i\theta(k)} \tilde{\psi}_S(-k). \quad (33)$$

TABLE I. Comparison between SPT phase transition and sub-SPT phase transition.

Property	SPT	Sub-SPT
Topological invariant	Z	$\bar{Z}$
Gap closing at critical point	Yes	No
Edge states	Two-fold degenerate	Nondegenerate
Parity change mechanism	Degeneracy point in adjacent bands	Ill-defined point in single band

Therefore, the Zak phase defined by $\tilde{\psi}_S(k)$ is quantized [45]:

$$\bar{Z}_j = i \int dk \langle \tilde{\psi}_S^j(k) | \partial_k \tilde{\psi}_S^j(k) \rangle = 0 \text{ or } \pi, \quad (34)$$

where j is the band index.

For a system with mirror symmetric wave function, it is well known that Zak phase can also be derived from the parity of wave functions at the high-symmetry points [46]:

$$Z_j = \pi [P_j(\pi) - P_j(0)] \bmod \pi, \quad (35)$$

where $P_j(k_0) = +1$ (-1) stands for even (odd) parity of the j th band wave function at the high-symmetry point $k_0 = 0, \pi$. In this sense, the topological phase transition manifests itself in the change of parity. Generally speaking, $P_j(k_0)$ remains unchanged under the continuous deformation of certain system parameter t unless the wave function $\phi_j(k_0)$ becomes ill-defined. For the conventional SPT phase transition, this is achieved when the j th band intersects with another band, i.e., there exists a gap-closing critical point $t = t_c$. As illustrated in Fig. 5(a), when t passes $t = t_c$, the wave functions of adjacent bands exchange their parities. Therefore, in SPT phases, the degenerate edge states always come in pairs and the Zak phases of adjacent bands always change simultaneously.

Meanwhile, for the normalized reduced wave function $\tilde{\psi}_S^j(k)$ defined in this section, we find another type of parity change in asymmetric systems. Equation (32) shows that $\tilde{\psi}_S^j(k_0)$ is also ill-defined when

$$\psi_S^j(k_0) = \begin{bmatrix} \phi_A^j(k_0) \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ \phi_B^j(k_0) \end{bmatrix}, \quad (36)$$

i.e., the band-edge state is the eigenstate of chiral operator $\Gamma = \sigma_z \otimes I$. As shown in Ref. [36], this is exactly the critical point $t = t_c$ of chiral edge state in Rice-Mele-like systems. Therefore, as illustrated in Fig. 5(b), accompanied by the appearance of chiral edge states, $\tilde{\psi}_S^j(k_0)$ changes its parity by passing the ill-defined point and the normalized Zak phase $\bar{Z}$ of the j th band takes an abrupt change. Compared with the conventional SPT phase transition, this type of parity change does not require gap closing. Moreover, the normalized Zak phase of each band changes independently and the edge states do not have to come in pairs. These features are in good match with the topological properties of sub-SPT chiral edge states in asymmetric systems, indicating the bulk-edge correspondence can be established. The differences between the SPT and sub-SPT phase transition are summarized in Table I. It is worth noting that $\bar{Z}$ equals to Z when additional mirror

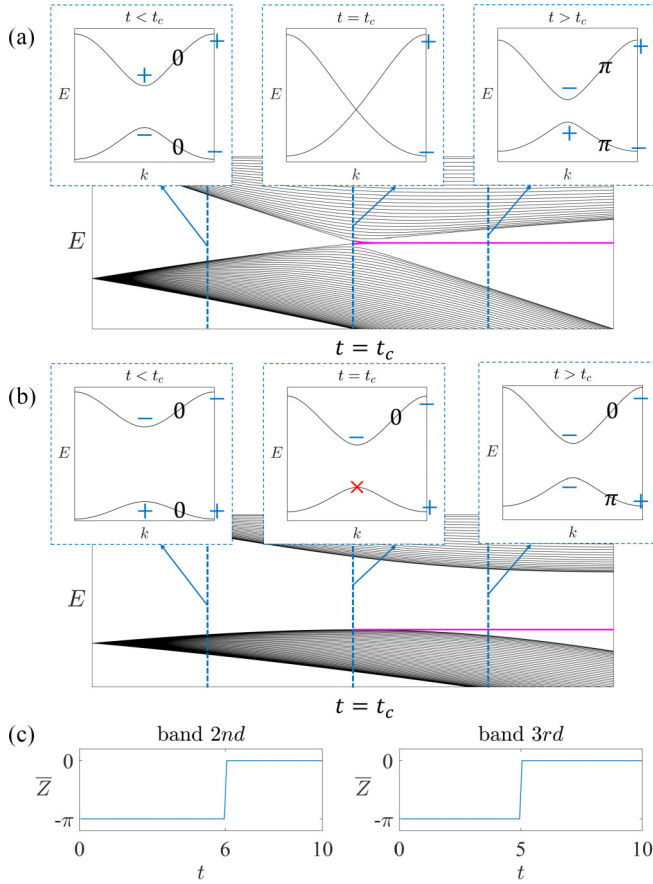


FIG. 5. Schematic of (a) SPT phase transition with gap closing and (b) sub-SPT phase transition without gap closing. Here, t stands for the system's parameter where t_c is the critical point. The corresponding topological edge states are marked by red lines. In the insets, we plot the bulk bands under different t . The parity of the wave function at the high-symmetry point $k = 0, \pi$ is marked by the sign $+$ ($-$) for even (odd) parity. The Zak phase or normalized Zak phase is marked on the corresponding bands in panels (a) or (b), respectively. The red cross in panel (b) stands for the high-symmetry point where the normalized reduced wave function is ill-defined. (c) The normalized Zak phase $\bar{Z}$ of the second and third bands for the asymmetric SSH ladder.

symmetry is present, indicating that our invariant defined by subsymmetry with redefined sublattices is still compatible with the conventional SPT phase.

To check the correction of our invariant, here we calculate $\bar{Z}$ for the asymmetric SSH ladder. As shown in Fig. 5(c), $\bar{Z}$ of the second (third) band exhibits an abrupt change at $t = 6$ ($t = 5$), which coincides with the appearance of the chiral edge states shown in Fig. 1(d). Moreover, the bulk-edge correspondence established by $\bar{Z}$ is robust against subsymmetry preserving disorder, which is shown in Appendix E.

Finally, we note that $\bar{Z}$ is a $\mathbb{Z}_2$ invariant. That is to say, $\bar{Z}$ remains unchanged when even number of edge states emerge simultaneously from the same band (e.g., the second band of the SSH3 model). In this sense, defining a $\mathbb{Z}$ invariant to completely describe the number of the sub-SPT edge states requires further investigation.

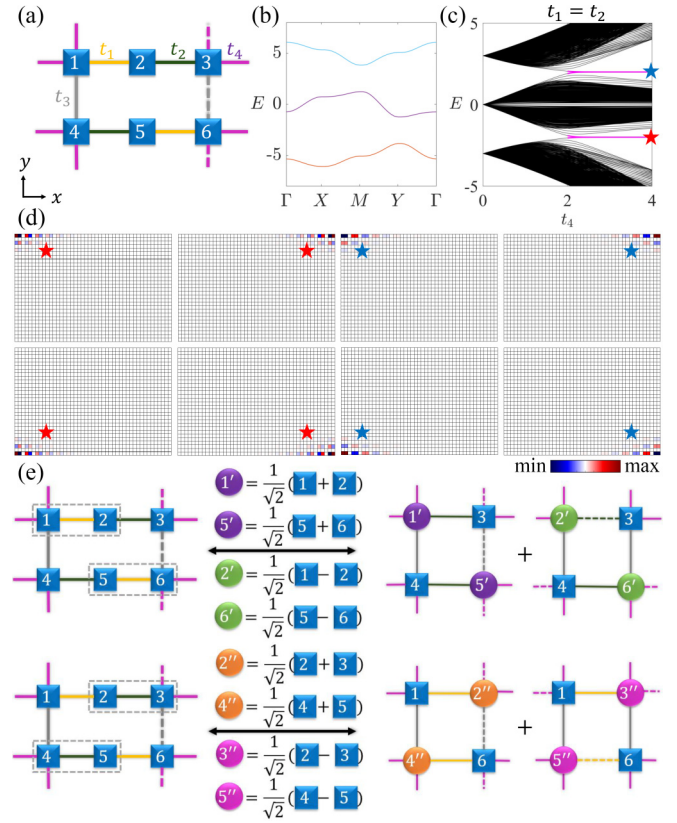


FIG. 6. (a) Schematic of the BBH3 model. The dashed lines indicate negative hoppings. (b) The bulk bands of the BBH3 model. All the bands are two-fold degenerate over the whole Brillouin zone. (c) The band structure vs t_4 under OBC with $t_1 = t_2 = 2$, and $t_3 = 1$. (d) The spatial distribution of the corner states, exhibiting typical characteristics of chiral corner states. (e) Schematics of the BBH3 model with two types of redefined sublattices.

IV. SUB-SPT CHIRAL CORNER STATES WITH REDEFINED SUBLATTICES IN 2D SYSTEMS

In this section, we would like to extend our theory into 2D systems where a new kind of higher-order topology can be found. With redefined sublattices, sub-SPT chiral corner states are discovered and topological phases can be defined. These corner states exhibit fundamental differences with the conventional quadrupole corner states. For example, they cannot be described by quantized quadrupole moment while they remain intact even when both chiral and spatial symmetry are fully broken. Here, we construct the 2D BBH3 model as a prototype to exhibit our findings. By breaking the spatial symmetry, higher-order topological phase transition without gap closing is observed. Even more, some of the chiral corner states can move into the bulk continuum, exhibiting typical characteristics of the TBICs. As to the topological invariant, we introduce $\bar{Z}$ into 2D and the edge-corner correspondence is established through the normalized edge polarization $\bar{p}^{\text{edge}}$.

A. BBH3 model

Here, we consider a 2D model whose unit-cell structure is shown in Fig. 6(a). It can be regarded as stacking the

SSH3 model and its mirror-symmetric counterpart alternatively along y direction. Mimic the BBH model [13], here we introduce a π flux in each plaquette by setting negative hoppings in y direction, which are marked by dashed lines in Fig. 6(a). In this sense, we define this model as the BBH3 model. The Hamiltonian of BBH3 model can be written as

$$\mathcal{H}_{\text{BBH3}}(k_x, k_y) = \begin{bmatrix} 0 & t_1 & h_1 & h_2 & 0 & 0 \\ t_1 & 0 & t_2 & 0 & 0 & 0 \\ h_1^* & t_2 & 0 & 0 & 0 & -h_2 \\ h_2^* & 0 & 0 & 0 & t_2 & h_1 \\ 0 & 0 & 0 & t_2 & 0 & t_1 \\ 0 & 0 & -h_2^* & h_1^* & t_1 & 0 \end{bmatrix}, \quad (37)$$

where $h_1 = t_4 e^{ik_x}$ and $h_2 = t_3 + t_4 e^{-ik_y}$. The bulk bands of the BBH3 model are shown in Fig. 6(b).

In terms of spatial symmetry, BBH3 model has projective inversion symmetry $P_I^{-1} \mathcal{H}_{\text{BBH3}}(k_x, k_y) P_I = \mathcal{H}_{\text{BBH3}}(-k_x, -k_y)$ with $P_I^2 = -1$. Along with the time-reversal symmetry T , it gives rise to two-fold band Kramers degeneracy over the whole Brillouin zone [47]. When $t_1 = t_2$, BBH3 model is also mirror symmetric in both x and y directions. Moreover, BBH3 model is neither chiral symmetric nor subsymmetric.

In the following, we would like to discuss the higher-order topological properties of the BBH3 model under three different settings. In Secs. IV B and IV C, we first consider the mirror-symmetric case, i.e., $t_1 = t_2$. In Sec. IV D, we investigate the general situation where $t_1 \neq t_2$. Additionally, in Sec. IV E, we introduce the asymmetric BBH3 model where all the spatial symmetry is broken.

