

Precursor behavior, bag formation, and lattice instabilities of confined hopfions in chiral magnetic films

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Topological particlelike excitations such as skyrmions and hopfions offer rich opportunities for spintronic and photonic applications. While skyrmions have been extensively studied, the stabilization mechanisms and phase behavior of three-dimensional hopfions remain largely unexplored. Here, we investigate the formation, stability, and interactions of hopfions in thin chiral magnetic films with surface anchoring, using three-dimensional micromagnetic simulations within a material-independent framework applicable to both magnetic and liquid crystalline systems. We identify four distinct types of isolated hopfions, generated by rotating bimeron and fingerlike solitons around a central axis. The metastability regions of these precursor textures closely follow the boundaries of modulated finger phases, enabling their size to be continuously tuned through anisotropy-driven inflation and collapse. Remarkably, we demonstrate that hopfions near their inflation threshold possess energies comparable with the homogeneous state, allowing them to enclose regions of modulated phases or other solitons, forming higher-order, baglike domains. In contrast, periodic hopfion lattices remain intrinsically unstable under confinement, spontaneously relaxing into finger phases. These findings establish general principles for stabilizing, tuning, and assembling three-dimensional topological solitons in confined chiral systems, suggesting experimentally accessible routes for texture engineering in liquid crystals via electric-field control.

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I. INTRODUCTION

The concept of topological solitons transcends disciplinary boundaries, arising in effective field theories describing systems from nuclear to condensed matter physics [1,2]. A soliton is a self-stabilizing, particlelike state in a continuous field [3,4], existing as a finite, localized excitation embedded within a uniform vacuum background, and maintaining its integrity through topological or dynamical conservation laws. Beyond isolated objects, solitons can self-organize through their mutual interactions into periodic arrays or crystallike assemblies. This gives rise to a form of *solitonic metamatter*, in which individual solitons function as quasiparticles of an emergent material [5].

Among topological solitons, two-dimensional (2D) particlelike skyrmions [6] have attracted significant attention in condensed matter systems [7]. For clarity, a compact glossary summarizing the various solitonic objects discussed in this work, together with their dimensionality and topological characteristics, is provided in the Supplemental Material [8].

Topologically, a skyrmion is a two-dimensional field configuration in which a smooth unit vector field $\mathbf{m}(\mathbf{r})$ extends

the compactified plane $\mathbb{R}^2 \cup \{\infty\} \simeq S^2$ to the order-parameter sphere S^2 . Such maps are classified by the second homotopy group $\pi_2(S^2) \cong \mathbb{Z}$ [9,10], and are therefore characterized by an integer-valued topological charge. This charge measures how many times the order parameter covers the unit sphere and can be expressed as

$$Q = \frac{1}{4\pi} \int \mathbf{m} \cdot (\partial_x \mathbf{m} \times \partial_y \mathbf{m}) dx dy, \quad (1)$$

which geometrically represents the total oriented area swept out on the sphere by the field configuration.

Skyrmions have been widely investigated in both chiral magnets (ChM) [11,12] and chiral liquid crystals (CLC) [13] due to their ability to exhibit physical phenomena and their potential for future spintronic and photonic applications [14,15]. Physically, $\mathbf{m}$ in Eq. (1) denotes the magnetization in ChM or the director field in CLC. In a prototypical skyrmion, the spins in the core are oriented antiparallel to the uniform background and rotate smoothly toward the far-field direction at large distances [Fig. 1(a)]. For such a texture, $|Q| = 1$, indicating that the magnetization wraps the unit sphere exactly once.

In bulk chiral magnets, e.g., in MnSi [16] and FeGe [17], axisymmetric skyrmionic filaments typically align parallel to an applied magnetic field or the easy axis of magnetic anisotropy [Fig. 1(a)]. When their orientation switches perpendicular to this axis, a skyrmion transforms into a pair of coupled merons, known as a bimeron (or a cholesteric finger CF-2 in CLC [13]), each carrying a fractional topological

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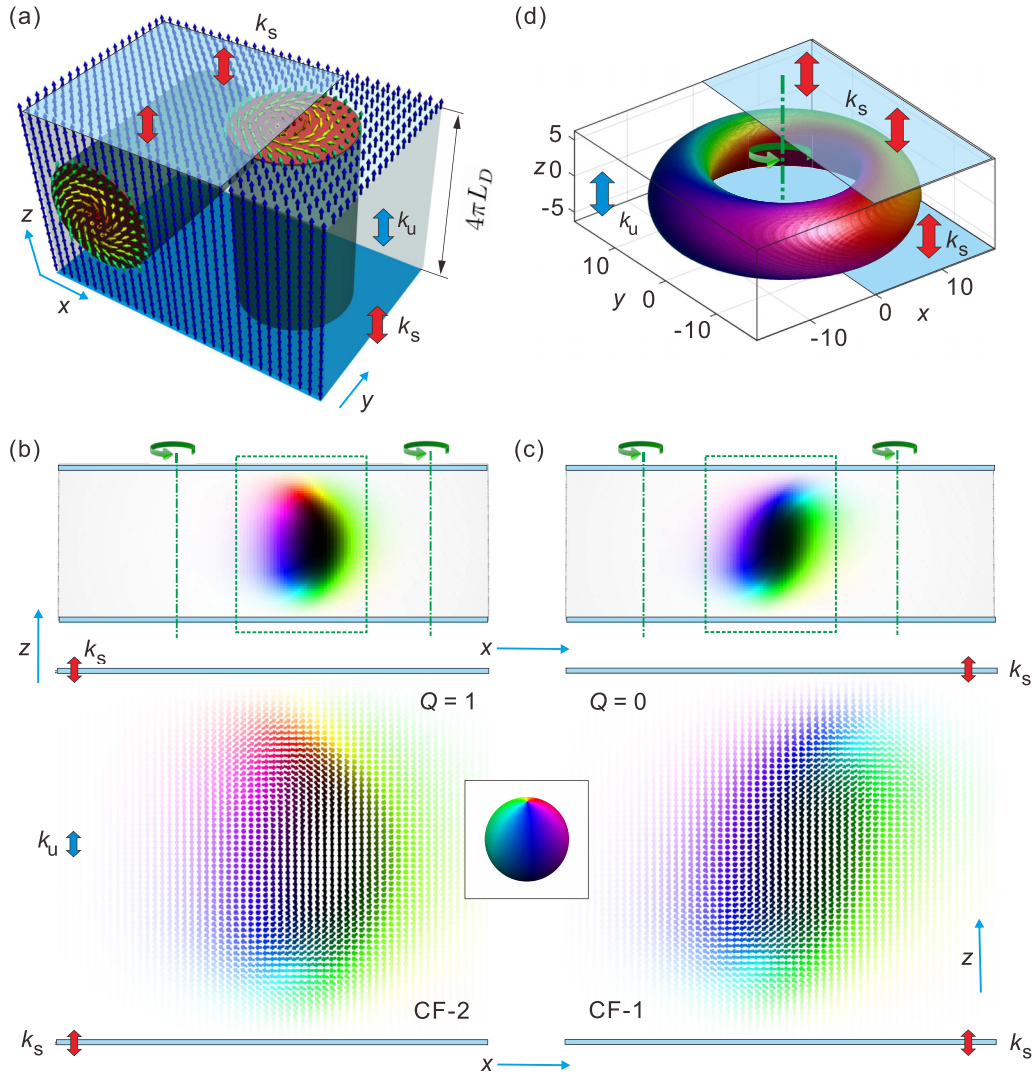


FIG. 1. Creation of hopfions by rotating CF-1 and CF-2 fingers. (a) Schematics of two skyrmion filaments: one aligned with the surface normal and another oriented perpendicular to it. To accommodate the surrounding homogeneous state, the axisymmetric skyrmion structure must continuously deform into the configuration shown in panel (b). (b) and (c) Spin distributions within the two-dimensional CF-1 and CF-2 solitons. The green dash-dotted lines mark the rotation axes used to generate the corresponding three-dimensional (3D) hopfion textures. (d) A representative 3D hopfion obtained through this rotational construction.

charge $Q = 1/2$, thereby preserving its overall topological identity while matching the surrounding homogeneous state [18,19] [Figs. 1(a) and 1(b), see also the Supplemental Material [8]].

A hopfion can be understood as a 3D extension of this idea [20,21]: A 2D bimeronlike structure continuously rotated around an axis in three dimensions, forming a toroidal, topologically nontrivial soliton characterized by a Hopf invariant [22,23],

$$Q_H = \frac{1}{(4\pi)^2} \int \mathbf{A} \cdot (\nabla \times \mathbf{A}) d^3r, \quad (2)$$

where $\mathbf{A}$ is a vector potential for the emergent magnetic (or helicity) field $\nabla \times \mathbf{A}$ associated with the hopfion [24]. These configurations are classified by the third homotopy group, $\pi_3(S^2) = \mathbb{Z}$ [25]. Unlike skyrmions and bimerons, hopfions are inherently 3D, with distinct topological, energetic, and dynamical properties [26–29] [Fig. 1(d)].

