

Phase-space representations of quantum error-correcting codes

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We connect the structure theorem for quasiprobability representations to quantum error correction. For a code space invariant under a subgroup of an extended Weyl-Heisenberg group, the Brif-Mann construction gives a semifunctorial representation whenever its kernel satisfies the Stratonovich-Weyl axioms and composes on channels covering full error-correction cycles. With phase-space parity available, a kernel other than the ordinary Wigner one exists iff the code space is invariant under a lattice of displacements: Among compact-phase-space bosonic codes, this is only the ideal Gottesman-Kitaev-Preskill family; in finite dimension, this is every qudit stabilizer code, whose negative volume for odd prime dimension is the syndrome-averaged logical magic. For rotation-symmetric, one-photon, non-Pauli qudit and spin codes, only the ordinary kernel remains, its negativity reflecting the physical carrier, not logical content; a sector-graded representation built from recovery isometries restores a resource reading. We add a channel atlas, a closed-form magic lifetime under displacement noise, a Kirkwood-Dirac layer, and finite-energy corrections.

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I. INTRODUCTION

Phase-space representations turn operator algebra into function theory. States become real functions on a phase space, channels become maps of those functions, and whatever separates the theory from a classical one shows up as negative values [1–3]. This raises the question of whether we could use such a representation to identify encoded resources for quantum computation in a quantum error-correction scheme. For this, we would need a representation adapted to the code used.

Brif and Mann [4] previously constructed a phase-space representation for any system with a Lie-group dynamical symmetry and a chosen isotropy subgroup, and Davis *et al.* [5] applied that construction to the Gottesman-Kitaev-Preskill (GKP) code [6], obtaining the Zak-Gross Wigner (ZGW) function, whose negativity is the magic of the encoded logical state. Separately, Schmid *et al.* [7] and Wagner *et al.* [8] proved structure theorems that fix the form of any linearity-preserving, empirically adequate (semi)functorial quasiprobability representation of a generalized probabilistic theory (GPT). Combining these presents a way to determine the action of an error channel on a code-adapted phase space, valid for whichever code the construction is fed.

However, as we show in this paper, putting these two ingredients together to obtain such a phase-space representation

for any quantum error-correcting code is not so simple. The Stratonovich-Weyl (SW) axioms, together with the requirement that the reconstruction map be idempotent and faithful on the code algebra, strongly restrict the kernel (Proposition III.3 lists the hypotheses in full, including a continuity condition on the kernel and, for non-normalizable code words, the form that we assume for the reconstruction map). Assuming that the code's symmetry makes the phase-space parity available, as it does for every family we consider, a code-adapted kernel differing from the ordinary Wigner kernel exists if and only if the code space is invariant under a lattice of displacements in the relevant Weyl-Heisenberg group.

In the continuous-variable (CV) setting, no finite-dimensional space of normalizable states can be displacement invariant, so we only get a code-adapted kernel differing from the ordinary Wigner kernel if the code words of that code are not elements of $L^2(\mathbb{R})$: The only bosonic codes with compact phase space that fulfill this (and so have a code-adapted kernel of their own) are the ideal, Dirac-comb GKP family. In the discrete-variable setting, however, this criterion is met generically, since the finite Weyl-Heisenberg group is two-step nilpotent and every subgroup containing the center is normal: Every $[[n, k]]_d$ stabilizer code of prime-power dimension has a code-adapted SW kernel, where the code's phase space factorizes into syndrome and logical sectors, errors act as rigid translations, and the corrected cycle is the identity. For odd prime d , the negative volume of an arbitrary state is then its syndrome-probability-weighted logical magic (for $d = 2$, this interpretation of the negative volume sadly does not hold).

Notably, our result also proves that we cannot obtain a code-adapted kernel differing from the ordinary Wigner kernel for rotation-symmetric bosonic codes such as the cat and binomial families, one-photon codes, finite-energy GKP code

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words, qudit codes stabilized by non-Pauli symmetries, or spin codes. For the first four, the SW axioms themselves, under the hypotheses of Propositions III.3 and V.1, leave only the standard kernel: with no freedom at all in the bosonic cases and up to a code-independent sign family in the finite-dimensional one. For those four, no code-adapted representation exists, so the negativity of their (ordinary Wigner kernel) representation just measures the nonclassicality of the physical carrier, rather than any property of the logical system. For spin codes, the averaging prescription supplies no code-adapted kernel at all, leaving the ordinary spherical kernel in place but not giving us the same axiom-direct no go as we have for the other codes.

However, even for codes for which we cannot construct a code-adapted kernel differing from the ordinary Wigner kernel, we show we can still identify encoded resources through another route. Building the representation from a code's recovery isometries instead of from the symmetry of its code space gives a sector-graded logical kernel that inherits the semifunctorial and compositional structure of the framework, and whose negativity is a syndrome-resolved witness of logical magic, two sided on pure logical states, for rotation codes encoding a qudit of odd prime dimension. However, in this representation, errors do not apply rigid translations, and the representation is defined relative to a recovery scheme. This is consistent with the no-go theorem, which requires covariance under the physical Weyl-Heisenberg group; the sector-graded kernel is not covariant in that sense.

The paper is organized as follows. Section II gives background on quasiprobability distributions and the normalization conventions we use, the real and complex structure theorems, and the SW axioms with the CV and discrete-variable (DV) Wigner functions. Section III sets up the code-adapted construction, classifies the kernels the axioms permit, and identifies the class of channels on which the representation composes correctly. Section IV covers codes that allow a code-adapted kernel differing from the ordinary Wigner kernel: ideal GKP codes, every qudit stabilizer code, how we identify resources for these codes, multimode and concatenated lattices, and the dissipation of encoded magic under displacement noise. Section V covers codes that do not allow a code-adapted kernel differing from the ordinary Wigner kernel, and Sec. VI gives the sector-graded rescue. Section VII gives the channel atlas: a reflection identity from which every k -photon jump kernel follows, the gain and dephasing channels, and a comparison between the cat and binomial codes at matched mean photon number. Section VIII covers finite-energy code words and the imperfections of a realistic cycle. Section IX builds a code-adapted Kirkwood-Dirac (KD) layer. Section X summarizes and discusses future works.

We take from previous work the real and complex structure theorems [7,8]; the Brif-Mann construction [4]; the Zak-Gross Wigner function for GKP codes together with its SW properties and its negative-volume interpretation [5]; the Gross Wigner function and the discrete Hudson theorem [2]; the lattice-theoretic account of GKP concatenation [9]; phase-space accounts of stabilizer states and qubit codes in a fixed finite-field Wigner function [10,11]; and the GKP bridge between the continuous and discrete Wigner functions [12].

New to this work are the proof that every Brif-Mann representation with a Stratonovich-Weyl kernel satisfies the structure-theorem conditions (Sec. III B); the identification of the class of transformations on which the representation is compositional (Sec. III E); the classification of the SW kernels available to a code (Proposition III.3 and Corollary III.1) and the no-go theorem that follows from it (Theorem V.1); the code-adapted quotient representation for qudit stabilizer codes and the resource identification of Theorem IV.1; the sector-graded logical representation (Sec. VI); the negative results for one-photon codes, spin codes, and non-Pauli qudit symmetries (Secs. V D–V F); the determination of the cat and binomial representations, and the analysis of the channels acting on them (Secs. V A, V B, and VII); the reflection identity (96) and the jump kernels it generates; the closed-form magic dissipation and lifetime of Proposition IV.5; the code-adapted Kirkwood-Dirac layer (Sec. IX); and the account of finite-energy code words and realistic syndrome extraction (Sec. VIII).

II. BACKGROUND

A. Quasiprobability distributions

A quasiprobability distribution relaxes at least one of the Kolmogorov axioms of probability theory [13], namely, non-negativity, unit total measure, and countable additivity, and the known representations of quantum mechanics [1,14–16] each violate at least one. Every representation used here relaxes only non-negativity: Each is real valued and linear in the state, so additivity holds, and each satisfies the Stratonovich-Weyl standardization axiom (Sec. II C), so that

$$\int_{\mathcal{X}^c} d\mu(\Omega) W_{\hat{\rho}}^c(\Omega) = \text{Tr}[\hat{\rho}] = 1 \quad (1)$$

for any normalized state, where $d\mu$ is the invariant measure on the relevant phase space (explicitly, $\frac{1}{d}dudv$ on the torus $\mathbb{T}_{dl}^2$ in the GKP case of Sec. IV B).

We represent conditional as well as unconditional processes, so we distinguish throughout a *normalized state* (a density operator of unit trace) and its *normalized quasiprobability distribution*, obeying Eq. (1), from an *unnormalized conditional state* (the output of a single Kraus operator, whose trace is the probability of that outcome) and its *subnormalized phase-space function*, whose integral against $d\mu$ is that probability rather than unity. We say at each point what is meant.