B. Sub-SPT chiral corner states in mirror-symmetric BBH3 model

In this section, we first consider the corner states in mirror-symmetric BBH3 model by setting $t_1 = t_2$. The band structure versus t_4 under OBC with $t_1 = t_2 = 2$ and $t_3 = 1$ is shown in Fig. 6(c). When t_4 passes the critical point, accompanied by the closing and reopening of both gaps, four-fold degenerate corner states localized at four corners can be observed in each gap, which are marked by magenta lines in Fig. 6(c). The indexes of the eight corner states are $2N^2 - 1$, $2N^2$, $2N^2 + 1$, $2N^2 + 2$ and $4N^2 - 1$, $4N^2$, $4N^2 + 1$, $4N^2 + 2$, respectively, indicating a filling anomaly [48], which further confirms their nontrivial topology. In Fig. 6(d), we exhibit the spatial distribution of these corner states. All of them show typical chiral characteristics in a way that the amplitudes of the wave functions at certain sublattices remain zero.

To investigate the topological origins of these corner states, we redefine the sublattices in a way similar to Sec. II B. As shown in Fig. 6(e), we first take the representation transformation to diagonalize the subspace composed of $\{|1\rangle, |2\rangle, |5\rangle, |6\rangle\}$, which results in two coupled 2D Rice-Mele models [36] sharing sublattices $|3\rangle$ and $|4\rangle$:

$$H'_{\text{BBH3}} = H_{\{1',3,4,5'\}}^{\text{2DRM}} + H_{\{2',3,4,6'\}}^{\text{2DRM}} \quad (38)$$

where $H_{\{\alpha,\beta,\gamma,\delta\}}^{\text{2DRM}}$ stands for the 2D Rice-Mele model defined on sublattices $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, |\delta\rangle$. It is known that 2D Rice-Mele model supports four chiral corner states occupying one of the sublattices. Therefore, the four chiral corner states occupying

the redefined sublattices $|1'\rangle, |2'\rangle, |5'\rangle, |6'\rangle$, respectively, are subsymmetry protected.

As to the other four corner states, we perform another representation transformation to diagonalize the subspace composed of $\{|2\rangle, |3\rangle, |4\rangle, |5\rangle\}$. The Hamiltonian after transformation reads as

$$H''_{\text{BBH3}} = H_{\{1,2'',4'',6\}}^{\text{2DRM}} + H_{\{1,3'',5'',6\}}^{\text{2DRM}} \quad (39)$$

and the four chiral corner states occupying the redefined sublattices $|2''\rangle, |4''\rangle, |5''\rangle, |6''\rangle$, respectively are also subsymmetry protected. Therefore, all of the eight corner states shown in Figs. 6(c) and 6(d) are chiral corner states with redefined sublattices.

C. The edge-corner correspondence for mirror-symmetric BBH3 model

Generally speaking, the higher-order topology can be described by two methods. One is the bulk-corner correspondence where the topological property of the corner state is illustrated by the quantized bulk invariant, such as the 2D Zak phase [14] and the quadrupole moment [46]. The other is the edge-corner correspondence that relies on the topological properties of the edge bands under ribbon structure, such as the edge polarization [47,49] and the effective boundary Hamiltonian [50]. In this section, we surprisingly find that the bulk-corner correspondence fails in the mirror-symmetric BBH3 model even though a quantized bulk invariant can be calculated. However, the edge-corner correspondence can still be established by the edge polarization.

Firstly, we would like to exhibit the failure of bulk topological invariant. The presence of mirror symmetry M_x, M_y and projective inversion symmetry P_I guarantees vanishing 2D Zak phase as well as quantized quadrupole moment q_{xy} [46]. In Fig. 7(a), we calculate q_{xy} versus t_4 with $t_1 = t_2 = 2$ and $t_3 = 1$ at $1/3$ filling, where an abrupt change from 0 to 0.5 can be observed at $t_4 = 1.86$. The change of q_{xy} is associated with the vanishing of the Wannier band gap shown in Fig. 7(b) as well as the change of nested Wilson loop p^v . The relationship between q_{xy} and p^v satisfies the typical characteristics of Type-I quadrupole topological insulator [49]:

$$q_{xy} = 2p_x^{v_y} p_y^{v_x}. \quad (40)$$

However, as shown in Fig. 6(c), the critical point of the degenerate corner states is at $t_4 = 2$, which is inconsistent with q_{xy} . That is to say, although a nontrivial bulk topological phase can be defined in mirror-symmetric BBH3 model, the bulk-corner correspondence cannot be established. This mismatch between corner states and q_{xy} indicates a fundamental difference between our chiral corner states and the conventional symmetric-protected quadrupole corner states.

With the failure of bulk-corner correspondence, next we would like to establish the edge-corner correspondence. As shown in Fig. 7(c), here we consider the y -open ribbon structure of the BBH3 model. From the band structure shown in Fig. 7(d), three pairs of edge bands marked by different colors can be observed. We note that all the edge bands are two-fold degenerate due to the projective inversion symmetry. When t_4

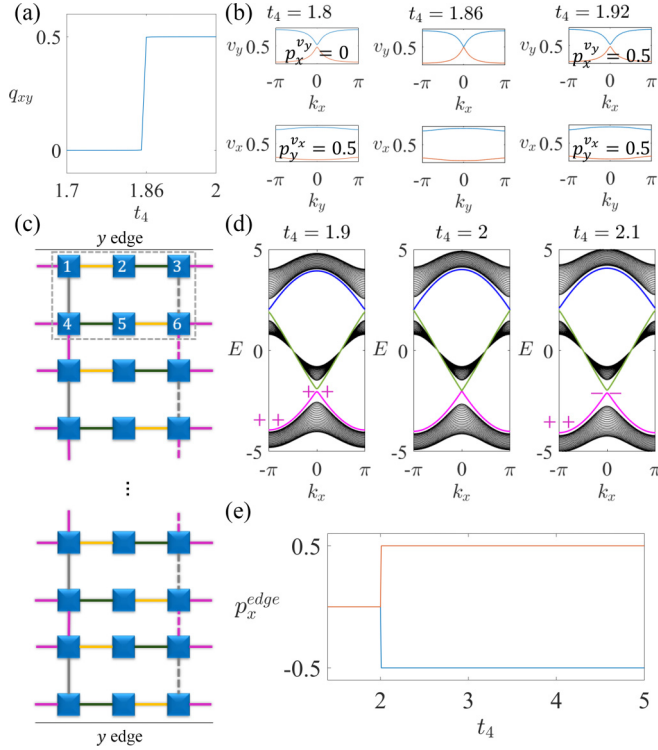


FIG. 7. The bulk-corner and edge-corner correspondence of the mirror-symmetric BBH3 model. (a) The quadrupole moment q_{xy} vs t_4 at $1/3$ filling. (b) The Wannier band $v_x(k_y)$ and $v_y(k_x)$ at $1/3$ filling, showing a topological phase transition at $t_4 = 1.86$. (c) Schematic of the y -open ribbon structure of the BBH3 model. (d) The band structure of the y -open system in panel (c). Three pairs of two-fold degenerate edge bands are plotted in different colors, while the parities of the lowest two edge bands at high-symmetry points are marked by $\pm$ signs. (e) The edge polarization of the lowest two edge bands with the y -open ribbon structure vs t_4 .

passes the critical value, the edge gaps close and reopen while the parity of the edge states at the high-symmetry point flips, indicating the topological phase transition of the edge bands.

To exhibit the topological property of the edge band, we can calculate its polarization:

$$p_x^{\text{edge}} = \frac{i}{2\pi} \int dk_x \langle \psi_e^j(k_x) | \partial_{k_x} \psi_e^j(k_x) \rangle, \quad (41)$$

where $\psi_e^j(k_x)$ stands for the wave function of the j th edge band. Figure 7(e) exhibits p_x^{edge} of the lowest two edge bands [marked by magenta lines in Fig. 7(d)] versus t_4 . When $t_4 = 2$, one can see an abrupt change from $p_x^{\text{edge}} = 0$ to $p_x^{\text{edge}} = \pm 0.5$, which is consistent with the appearance of the chiral corner states. In this way, the edge-corner correspondence can be established through the edge polarization p^{edge} .

We note that p^{edge} is quantized only in mirror-symmetric systems. Therefore, for the general BBH3 model discussed in the next section, new method is needed to illustrate its higher-order topology.

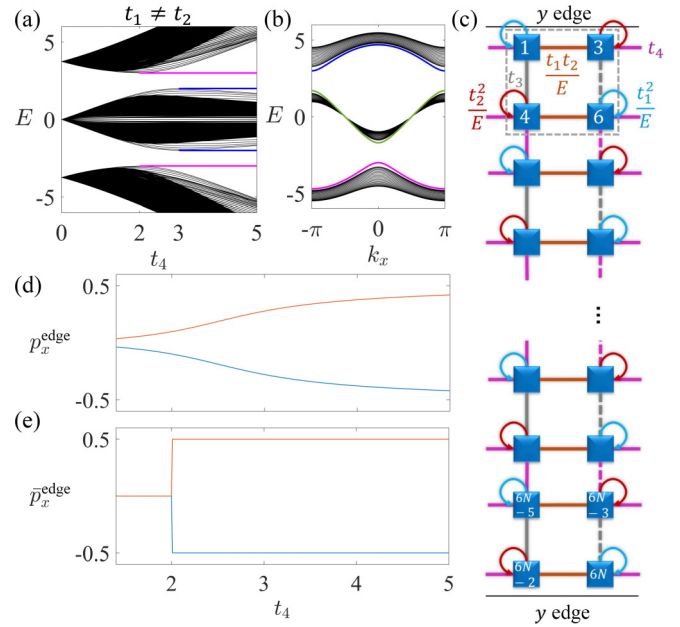


FIG. 8. (a) The band structure of the BBH3 model vs t_4 under OBC with $t_1 = 2$, $t_2 = 3$, and $t_3 = 1$. (b) The band structure of the y -open system. (c) Schematic of the effective Hamiltonian of the y -open ribbon structure of the BBH3 model. (d) The edge polarization of the lowest two edge bands vs t_4 . The edge polarization is no longer quantized due to the breaking of mirror symmetry. (e) The normalized edge polarization vs t_4 .

D. Chiral corner states in BBH3 model without mirror symmetry

By setting $t_1 \neq t_2$ in the BBH3 model, we can break the mirror symmetry in both x and y directions while preserving the spatial inversion symmetry. In this case, the band structure versus t_4 under OBC with $t_1 = 2$, $t_2 = 3$, and $t_3 = 1$ is plotted in Fig. 8(a). By breaking the mirror symmetry, the four-fold degenerate corner states in each gap split into two sets of two-fold degenerate corner states marked by different color with different energy and different critical point. Meanwhile, the emergence of the corner states is not accompanied by the closure of the band gaps, exhibiting similar characteristics of the asymmetric SSH3 model.

The split of the corner states can be understood through the redefined sublattices. Based on the Hamiltonian defined in Eq. (38) and Eq. (39), and the topological properties of 2D Rice-Mele model, we can deduce the energies and the critical points of chiral corner states as follows (Appendix F):

$$\begin{aligned} E &= \pm t_1, & t_4 &> \max\{t_2, t_3\}, \\ E &= \pm t_2, & t_4 &> \max\{t_1, t_3\}, \end{aligned} \quad (42)$$

which matches the band structure in Fig. 8(a).