Although the concept of hopfions has been theoretically proposed in various fields, their realization has been predominantly explored in confined geometries, such as thin magnetic [30] or liquid crystalline films [31]. Confinement plays a pivotal role in stabilizing hopfions. For instance, in bulk chiral magnets, hopfions tend to elongate along the direction of the applied magnetic field, eventually relaxing into spiral phases [20,21]. In contrast, thin film geometries impose finite-size effects and boundary conditions that inhibit such elongations and dramatically reshape the topological phase landscape. Notably, the conical spiral phase, typically dominant in bulk chiral magnets (the reason why, e.g., skyrmions were found only in small A-phase pockets near the Curie temperatures in chiral magnets [16]), loses rotational energy near surfaces where magnetization is pinned by surface anchoring [32], while bimeron-based chain configurations and their derivatives become energetically favorable. This reshaping of the phase diagram not only broadens the stability window for

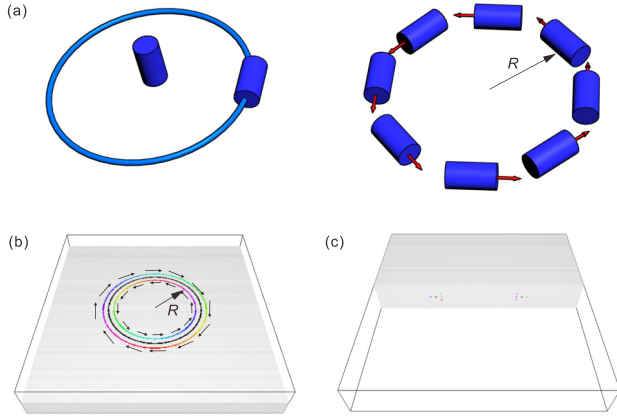


FIG. 2. Procedure for constructing initial hopfion configurations for relaxation in MUMAX3. (a) Elementary cylinders are positioned along prescribed circular trajectories and assigned magnetization vectors of specified orientation. (b) Positions of circular trajectories in the middle plane of the film. (c) Five circular trajectories visualized in xz plane cross sections.

bimeron-based hopfions but also permits the emergence of 3D textures based on CF-1 fingerlike structures with $Q = 0$ [13], which have no counterparts in bulk magnets [Fig. 1(c), see also the Supplemental Material [8]].

In this work, we systematically investigate the stabilization, structure, and interactions of hopfions in thin chiral magnetic films with surface anchoring. Through 3D micromagnetic simulations, we identify four distinct types of isolated hopfions, classified by their internal structure: Two formed by continuous rotation of bimerons and two derived from the rotation of CF-1 textures (Fig. 3). These configurations differ by the position of the rotation axis relative to the corresponding two-dimensional soliton, either to its left or right [green dash-dotted lines in Figs. 1(b) and 1(c)].

Isolated hopfions are found to exist in close proximity to the phase boundaries of the corresponding finger phases, revealing the essential role of confinement-induced topological frustration. By mapping the phase diagram in the plane of uniaxial anisotropy and surface anchoring strength—two parameters directly tunable in both magnetic [33,34] and liquid crystalline systems [13]—we demonstrate that varying the uniaxial anisotropy, equivalent to applying an electric field in chiral liquid crystals [13], induces qualitative transformations in the behavior of isolated hopfions: either gradual inflation and deformation into finger textures as the phase boundary is approached, or shrinkage and collapse into toronlike structures under increased anisotropy. Furthermore, we reveal that the distinct radii of different hopfion types (as the phase boundaries of CF-1 and CF-2 phases do not coincide) enable the formation of composite configurations, where a smaller hopfion can be encapsulated within a larger one, with their sizes mutually adjusted to minimize repulsive interactions. In particular, the external hopfion can substantially expand its radius to alleviate overlap with the internal soliton, offering a mechanism for engineering multisoliton states. More generally, the interior of a topological bag can be filled by a group of torons or the domain of a finger phase.

Finally, we demonstrate that periodic hopfion lattices (HLs) cannot be stabilized under these conditions, as their energy landscape does not exhibit a local minimum and instead relaxes toward the modulated finger phase. This instability can be traced to the structure of an isolated hopfion, which can be regarded as a closed, circular realization of a corresponding cholesteric finger. Once the energy of a finger becomes negative, a modulated finger phase can tile space with unit efficiency. In contrast, a hopfion lattice necessarily contains extended regions of homogeneous “vacuum,” both within the hopfion cores and in the interstitial regions between neighboring objects. This geometric inefficiency promotes a deformation of the hopfion textures, resulting in the collapse of their vacuum cores and their subsequent elongation into fingerlike structures.

Overall, the findings of the present paper introduce mechanisms for stabilizing, tuning, and combining three-dimensional topological solitons in confined chiral systems, providing practical avenues for topological texture engineering in both magnetic and liquid crystalline materials.

II. THE PHENOMENOLOGICAL MODEL

The magnetic energy density of a noncentrosymmetric ferromagnet that contains both bulk (Ω) and surface ($\partial\Omega$) terms is expressed as the sum of exchange, Dzyaloshinskii-Moriya interaction (DMI), and uniaxial anisotropy contributions [32]:

$$W(\mathbf{m}) = \int_{\Omega} ((\nabla \mathbf{m})^2 + \mathbf{m} \cdot (\nabla \times \mathbf{m}) - k_u (\mathbf{m} \cdot \mathbf{z})^2) d^3 \mathbf{r} - \int_{\partial\Omega} k_s t (\mathbf{m} \cdot \mathbf{z})^2 d^2 \mathbf{r}. \quad (3)$$

Here, we employ dimensionless units to generalize the results and allow direct mapping to various material systems. The characteristic length scale is $L_D = A/D$, and the dimensionless anisotropy constant is defined as $k_u = K_u A/D^2$. The system geometry corresponds to a thin film, infinite in the x and y directions (i.e., periodic boundary conditions are imposed along these directions), with a thickness along z equal to a single spiral pitch, $4\pi L_D$. Such a geometry is analogous to trilayered ChM systems, consisting of a single chiral ferromagnetic layer with weaker perpendicular magnetic anisotropy (PMA) sandwiched between two capping ferromagnetic layers with stronger PMA [35]. Both the bulk uniaxial anisotropy k_u and the surface anchoring strength k_s favor an easy-axis ferromagnetic state with the magnetization aligned along the z direction. The parameter $t = 4\pi/N_z \approx 0.2$ denotes the thickness of the surface layers in which the uniaxial anisotropy is set to k_s . Further implementation details are provided in the MUMAX3 script included in the Supplemental Material [8].

Importantly, the theoretical frameworks governing ChM and CLC are formally equivalent within the widely used one-constant approximation for liquid crystals [15,32]. In this approximation, the energy functionals describing both systems share the same mathematical form, differing only in the physical interpretation of the order parameter: the magnetization vector in chiral magnets and the director field in liquid crystals [13,15]. Consequently, theoretical predictions