B. Real and complex structure theorems

Reference [7] gave a structure theorem for real-valued, finite-dimensional, identity-preserving quasiprobability representations of generalized probabilistic theories, fixing any such functorial representation as $Q(T) = \chi_B \circ T \circ \chi_A^{-1}$ for every transformation $T : A \rightarrow B$, with χ_A the linear map taking an operational state of A into the ontological vector space. The structure theorem applies to maps that are empirically adequate [acting as the identity on every element of $\mathbf{G}(I, I)$, the set of scalars of the GPT, whose values are the outcome probabilities, equivalently, reproducing the outcome probabilities of the operational theory] and linearity preserving [$\Phi(\alpha T_1 + \beta T_2) = \alpha \Phi(T_1) + \beta \Phi(T_2)$ for all $\alpha, \beta \in \mathbb{R}$ and

$T_1, T_2 \in \mathbf{G}(A, B)$, the set of processes from A to B], and it holds for functors: maps preserving sequential composition and the identity on every system. A map preserving only the former is a semifunctor.

Reference [8] generalized this to complex-valued representations (through the complexification functor $\mathbb{C}$), which admits the Kirkwood-Dirac distribution [14,15] used in Sec. IX, and to semifunctors, which make the framework relevant here, since the code-adapted representations we use need not preserve the global identity. The resulting form is

$$Q(T) = \chi_B \circ \mathbb{C}(T) \circ \phi_A, \quad \phi_A = \underline{\chi}_A^{-1} \circ Q(\mathbb{1}_A), \quad (2)$$

where $\underline{\chi}_A$ is the surjective corestriction of χ_A (which is injective, so the corestriction is invertible). The theorem has already been used to study the dissipation of contextuality in a magic state under decoherence [17].

Both theorems are proved for tomographically local, finite-dimensional GPTs. Reference [8] allows the representation spaces to be infinite dimensional, but the operational theory itself must be finite dimensional, whereas our CV sections work on $\mathcal{B}(L^2(\mathbb{R}))$. We therefore do not invoke the structure theorems as black boxes in the infinite-dimensional setting. Following Ref. [5], we verify the properties they require (empirical adequacy, linearity preservation, and the composition law) directly for each representation we use (Secs. III B and III E) and use the structure-theorem form of the maps as the organizing principle. In the finite-dimensional setting of Sec. IV D, however, the theorems apply directly.

C. Wigner functions and the Stratonovich-Weyl axioms

The Wigner function W [1] is a phase-space representation. Acting on an operator $\hat{A}$ on a Hilbert space $\mathcal{H}$, it must satisfy the SW axioms to be a faithful phase-space representation of $\hat{A}$:

Axiom 1 (Linearity): $\hat{A} \mapsto W_{\hat{A}}$ is a linear map with $W_{\hat{A}} = W_{\hat{B}}$ iff $\hat{A} = \hat{B}$.

Axiom 2 (Reality): $W_{\hat{A}} = (W_{\hat{A}})^*$.

Axiom 3 (Standardization): $\int_X d\mu W_{\hat{A}} = \text{Tr}(\hat{A})$, with X the whole phase space.

Axiom 4 (Covariance): $W_{\pi(g)^{\dagger} \hat{A} \pi(g)}(\Omega) = W_{\hat{A}}(g \cdot \Omega)$ for all $\Omega \in X$ and $g \in G$, where π is an irreducible unitary representation on $\mathcal{H}$ of a finite-dimensional Lie group G , and $g \cdot \Omega$ is the natural action of G on X .

Axiom 5 (Traciality): $\int_X d\mu W_{\hat{A}} W_{\hat{B}} = \text{Tr}(\hat{A} \hat{B})$.

Axiom 4 is typically stated for a simply connected group. Nothing below depends on connectedness, since the axiom enters only through the action of G on $X^C = G/H^C$ and through the invariant measure $d\mu$, neither affected by a finite extension. We therefore read it for a finite-dimensional Lie group with finitely many components, as the rotation-symmetric codes of Secs. V A and V B need $[G = H_3(\mathbb{C}) \rtimes \mathbb{Z}_M, \text{ not connected}]$, and drop “simply” as well, for $\text{Mp}(2, \mathbb{R})$ in Proposition III.1 (connected but not simply connected).

Note that a code-adapted representation cannot satisfy Axioms 1 and 5 in the form just stated. Wherever the reconstruction map $\mathcal{E}_C = \phi_C \circ \chi_C$ of Sec. III B is a nontrivial twirl, $\hat{A} \mapsto W_{\hat{A}}^C$ is not injective, since $W_{\hat{A}}^C = W_{\mathcal{E}_C(\hat{A})}^C$ for every $\hat{A}$, and traciality fails on $\mathcal{B}(\mathcal{H})$ for the same reason, holding instead in the form $\int d\mu W_{\hat{A}} W_{\hat{B}} = \text{Tr}[\hat{A} \mathcal{E}_C(\hat{B})]$. We therefore

use throughout the *code-faithful* reading of Axioms 1 and 5: Injectivity in Axiom 1 and the pairing in Axiom 5 are required on the $\mathcal{E}_C$ -invariant subalgebra rather than on all of $\mathcal{B}(\mathcal{H})$. The representation resolves that subalgebra, and the subalgebra contains the code algebra $\mathcal{B}(C)$, which is what “faithful on the code” means. Injectivity in this form is not an extra assumption: Self-duality of the kernel makes χ_C injective on the span of the kernels, which is that subalgebra, so the clause we use in Proposition III.3 and Theorem V.1, that is, $\mathcal{E}_C$ is idempotent and acts as the identity on $\mathcal{B}(C)$, is all that this reading adds to Axioms 2–4. It is an assumption on the reconstruction map, not a consequence of those axioms: Appendix C gives the Husimi kernel as a counterexample. Proposition III.3 and Theorem V.1 need further hypotheses (the parity, continuity of the kernel’s characteristic function, and, where that characteristic function exists only as a distribution, either complete positivity of $\mathcal{E}_C$ or its form as a conditional expectation). Both results list their hypotheses in their statements and say which properties of $\mathcal{E}_C$ are derived and which are assumed. Axioms 2–4 are used unchanged. Where $\mathcal{E}_C = \mathbb{1}$, which by Proposition III.3 is every bosonic code with code words in $L^2(\mathbb{R})$, the two readings coincide and the representation is faithful on the whole algebra. Whenever we say below that a kernel satisfies the Stratonovich-Weyl axioms, we mean this reading (Proposition III.3 and Theorem V.1 state it in full).

The Wigner function was originally a CV object; a DV analog was later given by Gross [2]. Displacement operators play a key role in both:

$$\hat{D}(x, p) = e^{i \frac{px - xp}{\hbar}} = e^{-i \frac{xp}{\hbar}} e^{i \frac{px}{\hbar}} e^{-i \frac{xp}{\hbar}}, \quad x, p \in \mathbb{R},$$

$$\hat{D}_d(a, b) = \omega^{2^{-1}ab} \hat{X}^a \hat{Z}^b, \quad a, b \in \mathbb{Z}_d, \quad (3)$$

where $\hat{X}, \hat{Z}$ are the generalized Pauli operators, $\omega = e^{i2\pi/d}$, and $2^{-1} = \frac{d+1}{2}$ is the inverse of 2 in $\mathbb{Z}_d$ for odd d . We set $\hbar = 1$ throughout. The undisplaced parity operators are fixed by $\hat{\Pi}(0, 0)|x\rangle_{\hat{x}} = |-x\rangle_{\hat{x}}$ and $\hat{\Pi}_d(0, 0)|j\rangle = |-j\rangle$, with $\{|x\rangle_{\hat{x}}\}$ the position basis, $\{|j\rangle\}_{j=0}^{d-1}$ the computational basis, and $-j \cong d - j$; they are Fourier transforms of the displacements, $\hat{\Pi}(0, 0) = \frac{1}{4\pi} \int dz \hat{D}(z)$ and $\hat{\Pi}_d(0, 0) = \frac{1}{d} \sum_z \hat{D}_d(z)$, and are displaced by conjugation:

$$\begin{aligned} \hat{\Pi}(z) &= \hat{D}(z) \hat{\Pi}(0, 0) \hat{D}^\dagger(z) \\ &= \frac{1}{4\pi} \int dz' \hat{D}(z') e^{-iz^T \sigma z'}, \\ \hat{\Pi}_d(z) &= \hat{D}_d(z) \hat{\Pi}_d(0, 0) \hat{D}_d^\dagger(z), \end{aligned} \quad (4)$$

with $z = (x, p)^T \in \mathbb{R}^2$ and $z = (a, b)^T \in \mathbb{Z}_d^2$, respectively, and $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the standard symplectic matrix. The characteristic and Wigner functions are then

$$\begin{aligned} \chi_{\hat{\rho}}^{\text{CV}}(z) &= \text{Tr}[\hat{D}(z) \hat{\rho}], & W_{\hat{\rho}}^{\text{CV}}(z) &= \text{Tr}[2\hat{\Pi}(z) \hat{\rho}], \\ \chi_{\hat{\rho}}^{\text{DV}}(z) &= \text{Tr}[\hat{D}_d(z) \hat{\rho}], & W_{\hat{\rho}}^{\text{DV}}(z) &= \text{Tr}[\hat{\Pi}_d(z) \hat{\rho}]. \end{aligned} \quad (5)$$