In terms of the edge-corner correspondence, as shown in Fig. 8(b), three pairs of two-fold degenerate edge bands can still be observed. However, due to the breaking of mirror symmetry, the edge polarization in Fig. 8(d) is no longer quantized. Meanwhile, we notice that the y -open ribbon structure of the BBH3 model illustrated in Fig. 7(c) is effectively a 1D system, indicating that the methods introduced in Sec. III

might be utilized. Therefore, in Fig. 8(c) we illustrate the effective Hamiltonian $\tilde{\mathcal{H}}_S(k_x, E)$ of the ribbon structure with $S = \{|6n-5\rangle, |6n-3\rangle, |6n-2\rangle, |6n\rangle\}$, where $n = 1, \dots, N$ is the index of the unit cell in y direction. $\tilde{\mathcal{H}}_S(k_x, E)$ show typical properties of Rice-Mele-like structure with $A = \{|6n-5\rangle, |6n\rangle\}$ and $B = \{|6n-3\rangle, |6n-2\rangle\}$:

$$\tilde{\mathcal{H}}_S(k_x, E) = \begin{bmatrix} \frac{t_1^2}{E} I_N & h(k_x, E) \\ h^\dagger(k_x, E) & \frac{t_2^2}{E} I_N \end{bmatrix}. \quad (43)$$

Meanwhile, the effective Hamiltonian without onsite terms $\tilde{\mathcal{H}}_S^0(k_x)$ is mirror symmetric. Therefore, through the normalization of the reduced wave function in Eq. (32), similar to the normalized $\tilde{Z}$ defined in Sec. III B, we can calculate the normalized edge polarization:

$$\bar{p}_x^{\text{edge}} = \frac{i}{2\pi} \int dk_x \langle \tilde{\psi}_e^j(k_x) | \partial_{k_x} \tilde{\psi}_e^j(k_x) \rangle. \quad (44)$$

The $\bar{p}_x^{\text{edge}}$ of the lowest two edge bands are shown in Fig. 8(e). $\bar{p}_x^{\text{edge}}$ is quantized and an abrupt change from $\bar{p}_x^{\text{edge}} = 0$ to $\bar{p}_x^{\text{edge}} = \pm 0.5$ can be observed at $t_4 = 2$, which coincides with the band structure in Fig. 8(a).

It is worth noting that the normalized $\bar{p}_x^{\text{edge}}$ also has the $\mathbb{Z}_2$ feature similar to the normalized $\tilde{Z}$ defined in Sec. III B. That is to say, it cannot describe the case where even sets of corner states emerge simultaneously from the same edge band, such as the edge bands marked in green in Fig. 8(b).

E. Chiral corner states in asymmetric BBH3 model without any spatial symmetry

To take a step further, in this section we would like to break the inversion symmetry in BBH3 model. As illustrated in Fig. 9(a), we introduce the extra parameters t_5 and t_6 so that all the spatial symmetry is broken. The Hamiltonian of the asymmetric BBH3 model can be written as

$$\mathcal{H}_{\text{ASBBH3}}(k_x, k_y) = \begin{bmatrix} 0 & t_1 & h'_1 & h'_2 & 0 & 0 \\ t_1 & 0 & t_2 & 0 & 0 & 0 \\ h'_1 & t_2 & 0 & 0 & 0 & -h'_2 \\ h'_2 & 0 & 0 & 0 & t_5 & h'_1 \\ 0 & 0 & 0 & t_5 & 0 & t_6 \\ 0 & 0 & -h'_2 & h'_1 & t_6 & 0 \end{bmatrix}, \quad (45)$$

where $h'_1 = t_4 e^{ik_x}$ and $h'_2 = t_3 + t_4 e^{-ik_y}$. As a result, one can see in Fig. 9(b) that the two-fold degeneracy of the bulk band is lifted. Surprisingly, as to the band structure under OBC shown in Fig. 9(c), the eight in-gap corner states remain intact. Moreover, by turning the intercell hopping term t_6 , the corner states marked by red and magenta lines move into the bulk continuum while their wave functions in Fig. 9(e) are still localized at the corners, exhibiting the typical characteristics of the TBICs.

In Fig. 9(e), we also find that these corner states exhibit chiral characteristics. However, they can no longer be understood as the subsymmetry-protected chiral corner states with the redefined sublattices. When $t_1 \neq t_6$ and $t_2 \neq t_5$, as illustrated in Fig. 9(d), the system after representation transformation cannot be decomposed into two 2D Rice-Mele models since

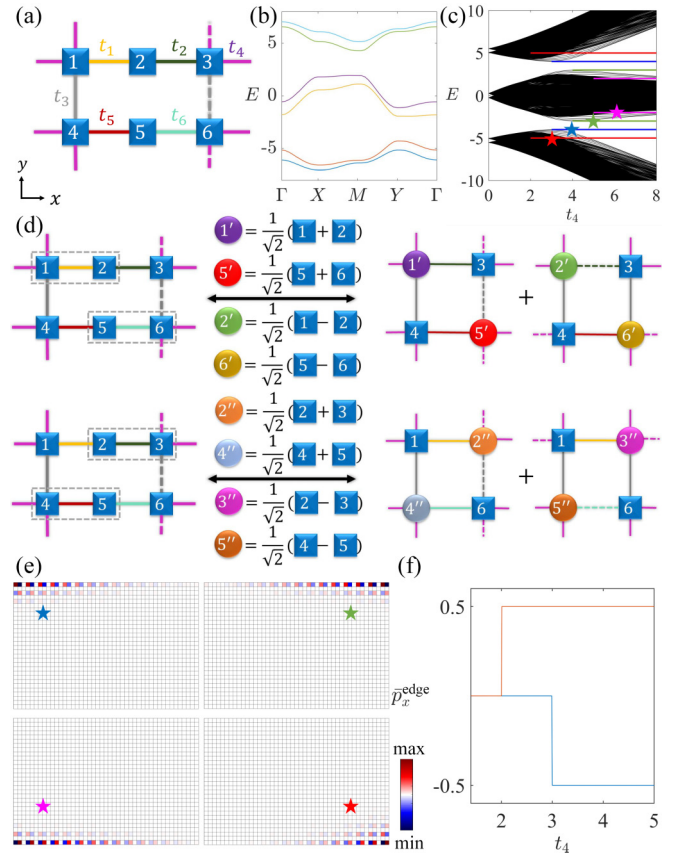


FIG. 9. (a) Schematic of the asymmetric BBH3 model without spatial inversion symmetry. (b) The bulk bands of the asymmetric BBH3 model. (c) The band structure vs t_4 under OBC with $t_1 = 2$, $t_2 = 5$, $t_3 = 1$, $t_5 = 4$, and $t_6 = 3$. (d) Schematics of the asymmetric BBH3 model with two types of redefined sublattices. The system can no longer be decoupled into two 2D Rice-Mele models since the redefined sublattices on the opposite positions share different onsite energies. (e) Spatial distributions of the corner states marked by stars in panel (c). (f) The normalized edge polarization of the two lowest edge bands vs t_4 .

the redefined sublattices on the opposite positions, for example $|1'\rangle$ and $|5'\rangle$, $|2'\rangle$ and $|6'\rangle$, etc., share different effective onsite energies. Instead, these corner states can be explained as the sub-SPT generalized chiral states [11,30] as they only occupy one of the redefined sublattices. Based on the analysis in Appendix G, we find that the energies and the critical points of these corner states can be written as

$$\begin{aligned} E &= \pm t_1, & t_4 &> \max\{t_2, t_3\}, \\ E &= \pm t_2, & t_4 &> \max\{t_1, t_3\}, \\ E &= \pm t_6, & t_4 &> \max\{t_5, t_3\}, \\ E &= \pm t_5, & t_4 &> \max\{t_6, t_3\}, \end{aligned} \quad (46)$$

which is consistent with the band structure in Fig. 9(c). Moreover, as exhibit in Appendix H, these TBICs are robust against subsymmetry preserving disorder, indicating that they are indeed topologically protected. As to the edge-corner correspondence, in Fig. 9(f), we plot the normalized edge polarization $\bar{p}_x^{\text{edge}}$ of the lowest two edge bands. Two abrupt

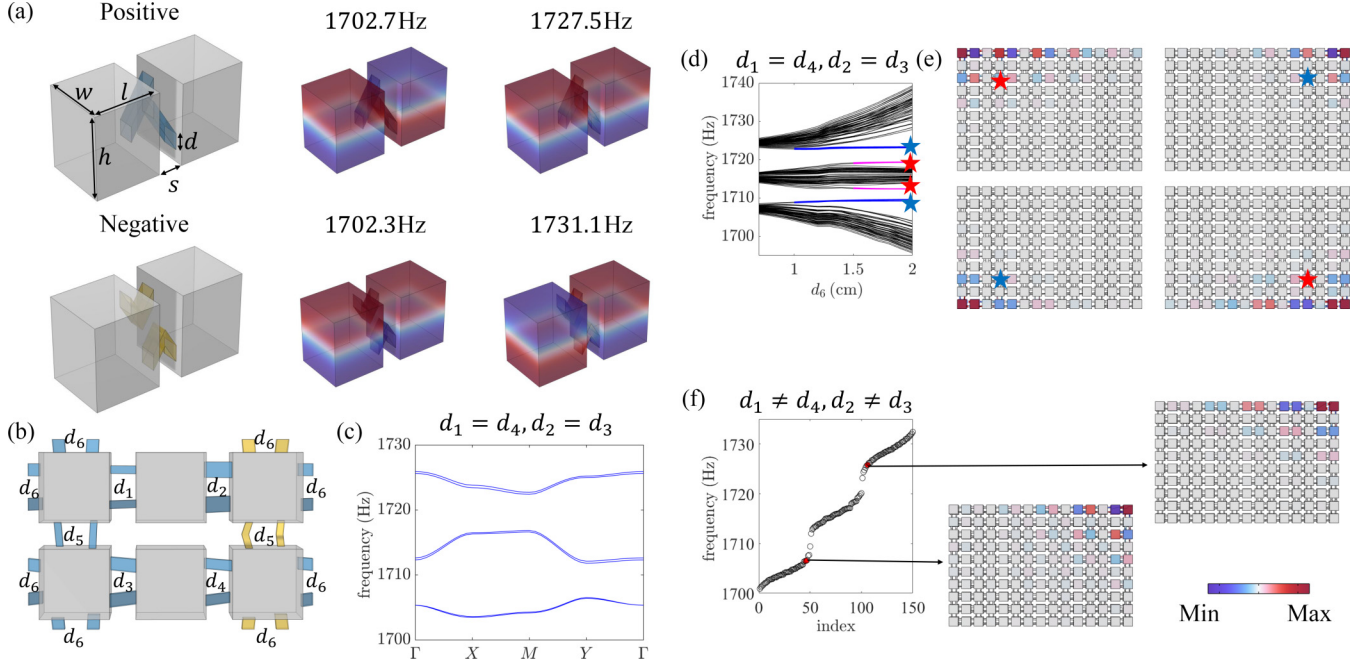


FIG. 10. (a) Acoustic design for the positive and negative hoppings. The size of the air cavity is set as $w = l = 8$ cm, $h = 10$ cm, $s = 3$ cm, and $d = 2$ cm. (b) Schematic of the acoustic BBH3 model. (c) The bulk bands of the acoustic BBH3 model with $d_1 = d_4 = 1$ cm, $d_2 = d_3 = 1.5$ cm, $d_5 = 0.5$ cm, and $d_6 = 1$ cm. (d) The band structure vs d_6 for a finite-sized (5×5 unit cells) sample under OBC with $d_1 = d_4 = 1$ cm, $d_2 = d_3 = 1.5$ cm, and $d_5 = 0.9$ cm. The corner states are marked by red and blue lines. (e) The spatial distribution of the corner states in acoustics BBH3 model. The other four corner states can be obtained by inversion symmetry. (f) The band structure of the asymmetric BBH3 model under OBC with $d_1 = 0.9$ cm, $d_2 = 1.8$ cm, $d_3 = 1.5$ cm, $d_4 = 1.2$ cm, $d_5 = 0.8$ cm, and $d_6 = 1.4$ cm. The spatial distribution of the chiral corner states marked by red points is plotted on the right side.

changes can be observed at $t_4 = t_1$ and $t_4 = t_6$ respectively, which exactly correspond to the critical points of the two lowest corner states.