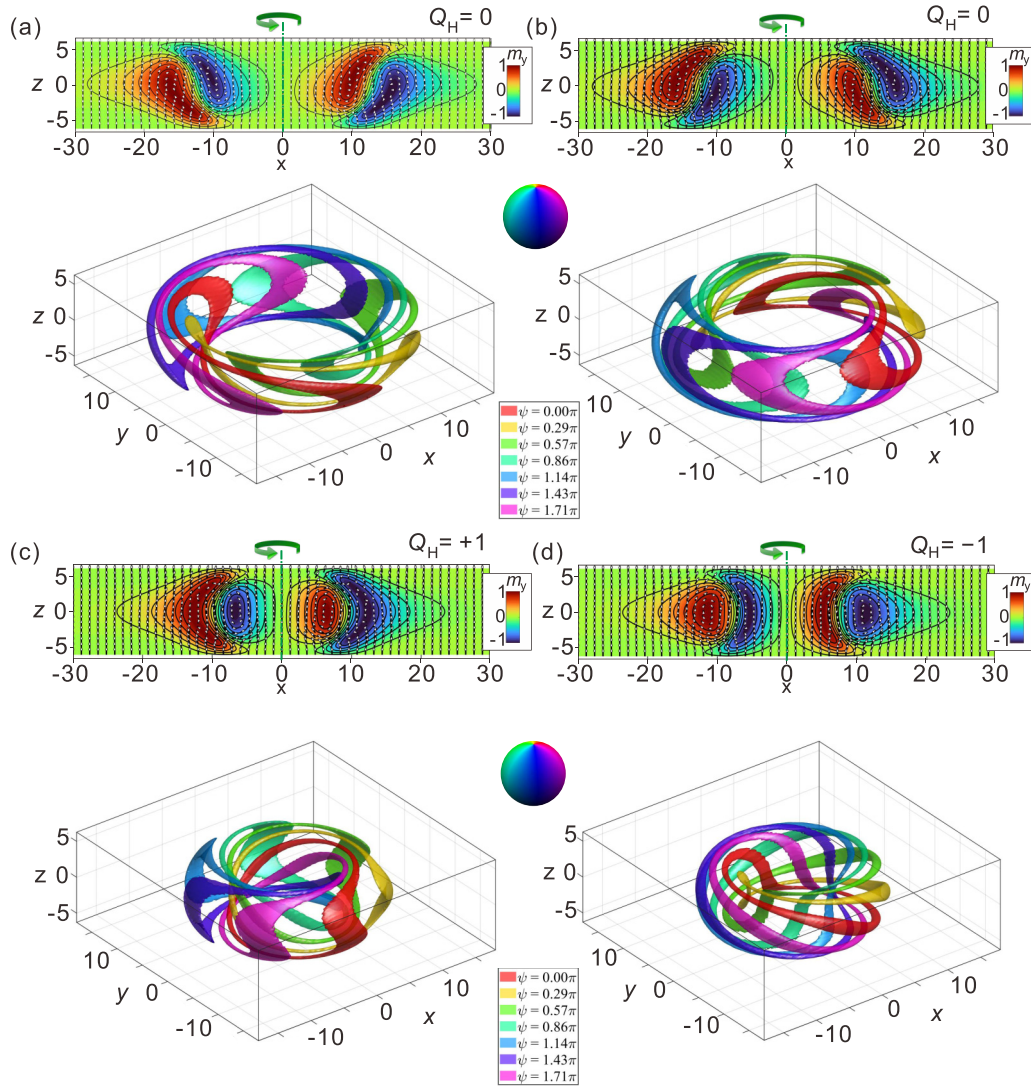


FIG. 3. Four types of isolated hopfions. [(a) and (b)] Trivial hopfions with Hopf index $Q_H = 0$. [(c) and (d)] Hopfions with $Q_H = +1$ and $Q_H = -1$, respectively. Upper panels show color maps of the m_y component in xz cross sections. Lower panels display preimages plotted for $\theta = \pi/2$ and several values of the azimuthal angle ψ .

made for one system can be directly transferred to the other, provided appropriate material parameters and boundary conditions are considered. We emphasize, however, that this minimal model is not intended to capture the full complexity of cholesteric liquid-crystal phase diagrams, which are known to depend sensitively on the elastic anisotropy encoded in the distinct Frank-Oseen constants K_1 , K_2 , and K_3 . Rather, it provides a controlled theoretical framework that isolates the essential topological and geometric mechanisms governing the formation of versatile solitonic states and their ordered metamatter.

CLC offer several advantages over magnetic systems for the investigation of inhomogeneous structures. Their system parameters can be tuned over a broad range to establish the desired experimental conditions, experiments are typically conducted at room temperature and are relatively straightforward, and, most importantly, the resulting structures are readily visualized—often with a degree of clarity that is difficult to achieve in magnetic systems. Nevertheless, in fer-

romagnetic materials, advanced techniques such as electron holography and x-ray magnetic imaging have recently enabled the experimental visualization of complex three-dimensional magnetization configurations [36]. This equivalence has enabled valuable cross-disciplinary insights, with liquid crystals serving as a versatile experimental platform for realizing and visualizing topological solitons originally proposed in magnetic systems, and vice versa [15,31].

The uniaxial anisotropy k_u in CLCs is typically induced by an applied electric field $\mathbf{E}$, giving rise to an energy contribution of the same functional form, $-\frac{\varepsilon_0 \Delta \varepsilon}{2} (\mathbf{m} \cdot \mathbf{E})^2$, where $\Delta \varepsilon$ denotes the dielectric anisotropy. This expression is, however, strictly valid only in the limit of weak dielectric anisotropy and under the assumption of a spatially uniform electric field. In general, the field-induced contribution to the free energy is more complex, as the electric field can generate nonlocal effects. Such nonlocal electrostatic contributions have been analyzed, for example, in the context of electric-field-driven cholesteric fingers in Refs. [37,38]. In the following, we ne-

glect these spatial inhomogeneities of the induced anisotropy, as well as higher-order surface contributions such as the K_{13} and K_4 terms [13,39]. This approximation allows us to retain a minimal bulk free-energy functional and to focus on the essential topological and geometric mechanisms underlying the formation and stability of the three-dimensional solitonic structures considered here.

The Zeeman energy term, $-\mathbf{m} \cdot \mathbf{h}$, is omitted throughout this work, as it has no direct analog in liquid crystal systems. In addition, such a linear coupling does not exist in standard apolar nematic liquid crystals, further justifying its exclusion. In principle, a Zeeman-like term can be substituted by the interplay of an electric field, which fulfills the role of a bulk easy-plane or easy-axis anisotropy, and the surface anchoring acting in an opposite way. Interestingly, in ferromagnetic CLCs formed by colloidal dispersion of magnetic monodomain nanoparticles, one may achieve a linear coupling and facile response to applied magnetic fields, as well [40,41]. Moreover, in recently discovered ferroelectric nematic phases, a spontaneous macroscopic polarization gives rise to a genuine linear coupling to an applied electric field, thereby providing a closer liquid-crystal analog of a Zeeman-type term [42].

In this work, we adopt a unified terminology motivated by both chiral magnetism and chiral liquid-crystal research. For instance, following established practice in chiral magnets, we refer to CF-2 cholesteric fingers as *bimerons*, since their cross-sectional structure, topological charge distribution, and characteristic double-core morphology are fully analogous to magnetic bimerons. In contrast, there is no direct magnetic analog or widely accepted term for CF-1 structures. Our goal is to bridge the vocabulary and conceptual frameworks used in chiral magnetism and chiral liquid crystals to facilitate a more coherent dialogue between the two communities.

III. ISOLATED HOPFIONS

A. Initial hopfion configurations

Initial hopfion configurations were generated in MUMAX3 [43] by arranging the magnetization vectors along several circular trajectories. First, a small cylindrical region was defined at the center of the numerical grid [Fig. 2(a)]. This cylinder was rotated by $\pi/2$ around the x axis and subsequently by an angle α around the z axis, where α was chosen such that the cylinder axis became tangent to a predefined circular trajectory of fixed radius.

These elementary cylinders were then translated and smoothly aligned along the corresponding circular path. The magnetization inside each cylinder was initialized so that, along the circle of radius R , the magnetization vector was coaligned with the local cylinder axis [Fig. 2(b)]. The parameter R defines the hopfion radius used consistently throughout the article for all hopfion types.

In practice, several such circular trajectories were imposed simultaneously [Fig. 2(c) shows five trajectories for a $Q_H = 1$ hopfion], with prescribed and pinned magnetization orientations. After energy minimization in MUMAX3, these constraints relaxed into smooth, defect-free hopfion configurations. For example, in Fig. 2(b), the magnetization

rotates counterclockwise along the inner circle, points downward on the middle circle, and rotates clockwise along the outer trajectory.

To verify numerical robustness, simulations were performed on grids up to $512 \times 512 \times 64$, with a cell size along the z direction of $4\pi/64$. The lateral dimensions were chosen sufficiently large to eliminate spurious interactions between the hopfion structures and the boundaries of the computational domain. Increasing the grid size beyond these values does not affect the qualitative behavior or energetic ordering of the states. A representative ready-to-use MUMAX3 script, in which the grid size can be varied straightforwardly (an example uses $264 \times 264 \times 64$), is provided in the Supplemental Material to ensure reproducibility [8].

This construction protocol does not rely on any analytical ansatz, such as those employed in Ref. [35], and can be readily generalized to generate hopfions of higher topological complexity. Different initial constructions that correspond to the same pinned circular topology consistently relax to identical smooth hopfion states, confirming the stability of the obtained configurations with respect to initialization details.

To obtain the energy dependencies presented in Figs. 8(a) and 8(b), only the magnetization along the circular trajectory of radius R was pinned while computing the eigen hopfion energy $W(R)$. In a subsequent step, the pinning constraints were removed entirely to verify that the relaxed configurations preserve their identity and follow the same qualitative energy trends.

B. Internal structure of isolated hopfions

To elucidate the internal structure of the four types of isolated hopfions identified in our simulations, we computed and visualized several key field characteristics in the xz plane, including the magnetization distribution (upper panel in Fig. 3), the topological charge density ρ_Q , and the local energy density w (Fig. 4). Additionally, we reconstructed preimage surfaces for selected magnetization directions (with $\theta = \pi/2$ and varying azimuthal angle ψ) to reveal the global topology of these solitons (lower panel in Fig. 3).