The measure of Eq. (1) is $dz = dx dp$, with $d\mu = dz/2\pi$; since $\int dz \hat{\Pi}(z) = \pi \mathbb{1}$, this gives $\int d\mu W_{\hat{\rho}}^{\text{CV}} = \text{Tr}[\hat{\rho}]$, and the kernel $2\hat{\Pi}(z)$ is self-dual with respect to it, so Axiom 5 holds in the form $\int d\mu W_{\hat{A}} W_{\hat{B}} = \text{Tr}[\hat{A} \hat{B}]$. In the DV case, the measure is $\frac{1}{d} \sum_z$, with kernel $\hat{\Pi}_d(z)$. From Sec. III onward, we use complex-plane notation, in which $dz = 2d^2\beta$,

$d\mu = d^2\beta/\pi$, and $\hat{\Delta}_{\text{std}}(\beta) = 2\hat{\Pi}(\beta)$: the same kernel and measure in other coordinates. The unnormalized object $W_{\hat{\rho}}^{\text{std}}(\beta) = \frac{2}{\pi} \text{Tr}[\hat{\Pi}(\beta)\hat{\rho}] = W_{\hat{\rho}}^{\text{CV}}/\pi$, which integrates to $\text{Tr}[\hat{\rho}]$ against $d^2\beta$, is the standard Wigner function; we keep both and relate them at Eq. (78).

We also use the following properties of discrete-variable systems. For odd d , the Gross Wigner function is the unique covariant representation that is non-negative on all stabilizer states [2], already under permutation symmetry alone [18], and its negativity is a magic monotone [3,19,20], which means the resource statements below are two sided on pure logical states and one sided on mixed ones, for the reasons we give in Sec. IV B [2,3,21]. Where we write $A(u)$ for the DV phase-point operators, we mean $A(0) = d^{-n} \sum_{v \in \mathbb{Z}_d^{2n}} D(v)$, the parity $\sum_j | -j \rangle \langle j |$, with $A(u) = D(u)A(0)D^\dagger(u)$, $W_{\hat{\rho}}(u) = d^{-n} \text{Tr}[A(u)\hat{\rho}]$, $\text{Tr}[A(u)A(u')] = d^n \delta_{uu'}$, and $\sum_u A(u) = d^n \mathbb{1}$; this is Eq. (5) written multiplicatively for n qudits, with d^{-n} made explicit, so that $\sum_u W_{\hat{\rho}}(u) = \text{Tr} \hat{\rho}$ against counting measure. Where the two normalizations meet, in Proposition IV.4, we use this one.

We use N in this paper in two different ways: as the number of coherent-state components wherever a cat code is under discussion and as the order $\max\{L, G, 2D\}$ of the code word construction wherever a binomial code is; the binomial code is always written with its pair (N, S) , and where both families appear in one statement, we name the harmonic or the component count rather than the symbol. $\mathcal{M}$ is the stabilizer lattice of a qudit stabilizer code and $\mathbb{Z}_M$ the rotation group of a rotation-symmetric bosonic code, M being its order; the sector modulus of Sec. VI is written as N/d . The central Stratonovich-Weyl kernel is $\hat{K}$ throughout, the jump operators of the amplitude-damping channel are $\hat{J}_\ell$, and the quadrature operator of Lemma III.1 is $\hat{K}_\theta$. Q with a map as its argument is the representation semifunctor of Eq. (12); the objects $Q_{\hat{\rho}}$ and $Q(s_1, s_2)$ of Sec. IX are distributions, not values of it.

Finally, the negative volume of a normalized quasiprobability distribution is

$$\mathcal{N}_{\hat{\rho}}^{\mathcal{C}} := \int_{X^{\mathcal{C}}} d\mu(\Omega) |W_{\hat{\rho}}^{\mathcal{C}}(\Omega)| - 1, \quad (6)$$

the Kenack-Życzkowski convention [22]. Since $\int d\mu |W| \geq |\int d\mu W| = 1$ for a normalized distribution, $\mathcal{N}_{\hat{\rho}}^{\mathcal{C}} \geq 0$, vanishing on non-negatively represented states. It has no universal upper bound.

III. CODE-ADAPTED PHASE-SPACE REPRESENTATIONS

Section III A gives the Brif-Mann prescription and fixes which subgroup of the symmetry group the phase space is quotiented by. Section III B shows that every representation the prescription produces, whose kernel satisfies the Stratonovich-Weyl axioms, meets the conditions the structure theorem requires. Section III C shows that the natural group-averaging recipe for the kernel yields a Stratonovich-Weyl kernel when the code satisfies a normality condition (which every rotation-symmetric bosonic code fails and every stabilizer code satisfies). Section III D then drops the averaging recipe and asks what the axioms permit directly, which turns out to cover most cases. Section III E identifies the class

of channels on which the resulting representation composes correctly.

A. The construction

Consider a Hilbert space $\mathcal{H}$ carrying an irreducible unitary representation π of a symmetry group G . Let us choose a code space $\mathcal{C} \subset \mathcal{H}$, which encodes the logical information. This in turn defines a subgroup $H_0^{\mathcal{C}} \subset G$, the set of elements h such that

$$\pi(h)|\psi\rangle\langle\psi|\pi^\dagger(h) = |\psi\rangle\langle\psi|, \quad \forall |\psi\rangle \in \mathcal{C}. \quad (7)$$

Equation (7) defines the subgroup $H_0^{\mathcal{C}}$ that fixes every state of $\mathcal{C}$, i.e., the subgroup of which $\mathcal{C}$ is (part of) the fixed-point space. The Brif-Mann construction of the phase space instead uses the isotropy subgroup of the chosen reference state,

$$H^{\mathcal{C}} := \{h \in G : \pi(h)|\psi_0\rangle\langle\psi_0|\pi^\dagger(h) = |\psi_0\rangle\langle\psi_0|\} \supseteq H_0^{\mathcal{C}}, \quad (8)$$

since $H^{\mathcal{C}}$, not $H_0^{\mathcal{C}}$, must be quotiented out for the coherent states $|\Omega\rangle = \pi(\Omega)|\psi_0\rangle$ to be labeled by the points of $X^{\mathcal{C}} = G/H^{\mathcal{C}}$. For the GKP code, the two coincide for a generic reference code word, but not the stabilizer code words: $|\bar{0}\rangle$ is fixed at the projector level by the logical $\hat{Z} = e^{i\hat{x}}$ as well as by the stabilizers (its position comb sits at $x \in d\mathbb{Z}$ and $l \cdot dl = 2\pi$), so its isotropy subgroup is the rectangular lattice $d\mathbb{Z} \times l\mathbb{Z}$, strictly larger than the stabilizer lattice. The ZGW construction quotients by the stabilizer lattice $H_0^{\mathcal{C}}$ (the isotropy subgroup of a reference code word that is not an eigenstate of any logical displacement), which it is allowed to do, as the resulting kernel does not depend on the reference state at all [5]. The two subgroups also differ for Secs. V A and V B's rotation-symmetric codes: There, $H^{\mathcal{C}}$, the isotropy of the character code word we take as reference, contains the rotation acting on the code space as the logical $\hat{Z}$, and $H_0^{\mathcal{C}}$ is its index-two subgroup. We write $H^{\mathcal{C}}$ for the subgroup by which the phase space is quotiented (the reference-state isotropy for the rotation-symmetric codes, $H_0^{\mathcal{C}}$ for GKP).

We then follow Ref. [4]. Pick a reference state $|\psi_0\rangle \in \mathcal{C}$; this gives a Wigner function depending on the choice of $\mathcal{C}$ and defined over the phase space $X^{\mathcal{C}} = G/H^{\mathcal{C}}$, with coherent states $|\Omega\rangle := \pi(\Omega)|\psi_0\rangle$ for $\Omega \in X^{\mathcal{C}}$. The identity on the full Hilbert space $\mathcal{H}$ does not need to be preserved by the resulting function, making the construction a semifunctor rather than a functor in general; Proposition III.3 shows that it fails to be preserved for codes that allow a code-adapted representation.