V. TOPOLOGICAL CHIRAL CORNER STATES IN 2D ASYMMETRIC ACOUSTIC CRYSTALS

Mimicking the experimental realization of the acoustic BBH model [17,51], in this section we propose an experimental design to realize the asymmetric BBH3 model and the corresponding chiral corner states. Here, we utilize the air cavity working at the P_z -dipole modes to emulate the sublattices of the tight-binding model. As shown in Fig. 10(a), the size of the air cavity is set to $w = l = 8$ cm, $h = 10$ cm, which presents a P_z -dipole mode with eigenfrequency $\omega_0 = 1716$ Hz. By connecting two identical air cavities with tubes, the effective Hamiltonian of the composed system can be written as [17]

$$H = \begin{bmatrix} \omega_0 & \kappa \\ \kappa & \omega_0 \end{bmatrix}, \quad (47)$$

where κ stands for the coupling strength between the P_z modes. For positive (negative) coupling, the $[1, 1]^T$ ($[1, -1]^T$) eigenmode has a higher frequency. As shown in Fig. 10(a), the positive and negative hoppings can be obtained through different types of connecting tubes between the air cavities, i.e., the straight tubes marked in blue and the twisted tubes marked in yellow. Meanwhile, the coupling strength can be tuned by the width of the connecting tube d . Based on these

results, we design the acoustic BBH3 model, whose unit cell structure containing six air cavities is shown in Fig. 10(b).

We first consider the special case with $d_1 = d_4$ and $d_2 = d_3$, which corresponds to the BBH3 model discussed in Sec. IV D. The six dipole bulk bands are plotted in Fig. 10(c). Similar to the tight-binding model, the bands are almost two-fold degenerate. In order to demonstrate the corner states, in Fig. 10(d), we plot the band structure versus d_4 for a finite-sized (5×5 unit cells) sample under OBC. As expected, two pairs of two-fold degenerate in-gap corner states marked by different colors can be observed with the critical points $d_6 = d_1$ and $d_6 = d_2$ respectively, which shares similar property with the band structure of the BBH3 model in Fig. 8(a). Figure 10(e) shows the spatial distributions of the acoustic corner states. They also exhibit typical chiral characteristics in a way that only some of the air cavities in each unit cell are occupied. At last, we consider the general case with $d_1 \neq d_4$ and $d_2 \neq d_3$, which corresponds to the asymmetric BBH3 model discussed in Sec. IV E. In Fig. 10(f), we find that two eigenstates marked by red circles are still chiral corner states even though they move into the bulk continuum. Therefore, all the topological chiral corner states obtained in Sec. IV can be realized in the acoustic crystals proposed in this section.

VI. CONCLUSION

In summary, utilizing redefined sublattices, we extend the sub-SPT phases to three different multigap asymmetric models, i.e., the SSH m model, the asymmetric SSH ladder, and the

2D BBH3 model. With redefined sublattices, each system can be divided into several coupled subspaces with subsymmetry. Due to the effective onsite energy of the redefined sublattices, nonzero chiral boundary states can be identified. Moreover, these nonzero boundary states exhibit some exotic properties. For example, the corner states in the BBH3 model can move into the bulk continuous spectrum with proper parameters, showing the characteristics of TBICs. In terms of topological invariant, we define a universal topological invariant, namely the normalized Zak phase $\bar{Z}$, through the Rice-Mele-like effective Hamiltonian. Based on the chiral property of the boundary states, the bulk-edge correspondence can be properly established. Finally, we propose an experimental realization of the acoustic BBH3 model and the nonzero chiral corner states are numerically observed. Our work discovers the sub-SPT chiral characteristics of the boundary states in various asymmetric systems. By redefining sublattices, we extend the concept of sub-SPT phases to multi-gap systems with nonzero edge states. Moreover, we find that models with entirely different structures might share the same topological origins. In this sense, we believe our findings can advance the understanding of the topological properties of the asymmetric systems.

ACKNOWLEDGMENT

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: NON-BLOCH THEORY FOR SSH m MODEL

In this Appendix, we would like to analytically derive the wave functions of the possible edge states in SSH m model. For the SSH m model with N unit cells under OBC, its eigenstates take the form $|\Psi\rangle = 1/A \sum_{n=1}^N \sum_{j=1}^m \psi_{n,j} |n, j\rangle$ with $\psi_{n,j} \equiv \psi_{n,j}(z_1, z_2) = c_1 \phi_j^{(1)} z_1^n + c_2 \phi_j^{(2)} z_2^n$, where A is the normalization coefficient, $z_\alpha \in C$ is the generalized complex momentum with $\alpha = 1, 2$. $|\Psi\rangle$ should satisfy the bulk condition, i.e., $\{z_\alpha, \phi_j^{(\alpha)}\}$ are the solutions of the following bulk equation with a given E :

$$\mathcal{H}_m(z) \begin{bmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_m \end{bmatrix} = E \begin{bmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_m \end{bmatrix}, \quad (\text{A1})$$

where $\mathcal{H}_m(z)$ can be obtained by applying the substitution $e^{ik} \rightarrow z$ to the momentum-space Hamiltonian $\mathcal{H}_m(k)$ of the

SSH m model:

$$\mathcal{H}_m(z) = \begin{bmatrix} 0 & t_1 & & & & & t_m/z \\ t_1 & 0 & t_2 & & & & \\ & t_2 & 0 & t_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & t_{m-3} & 0 & t_{m-2} & \\ & & & & t_{m-2} & 0 & t_{m-1} \\ t_m z & & & & & t_{m-1} & 0 \end{bmatrix}. \quad (\text{A2})$$

From Eq. (A1), we have $z_1 z_2 = 1$. Without loss of generality, let $|z_1| \leq |z_2|$. On the other hand, $|\Psi\rangle$ should also satisfy the boundary condition:

$$M_b [c_1, c_2]^T = 0, \quad (\text{A3})$$

with

$$M_b = \begin{bmatrix} P(z_1) & P(z_2) \\ z_1^{N+1} Q(z_1) & z_2^{N+1} Q(z_2) \end{bmatrix}, \quad (\text{A4})$$

where $P(z_\alpha) = -t_m \phi_m^{(\alpha)}$ and $Q(z_\alpha) = -t_m \phi_1^{(\alpha)}$. When $\det[M_b] = 0$, the nontrivial solutions of Eq. (A3) are exactly the eigenstates of the SSH m model under OBC.

For $|\Psi\rangle$ representing the localized edge states, we have $|z_1| < 1 < |z_2|$. Together with $z_1^{N+1} \rightarrow 0$, one can obtain

$$\det[M_b] = -z_2^{N+1} P(z_1) Q(z_2) = 0. \quad (\text{A5})$$

Two solutions can be obtained from Eq. (A5), i.e., $P(z_1) = 0$ and $Q(z_2) = 0$.

For $P(z_1) = 0$, it follows that $\phi_m^{(1)} = 0$. Substituting into Eq. (A1) yields

$$\begin{aligned} \mathcal{H}_{m-1}^L [\phi_1^{(1)}, \dots, \phi_{m-1}^{(1)}]^T &= E [\phi_1^{(1)}, \dots, \phi_{m-1}^{(1)}]^T, \\ t_m \phi_1^{(1)} z_1 + t_{m-1} \phi_{m-1}^{(1)} &= 0, \end{aligned} \quad (\text{A6})$$

where

$$\mathcal{H}_{m-1}^L = \begin{bmatrix} 0 & t_1 & & & 0 \\ t_1 & 0 & t_2 & & \\ & \ddots & \ddots & \ddots & \\ & & t_{m-3} & 0 & t_{m-2} \\ 0 & & & t_{m-2} & 0 \end{bmatrix} \quad (\text{A7})$$

is the Hamiltonian of the subspace composed of the sublattices 1 to $m-1$. Moreover, Eq. (A6) indicates that $[\phi_1^{(1)}, \dots, \phi_{m-1}^{(1)}]^T$ is the eigenstate of $\mathcal{H}_{m-1}^L$. When $|z_1| = |t_{m-1} \phi_{m-1}^{(1)} / t_m \phi_1^{(1)}| < 1$, $|\Psi\rangle$ stands for the edge state localized at the left boundary of the system:

$$|L\rangle = A_L \sum_{n=1}^N z_1^{n-1} \sum_{j=1}^m \phi_j^{(1)} |n, j\rangle. \quad (\text{A8})$$

Since $\mathcal{H}_{\text{sub-1}}$ is $(m-1)$ -dimensional matrix, there are $m-1$ different eigenstates. Therefore, SSH m model has at most $m-1$ left-boundary edge states.

Similarly, for $Q(z_2) = 0$, it follows that $\phi_1^{(2)} = 0$, which yields

$$\begin{aligned} \mathcal{H}_{m-1}^R [\phi_2^{(2)}, \dots, \phi_m^{(2)}]^T &= E [\phi_2^{(2)}, \dots, \phi_m^{(2)}]^T, \\ t_m \phi_m^{(2)} + t_1 \phi_2^{(2)} z_2 &= 0, \end{aligned} \quad (\text{A9})$$

where $\mathcal{H}_{m-1}^R$ is the Hamiltonian of the subspace composed of the sublattices 2 to m . When $|z_2| = |t_m \phi_m^{(2)} / t_1 \phi_2^{(2)}| > 1$, $|\Psi\rangle$ stands for the edge state localized at the right boundary of the system:

$$|R\rangle = A_R \sum_{n=1}^N z_2^{n-1} \sum_{j=1}^m \phi_j^{(2)} |n, j\rangle. \quad (\text{A10})$$

Since $\mathcal{H}_{m-1}^R$ is also $(m-1)$ -dimensional, at most $m-1$ right-boundary edge states can be obtained. In summary, all the possible edge states of the SSH m model are given by Eqs. (A8) and (A10). It is worth noting that similar methods can also be utilized to analyze the topological chiral edge states in the disordered SSH model [52].

APPENDIX B: THE BREAKING OF SUBSYMMETRY BY INTERCELL LONG-RANGE HOPPINGS IN THE EXTENDED SSH3 MODEL

In recent years, the topological property of the systems with long-range hoppings has attracted much attention [19,53–56], including the extended SSH3 model [25,26,57]. In previous study of the extended SSH3 model, the intracell and intercell long-range hoppings are equally treated, while their differences are not highlighted. From the perspective of subsymmetry-protected topology with redefined sublattices, in this Appendix we would like to distinguish these two kinds of long-range hoppings from the perspective of sub-SPT phases.

As illustrated in Fig. 11(a), here we consider the topological properties of the extended SSH3 model with only the intracell long-range hopping t_3 and the one with only the intercell long-range hopping t_4 , respectively. The band structure under OBC is shown in Fig. 11(b), where two pairs of two-fold degenerate edge states can be observed for both models.

However, we find that the topological properties of the edge states in these two cases are quite different. Since both of the extended SSH3 models considered here are mirror symmetric, we can analyze their topological properties from filling anomaly and the quantized Zak phases [48]. In Fig. 11(b), we find that the indexes of the edge states with t_3 are N , $N+1$, $2N$, and $2N+1$, which satisfies the filling anomaly. In contrary, the indexes of the edge states with t_4 are $N-1$, N , $2N+1$, and $2N+2$ and the filling anomaly is violated. In Fig. 11(c), we also calculate the Zak phase for both cases, which further confirms that the edge states with t_3 are topologically nontrivial while the ones with t_4 are trivial.