Remarkably, while the hopfions based on CF-1 fingers are energetically degenerate and possess a zero Hopf index, $Q_H = 0$ [Figs. 3(a) and 3(b)], the two hopfions constructed from CF-2 fingers [Figs. 3(c) and 3(d)] represent genuinely distinct topological solitons with $Q_H = \pm 1$.

Note that we also refer to the solitons generated by rotating the CF-1 fingers as hopfions, even though their Hopf index formally vanishes. This terminology is consistent with Ref. [44], where the term “hopfion” is applied to a broader class of three-dimensional, toruslike solitons characterized by a closed-loop preimage structure. The CF-1-derived textures preserve the essential geometrical features associated with hopfions—namely, a twisted toroidal core and mutually linked isosurfaces—despite having $Q_H = 0$. Thus, their classification as hopfions emphasizes their morphological and dynamical similarity to genuine Hopf-charge-carrying configurations, and aligns our nomenclature with the emerging literature in chiral magnetic and liquid-crystal systems.

Experimentally, different types of hopfionic textures are well distinguished by their characteristic geometric and op-

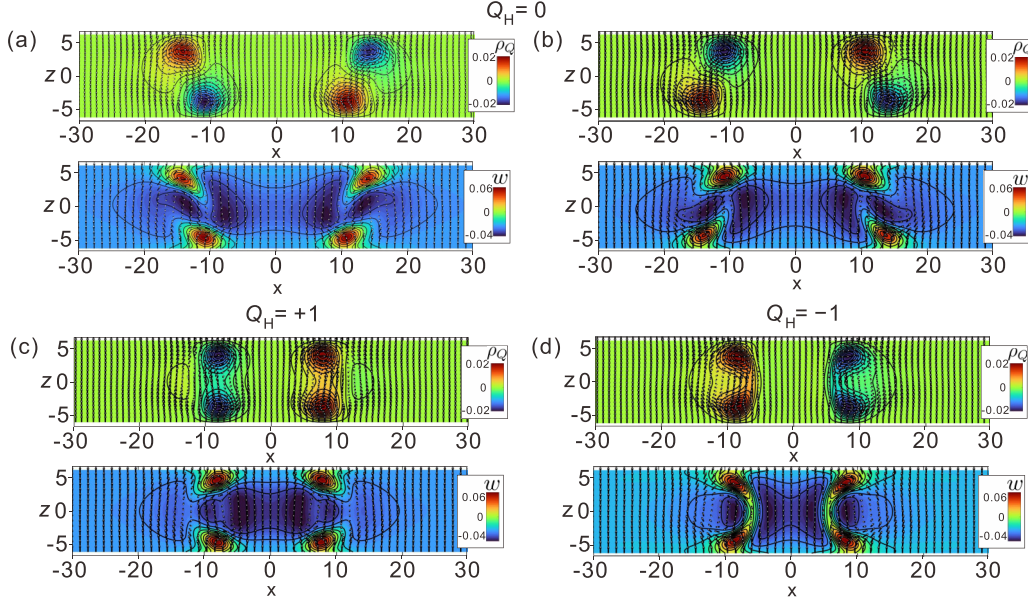


FIG. 4. Topological charge density ρ_Q and local energy density w for isolated hopfions. Color maps show the distributions of ρ_Q and w in the xz plane for all hopfion varieties at $k_u = 0.22$ and $k_s = 50$.

tical signatures. In particular, the three-dimensional director-field configurations associated with CF-1 and CF-2-derived solitons produce distinct patterns under polarizing optical microscopy. However, the structural details of CF-1 and CF-2 cholesteric fingers have been established primarily using fluorescence confocal polarizing microscopy, which enables direct three-dimensional reconstruction of the director field [44,45]. Such studies reveal variations in the internal twist, core morphology, and linking structure of preimages, leading to clearly identifiable experimental fingerprints. This ensures that, despite differences in their topological indices or origins, all hopfion types considered here can be reliably differentiated in practice.

C. Transformation among hopfions

In the present section, we investigate possible pathways for transformations between different hopfion configurations, such as between two trivial hopfions with $Q_H = 0$ and between hopfions with $Q_H = \pm 1$ (see Videos S1–S4 in the Supplemental Material for further details and Ref. [46]). As shown in Fig. 5, hopfion states are separated by significant energy barriers, which are much larger than the energies of the metastable hopfions themselves. Color plots display the spin configurations in the xz plane for trivial hopfions corresponding to the energy minima and the saddle-point configuration (middle panel). The interpolation between hopfions proceeds via the formation of localized defects. In this respect, the spin texture cross section at the saddle point resembles the structure of a toron.

To construct initial transition paths between two metastable hopfion configurations, we employed a direct geodesic interpolation method in spin space. The spin configurations $\{\mathbf{m}_i^A\}$ and $\{\mathbf{m}_i^B\}$ at each lattice site i were obtained from relaxed micromagnetic simulations using MUMAX3. For each lattice site, the intermediate magnetization vector at

interpolation parameter $t \in [0, 1]$ was computed via spherical linear interpolation (slerp) according to

$$\mathbf{m}_i(t) = \frac{\sin[(1-t)\theta_i]}{\sin\theta_i} \mathbf{m}_i^A + \frac{\sin(t\theta_i)}{\sin\theta_i} \mathbf{m}_i^B, \quad (4)$$

where $\theta_i = \arccos(\mathbf{m}_i^A \cdot \mathbf{m}_i^B)$ is the angle between the corresponding spins in the two configurations. This interpolation preserves the unit length of the magnetization vectors at each site.

The total number of intermediate images generated along the path was typically $N = 100$. Each interpolated configuration was then optionally relaxed using overdamped Landau-Lifshitz-Gilbert dynamics (with damping constant $\alpha = 1$) in an attempt to remove sharp local defects without allowing the system to fully collapse into a local energy minimum. The effective magnetic field used during the relaxation included contributions from dimensionless exchange, bulk DMI, uniaxial anisotropy, and surface anisotropy terms, consistent with the micromagnetic model employed in MUMAX3.

This approach provided a continuous set of physically reasonable intermediate states suitable for subsequent energy profile evaluation along the transition path.

IV. MODULATED FINGER PHASES

A. Anisotropy-driven expansion of modulated finger phases

Since the internal structure of isolated hopfions is directly related to the structure of the underlying finger textures, it is instructive to investigate the stability regions of these finger phases alongside their anisotropy-driven evolution. In the broad region of control parameters k_s and k_u [Fig. 6(a)], the modulated CF-1 phase with an equilibrium period p represents the local energy minimum of the functional (3). This stability is particularly pronounced for large values of the

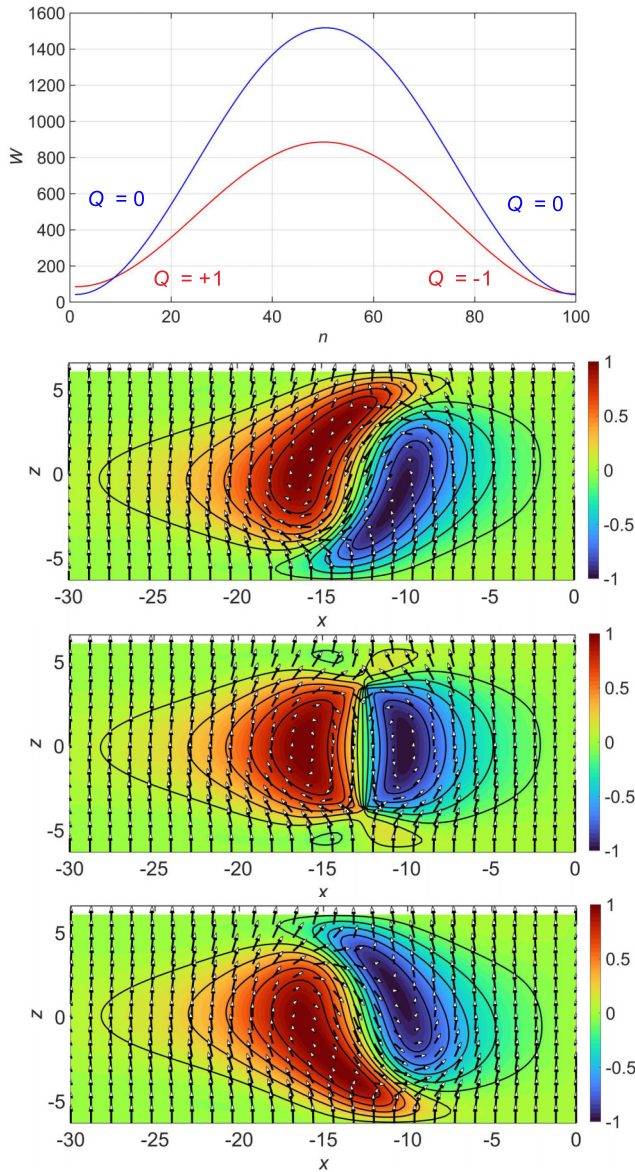


FIG. 5. Energy barriers between different hopfion states. The blue curve corresponds to $Q_H = 0$ hopfions, while the red curve represents $Q_H = \pm 1$ hopfions. Color plots depict three characteristic spin configurations for trivial $Q_H = 0$ hopfions along the transition pathway.

surface anchoring strength, typical for chiral liquid crystals [blue-shaded region in Fig. 6(a)].