The function obtained for a state $\hat{\rho}$ is

$$W_{\hat{\rho}}^{\mathcal{C}}(\Omega) = \text{Tr}[\hat{\Delta}^{\mathcal{C}}(\Omega)\hat{\rho}], \quad (9)$$

where $\hat{\Delta}^{\mathcal{C}}(\Omega)$ is the kernel associated with the code space $\mathcal{C}$ and is defined as

$$\hat{\Delta}^{\mathcal{C}}(\Omega) = \sum_{\nu} Y_{\nu}(\Omega) \hat{D}_{\nu}^{\dagger}, \quad (10)$$

where $\hat{D}_{\nu}$ are the tensor operators and Y_{ν} are the harmonic functions over $L^2(X^{\mathcal{C}}, \mu)$, with $d\mu$ the invariant measure over $X^{\mathcal{C}}$, and where ν ranges over the spectrum of the Laplace-Beltrami operator over $L^2(X^{\mathcal{C}}, \mu)$. The harmonic functions $Y_{\nu}(\Omega)$ are its eigenfunctions, a complete orthonormal basis for square-integrable functions on the phase space. The

tensor operators $\hat{D}_v$ are orthogonal in $\mathcal{B}(\mathcal{H})$ and span its $\mathcal{E}_C$ -invariant subalgebra, which is all the representation resolves and which contains $\mathcal{B}(\mathcal{C})$; they are a basis of $\mathcal{B}(\mathcal{H})$ itself when $\mathcal{E}_C = \mathbb{1}$. The kernel is applied to arbitrary $\hat{\rho} \in \mathcal{B}(\mathcal{H})$ throughout, including to operators supported outside the code space. The tensor operators are defined so they transform under the symmetries of the space in the same way that the harmonic functions do. They can be recovered from the harmonic functions by inverting an overlap relation [4], with the phase freedom fixed so that the resulting Wigner function depends only on the symmetry of the code space and not on a reference state. We do not need that inversion: Sections III C and III D obtain the kernel directly.

B. Semifunctoriality of the construction

This construction works for the class of symmetry-defined codes, whose code space is invariant under a subgroup of a group G acting irreducibly on the carrier space. Reference [4] builds phase-space representations, not codes, and not every code arises from such a symmetry (e.g., numerically optimized and nongroup bosonic codes do not). We show here that any representation whose kernel satisfies the Stratonovich-Weyl axioms, in the code-faithful reading of Sec. II C, has the properties the structure theorem requires in the sense given in Sec. II B. This does not necessarily mean a given prescription produces such a kernel, as we see in Secs. III C and III D: For the GKP code, for every qudit stabilizer code of prime-power dimension, and for the cat and binomial codes, a Stratonovich-Weyl kernel exists and is identified there, so the conclusion applies to all of them, though for the rotation-symmetric families the kernel is not the one the averaging prescription would give.

Because the two notions of Eqs. (7) and (8) imply different things, we must specify what that invariance means. The GKP code space is the fixed-point space of its stabilizer lattice. The binomial-code space is a two-dimensional subspace of the (infinite-dimensional) fixed-point space of $\hat{R}_{S+1}$. The four-component cat-code space is invariant under $\mathbb{Z}_4$ but is not contained in the fixed-point space of $\hat{R}_4$, which acts there as the logical $\hat{Z}$, only in that of $\hat{R}_4^2 = (-1)^{\hat{n}}$. So “fixed-point space of a subgroup” is correct only for GKP, and we use the weaker “invariant under a subgroup” for the class as a whole. Two qualifications keep the weaker phrase from being empty: The subgroup must be nontrivial modulo the center, since every code space is invariant under the global phases, and the phase space must be the quotient $X^C = G/H^C$ built from the isotropy subgroup of an actual code word (for ideal GKP, from the pointwise stabilizer H_0^C , as Sec. III A explains). A code specified by an arbitrary list of Fock amplitudes has H^C equal to the center alone, $X^C \simeq \mathbb{C}$, and no code-specific structure (Proposition III.3 and Corollary III.1).

Every representation generated by a kernel satisfying the Stratonovich-Weyl axioms is an empirically adequate and linearity-preserving semifunctor. Empirical adequacy follows from Axiom 5, which ensures that outcome probabilities follow Born’s rule; linearity preservation follows from the linearity of the trace in Eq. (9). Whether the representation is more than a semifunctor depends on the reconstruction map: It is a functor when the reconstruction returns $\hat{\rho}$ itself, as

for the standard Wigner function, and a genuine semifunctor when it returns the twirled operator $\mathcal{E}_C(\hat{\rho})$, preserving the identity only on the subtheory defined by the code space. Which of these is the case is settled by Proposition III.3, under its hypotheses: The reconstruction map is a nontrivial twirl, and the representation a genuine semifunctor, iff the code space is invariant under a lattice of displacements. In the single-mode continuous-variable setting that requires non-normalizable code words, so for every bosonic code with code words in $L^2(\mathbb{R})$, the twirl is the identity and the representation is a functor; in finite dimension, the lattice condition is met by every stabilizer code (Proposition III.2) and the twirl is nontrivial there.

The structure theorem can therefore be applied to any code in this class. (It is *not* available for a code with no such symmetry.) For such a code, the construction provides a subgroup H^C , a homogeneous phase space $X^C = G/H^C$ with unique invariant measure $d\mu(\Omega)$, and, wherever an admissible kernel exists, a code-adapted self-dual kernel $\hat{\Delta}^C(\Omega)$ [4]. The forward map χ_C traces an operator against the kernel [Eq. (9)]; the inverse map ϕ_C is built from the twirling map

$$\mathcal{E}_C(\hat{\rho}) = \int_{X^C} d\mu(\Omega) W_{\hat{\rho}}^C(\Omega) \hat{\Delta}^C(\Omega). \quad (11)$$

For any operational transformation T , its phase-space representation $Q(T)$ evaluated on an input distribution $W(\Omega)$ yields a new distribution at coordinate Ω' :

$$[Q(T)W](\Omega') = \text{Tr} \left[\hat{\Delta}^C(\Omega') T \left(\int_{X^C} d\mu(\Omega) W(\Omega) \hat{\Delta}^C(\Omega) \right) \right]. \quad (12)$$

Substituting the isotropy subgroup H^C of any code in the class just described, together with the invariant measure $d\mu(\Omega)$ on $X^C = G/H^C$, adapts it to that code.

C. Group averaging

We can make this construction explicit, in the same form for every code in the class. Let

$$\hat{\Delta}_{\text{std}}(\Omega) = 2\hat{\Pi}(\Omega), \quad \hat{\Pi}(\Omega) = \hat{D}(\Omega)(-1)^{\hat{n}}\hat{D}^\dagger(\Omega) \quad (13)$$

be the standard Stratonovich-Weyl kernel from Sec. II C, for a single mode. For m modes, the same construction gives $\hat{\Delta}_{\text{std}}(\Omega) = 2^m \hat{\Pi}(\Omega)$, with $\hat{\Pi}(\Omega) = \hat{D}(\Omega)(-1)^{\hat{n}_{\text{tot}}}\hat{D}^\dagger(\Omega)$, since $\text{Tr}[(-1)^{\hat{n}_{\text{tot}}}] = 2^{-m}$ and standardization requires unit trace (we set $m = 1$ everywhere except in Theorem V.2 and Sec. IV F). A natural candidate for a code-adapted kernel is the average of $\hat{\Delta}_{\text{std}}$ over the isotropy subgroup H^C ,

$$\hat{\Delta}^C(\Omega) = \frac{1}{|H^C|} \sum_{h \in H^C} \pi(h) \hat{\Delta}_{\text{std}}(\Omega) \pi^\dagger(h) \quad (14)$$

(the sum becoming an integral for continuous H^C). We now have to check whether our H^C possesses three properties. The latter two are held by every H^C , but the first is not:

(1) *Covariance*: Each $\pi(h)$ belongs to the extended Weyl-Heisenberg group, so $\pi(h)\hat{\Pi}(\Omega)\pi^\dagger(h) = \hat{\Pi}(h \cdot \Omega)$, where $h \cdot \Omega$ is the induced action on X^C , so $\hat{\Delta}^C(\Omega) = \frac{1}{|H^C|} \sum_h \hat{\Delta}_{\text{std}}(h \cdot \Omega)$. Averaging does not preserve covariance in general: $\pi(g)\hat{\Delta}^C(\Omega)\pi^\dagger(g)$ averages $\hat{\Delta}_{\text{std}}$ over the coset

gH^C , while $\hat{\Delta}^C(g \cdot \Omega)$ averages it over $H^C g$, so $\hat{\Delta}^C(g \cdot \Omega) = \pi(g)\hat{\Delta}^C(\Omega)\pi^\dagger(g)$ holds for all $g \in G$ iff H^C is normal in G . For the GKP code, H^C is a lattice of displacements inside the Abelian translation group and is normal, so covariance holds; for a rotation-symmetric code, H^C contains a rotation, and a rotation conjugated by a displacement is not a rotation, so covariance fails. Equation (14) is therefore valid only in the first case, so we do not use it in the second.