The difference between intracell and intercell long-range hopping can be understood from the perspective of subsymmetry. In Fig. 11(d), we plot the schematics of both models with the redefined sublattices in Sec. II A. It clearly shows that the intracell long-range hopping t_3 only affects the value of effective hoppings while the subsymmetry remains intact. The intercell long-range hopping t_4 , on the other hand, introduces long-range hoppings in the redefined sublattices and therefore breaks the subsymmetry. The difference between t_3 and t_4 also manifests itself in the spatial distribution of the edge states. As shown in Fig. 11(e), the edge states of the extended SSH3 model with t_3 exhibit clear evidence of topological chiral edge states as the amplitude of the wave function on certain

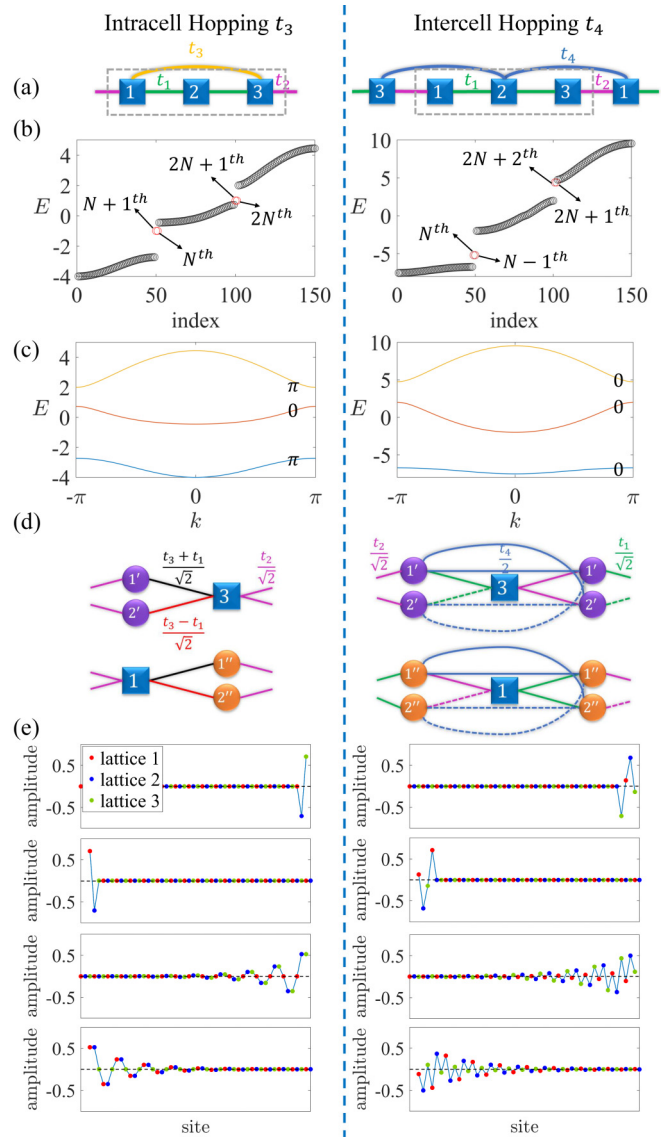


FIG. 11. (a) Schematics of the extended SSH3 models. In the left column, only the intracell hopping t_3 is considered. In the right column, only the intercell hopping t_4 is considered. In this panel, the parameters are set to $t_1 = 1$, $t_2 = 3$, and $t_3 = 1$ for the left column and $t_1 = 1$, $t_2 = 2$, and $t_4 = 5$ for the right column. (b) The energy spectrum of the extended SSH3 models. The edge states are colored in red, while their indexes are marked in the figure. (c) The bulk bands of the extended SSH3 models. The Zak phases are marked on the corresponding bands. (d) The schematics of the models with redefined sublattices. The dashed lines indicate negative hoppings. (e) The spatial distribution of the edge states.

sublattice is pinned to zero. For the edge states with t_4 , the chiral characteristics cannot be observed.

It is worth noting that same conclusion can be drawn for any extended SSH m model: As long as the long-range hoppings are kept intracell, the model can always be transformed into the form shown in Figs. 2(a) and 2(b) while chiral edge states with zero amplitude at sublattice 1 or m can be observed.

APPENDIX C: DETAILED DERIVATION OF THE SUB-SPT CHIRAL EDGE STATES IN ASYMMETRIC SSH LADDER

In this Appendix, we would like to give a detailed form of Eq. (18) and show that the edge states are subsymmetry protected. H'_{ladder} can be written as

$$H'_{\text{ladder}} = H_{\text{ladder}}^{(1)} + H_{\text{ladder}}^{(2)} + t_6 P_{1',4'} - t_6 P_{3',6'}, \quad (\text{C1})$$

where

$$\begin{aligned} H_{\text{ladder}}^{(1)} &= \sum_n \frac{t_1}{\sqrt{2}} c_{n,2} c_{n,1'}^\dagger + \frac{t_2}{\sqrt{2}} c_{n,2} c_{n+1,1'}^\dagger + \frac{t_5}{\sqrt{2}} c_{n,5} c_{n,1'}^\dagger \\ &\quad + \frac{t_5}{\sqrt{2}} c_{n,2} c_{n,4'}^\dagger + \frac{t_3}{\sqrt{2}} c_{n,5} c_{n,4'}^\dagger + \frac{t_4}{\sqrt{2}} c_{n+1,5} c_{n,4'}^\dagger + \text{H.c.}, \\ H_{\text{ladder}}^{(2)} &= \sum_n \frac{t_1}{\sqrt{2}} c_{n,2} c_{n,3'}^\dagger + \frac{t_2}{\sqrt{2}} c_{n,2} c_{n+1,3'}^\dagger - \frac{t_5}{\sqrt{2}} c_{n,5} c_{n,3'}^\dagger \\ &\quad + \frac{t_5}{\sqrt{2}} c_{n,2} c_{n,6'}^\dagger - \frac{t_3}{\sqrt{2}} c_{n,5} c_{n,6'}^\dagger - \frac{t_4}{\sqrt{2}} c_{n+1,5} c_{n,6'}^\dagger + \text{H.c.}, \end{aligned} \quad (\text{C2})$$

and $P_{\alpha,\beta}$ stands for the projection operator over sublattice $|\alpha\rangle$, $|\beta\rangle$. We note that the relationship between Eqs. (18) and (C1) reads as

$$\begin{aligned} H_{\{1',2,5,4'\}}^{\text{RM}} &= H_{\text{ladder}}^{(1)} + t_6 P_{1',4'}, \\ H_{\{3',2,5,6'\}}^{\text{RM}} &= H_{\text{ladder}}^{(2)} - t_6 P_{3',6'}. \end{aligned} \quad (\text{C3})$$

For $H_{\text{ladder}}^{(1)} + H_{\text{ladder}}^{(2)}$, the subsymmetry over $|1'\rangle$, $|4'\rangle$ and $|3'\rangle$, $|6'\rangle$ are satisfied:

$$\begin{aligned} \Gamma_{(1)}^\dagger [H_{\text{ladder}}^{(1)} + H_{\text{ladder}}^{(2)}] \Gamma_{(1)} P_{1',4'} &= -[H_{\text{ladder}}^{(1)} + H_{\text{ladder}}^{(2)}] P_{1',4'}, \\ \Gamma_{(2)}^\dagger [H_{\text{ladder}}^{(1)} + H_{\text{ladder}}^{(2)}] \Gamma_{(2)} P_{3',6'} &= -[H_{\text{ladder}}^{(1)} + H_{\text{ladder}}^{(2)}] P_{3',6'}, \end{aligned} \quad (\text{C4})$$

where $\Gamma_{(1)} = P_{1',4'} - P_{2,5}$ and $\Gamma_{(2)} = P_{3',6'} - P_{2,5}$. Therefore, the zero-energy chiral edge states occupying sublattices $|1'\rangle$, $|4'\rangle$ and $|3'\rangle$, $|6'\rangle$ are subsymmetry protected:

$$\begin{aligned} \sum_{i=1}^2 H_{\text{ladder}}^{(i)} |R_{1',4'}\rangle &= \sum_{i=1}^2 H_{\text{ladder}}^{(i)} |R_{3',6'}\rangle = 0, \\ P_{1',4'} |R_{1',4'}\rangle &= |R_{1',4'}\rangle, \quad P_{3',6'} |R_{3',6'}\rangle = |R_{3',6'}\rangle. \end{aligned} \quad (\text{C5})$$

Together with Eq. (C1), we have

$$H'_{\text{ladder}} |R_{1',4'}\rangle = t_6 |R_{1',4'}\rangle, \quad H'_{\text{ladder}} |R_{3',6'}\rangle = -t_6 |R_{3',6'}\rangle, \quad (\text{C6})$$

i.e., chiral edge states can be found in asymmetric SSH ladder at $E = \pm t_6$.

APPENDIX D: THE SYMMETRY OF THE NORMALIZED REDUCED WAVE FUNCTION

In this Appendix, we would like to show that the reduced wave function of the effective Hamiltonian satisfying Eqs. (28) and (31) can reconstruct its symmetry through normalization. The generalized eigenequation in Eq. (29) can be

decoupled into two parts:

$$\begin{aligned} h(k, E) h^\dagger(k, E) \phi_A(k) &= M(E) \phi_A(k), \\ h^\dagger(k, E) h(k, E) \phi_B(k) &= M(E) \phi_B(k), \end{aligned} \quad (\text{D1})$$

with $M(E) = [E - v_1(E)][E - v_2(E)]$. Meanwhile, from Eq. (31) one can get

$$h(k, E) = h^\dagger(-k, E). \quad (\text{D2})$$

Since all the systems under consideration here have time-reversal symmetry, we have $\phi_A(k) = \phi_A^*(-k)$, $\phi_B(k) = \phi_B^*(-k)$. Therefore, the second equation in Eq. (D1) can be rewritten as

$$h(-k, E) h^\dagger(-k, E) \phi_B(k) = M(E) \phi_B(k), \quad (\text{D3})$$

which indicates that the difference between $\phi_B(k)$ and $\phi_A(-k)$ is up to a constant

$$\phi_B(k) = |C_k| e^{i\theta(k)} \phi_A(-k). \quad (\text{D4})$$

Following the normalization performed in Eq. (32), we have $\tilde{\phi}_B(k) = e^{i\theta(k)} \tilde{\phi}_A(-k)$, $\tilde{\phi}_A(-k) = \tilde{\phi}_A^*(k)$, and $\tilde{\phi}_B(-k) = \tilde{\phi}_B^*(k)$. It is easy to see that the normalized reduced wave function $\tilde{\psi}_S(k)$ is mirror symmetric:

$$\begin{aligned} M_x \tilde{\psi}_S(k) &= \begin{bmatrix} I & I \\ I & I \end{bmatrix} \begin{bmatrix} \tilde{\phi}_A(k) \\ e^{i\theta(k)} \tilde{\phi}_A(-k) \end{bmatrix} \\ &= e^{i\theta(k)} \begin{bmatrix} \tilde{\phi}_A(-k) \\ e^{-i\theta(k)} \tilde{\phi}_A(k) \end{bmatrix} = e^{i\theta(k)} \tilde{\psi}_S(-k). \end{aligned} \quad (\text{D5})$$

Furthermore, as long as $\tilde{\psi}_S(k)$ is well defined, it has a definite parity at the high-symmetry point $k_0 = 0, \pi$:

$$M_x \tilde{\psi}_S(k_0) = \pm \tilde{\psi}_S(k_0). \quad (\text{D6})$$

APPENDIX E: ROBUSTNESS ANALYSIS OF THE CHIRAL EDGE STATES AND NORMALIZED ZAK PHASE $\tilde{Z}$ AGAINST DISORDER

In this Appendix, we would like to investigate the robustness of the chiral edge states of the asymmetric SSH ladder introduced in Sec. II B as well as the behavior of the normalized Zak phase $\tilde{Z}$ against random disorder. Here, we introduce the following disorder Hamiltonian $\mathcal{H}''$ to the system:

$$\mathcal{H}'' = \mathcal{H}''_{\text{horizontal}} + \mathcal{H}''_{\text{vertical}}, \quad (\text{E1})$$

where $\mathcal{H}''_{\text{horizontal}} = \sum_n \delta t_1^n c_{n,1}^\dagger c_{n,2} + \delta t_2^n c_{n+1,1}^\dagger c_{n,2} + \delta t_3^n c_{n,3}^\dagger c_{n,4} + \delta t_4^n c_{n+1,3}^\dagger c_{n,4} + \text{H.c.}$ stands for disorder in the horizontal hopping terms and $\mathcal{H}''_{\text{vertical}} = \sum_n \delta t_6^n c_{n,1}^\dagger c_{n,3} + \delta t_5^n c_{n,3}^\dagger c_{n,5} + \delta t_5^n c_{n,2}^\dagger c_{n,4} + \delta t_6^n c_{n,4}^\dagger c_{n,6} + \text{H.c.}$ stands for disorder in the vertical hopping terms. The disorder terms $\{\delta t_i^n\}$ are random numbers that follow a uniform distribution in $[-w, w]$, where w is the disorder strength.