As the uniaxial anisotropy k_u increases, the period of the CF-1 phase initially decreases [blue solid line in Fig. 6(b)] and subsequently diverges at a critical value of k_u (e.g., $k_u \approx 0.216$ for $k_s = 50$). This behavior closely resembles the magnetic field-driven evolution of spiral and skyrmion states in chiral magnets [47,48]. Beyond this critical anisotropy, isolated fingers become metastable solitons with positive energy relative to the homogeneous state.

Bimerons exhibit a qualitatively similar evolution, although the critical value of k_u at which isolated bimerons are released is slightly lower [dashed red line in Fig. 6(b); for instance, $k_u \approx 0.212$ at $k_s = 50$].

Representative magnetization distributions of finger textures are presented in Figs. 6(c) and 6(d).

B. Phase diagram of states

We also note that other phases present in the system can be safely omitted from the present analysis, although toron lattices and spiral states represent lower-energy configurations (Fig. 7). This omission is justified by the topological independence of each modulated phase: Geometrical and topological constraints prevent continuous transformations between them under confinement.

The phase diagram of states in the space of control parameters (k_u, k_s) is presented in Fig. 7 for both signs of the surface and bulk anisotropies [32]. The regions of thermodynamic stability for different phases are indicated by distinct colors. The hatched region marks the domain where the toron lattice represents the global energy minimum. In the present work, we focus exclusively on the upper-right quadrant of this phase diagram.

Notably, although the finger phases are overshadowed by toron lattices at large k_s values and by spiral phases at small k_s , there exists no continuous topological path allowing their transformation into these more energetically favorable states. This topological isolation stabilizes the finger phases as metastable textures within their respective regions of the phase diagram. We note that the phase diagram presented in Ref. [35] for trilayered systems omits several of the phases discussed here.

C. Isolated hopfions: from collapse to expansion

The critical lines of the finger phases in Fig. 6(a) serve as reliable guides for identifying the stability regions of metastable hopfions. The eigenenergies W of isolated hopfions, plotted in Figs. 8(a) and 8(b) for a fixed surface anchoring strength k_s and varying uniaxial anisotropy k_u , exhibit distinct minima at equilibrium radii R_0 located in the immediate vicinity of the corresponding finger phase boundaries. As stated in the caption of Fig. 8, for clarity of presentation all energy curves except the lowest one are vertically shifted by individual offsets ΔW so that they coincide at $R = 4$; these shifts do not affect the positions of the minima or the physical conclusions. This behavior confirms that metastable hopfions act as precursor textures stabilized by proximity to modulated finger phases. Notably, the equilibrium radii differ between hopfion types, providing a practical criterion for their experimental identification based on size.

As shown by the dependence of $R_0(k_u)$ in Fig. 8(c), all hopfion types collapse into toronlike structures with increasing electric field [Fig. 8(d)]. In this regime, the specific identity of the initial hopfion type is effectively erased, as all configurations converge to a common toron topology. Conversely, as the system approaches the stability regions of the CF-1 and CF-2 phases, hopfions inflate without bound.

When the electric field is reduced beyond the metastability window, isolated hopfions become elliptically unstable and begin to elongate in order to occupy the available

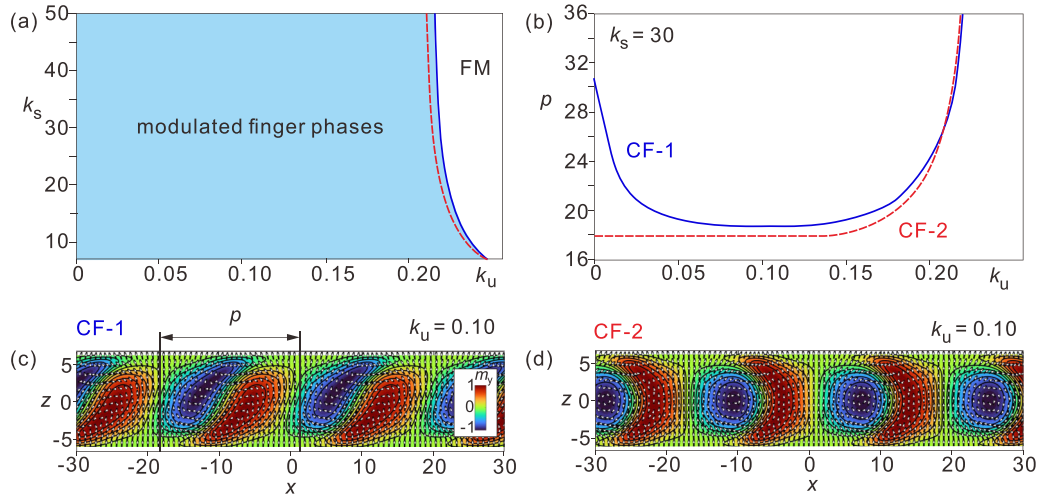


FIG. 6. Phase diagram and internal structure of modulated finger phases. (a) Phase diagram in the space of control parameters (k_u, k_s) . To the left of the solid blue (dashed red) line, the energy of a periodic CF-1 (CF-2) phase becomes lower than that of the homogeneous state. (b) Equilibrium periods of the finger phases as a function of k_u for a fixed surface anchoring $k_s = 30$, illustrating their expansion and the release of isolated fingers. (c) and (d) Magnetization distributions within two-dimensional CF-1 and CF-2 phases, respectively.

space, thereby reconstructing the corresponding modulated finger phase whose equilibrium period was calculated in Fig. 6. An example of such a distorted hopfion is shown in Fig. 8(e). At the initial stage of this deformation, the hopfion does not immediately lose its identity: It still exhibits linked preimages, indicating that its core topological structure remains intact. However, as k_u decreases further, the equilibrium period of the finger phase shrinks, requiring increasingly dense packing of fingers. This increasing geometric frustration is expected to eventually induce ruptures in the hopfion profile, causing a breakdown of the solitonic configuration. Formation of mazelike hopfions was also observed in Ref. [35].

Among the isolated hopfions, those with $Q_H = +1$ consistently exhibit the highest eigenenergy [Fig. 8(f)]. In contrast, the eigenenergies of hopfions with $Q_H = 0$ and $Q_H = -1$ display a crossover behavior. This energetic ordering correlates with the internal structure and topological character of each configuration: $Q_H = +1$ hopfions incur a higher energy cost due to their fully linked preimage topology, while the lower

energy of $Q_H = -1$ hopfions results from the compactness of their constituent bimeron textures, whose crescent-shaped parts face inward, thereby minimizing distortions in the surrounding homogeneous field.

V. HOPFION BAGS

The dependence $R_0(k_u)$ in Fig. 8(c) also indicates the possibility of assembling composite hopfions (Fig. 9), consistent with an effective repulsive interaction between individual hopfions. Before analyzing such composite structures, we first discuss the nature of the repulsion between isolated hopfions.

A. Repulsive interactions between isolated hopfions

The repulsive interhopfion interaction originates from the characteristic energy density asymptotics within hopfion structures (Fig. 10). Both the energy distribution averaged over the film thickness and the local energy density in individual cross sections exhibit negative values at the hopfion outskirts. This implies that any spatial overlap of the energy profiles from neighboring hopfions results in an energy increase, thereby generating an effective repulsive interaction.

The interhopfion interaction potential can be quantified by preparing two hopfion configurations at controlled separations and computing the total system energy as a function of their distance. Direct simulations of this type, however, are computationally demanding, as they require large simulation domains and fine spatial resolution to avoid boundary-induced artifacts. As an efficient alternative, the interaction potential can be estimated from the asymptotic decay of the energy-density profiles associated with an isolated hopfion. By analyzing the overlap of these tails, one can infer the effective interaction strength without performing full two-hopfion simulations.

Composite hopfions (Fig. 11) provide a complementary manifestation of this repulsive interaction, where the formation of nested multisoliton configurations necessarily involves

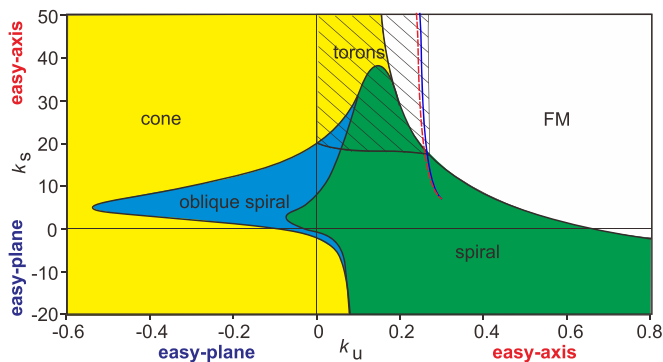


FIG. 7. Phase diagram of states. The regions of thermodynamic stability for different phases are shown in distinct colors. A detailed description of the phase transitions between the modulated phases is provided in Ref. [32].