(2) *Standardization*: Using $\frac{1}{\pi} \int d^2\Omega \hat{D}(\Omega) \hat{X} \hat{D}^\dagger(\Omega) = \text{Tr}[\hat{X}] \mathbb{1}$ together with $\text{Tr}[(-1)^{\hat{n}}] = 1/2$,

$$\int d\mu(\Omega) \hat{\Delta}^C(\Omega) = \text{Tr}[\hat{\Delta}_{\text{std}}(0)] \mathbb{1} = \mathbb{1} \quad (15)$$

for every code, meaning the group average leaves the trace unchanged. When the kernel is covariant, so that $\hat{\Delta}^C(\Omega) = \hat{D}(\Omega) \hat{K} \hat{D}^\dagger(\Omega)$, and $\hat{K}$ is trace class, this is equivalent to $\text{Tr}[\hat{K}] = \text{Tr}[\hat{\Delta}^C(0)] = 1$ (in which form we check it below). This is a necessary condition: For the ideal GKP code, $\hat{K}$ is a lattice sum of displacement operators, meaning that $\text{Tr}[\hat{K}]$ diverges, so we instead have to verify standardization through the integral given in Eq. (41).

(3) *The reconstruction map is the twirl over H^C* : The standard kernel is self-dual, $\hat{A} = 2 \int d^2\Omega W_{\hat{A}}^{\text{std}}(\Omega) \hat{\Gamma}(\Omega)$, and the standard Wigner function is covariant, $W_{\hat{A}}^{\text{std}}(g \cdot \Omega) = W_{\pi^\dagger(g)\hat{A}\pi(g)}^{\text{std}}(\Omega)$. Substituting Eq. (14) into $\phi_C \circ \chi_C$ and changing integration variable in each term of the resulting double sum leaves a single sum over the group,

$$\phi_C(\chi_C(\hat{A})) = \mathcal{E}_C(\hat{A}) = \frac{1}{|H^C|} \sum_{h \in H^C} \pi^\dagger(h) \hat{A} \pi(h). \quad (16)$$

Because H^C is a group, $\mathcal{E}_C$ is trace preserving and idempotent, projecting onto the H^C -invariant subalgebra. This is true for every H^C , normal or not; the kernel must also be covariant for $\mathcal{E}_C$ to be the reconstruction map of an admissible Stratonovich-Weyl representation, making $Q(\mathbb{1})$ the idempotent required by the structure theorem.

This reproduces the known GKP case. For the GKP code, $H^{C_{\text{GKP}}}$ is the stabilizer lattice of displacements by multiples of $d\mathbb{1}$, which is normal in G , so averaging the displaced parity over that lattice gives a sum of displacement operators over the symplectic dual lattice [Eq. (35)] (Ref. [5]’s Zak-Gross kernel). For rotation-symmetric codes (Secs. V A and V B), the subgroup is $U(1) \times \mathbb{Z}_M$, meaning that Eq. (14) does not give a Stratonovich-Weyl kernel at all. Appendix A shows why it fails. We replace Eq. (14) with Proposition III.3 there, which fixes the kernel from the axioms directly, so requires no reference-state expansion (however, for a non-Gaussian reference state, this produces a kernel that is neither covariant nor self-dual).

Nothing in items (i)–(iii) required the elements of H^C to be displacements. All three properties only require that $\pi(h)$ act on phase space by a measure-preserving affine symplectic map and commute with the parity operator. These requirements hold throughout the metaplectic extension of the Weyl-Heisenberg group: The quadratic Hamiltonians that generate rotations and squeezes are even in $\hat{a}$ and $\hat{a}^\dagger$, so commute with $(-1)^{\hat{n}}$, and symplectic maps have unit determinant, and therefore preserve $d^2\Omega$.

Proposition III.1. Let $G = H_3(\mathbb{C}) \rtimes \text{Mp}(2, \mathbb{R})$ and $H^C \subseteq G$ be a compact isotropy subgroup of a reference code word, so that the average in Eq. (14) is taken against a normalized Haar measure. Then, the kernel of Eq. (14) satisfies the Stratonovich-Weyl standardization axiom and induces the twirl of Eq. (16). It satisfies the covariance axiom on the normalizer of H^C in G , and hence on all of G iff H^C is normal in G . The only compact normal subgroups of this G lie in its central $U(1)$ of phases, which acts trivially by conjugation; so for a compact isotropy group, Eq. (14) yields a Stratonovich-Weyl kernel only when the average does nothing (i.e., never for a rotation-symmetric code).

We prove this in Appendix C. The averaging prescription is therefore never useful for a compact isotropy group: Whatever kernel the code admits must be found from the axioms themselves, which we do in Sec. III D.

Let us now consider finite-dimensional codes. Let P_n be the n -qudit Weyl-Heisenberg group, generated by the displacements $D(v)$, $v \in \mathbb{Z}_d^{2n}$, together with its center $\mathcal{Z}$ of scalar phases $\{\omega^e\}$ for odd d , $\{i^e\}$ at $d = 2$, and $\mathcal{C}$ the code space of a stabilizer code $[[n, k]]_d$, with stabilizer group $\mathcal{S} = \{D(m) : m \in \mathcal{M}\}$ for an isotropic subgroup $\mathcal{M} \subset \mathbb{Z}_d^{2n}$ of order d^{n-k} . (For odd d in symmetric ordering, a stabilizer group with phase-free generators is phase-free as a set, since commuting displacements compose without phases; for $d = 2$, this fails, but we only use phase-freeness in Proposition IV.2 and Theorem IV.1, which are restricted to odd d anyway (Sec. IV E). In both cases, $\mathcal{M} = -\mathcal{M}$, since $\mathcal{M}$ is a subgroup.)

Proposition III.2 (Normality). Conjugation in P_n acts by $gD(v)g^{-1} = \omega^{[u,v]}D(v)$ for $g = zD(u)$, with $z \in \mathcal{Z}$, and $[u, v]$ the symplectic form. Every subgroup of P_n that contains the center is therefore normal; this means that the projective isotropy group $H^C = \langle \mathcal{Z}, \mathcal{S} \rangle$ of a generic code word is normal in $G = P_n$, so Eq. (14)’s averaging prescription is covariant on all of G . Proposition IV.1 verifies it meets the remaining Stratonovich-Weyl axioms, as per Sec. II C’s code-faithful reading.

Proposition III.1’s obstruction cannot arise in P_n because there are no rotations to conjugate: A stabilizer group is a displacement subgroup by definition, and two-step nilpotency makes it normal once the center is adjoined (Proposition III.2). Every DV stabilizer code is in this sense “GKP-like”: Its code space is the fixed-point space of a lattice of displacements, so the twirl below is a finite-group average, so a genuine channel, and the finite-dimensional structure theorems [7,8] apply as black boxes. We explore the consequences of this in Sec. IV D and cover non-Pauli analogs in Sec. V D.

As in the CV case, we quotient by the isotropy group of a generic code word (i.e., not an eigenstate of any logical Pauli), meaning that $H^C = \langle \mathcal{Z}, \mathcal{S} \rangle$ is the center together with the stabilizer lattice; for the three-qutrit repetition code, numerics confirm that the isotropy of a Haar-random code word is $\langle \mathcal{Z}, \mathcal{S} \rangle$, whose image in $\mathbb{Z}_d^{2n}$ is the nine-element lattice $\mathcal{M}$.

D. What the axioms allow

The two requirements just used (covariance and an idempotent reconstruction map) are restrictive enough, together with the further hypotheses listed in Proposition III.3, to

determine the kernel of every code with normalizable code words. Covariance forces the kernel to be the displaced conjugate of a single operator, $\hat{\Delta}^C(\Omega) = \hat{D}(\Omega)\hat{K}\hat{D}^\dagger(\Omega)$, with $\hat{K}$ invariant under H^C . For such a kernel, the reconstruction map acts on characteristic functions by pointwise multiplication,

$$\chi_{\mathcal{E}_C(\hat{A})}(\xi) = m(\xi) \chi_{\hat{A}}(\xi), \quad m(\xi) := |\chi_{\hat{K}}(\xi)|^2, \\ \chi_{\hat{A}}(\xi) = \text{Tr}[\hat{A}\hat{D}(\xi)], \quad (17)$$

and standardization fixes $\chi_{\hat{K}}(0) = \text{Tr}[\hat{K}] = 1$ whenever $\hat{K}$ is trace class. [When it is not trace class, use the integral in Eq. (15).]