To calculate the topological invariant in disordered system, we also introduce the real-space form of the j th band normalized Zak phase $\tilde{Z}_j$, which can be expressed in a Bott index style:

$$\tilde{Z}_j = \text{Im}\{\text{In}[\det(\tilde{U}_{S,j}^\dagger X \tilde{U}_{S,j})]\}, \quad (\text{E2})$$

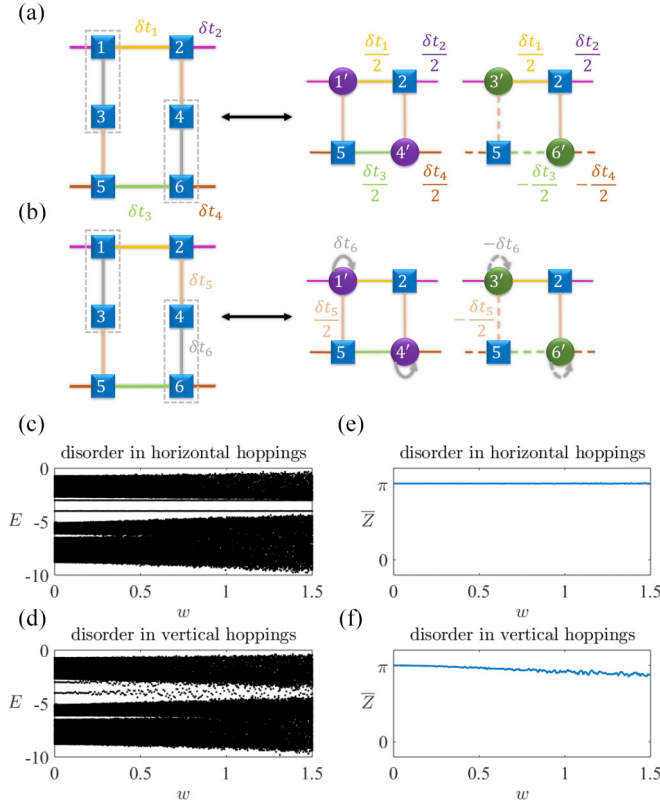


FIG. 12. The robustness of edge states and the behavior of the normalized Zak phase against random disorder. Schematic of the effects of the disorder terms with the redefined sublattices with the disorder (a) in the horizontal hopping terms or (b) in the vertical hopping terms. Here, δt_i stands for the disorder of the t_i hopping terms. The robustness of edge states in the second gaps against two kinds of random disorder. Parameters are set to $t_1 = 5, t_2 = 1, t_3 = 2, t_4 = 4, t_5 = 3$, and $t_6 = 4$. (e), (f) The corresponding real-space normalized Zak phase $\bar{Z}$ of the third band vs the disorder strength w .

where $\tilde{U}_{S,j} = [\dots, |\tilde{\psi}_S\rangle, \dots]$ denotes a set of the normalized reduced eigenstate over set S of the j th band. X is the position operator:

$$X = \sum_{l=1}^N e^{i\frac{2\pi}{N}l} \sum_{i \in S} |l, i\rangle \langle l, i|, \quad (\text{E3})$$

where l is the index of the unit cell, i is the index of the sublattice over set S , and N is the total number of the unit cells in the system. In the following calculation, we take $N = 50$.

In Fig. 12, we exhibit the spectrum versus w under OBC and the corresponding real-space $\bar{Z}$ with two different kinds of disorder. In Fig. 12(a), disorder only exists in the horizontal hopping terms $\mathcal{H}'' = \mathcal{H}''_{\text{horizontal}}$. In Fig. 12(b), we have $\mathcal{H}'' = \mathcal{H}''_{\text{vertical}}$. Based on the analysis in Sec. II B, $\mathcal{H}''_{\text{horizontal}}$ stands for the disorder of the effective hopping terms, which preserves the subsymmetry of the redefined sublattices. On the other hand, $\mathcal{H}''_{\text{vertical}}$ introduces disorder to the effective onsite energy terms, which breaks the subsymmetry. As a result, the edge states in Fig. 12(c) are pinned to $E = -t_5, -t_6$ while the edge states in Fig. 12(d) deviate considerably from their original values as the disorder strength grows. As to the topological invariant, one can see that $\bar{Z}$ sticks to π in

Fig. 12(e) and $\bar{Z}$ fluctuates dramatically in Fig. 12(f). These results clearly show that the chiral edge states are robust against the symmetry-preserving disorder while our topological invariant can establish the bulk-edge correspondence even in the presence of strong disorder (i.e., $w > 1$).

APPENDIX F: THE DETAILED TOPOLOGICAL PROPERTIES OF THE CHIRAL CORNER STATES IN BBH3 MODEL

In this Appendix, we would like to deduce the energies and the critical points of the chiral corner states in BBH3 model. The two 2D Rice-Mele models defined in Eq. (38) can be written as

$$\mathcal{H}_{\{1',3,4,5'\}}^{\text{2DRM}} = \begin{bmatrix} t_1 I_2 & \mathcal{H}'_1 \\ \mathcal{H}_1'^\dagger & 0 \end{bmatrix}, \quad \mathcal{H}_{\{2',3,4,6'\}}^{\text{2DRM}} = \begin{bmatrix} -t_1 I_2 & \mathcal{H}'_2 \\ \mathcal{H}_2'^\dagger & 0 \end{bmatrix} \quad (\text{F1})$$

with

$$\mathcal{H}'_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} t_2 + t_4 e^{ik_x} & t_3 + t_4 e^{-ik_y} \\ -t_3 - t_4 e^{ik_y} & t_2 + t_4 e^{-ik_x} \end{bmatrix},$$

$$\mathcal{H}'_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -t_2 + t_4 e^{ik_x} & t_3 + t_4 e^{-ik_y} \\ t_3 + t_4 e^{ik_y} & t_2 - t_4 e^{-ik_x} \end{bmatrix}. \quad (\text{F2})$$

Based on the topological property of 2D Rice-Mele model [36], for $\mathcal{H}_{\{1',3,4,5'\}}^{\text{2DRM}}$, when the intercell hopping exceeds the maximum of intra-cell hoppings, i.e., $t_4 > \max\{t_2, t_3\}$, there exist chiral corner states occupying sublattice $|1'\rangle, |5'\rangle$ with $E = t_1$. For $\mathcal{H}_{\{2',3,4,6'\}}^{\text{2DRM}}$, chiral corner states occupying sublattice $|2'\rangle, |6'\rangle$ with $E = -t_1$ can be observed also when $t_4 > \max\{t_2, t_3\}$.

The two 2D Rice-Mele models defined in Eq. (39) can be written as

$$\mathcal{H}_{\{1,2'',4'',6\}}^{\text{2DRM}} = \begin{bmatrix} 0 & \mathcal{H}''_1 \\ \mathcal{H}_1''^\dagger & t_2 I_2 \end{bmatrix}, \quad \mathcal{H}_{\{1,3'',5'',6\}}^{\text{2DRM}} = \begin{bmatrix} 0 & \mathcal{H}''_2 \\ \mathcal{H}_2''^\dagger & -t_2 I_2 \end{bmatrix}, \quad (\text{F3})$$

with

$$\mathcal{H}''_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} t_1 + t_4 e^{ik_x} & t_3 + t_4 e^{-ik_y} \\ -t_3 - t_4 e^{ik_y} & t_1 + t_4 e^{-ik_x} \end{bmatrix},$$

$$\mathcal{H}''_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} t_1 - t_4 e^{ik_x} & t_3 + t_4 e^{-ik_y} \\ t_3 + t_4 e^{ik_y} & -t_1 + t_4 e^{-ik_x} \end{bmatrix}. \quad (\text{F4})$$

Therefore, when $t_4 > \max\{t_1, t_3\}$, there exist chiral corner states occupying sublattice $|2''\rangle, |4''\rangle$ ($|3''\rangle, |5''\rangle$) with $E = t_2$ ($E = -t_2$).

APPENDIX G: THE GENERALIZED CHIRAL CORNER STATES IN ASYMMETRIC BBH MODEL

In this Appendix, we would like to analyze the properties of the generalized chiral corner states in the asymmetric BBH model. As shown in Fig. 13(a), the Hamiltonian of the asymmetric BBH model can be written as

$$\mathcal{H}_{\text{ASBBH}}(k_x, k_y) = \begin{bmatrix} 0 & m_1 & m_2 & 0 \\ m_1^* & 0 & 0 & m_3 \\ m_2^* & 0 & 0 & m_4 \\ 0 & m_3^* & m_4^* & 0 \end{bmatrix}, \quad (\text{G1})$$

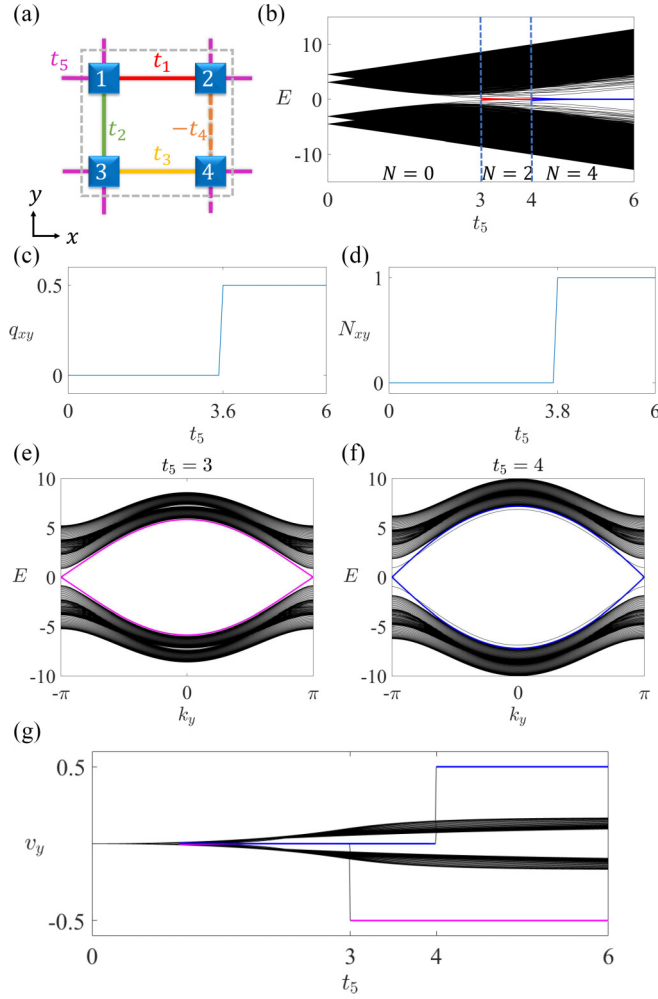


FIG. 13. (a) Schematic of the asymmetric BBH model. (b) The band structure vs t_5 under OBC with $t_1 = 1$, $t_2 = 3$, $t_3 = 2$, and $t_4 = 4$. The spectrum is divided into three regions according to the number of zero-energy corner states. (c) The quadrupole moment q_{xy} vs t_5 . (d) The quadrupole winding number N_{xy} vs t_5 . (e), (f) The band structure of the x -open ribbon structure at (e) $t_5 = 3$ and (f) $t_5 = 4$. (g) The Wannier band vs t_5 at half filling. The edge polarization p_y^{edge} is marked by red and blue lines.

where $m_1 = t_1 + t_5 e^{ik_x}$, $m_2 = t_2 + t_5 e^{-ik_y}$, $m_3 = -t_4 - t_5 e^{-ik_y}$, and $m_4 = t_3 + t_5 e^{ik_x}$. With the unequal intracell hopping terms, both the mirror symmetry and spatial inversion symmetry are broken. As to the nonspatial symmetry, the asymmetric BBH model has time-reversal symmetry and chiral symmetry. The band structure versus t_5 under OBC with $t_1 = 1$, $t_2 = 3$, $t_3 = 2$, and $t_4 = 4$ is plotted in Fig. 13(b), which can be divided into three regions according to the gap-closing points $t_5 = 3$ and $t_5 = 4$. In these three regions, the number of the zero-energy corner states is $N = 0$, $N = 2$, and $N = 4$, respectively. This is quite different from the conventional BBH model [13] and the extended BBH model [55], where the number of the zero-energy corner states has to be a multiple of 4. The difference is also verified through the bulk invariant. Although both the quadrupole moment q_{xy} and the quadrupole winding number N_{xy} are quantized due to chiral symmetry

[55,58], neither of them is able to establish the bulk-corner correspondence.