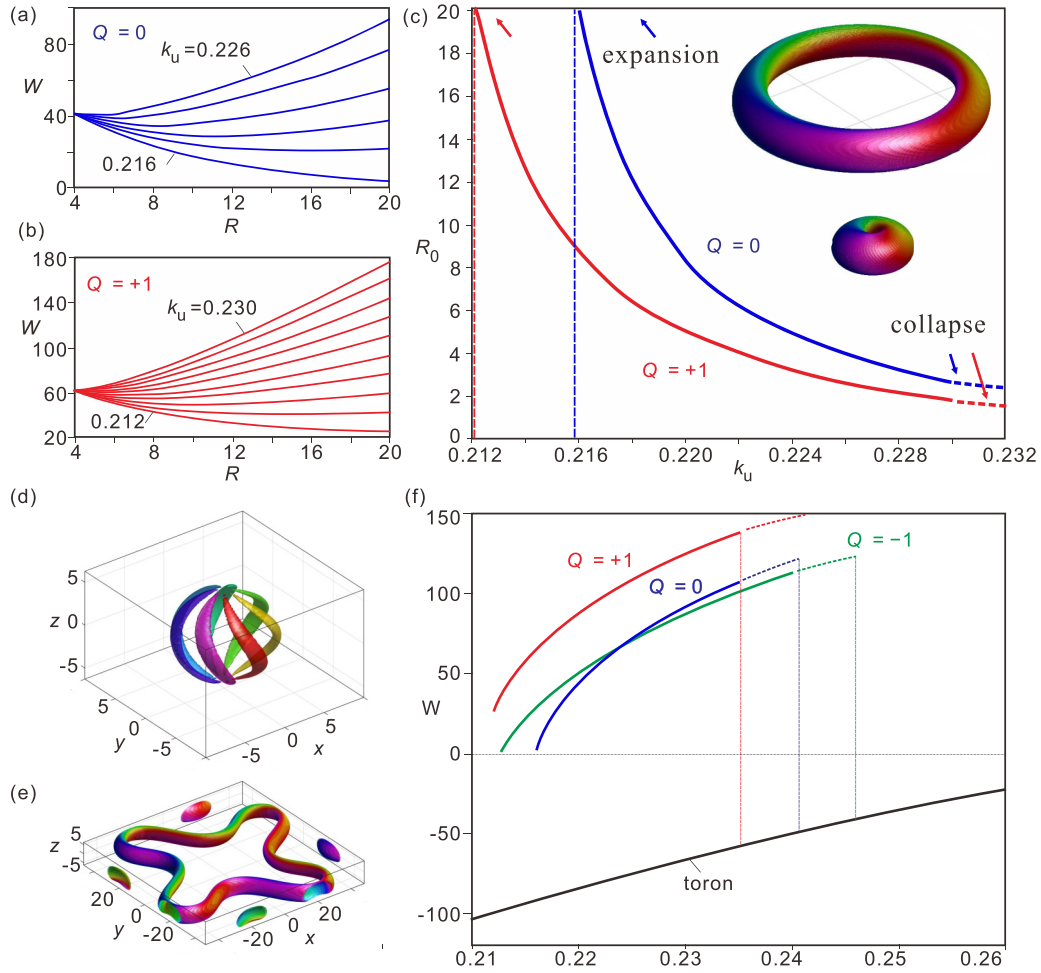


FIG. 8. Energetic and structural characteristics of isolated hopfions. [(a) and (b)] Eigenenergies of isolated hopfions computed relative to the FM state and plotted as a function of the pinned hopfion radius R for various values of k_u . For illustrative purposes, all curves except the bottom one are vertically shifted by individual energy offsets ΔW such that their values coincide at $R = 4$. (c) Equilibrium radii R_0 as a function of k_u , illustrating the inflation of hopfions near the phase boundary and their collapse into torons at large k_u values. (d) Preimage structure corresponding to a toron. (e) Transformation of an isolated hopfion into a modulated finger phase once k_u exceeds the metastability range. (f) Eigenenergies of hopfions as a function of k_u .

energy penalties due to profile overlap and deviations from the equilibrium radii of the constituent hopfions.

B. Composite hopfions

The most natural configuration within a composite hopfion involves placing a $Q_H = \pm 1$ hopfion inside the interior of an inflating $Q_H = 0$ hopfion [Fig. 9(a)]. Since the eigenenergy of the $Q_H = 0$ hopfion remains relatively close to that of the homogeneous state—vanishing as k_u approaches its critical value—it preferentially increases its radius to minimize interhopfion repulsion, while the internal $Q_H = \pm 1$ hopfion largely retains its structure.

For $k_u = 0.22$, the eigenenergies of isolated hopfions are $W(Q_H = 0) = 42.06$ and $W(Q_H = 1) = 86.45$ [Fig. 8(f)]. The total energy of the corresponding composite hopfion configuration is $W = 168.04$. This value allows us to estimate the excess energy $\Delta W = 39.53$, arising from the mismatch of the hopfion radii relative to their equilibrium sizes and from

the intersoliton interactions within the composite structure. The dominant contribution to this excess energy originates from the overlap of the magnetization profiles in the region between the nested solitons.

When this excess energy exceeds the energy barrier separating the internal hopfion state from a toron configuration, a collapse of the inner hopfion can occur, resulting in the formation of a composite texture where a $Q_H = 0$ hopfion encloses a toron [Fig. 9(d)]. The same mechanism can be extended to encompass multiple torons within a $Q_H = 0$ hopfion, which, being near its inflation point and possessing minimal eigenenergy, imposes the least pressure on its interior.

The opposite ordering of hopfions, with a $Q_H = 0$ hopfion placed inside a $Q_H = +1$ hopfion, results in a significantly more stressed composite configuration with a total energy $W = 223.82$ [Fig. 9(c)]. In this arrangement, both hopfions possess radii markedly displaced from their respective equilibrium values: The outer $Q_H = +1$ hopfion tends to shrink, while the inner $Q_H = 0$ hopfion seeks to expand. This mutual

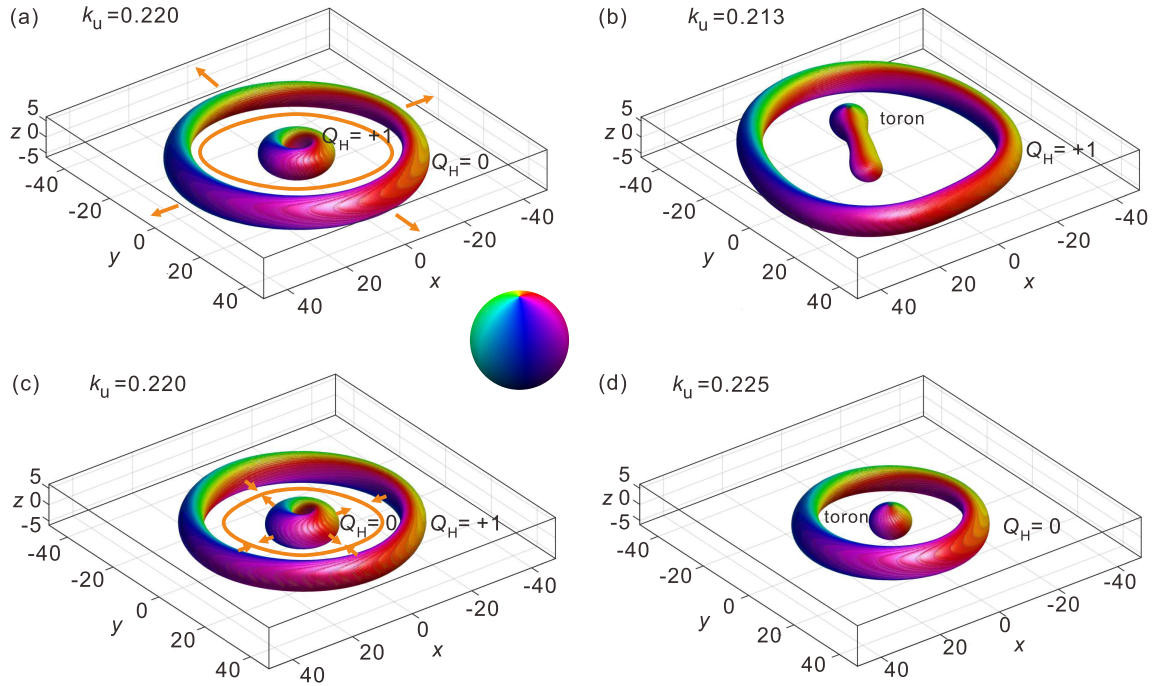


FIG. 9. Hopfion bags. [(a) and (b)] Hopfion bag configurations formed by outer hopfions with vanishing eigenenergies, allowing their radii to flexibly adjust and minimize overlap with the enclosed structures. In panel (a), the interior is occupied by another hopfion, while in panel (b) it consists of an elongating toron, which nucleates a CF-1 finger phase. [(c) and (d)] Hopfion bags stabilized as a compromise between overlapping hopfions with nonequilibrium radii. In panel (d), the inner hopfion collapses into a toron due to excessive overlap.