The classification rests on one fact about which subgroups of the displacement group can leave a code space alone.

Lemma III.1. Let $\mathcal{C} \subset L^2(\mathbb{R}^m)$ be a finite-dimensional subspace, for any number of modes $m \geq 1$. Then, $\hat{D}(\beta)\mathcal{C} = \mathcal{C}$ implies $\beta = 0$.

We prove this in Appendix C: To summarize, $\hat{D}(\beta)$ restricted to $\mathcal{C}$ would be a unitary on a finite-dimensional space, so would have an eigenvector, whereas $\hat{D}(\beta) = e^{i\lambda\hat{K}_\theta}$, with $\hat{K}_\theta$ a quadrature operator (not the central kernel $\hat{K}$), after a passive rotation taking β into a single mode, whose spectrum is purely absolutely continuous, so it has no eigenvectors in $L^2(\mathbb{R}^m)$ for $\beta \neq 0$. The same argument excludes a non-trivial squeeze (whose generator is the dilation operator), but not rotations $e^{i\theta\hat{n}}$, since the Fock states are their eigenvectors, which is why the rotation-symmetric codes exist. The ideal GKP code instead escapes the lemma as its code words are Dirac combs [i.e., tempered distributions rather than elements of $L^2(\mathbb{R})$], so $\mathcal{C}_{\text{GKP}}$ is not a subspace of $L^2(\mathbb{R})$ at all.

Proposition III.3. Let G contain the displacements of $m \geq 1$ modes, and let the kernel $\hat{\Delta}^C$, which defines both χ_C and ϕ_C (Sec. III B), satisfy three hypotheses:

- (H1) Axioms 2–4, as stated in Sec. II C;
- (H2) the code-faithful form of Axioms 1 and 5: the reconstruction map $\mathcal{E}_C = \phi_C \circ \chi_C$ is idempotent and acts as the identity on the code algebra $\mathcal{B}(\mathcal{C})$ (i.e., the representation is faithful on the code);
- (H3) if $\chi_{\hat{K}}(\xi) = \text{Tr}[\hat{K}\hat{D}(\xi)]$ is a function, it is continuous (for a bounded, non-trace-class $\hat{K}$, we mean the trace in the Abel-regularized sense; $\chi_{\hat{K}}$ is a comb rather than a function in the ideal GKP case).

(H1) alone gives $\hat{\Delta}^C(\Omega) = \hat{D}(\Omega)\hat{K}\hat{D}^\dagger(\Omega)$, with $\hat{K}$ Hermitian, and, $d\mu$ being invariant, makes the linear map $\mathcal{E}_C$ commute with conjugation by every displacement. If $\chi_{\hat{K}}$ is a function, (H1) also makes $\mathcal{E}_C$ trace preserving, and $\mathcal{E}_C$ acts on characteristic functions by multiplication by $m = |\chi_{\hat{K}}|^2$ [in the distributional case, where the square of the comb is undefined, m is taken from the twirl of setting (B) below]. These properties of $\mathcal{E}_C$ are derived. Its complete positivity, and its form as an average of conjugations by displacements (the twirl form), we derive from (H1)–(H3) only when $\chi_{\hat{K}}$ is a function, through part (i) below. Where we use them without that, they enter through one of two settings, and we say in each what is assumed:

(A) $\mathcal{E}_C$ is a completely positive, trace-preserving map on $\mathcal{B}(L^2(\mathbb{R}^m))$. Complete positivity is the added assumption here (trace preservation is derived if $\chi_{\hat{K}}$ is a function, and is otherwise part of the setting). With the derived covariance, $\mathcal{E}_C$

is then a displacement-covariant channel, and the twirl form is derived: $\mathcal{E}_C(\hat{A}) = \int d\nu(\beta)\hat{D}(\beta)\hat{A}\hat{D}^\dagger(\beta)$ for a probability measure ν , and we write $\Lambda^\perp \subseteq \mathbb{R}^{2m}$ for the closed subgroup generated by the support of ν .

(B) $\mathcal{E}_C$ is the average over a closed group $\Lambda^\perp \subseteq \mathbb{R}^{2m}$ of displacements, that is, the conditional expectation onto its commutant, acting on the code algebra of a rigged Hilbert space. The twirl form is the added assumption here, taken as in Ref. [5]. It cannot be derived as in (A), since for nontrivial $\Lambda^\perp$ this $\mathcal{E}_C$ is not a trace-preserving channel on $\mathcal{B}(L^2(\mathbb{R}^m))$ (Appendix C).

In both settings, (H2) makes $\Lambda^\perp$ leave $\mathcal{C}$ invariant. (We write $\Lambda^\perp$ because for the ideal GKP code it is the symplectic dual of the lattice Λ supporting m . We write the single-mode case below, but nothing in the statement or its proof depends on the mode count.)

(1) In setting (A), $\Lambda^\perp = \{0\}$ and $\mathcal{E}_C = \mathbb{1}$. The conclusion $\mathcal{E}_C = \mathbb{1}$ also holds with neither setting assumed if $\chi_{\hat{K}}$ is a function, and hence continuous by (H3): (H2) gives $m^2 = m$, so $m = |\chi_{\hat{K}}|^2$ is a continuous function with values in $\{0, 1\}$ and $m(0) = 1$; hence, $m \equiv 1$ on the connected space $\mathbb{R}^{2m}$. Complete positivity of $\mathcal{E}_C = \mathbb{1}$ is then a conclusion and not a hypothesis. This is the route Theorem V.1 takes. The route through (A) does not use (H3), and the continuity route does not use complete positivity.

(2) If, in addition to $\mathcal{E}_C = \mathbb{1}$ holding by either route of (i), G contains the phase-space inversion $\Omega \mapsto -\Omega$, then [since $\mathcal{E}_C = \mathbb{1}$ means $m \equiv 1$, so that $\chi_{\hat{K}}$ is a function of unit modulus, continuous by (H3)]

$$\hat{K} = 2^m(-1)^{\hat{n}_{\text{tot}}}, \quad \hat{\Delta}^C(\Omega) = \hat{\Delta}_{\text{std}}(\Omega), \quad (18)$$

meaning that the only code-adapted representation available is the ordinary Wigner function.

(3) Whenever $\mathcal{C}$ is a finite-dimensional subspace of $L^2(\mathbb{R}^m)$, Lemma III.1 forces $\Lambda^\perp = \{0\}$ directly, since $\Lambda^\perp$ leaves $\mathcal{C}$ invariant. Given the twirl form, whether derived as in (A) or assumed as in (B), conclusion (i) then follows with no further use of complete positivity, and conclusion (ii) follows as well once its parity assumption is granted. We do not claim (iii) for a reconstruction map in either setting, for which the twirl form itself is not available [the continuity route of (i) does not need it].

In setting (B), $\Lambda^\perp$ may be a nontrivial lattice of displacements, and the ideal GKP code is the instance of it. Its m is the indicator function of the symplectic dual lattice, which takes the values $\{0, 1\}$ that (H2) demands but is not continuous, and its $\mathcal{E}_C$ is not a channel on $\mathcal{B}(L^2(\mathbb{R}))$, so neither route of (i) applies to it.

We prove this in Appendix C: To summarize setting (A), (H1) makes $\mathcal{E}_C$ displacement covariant, so as a completely positive, trace-preserving map it is a random-displacement channel $\mathcal{E}_C(\hat{A}) = \int d\nu(\beta)\hat{D}(\beta)\hat{A}\hat{D}^\dagger(\beta)$; idempotence makes ν an idempotent probability measure, so (by Kawada-Itô) the normalized Haar measure of a compact subgroup; and $\mathbb{R}^{2m}$ has no compact subgroup but $\{0\}$. The kernel is then fixed by $|\chi_{\hat{K}}| \equiv 1$ together with two reflections of the same symmetry: Hermiticity of $\hat{K}$ gives $\chi_{\hat{K}}(-\xi) = \chi_{\hat{K}}(\xi)^*$, and covariance under phase-space inversion gives $\chi_{\hat{K}}(-\xi) = \chi_{\hat{K}}(\xi)$, so $\chi_{\hat{K}}$ is real and unimodular, hence ± 1 , and thus $\equiv 1$ by continuity and $\chi_{\hat{K}}(0) = 1$.

TABLE I. Group-theoretic data for the three code families considered here. H^C is the subgroup by which the phase space G/H^C is quotiented: for the rotation-symmetric codes, the isotropy subgroup of the character reference code word [Eq. (8)], and for the ideal GKP code, the stabilizer lattice H_0^C (see Sec. III A). The fourth column is the subgroup of displacements over which the reconstruction map twirls, which grades Eq. (19); the fifth is the syndrome the experimenter measures, which is the grading of the subgroup H_0^C fixing the code space pointwise [Eq. (7)]. The two agree for the ideal GKP code but differ for the rotation-symmetric codes, where the reconstruction map is the identity and does not grade at all.