The properties of the zero-energy corner states can be explained through the generalized chiral symmetry. For a system composed of n sublattices in each unit cell, the generalized chiral symmetry is represented as [11,59]

$$\Gamma_n^{-1} \mathcal{H}_i \Gamma_n = \mathcal{H}_{i+1}, \quad i = 0, \dots, n-2, \quad (G2)$$

$$\sum_{j=0}^{n-1} \mathcal{H}_j = 0,$$

where $\mathcal{H}_0$ is the system Hamiltonian and Γ_n is the generalized chiral operator:

$$\Gamma_n = \begin{bmatrix} 1 & & & \\ & e^{i\frac{2\pi}{n}} & & \\ & & \ddots & \\ & & & e^{i\frac{2\pi(n-1)}{n}} \end{bmatrix}. \quad (G3)$$

For the zero-energy corner state protected by the Γ_n , its wave function only occupies one of the sublattices [11]. Obviously, $\mathcal{H}_{\text{ASBBH}}(k_x, k_y)$ possesses the generalized chiral symmetry Γ_4 . Therefore, the zero-energy corner states $|C_\alpha\rangle$ occupying sublattices $\alpha = 1, 2, 3, 4$ should satisfy

$$\langle \beta | C_\alpha \rangle = \delta_{\alpha\beta}, \quad \beta = 1, 2, 3, 4, \quad (G4)$$

$$\mathcal{H}_{\text{ASBBH}} | C_\alpha \rangle = 0.$$

Therefore, we have

$$\begin{aligned} |C_1\rangle &: t_5 > \max\{t_1, t_2\}, \\ |C_2\rangle &: t_5 > \max\{t_1, t_4\}, \\ |C_3\rangle &: t_5 > \max\{t_2, t_3\}, \\ |C_4\rangle &: t_5 > \max\{t_3, t_4\}, \end{aligned} \quad (G5)$$

which coincides with the band structure exhibited in Fig. 13(b).

The topological property of the asymmetric BBH model can be described through the edge polarization p^{edge} . In Figs. 13(e) and 13(f), we plot the band structure of the x -open ribbon structure. Due to the breaking of spatial inversion symmetry, the degeneracy of the edge bands marked in blue and red is lifted. As a result, they share different topological transition points, which corresponds to the gap-closing points in Fig. 13(b). It is worth noting that the edge bands are not isolated from the bulk bands, so the p^{edge} cannot be directly calculated by Eq. (41). Instead, in Fig. 13(g), we calculate the Wannier spectrum v_y at half filling. The edge states in v_y marked in color stands for p^{edge} , where the abrupt change takes place at $t_5 = 3$ and $t_5 = 4$, respectively, indicating the edge-corner correspondence is established.

As to the subsymmetry property, the chiral corner states occupying sublattice $|\alpha\rangle$ is preserved as long as the following relationship is satisfied [30]:

$$\sum_{j=0}^{n-1} \mathcal{H}_j P_\alpha = 0, \quad (G6)$$

where $\mathcal{H}_j$ is defined in Eq. (G2), P_α is the projection operator $P_\alpha = |\alpha\rangle\langle\alpha|$. Meanwhile, the onsite term over sublattice

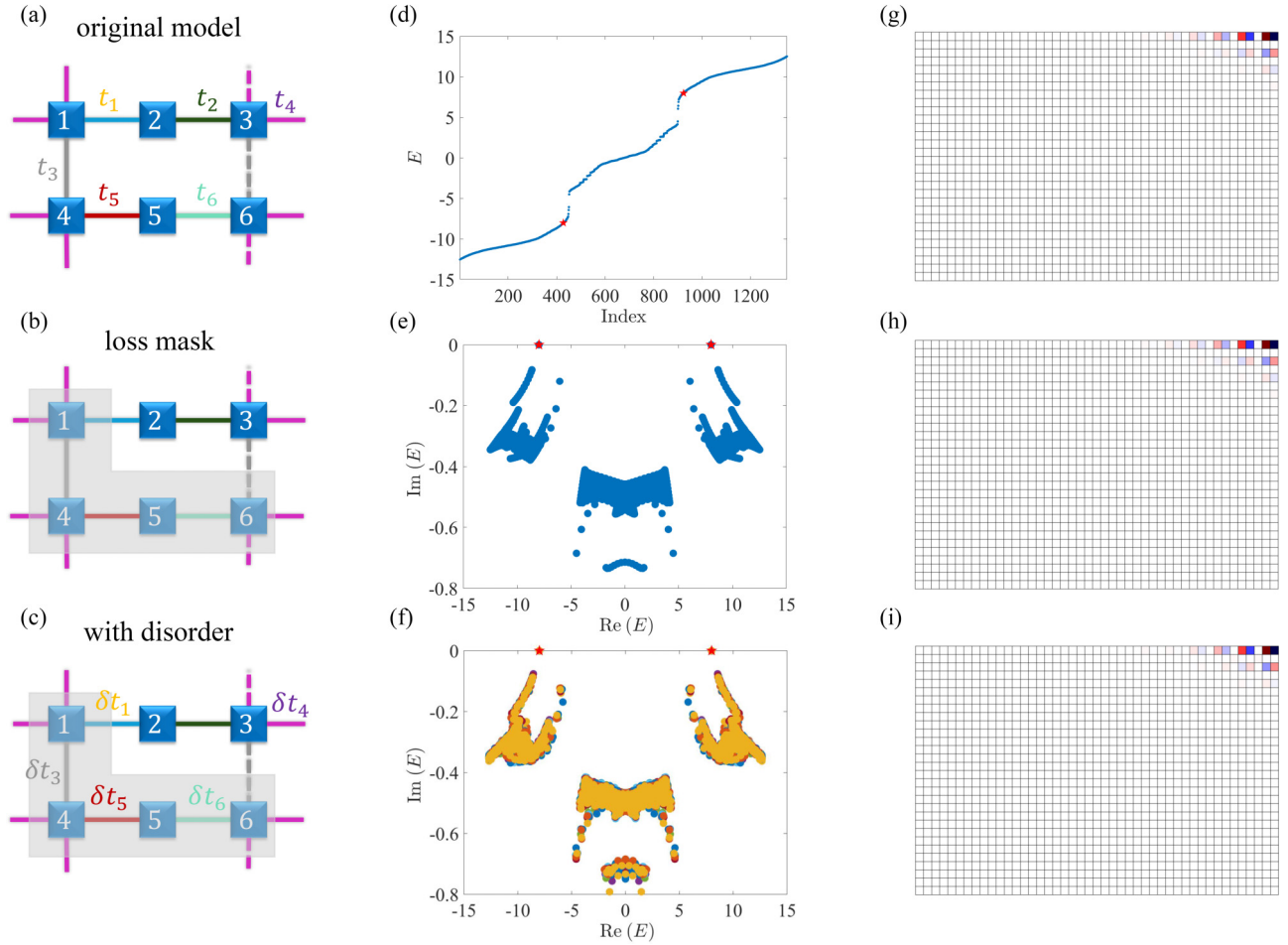


FIG. 14. Robustness analysis of the TBICs. Schematic of (a) the original asymmetric BBH3 model, (b) the asymmetric BBH3 model with loss mask, and (c) the asymmetric BBH3 model with loss mask and subsymmetry-preserving disorder. The parameters are set to $t_1 = 2$, $t_2 = 8$, $t_3 = 6$, $t_4 = 5$, $t_5 = 1$, and $t_6 = 5$. The spectra of (d) the original BBH3 model, (e) the BBH3 model with loss mask, and (f) the BBH3 model with loss mask and disorder under OBC. In panel (f), the spectra of ten different configurations are presented. The TBICs pinned at the real axis are marked by red stars. (g)–(i) Spatial distributions of the TBICs with $E = -8$ marked in panels (d)–(f), respectively.

α , V_α only shifts the energy of the chiral corner states without affecting its spatial distribution. Therefore, for the asymmetric BBH3 model with redefined sublattices shown in Fig. 9(d), the generalized chiral corner states occupying sublattices $|1'\rangle$, $|2'\rangle$, $|5'\rangle$, $|6'\rangle$, $|2''\rangle$, $|3''\rangle$, $|4''\rangle$, and $|5''\rangle$ are preserved, which are exactly the eight corner states shown in Figs. 9(c) and 9(e).

APPENDIX H: ROBUSTNESS ANALYSIS OF THE GENERALIZED CHIRAL TBICs PROTECTED BY SUBSYMMETRY

In this Appendix, we would like to show that the TBICs observed in Sec. IV E is protected by the subsymmetry with the redefined sublattices. To verify this point, we utilize the method introduced in Ref. [31] to test the robustness of the TBICs against symmetry-preserving perturbations. For the asymmetric BBH3 model with $t_1 = 2$, $t_2 = 8$, $t_3 = 6$, $t_4 = 5$, $t_5 = 1$, and $t_6 = 5$, two TBICs marked by red stars can be found in Fig. 14(d). We first use a “loss mask” to cover the asymmetric BBH3 model, shown as the gray shadow region

in 14(b). Here, the loss mask means that we introduce onsite losses (i.e., imaginary onsite energy) at every sublattice in the shadow area. As exhibited in Fig. 14(e), all the modes that overlap with the shadow area are decaying (i.e., the energy of the eigenstate has nonzero imaginary part) except for the TBICs marked by red stars. This is a natural result as the loss mask is designed to avoid the TBICs.

Next, we consider random disorder in the hopping terms. Since the TBICs shown in Fig. 14(h) occupy the redefined sublattices $|3''\rangle$, to preserve the subsymmetry, the hopping term t_2 is set to be unperturbed. For other hopping terms, the disorder is introduced as $t'_i = t_i + \delta t_i$, where δt_i are random numbers that follow a uniform distribution in $[-w, w]$, where w is the disorder strength. In Fig. 14(f), we plot the overlap of the complex spectrum of the disordered BBH3 model under ten different configurations with $w = 1$. One can easily see that the TBICs marked by red stars are still pinned at the real axis, exhibiting the fact that these localized states do not couple to the decaying bulk states. In summary, we show that these TBICs are robust even in the presence of loss mask and disorder, which confirms that these TBICs are topological and protected by subsymmetry.