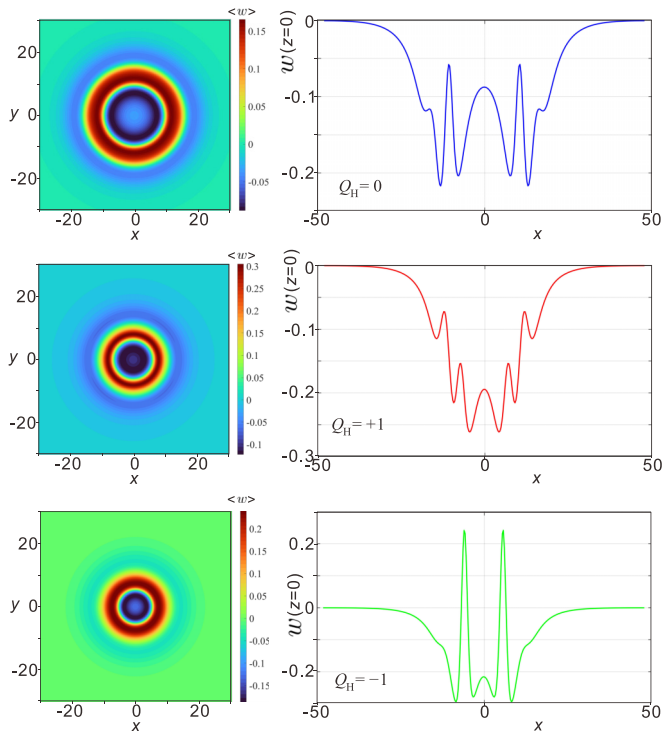


FIG. 10. Energetic fingerprints of isolated hopfions. The energetic characteristics of isolated hopfions are presented as color plots of the energy density on the xy plane, averaged over the film thickness, together with the corresponding energy profiles along the middle cross section of the film.

constraint increases their spatial overlap, leading to a substantial excess energy $\Delta W = 95.31$, primarily originating from the distorted profiles and enhanced intersoliton interaction.

Interestingly, one can envision yet another bag-type configuration realized between the critical anisotropy values for the two hopfion types, for example at $k_u = 0.213$ [Fig. 9(b)], where the outer $Q_H = +1$ hopfion inflates without energy cost, while the interior accommodates the CF-1 finger phase. Notably, this modulated interior phase can be nucleated through the elongation of a toron.

The proposed bag configurations extend the conventional concept of a bag, which typically refers to a bundle of skyrmions enclosed by a circular domain wall [49,50].

VI. UNSTABLE HOPFION LATTICES: ELONGATION INTO FINGER PHASES

Another important conclusion emerging from the $R_0(k_u)$ behavior in Fig. 8(c) is the fundamental instability of hopfion lattices in this system. In particular, the eigenenergy of an expanding hopfion never becomes negative and therefore does not trigger condensation into a periodic modulated phase. This is in stark contrast to spiral and skyrmion states, where isolated solitons retain a finite equilibrium size, and the point at which their eigenenergy crosses zero marks the onset of lattice formation [47]. Hence, isolated hopfions cannot serve as stable building blocks for periodic hopfion lattices in the present confined geometry. Any attempt to realize a hopfion lattice must therefore occur within the intrinsic stability region of the corresponding finger phases.

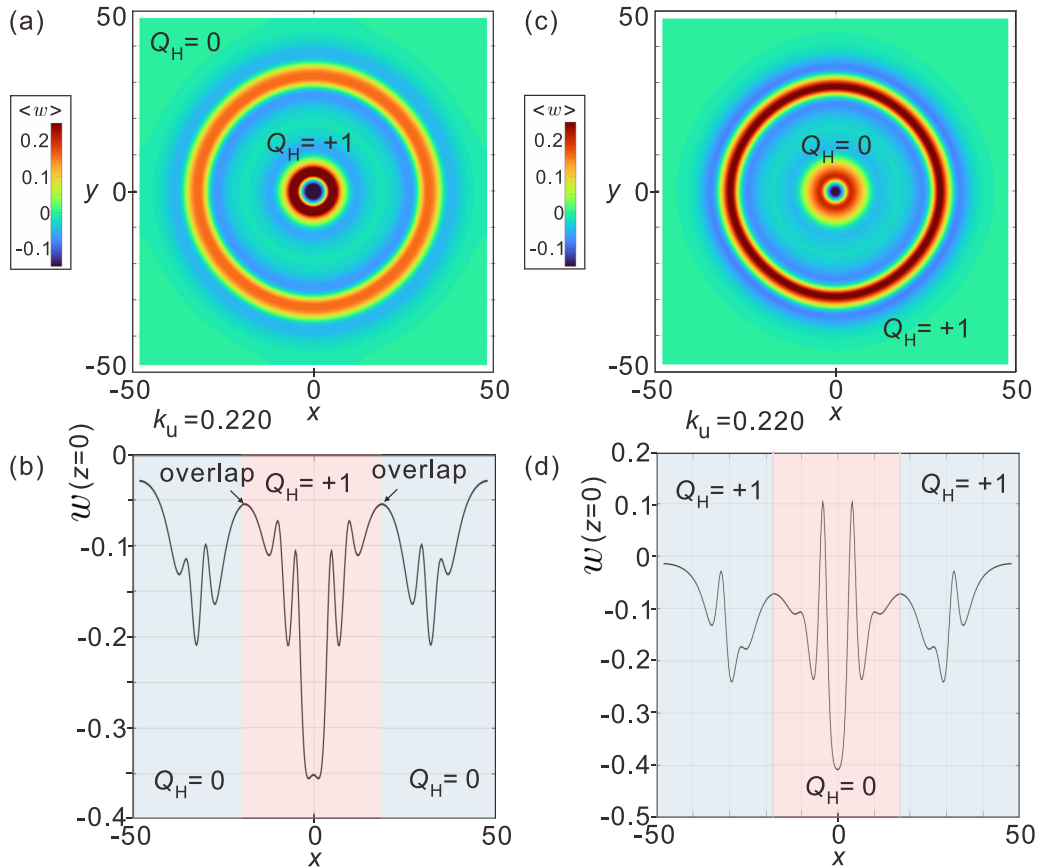


FIG. 11. Energetic fingerprints of composite hopfions. The energetic characteristics of composite hopfions are presented as color plots of the energy density on the xy plane, averaged over the film thickness, together with the corresponding energy profiles along the middle cross section of the film.

To explore this scenario, we computed the total energy of a $Q_H = 1$ hopfion lattice while systematically varying the unit-cell dimensions p_x and p_y (the orange rectangles in Fig. 12). For a square unit cell [Fig. 12(a)], the energy as a function of side length appears to exhibit a minimum along this one-dimensional cut of the parameter space [red curve in Fig. 12(c)]. A comparable behavior is found for a hexagonal cell [Fig. 12(b)], where the ratio $p_y = \sqrt{3}p_x$ is held fixed. However, when the full two-dimensional space of (p_x, p_y) is explored, it becomes clear that these apparent minima are not true minima of the total energy: The square-cell minimum corresponds to a saddle point, while the minimum along the $\sqrt{3}$ ratio lies on a descending slope of the actual three-dimensional energy landscape.

The genuine global minimum is obtained only when the unit cell is elongated, forcing the hopfions themselves to stretch and deform [Fig. 12(d)]. This elongation ultimately transforms the nominal hopfion lattice into a modulated finger phase whose period is fixed by the system parameters. This behavior highlights a key distinction between skyrmion-based and hopfion-based systems: The inherently three-dimensional topology of hopfions prevents the formation of stable periodic lattices under confinement, as any periodic arrangement inevitably relaxes into the lower-energy finger state.

Interestingly, a hexagonal hopfion lattice was theoretically reported in Ref. [51] under applied magnetic fields. The sta-

bility of such periodic states, however, must be assessed by minimizing the energy density with respect to all independent unit-cell parameters (p_x, p_y) and other relevant degrees of freedom. As discussed above, restricting the geometry to $p_y = \sqrt{3}p_x$ may identify a stationary point along a constrained path, which does not necessarily correspond to a true minimum of the full energy surface. Therefore, a careful examination of the complete parameter space is required to establish thermodynamic stability.