Code	G	H^C	$\Lambda^\perp$ (grades the sectors of $\mathcal{E}_C$)	Syndrome measured
GKP (ideal)	$H_3(\mathbb{C})^a$	$U(1) \times \mathbb{Z} \times \mathbb{Z}$	Stabilizer lattice	$(s, t) \in [-1/2, 1/2]^2$
Cat ($N = 4$)	$H_3(\mathbb{C}) \rtimes \mathbb{Z}_4$	$U(1) \times \mathbb{Z}_4$	$\{0\}$, so $\mathcal{E}_C = \mathbb{1}$	$\hat{n} \bmod 2$ (parity)
Binomial	$H_3(\mathbb{C}) \rtimes \mathbb{Z}_{2(S+1)}$	$U(1) \times \mathbb{Z}_{2(S+1)}$	$\{0\}$, so $\mathcal{E}_C = \mathbb{1}$	$\hat{n} \bmod (S + 1)$

^aThis is the smallest group that this construction needs. Where we use Proposition III.3's parity requirement for this code, we adjoin $(-1)^{\hat{n}}$ to G , which is valid because parity preserves the inversion-symmetric GKP code space.

Note that the requirement that G contains the phase-space inversion $\Omega \mapsto -\Omega$ applies for the code families we consider. We show in Sec. V A and Appendix A that for the rotation-symmetric codes considered here the isotropy group of a reference code word is $U(1) \times \mathbb{Z}_M$, with M even [$M = N$ for the N -component cat code, with N even as Eq. (70) requires; $M = 2(S + 1)$ for the binomial code]. This ensures that $\hat{K}$ commutes with the parity, which H^C requires whenever it contains a rotation of even order: $\hat{K}$ must commute with $\pi(H^C)$ for the kernel to be well defined on $X^C = G/H^C$, and $(-1)^{\hat{n}} = \hat{K}_M^{M/2} \in \pi(H^C)$, so $\chi_{\hat{K}}(-\xi) = \text{Tr}[(-1)^{\hat{n}} \hat{K} (-1)^{\hat{n}} \hat{D}(\xi)] = \chi_{\hat{K}}(\xi)$. Without either route (no inversion in G and no even-order rotation in H^C), Hermiticity alone leaves $\chi_{\hat{K}} = e^{if}$, with f real, continuous, odd, and vanishing at the origin. This is a strictly larger family than the displaced parities $\hat{K} = 2\hat{\Gamma}(\gamma)$, which also require f to be linear. A code in that situation has at most odd-order rotational symmetry; the codes that motivate the case, numerically optimized ones, have H^C equal to the center and so no code-specific phase space at all (as discussed above).

In contrast to this is the case of the ideal GKP code. There, the code words are not normalizable, Lemma III.1 does not apply, and $\Lambda^\perp$ may be a nontrivial closed subgroup of displacements leaving $\mathcal{C}$ invariant. $\Lambda^\perp$ cannot contain a line: $\mathcal{E}_C$ must act as the identity on the code algebra, so $\mathcal{B}(\mathcal{C})$ lies in the commutant of $\Lambda^\perp$, and the commutant of a one-parameter group of displacements $\{e^{it\hat{K}_\theta}\}$, with $\hat{K}_\theta$ a quadrature, is the maximal Abelian algebra generated by $\hat{K}_\theta$, which cannot contain the noncommuting logical operators of a qubit. This leaves a lattice of rank 1 or 2. Rank 2 is the GKP case: $\Lambda^\perp$ is the stabilizer lattice, $\mathcal{E}_C$ is the twirl of Eq. (43), X^C is the compact torus $\mathbb{T}_{dI}^2$, and the Zak-Gross kernel of Eq. (35) is an admissible kernel. We do not prove it is the only one: The uniqueness step of the proof (in Appendix C) uses continuity of $\chi_{\hat{K}}$, which fails in the distributional case, where Hermiticity and parity fix the lattice coefficients only up to a sign pattern, the one in Eq. (35) being an admissible choice; Ref. [2]'s uniqueness argument uses Clifford covariance, which we do not assume. Rank 1 would describe a code invariant under displacements along a single direction, for which X^C is a noncompact cylinder. Both nontrivial cases are inherently distributional, in the sense of Sec. IV A, and we treat them in setting (B), the conditional-expectation setting of Ref. [5].

Corollary III.1. Let the kernel satisfy hypotheses (H1)–(H3) of Proposition III.3 and belong to the class covered by that proposition: $\chi_{\hat{K}}$ is a function, or $\mathcal{E}_C$ is in setting (A) or (B). Let the parity be available, in either of the two senses set out above (G contains the phase-space inversion, or H^C contains a rotation of even order), and let $\mathcal{C}$ have a logical dimension at least 2. Then, a code-adapted phase-space representation differing from the ordinary Wigner function requires the code space to be invariant under a nontrivial lattice of displacements, and hence, in the continuous-variable setting, forbids $\mathcal{C}$ from being a finite-dimensional subspace of $L^2(\mathbb{R})$ at all (Lemma III.1): Its code words cannot be normalizable. Among bosonic codes whose phase space is compact, or equivalently, whose invariance lattice has rank 2, this only holds for the ideal GKP family, whose code space is the fixed-point space of its stabilizer lattice. As far as we are aware, no one has yet made a bosonic code realizing the rank-1 case, which would give a noncompact cylinder, so we do not consider this further. The statement is made within the class just named. A kernel whose $\chi_{\hat{K}}$ is not a function, and whose reconstruction map is neither a completely positive, trace-preserving map on $\mathcal{B}(L^2(\mathbb{R}))$ nor a conditional expectation onto the commutant of a group of displacements, lies outside that class, and we make no claim about it.

The parity is available for every code considered in this paper: for the rotation-symmetric codes through the even-order rotation in H^C , and for the ideal GKP code by adjoining $(-1)^{\hat{n}}$ to G , which is valid as parity maps the code word $|\bar{j}\rangle$ to $|-j\rangle$, so preserves the inversion-symmetric code space; the groups are listed in Table I. This has two consequences. First, for a bosonic code with normalizable code words (the cat and binomial codes of Secs. V A and V B, and finite-energy GKP code words), the Stratonovich-Weyl axioms, in the form of hypotheses (H1)–(H3) and within the class of Corollary III.1, only admit the ordinary Wigner kernel. Sections V A and V B therefore look at what the ordinary representation shows about the code words of each family. Second, the Zak-Gross representation of the GKP code is exact only in the ideal limit; we return to what remains at finite energy in Sec. VIII.

As noted in Sec. III C, the construction also requires a homogeneous phase space $X^C = G/H^C$ carrying an action of G , because that gives the covariance axiom its content, and covariance produces the rigid action of errors in Eq. (48), and hence the usefulness of Eq. (12). A code space invariant under

no subgroup of G beyond the center has no code-specific phase space: The construction still runs, but with H^C the center it returns $X^C \simeq \mathbb{C}$ and the ordinary Wigner function, so the structure theorem tells us nothing about the code beyond what it already tells us about the mode. This limits our argument: outside the symmetry-defined codes, there is no code-adapted phase space for the structure theorem to act on, and a construction that claimed to supply one would not be a Stratonovich-Weyl representation.

E. The class of transformations represented by Eq. (12)

The reconstruction map ϕ_C returns not $\hat{\rho}$ but the twirled operator $\mathcal{E}_C(\hat{\rho})$. Before we can use Eq. (12) to represent a composite process (an error, followed by a syndrome measurement, followed by a recovery), we therefore have to say for which transformations the insertion of $\mathcal{E}_C$ is operationally irrelevant.

Proposition III.3 leaves only two cases. For a bosonic code with code words in $L^2(\mathbb{R})$, $\mathcal{E}_C = \mathbb{1}$ and every channel qualifies, so nothing in this subsection restricts anything. The restriction applies only when $\Lambda^\perp$ is a nontrivial displacement lattice and $\mathcal{E}_C$ is a genuine twirl: i.e., the ideal GKP code in the continuous-variable setting and every stabilizer code in the finite-dimensional one (Sec. IV D).