- [1] X.-L. Qi and S.-C. Zhang, Topological insulators and superconductors, *Rev. Mod. Phys.* **83**, 1057 (2011).
- [2] A. Bansil, H. Lin, and T. Das, Colloquium: Topological band theory, *Rev. Mod. Phys.* **88**, 021004 (2016).
- [3] T. Ozawa, H. M. Price, A. Amo, N. Goldman, M. Hafezi, L. Lu, M. C. Rechtsman, D. Schuster, J. Simon, O. Zilberberg, and I. Carusotto, Topological photonics, *Rev. Mod. Phys.* **91**, 015006 (2019).
- [4] M. Kim, Z. Jacob, and J. Rho, Recent advances in 2D, 3D and higher-order topological photonics, *Light Sci. Appl.* **9**, 130 (2020).
- [5] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Quantum spin Hall effect and topological phase transition in HgTe quantum wells, *Science* **314**, 1757 (2006).
- [6] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Quantum spin Hall insulator state in HgTe quantum wells, *Science* **318**, 766 (2007).
- [7] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Classification of topological insulators and superconductors in three spatial dimensions, *Phys. Rev. B* **78**, 195125 (2008).
- [8] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Classification of topological quantum matter with symmetries, *Rev. Mod. Phys.* **88**, 035005 (2016).
- [9] L. Fu, Topological crystalline insulators, *Phys. Rev. Lett.* **106**, 106802 (2011).
- [10] T. L. Hughes, E. Prodan, and B. A. Bernevig, Inversion-symmetric topological insulators, *Phys. Rev. B* **83**, 245132 (2011).
- [11] X. Ni, M. Weiner, A. Alu, and A. B. Khanikaev, Observation of higher-order topological acoustic states protected by generalized chiral symmetry, *Nat. Mater.* **18**, 113 (2019).
- [12] B. Xie, H.-X. Wang, X. Zhang, P. Zhan, J.-H. Jiang, M. Lu, and Y. Chen, Higher-order band topology, *Nat. Rev. Phys.* **3**, 520 (2021).
- [13] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Quantized electric multipole insulators, *Science* **357**, 61 (2017).
- [14] F. Liu and K. Wakabayashi, Novel topological phase with a zero Berry curvature, *Phys. Rev. Lett.* **118**, 076803 (2017).
- [15] X.-D. Chen, W.-M. Deng, F.-L. Shi, F.-L. Zhao, M. Chen, and J.-W. Dong, Direct observation of corner states in second-order topological photonic crystal slabs, *Phys. Rev. Lett.* **122**, 233902 (2019).
- [16] B.-Y. Xie, G.-X. Su, H.-F. Wang, H. Su, X.-P. Shen, P. Zhan, M.-H. Lu, Z.-L. Wang, and Y.-F. Chen, Visualization of higher-order topological insulating phases in two-dimensional dielectric photonic crystals, *Phys. Rev. Lett.* **122**, 233903 (2019).
- [17] Y. Qi, C. Qiu, M. Xiao, H. He, M. Ke, and Z. Liu, Acoustic realization of quadrupole topological insulators, *Phys. Rev. Lett.* **124**, 206601 (2020).
- [18] L. He, Z. Addison, E. J. Mele, and B. Zhen, Quadrupole topological photonic crystals, *Nat. Commun.* **11**, 3119 (2020).
- [19] D. Wang, Y. Deng, J. Ji, M. Oudich, W. A. Benalcazar, G. Ma, and Y. Jing, Realization of a $\mathbb{Z}$ -classified chiral-symmetric higher-order topological insulator in a coupling-inverted acoustic crystal, *Phys. Rev. Lett.* **131**, 157201 (2023).
- [20] V. M. Martinez Alvarez and M. D. Coutinho-Filho, Edge states in trimer lattices, *Phys. Rev. A* **99**, 013833 (2019).
- [21] Y. V. Kartashov, A. A. Arkhipova, S. A. Zhuravitskii, N. N. Skryabin, I. V. Dyakonov, A. A. Kalinkin, S. P. Kulik, V. O. Kompanets, S. V. Chekalin, L. Torner, and V. N. Zadkov, Observation of edge solitons in topological trimer arrays, *Phys. Rev. Lett.* **128**, 093901 (2022).
- [22] A. Anastasiadis, G. Styliaris, R. Chaunsali, G. Theocharis, and F. K. Diakonov, Bulk-edge correspondence in the trimer Su-Schrieffer-Heeger model, *Phys. Rev. B* **106**, 085109 (2022).
- [23] W. Yan, W. Cheng, W. Liu, and F. Chen, Topological edge states in photonic decorated trimer lattices, *Opt. Lett.* **48**, 1802 (2023).
- [24] I. Ioannou Sougleridis, A. Anastasiadis, O. Richoux, V. Achilleos, G. Theocharis, V. Pagneux, and F. K. Diakonov, Existence and characterization of edge states in an acoustic trimer Su-Schrieffer-Heeger model, *Phys. Rev. B* **110**, 174311 (2024).
- [25] S. Verma and T. K. Ghosh, Bulk-boundary correspondence in extended trimer Su-Schrieffer-Heeger model, *Phys. Rev. B* **110**, 125424 (2024).
- [26] T. Du, Y.-X. Li, H.-L. Lu, and H. Zhang, One-dimensional extended Su-Schrieffer-Heeger models as descendants of a two-dimensional topological model, *New J. Phys.* **26**, 023044 (2024).
- [27] Z. Guo, X. Wu, S. Ke, L. Dong, F. Deng, H. Jiang, and H. Chen, Rotation controlled topological edge states in a trimer chain composed of meta-atoms, *New J. Phys.* **24**, 063001 (2022).
- [28] Q.-H. Guo, Y. Zhang, X.-H. Wan, and L.-Y. Zheng, Isospectral reduction of the trimer Su-Schrieffer-Heeger lattice and its bulk-edge correspondence, *Phys. Rev. Appl.* **23**, L031001 (2025).
- [29] J. Zhong, H. Wang, A. N. Poddubny, and S. Fan, Topological nature of edge states for one-dimensional systems without symmetry protection, *Phys. Rev. Lett.* **135**, 016601 (2025).
- [30] Z. Wang, X. Wang, Z. Hu, D. Bongiovanni, D. Jukić, L. Tang, D. Song, R. Morandotti, Z. Chen, and H. Buljan, Subsymmetry-protected topological states, *Nat. Phys.* **19**, 992 (2023).
- [31] X. Wang, D. Bongiovanni, Z. Wang, A. Abdrabou, Z. Hu, D. Jukić, D. Song, R. Morandotti, R. El-Ganainy, Z. Chen, and H. Buljan, Construction of topological bound states in the continuum via subsymmetry, *ACS Photonics* **11**, 3213 (2024).
- [32] M. Kang, M. Lee, and S. Cheon, Subsymmetry protected topology in topological insulators and superconductors, *Phys. Rev. Res.* **6**, 033323 (2024).
- [33] L. Liu, T. Li, Q. Zhang, M. Xiao, and C. Qiu, Universal mirror-stacking approach for constructing topological bound states in the continuum, *Phys. Rev. Lett.* **130**, 106301 (2023).
- [34] J. Guo, Z. Gu, and J. Zhu, Realization of merged topological corner states in the continuum in acoustic crystals, *Phys. Rev. Lett.* **133**, 236603 (2024).
- [35] X.-C. Sun, J.-B. Wang, C. He, and Y.-F. Chen, Non-Abelian topological phases and their quotient relations in acoustic systems, *Phys. Rev. Lett.* **132**, 216602 (2024).
- [36] Y. Li, J. Chen, X. Qi, L. Xiong, X. Wang, Y. Liu, F. Guan, L. Shi, and X. Jiang, Topology reconstruction for asymmetric systems by homomorphic mapping or perturbation approximation, *Phys. Rev. B* **110**, 104106 (2024).
- [37] A. Marques and R. Dias, Analytical solution of open crystalline linear 1D tight-binding models, *J. Phys. A: Math. Theor.* **53**, 075303 (2020).
- [38] S. Verma and M. J. Park, Non-Bloch band theory of subsymmetry-protected topological phases, *Phys. Rev. B* **110**, 035424 (2024).

- [39] S. Verma and M. J. Park, Topological phase transitions of generalized Brillouin zone, *Commun. Phys.* **7**, 21 (2024).
- [40] T. Kawarabayashi and Y. Hatsugai, Bulk-edge correspondence with generalized chiral symmetry, *Phys. Rev. B* **103**, 205306 (2021).
- [41] M. Röntgen, M. Pyzh, C. V. Morfonios, N. E. Palaodimopoulos, F. K. Diakonov, and P. Schmelcher, Latent symmetry induced degeneracies, *Phys. Rev. Lett.* **126**, 180601 (2021).
- [42] M. Röntgen, X. Chen, W. Gao, M. Pyzh, P. Schmelcher, V. Pagneux, V. Achilleos, and A. Coutant, Topological states protected by hidden symmetry, *Phys. Rev. B* **110**, 035106 (2024).
- [43] L. Eek, M. Röntgen, A. Moustaj, and C. M. Smith, Higher-order topology protected by latent crystalline symmetries, *SciPost Phys.* **18**, 061 (2025).
- [44] Q.-H. Guo, Y. Zhang, X.-H. Wan, and L.-Y. Zheng, Observation of chiral edge state pairs in an acoustic trimer waveguide, *Appl. Phys. Lett.* **126**, 133102 (2025).
- [45] J. Lu, L. He, Z. Addison, E. J. Mele, and B. Zhen, Floquet topological phases in one-dimensional nonlinear photonic crystals, *Phys. Rev. Lett.* **126**, 113901 (2021).
- [46] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Electric multipole moments, topological multipole moment pumping, and chiral hinge states in crystalline insulators, *Phys. Rev. B* **96**, 245115 (2017).
- [47] J. Hu, S. Zhuang, and Y. Yang, Higher-order topological insulators via momentum-space nonsymmorphic symmetries, *Phys. Rev. Lett.* **132**, 213801 (2024).
- [48] S. Vaidya, A. Ghorashi, T. Christensen, M. C. Rechtsman, and W. A. Benalcazar, Topological phases of photonic crystals under crystalline symmetries, *Phys. Rev. B* **108**, 085116 (2023).
- [49] Y.-B. Yang, K. Li, L.-M. Duan, and Y. Xu, Type-II quadrupole topological insulators, *Phys. Rev. Res.* **2**, 033029 (2020).
- [50] M. Ezawa, Edge-corner correspondence: Boundary-obstructed topological phases with chiral symmetry, *Phys. Rev. B* **102**, 121405(R) (2020).
- [51] H. Xue, Y. Ge, H.-X. Sun, Q. Wang, D. Jia, Y.-J. Guan, S.-Q. Yuan, Y. Chong, and B. Zhang, Observation of an acoustic octupole topological insulator, *Nat. Commun.* **11**, 2442 (2020).
- [52] The analytic solution of the topological chiral edge states in the disordered SSH model utilizing the method in Sec. II B will be discussed in our other work, which will be published on other academic journal
- [53] Y. Li, J.-H. Zhang, F. Mei, B. Xie, M.-H. Lu, J. Ma, L. Xiao, and S. Jia, Large-chiral-number corner modes in $\mathbb{Z}$ -class higher-order topoelectrical circuits, *Phys. Rev. Appl.* **20**, 064042 (2023).
- [54] Z.-Q. Jiao, S. Longhi, X.-W. Wang, J. Gao, W.-H. Zhou, Y. Wang, Y.-X. Fu, L. Wang, R.-J. Ren, L.-F. Qiao, and X.-M. Jin, Experimentally detecting quantized Zak phases without chiral symmetry in photonic lattices, *Phys. Rev. Lett.* **127**, 147401 (2021).
- [55] W. A. Benalcazar and A. Cerjan, Chiral-symmetric higher-order topological phases of matter, *Phys. Rev. Lett.* **128**, 127601 (2022).
- [56] R. K. Malakar and A. K. Ghosh, Engineering topological phases of any winding and Chern numbers in extended Su-Schrieffer-Heeger models, *J. Phys.: Condens. Matter* **35**, 335401 (2023).
- [57] M. Ghuneim and R. W. Bomantara, Topological phases of tight-binding trimer lattice in the BDI symmetry class, *J. Phys.: Condens. Matter* **36**, 495402 (2024).
- [58] C.-A. Li, B. Fu, Z.-A. Hu, J. Li, and S.-Q. Shen, Topological phase transitions in disordered electric quadrupole insulators, *Phys. Rev. Lett.* **125**, 166801 (2020).
- [59] Y.-Z. Li, Z.-F. Liu, X.-W. Xu, Q.-P. Wu, X.-B. Xiao, M.-R. Liu, L.-L. Chang, and R.-L. Zhang, Zero-energy corner states protected by generalized chiral symmetry in C_4 symmetric crystals, *New J. Phys.* **23**, 043010 (2021).