A. Unstable skyrmionium lattices

The same phenomenon of metamatter instability was recently reported also for skyrmioniums in quasi-2D chiral magnets [5], where both hexagonal and square skyrmionium lattices were shown to elongate and eventually transform into spiral states. This parallel is not accidental: Skyrmioniums represent the natural two-dimensional cross sections of hopfions, and thus their behavior encodes important aspects of the underlying three-dimensional topology.

The geometric origin of this instability can be understood from a simple packing argument. Modeling the skyrmionium lattice as a hexagonal array, one can estimate the fraction of the sample area occupied by domain walls as

$$\eta = \frac{2\pi(p_x l - l^2)}{\sqrt{3}p_x^2}, \quad (5)$$

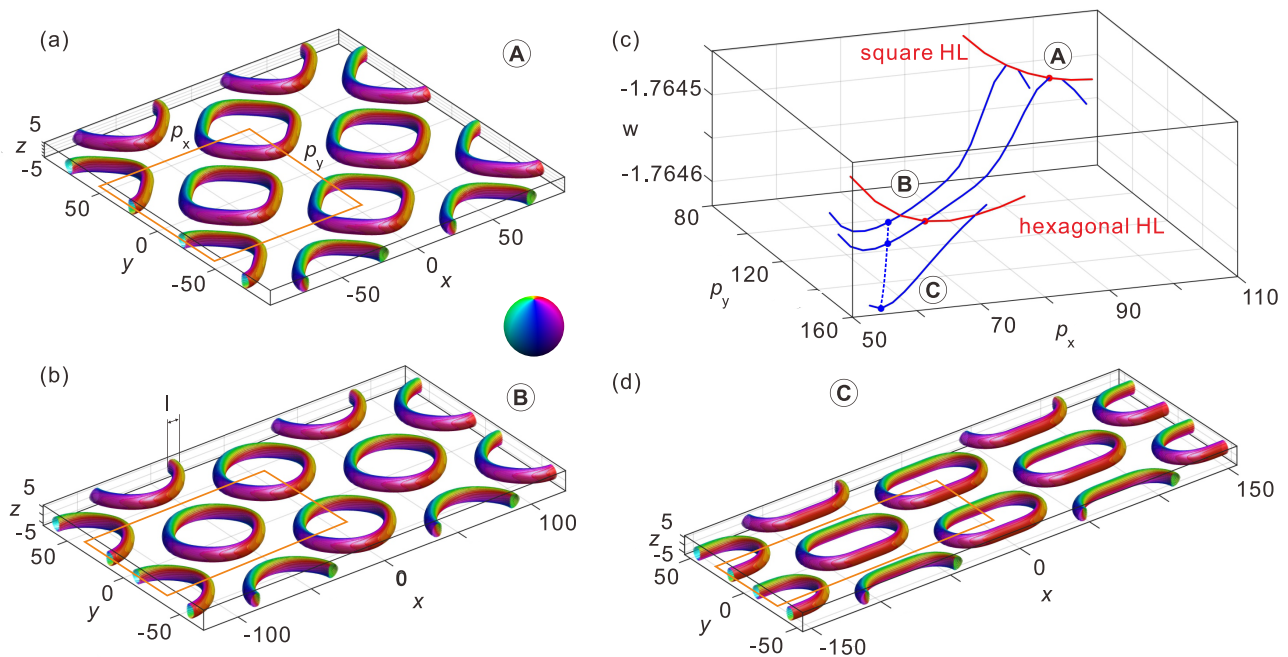


FIG. 12. Instability of hopfion lattices. [(a) and (b)] Preimage structures of square and hexagonal HLs, each corresponding to energy minima when the unit cell geometry is fixed. (c) Energy densities of square and hexagonal HLs, plotted as red curves with red dots marking energy minima. Blue curves show the energy variation with changing unit cell size, revealing the instability of HLs under lattice deformation. (d) Elongated HL structure attempting to reach the global energy minimum by transitioning toward the corresponding modulated finger phase.

where l denotes the domain-wall width and p_x is the lattice constant (Fig. 12). Even in the most favorable limiting case $l = p_x/2$, where the internal “vacuum” of the skyrmionium core vanishes, the maximal packing fraction remains bounded by $\eta_{\max} = \pi/(2\sqrt{3}) \approx 0.91$. Therefore, a skyrmionium lattice is intrinsically unable to fill space completely.

Consequently, once straight domain walls become energetically advantageous, the system can further reduce its energy by eliminating the geometrically unavoidable vacuum regions. This occurs via deformation of the skyrmionium textures: Their cores collapse and the circular walls progressively elongate into spirallike structures. Because additional anisotropic terms modify the energies of straight and curved walls in a comparable manner, they do not remove this geometric constraint. The instability of a pure skyrmionium lattice thus follows from a robust packing limitation rather than from fine details of the model and reflects a more general and universal mechanism.

To summarize, in both cases—skyrmioniums in thin films and hopfions under confinement—the constituent solitons possess internal degrees of freedom that make them susceptible to elongation rather than isotropic packing. As a result, periodic arrays of such solitons do not form true equilibrium states; instead, they continuously deform into the corresponding lower-energy modulated phases (spirals in 2D, finger phases in 3D). This universality highlights an important organizing principle: When the soliton’s internal structure is compatible with a continuously deformable modulated state, lattice formation becomes unstable, and the system naturally relaxes toward an extended periodic texture.

Thus, the instability of hopfion lattices can be understood as the three-dimensional analog of skyrmionium-lattice in-

stability: In both cases, the solitons are not rigid entities but can stretch along a preferred direction, enabling a smooth transformation into the energetically favorable modulated background. This analogy further strengthens the interpretation of hopfions as 3D counterparts of skyrmioniums and points to a unifying framework for understanding metastable “topological metamatter” across different dimensionalities and material platforms.

Remarkably, there exist parameter regimes in which a two-dimensional hexagonal skyrmion lattice becomes unstable with respect to the spiral state, demonstrating that the localized two-dimensional twist within the skyrmion cores does not by itself guarantee their stability. In Ref. [5], it was shown that, at zero magnetic field, the skyrmion lattice elongates into spiral structures irrespective of the strength of the bulk uniaxial anisotropy.

VII. DISCUSSION AND CONCLUSIONS

In this work, we have systematically explored the stabilization mechanisms, structural diversity, and interactions of three-dimensional hopfions in confined chiral magnetic films with surface anchoring. Our results reveal that the metastability regions of distinct hopfion types are closely linked to the phase boundaries of the underlying modulated finger textures. As the uniaxial anisotropy decreases (corresponding to a reduced electric field in liquid crystalline analogs), isolated hopfions inflate and progressively deform, eventually transforming into extended modulated phases: CF-1 phases for $Q_H = 0$ hopfions and CF-2 phases for $Q_H = \pm 1$ types.

This behavior exemplifies a broader principle whereby isolated solitons embedded within a homogeneous ferromag-

netic background act as sensitive probes of nearby modulated phases in the topological phase diagram. Torons, when prevented from assembling into periodic lattices, likewise elongate and fill the available space, reproducing the CF-1 phase [Fig. 9(b)]. Similarly, defect-free isolated skyrmions stabilized at low surface anchoring strengths destabilize through elliptical instabilities, initiating the formation of spiral states [52]. Notably, composite hopfions in such parameter regimes generate mixed CF-1 and CF-2 phases, with their equilibrium periods determined by the energy minima of the corresponding modulated textures.

A key conclusion of this study is the intrinsic instability of periodic hopfion lattices under the confinement conditions considered here. Whereas lattices of two-dimensional solitons, such as skyrmions and torons, are stabilized by mutual interactions and confinement effects, hopfion lattices lack a local energy minimum and spontaneously relax into modulated finger phases. However, this limitation could potentially be overcome by modifying the energetic hierarchy of competing modulated phases. For example, if the conical phase were stabilized as the global energy minimum, hopfions could exist as localized inclusions within a conical background. In such a setting, hopfions would experience an effective attraction mediated by the surrounding conical phase, enabling the formation of stable clusters or even periodic hopfion lattices embedded in a conical matrix.

Our findings thus clarify the delicate interplay of energetic and topological constraints governing the stability of

three-dimensional solitons in confined chiral systems and suggest practical routes for topological phase control via external fields, elastic tuning, or confinement engineering. Moreover, the demonstrated capacity to stabilize multisoliton configurations through hierarchical nesting introduces a versatile strategy for constructing tunable, metastable textures with potentially reconfigurable properties. Future work should address the dynamical stability, nucleation pathways, and intersoliton interactions in time-dependent or nonequilibrium conditions, opening opportunities for topological spintronic and photonic devices based on three-dimensional solitonic matter.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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