$\mathcal{E}_C$ is the conditional expectation onto the commutant of $\Lambda^\perp$. Writing the physical Hilbert space as a direct sum (or, for GKP, a direct integral) over the joint eigenspaces of $\Lambda^\perp$, which are the syndrome sectors,

$$\mathcal{H} = \bigoplus_s \mathcal{H}_s, \quad \mathcal{H}_s = \hat{\Pi}_s \mathcal{H}, \quad (19)$$

(where $\hat{\Pi}_s$ are the displaced code-space projectors $\hat{\Pi}_{(s,t)}$ of Sec. IV A in the GKP case, and where there is a single sector, $\hat{\Pi}_s = \mathbb{1}$, in the cat and binomial cases, because $\Lambda^\perp = \{0\}$ there) the twirl is the projection onto the block-diagonal part of the operator algebra,

$$\mathcal{E}_C(\hat{\rho}) = \sum_s \hat{\Pi}_s \hat{\rho} \hat{\Pi}_s, \quad (20)$$

the sum being the integral of Eq. (43) in the GKP case. $\mathcal{E}_C$ therefore removes coherences *between* distinct sectors and leaves everything *within* each sector untouched. This makes the framework usable for error correction: An error moves logical information from one sector to another, but does not store that information in intersector coherences, and $\mathcal{E}_C$ only destroys these coherences. The sectors of Eq. (19) do not need to coincide with the sectors the experimenter measures: For the cat and binomial codes, the measured syndrome is the grading of H_0^C [photon-number parity for the four-component cat code, $\hat{n} \bmod (S+1)$ for the binomial code] but $\mathcal{E}_C$ does not grade by it, because $\mathcal{E}_C$ is the identity there. We separate the two in Table I.

Definition III.1. A channel T is *syndrome graded* with respect to the decomposition of Eq. (19) if $\mathcal{E}_C \circ T \circ \mathcal{E}_C = T \circ \mathcal{E}_C$. We write $\mathcal{G}_C$ for the set of such channels.

A sufficient condition for a channel being syndrome graded is that T carry sectors to sectors, i.e.,

$$T(\hat{\Pi}_s \cdot \hat{\Pi}_{s'}) \subseteq \hat{\Pi}_{\zeta(s)} \cdot \hat{\Pi}_{\zeta(s')}, \quad (21)$$

for some map ζ of the syndrome label set (we write ζ rather than π here, which is our unitary representation of G).

Proposition III.4. $\mathcal{G}_C$ contains the identity and is closed under composition, under convex combination, and under tensor product with a syndrome-graded channel on an ancilla.

We prove this in Appendix C.

Proposition III.5. For $T_1, \dots, T_n \in \mathcal{G}_C$,

$$Q(T_n \circ \dots \circ T_1) = Q(T_n) \circ \dots \circ Q(T_1). \quad (22)$$

We prove this in Appendix C: Each $\phi_C \circ \chi_C = \mathcal{E}_C$ inserted between adjacent channels is absorbed into the channel before it, by Proposition III.4.

On the subtheory generated by $\mathcal{G}_C$, Q preserves composition, and where $\mathcal{E}_C = \mathbb{1}$ it preserves the identity too, so it is a functor there and a semifunctor in general (as per Refs. [7,8]). Outside $\mathcal{G}_C$, composition fails: $Q(T_2) \circ Q(T_1) = Q(T_2 \circ \mathcal{E}_C \circ T_1)$, which differs from $Q(T_2 \circ T_1)$ in general. The requirement is therefore not an extra assumption on the structure theorem, but a statement of which subtheory it is applied to.

Every map used in this paper lies in $\mathcal{G}_C$. For the cat and binomial codes, this holds because $\mathcal{E}_C = \mathbb{1}$. For the GKP code, the displacement error and the recovery displacement permute the displaced code spaces, so ζ is a translation of the syndrome label; the syndrome measurement is $\mathcal{E}_C$ itself, up to a relabeling of outcomes; the Gaussian random-displacement channel of Eq. (52) is a convex combination of displacements; and every logical Pauli acts within each sector, i.e., with ζ the identity. Table I collects the group-theoretic data for the three families.

Note that channels creating coherence between sectors are not syndrome graded, and for them Eq. (12) represents $\mathcal{E}_C \circ T \circ \mathcal{E}_C$ rather than $T \circ \mathcal{E}_C$. Since $\chi_C \circ \mathcal{E}_C = \chi_C$, this is invisible on a single channel and shows up on composition, which fails when the earlier channel is not graded. Since $\mathcal{E}_C = \mathbb{1}$ for codes where the stabilizer is not a lattice of displacements in a Weyl-Heisenberg group, the continuous-variable examples are necessarily GKP ones, of which two are physically important. A Gaussian unitary that is not a displacement (e.g., a residual squeeze or a rotation misaligned with the code lattice) takes a code word to a coherent superposition spread across syndrome sectors, whose intersector coherences are removed by $\mathcal{E}_C$, and an unsharp syndrome extraction, whose ancilla measurement does not fully resolve (s, t) , likewise leaves coherence between the unresolved sectors. The restriction has little practical effect, because a full cycle begins with a sharp syndrome measurement, which destroys those coherences physically, meaning that $\mathcal{E}_C$ discards no more than the protocol itself does; but it remains a restriction: Equation (12) is a statement about $\mathcal{G}_C$, rather than arbitrary channels.

This treatment gives $Q(T_{\text{QEC}}) = \mathbb{1}$ exactly for the ideal GKP code [Eq. (50)] and, after dividing out the jump weight $\langle \hat{n} \rangle$, exactly for the binomial code and up to corrections exponentially small in $|\alpha|^2$ for the cat code [Eqs. (103) and (111)]. Note that this means the logical phase-space distribution is returned to itself, not that the physical process has been reversed: The environment keeps the lost photon, and the recovery uses the syndrome to map the error sector back onto the code space.

IV. STABILIZER-IS-WH-DISPLACEMENT-LATTICE CODES

Corollary III.1 implies that, within the class of kernels it covers, a code whose symmetry makes the phase-space parity available, as every code considered here does, has a phase-space representation of its own iff its code space is invariant under a lattice of displacements. In the continuous-variable setting, Lemma III.1 makes that a strong requirement, met only by the ideal, Dirac-comb GKP family; in the discrete-variable setting, Proposition III.2 makes it a weak one, met by every stabilizer code. The two cases involve a quotient phase space factorizing into syndrome and logical parts, errors acting as rigid translations, a corrected cycle acting as the identity, and, for odd prime logical dimension, a negative volume equal to the syndrome-averaged logical magic. They also compose, since a GKP code concatenated with a qudit stabilizer code is again a lattice code.

A. Ideal GKP codes

GKP codes [6] encode a logical qudit into the continuous degrees of freedom of a single mode (e.g., the field quadratures of an optical mode), trading the redundancy of many physical qubits for a single continuous-variable system (while difficult to create, maintain, and manipulate, developments in hardware and control have made these codes realizable [23–26]). They protect against both small shifts in the quadratures and losses such as photon subtraction [27]. Recent experiments include autonomous error correction of GKP states in a superconducting cavity [25], error-corrected entanglement between bosonic logical qubits [28], and fault-tolerant schemes using discrete-variable ancillae [29].

Throughout this section, and in Sec. IV C, we work with ideal GKP code words (as does Ref. [5]). These are Dirac combs, so not elements of $L^2(\mathbb{R})$, but tempered distributions in the Gel'fand triple $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$. The phase-space objects we construct from them are nonetheless well defined, because the Zak-Gross Wigner function is built entirely from the quantities $\text{Tr}[\hat{D}(nl, ml)\hat{A}]$, which are well-defined distributional pairings for $\hat{A}$ in the code algebra, while the twirling and reconstruction maps are defined through the continuum-normalized projectors $\hat{\Pi}_{(s,t)}$ (in the same way, one writes $\int dx |x\rangle\langle x| = \mathbb{1}$ for the position basis). Section VIII considers the modifications produced by a finite-energy envelope.

We take d odd throughout this section and Secs. IV B and IV C, since the discrete-variable formalism of Sec. II C requires it (Sec. IV E identifies which results still hold at $d = 2$). In general, a GKP code encodes a d -dimensional logical subspace in n bosonic modes. To describe the code, we define two logical operations

$$\hat{Z}^d = e^{il\hat{x}}, \quad \hat{X}^d = e^{-il\hat{p}} \quad (23)$$

and stabilizer generators

$$\hat{Z}^d = e^{idl\hat{x}}, \quad \hat{X}^d = e^{-idl\hat{p}}, \quad (24)$$

where $l = \sqrt{\frac{2\pi}{d}}$ is the logical step size; we take the square code here (rectangular and hexagonal lattices are covered by the same argument; see Sec. IV B). The generators commute

and generate the Abelian stabilizer group $\mathcal{S} = \langle \hat{Z}^d, \hat{X}^d \rangle$. The GKP logical code space is the simultaneous $+1$ eigenspace of the two generators, $\{|\psi\rangle \in \mathcal{H} : U|\psi\rangle = |\psi\rangle \forall U \in \mathcal{S}\}$. We can define a computational basis of encoded states with elements

$$|\bar{j}\rangle = \frac{1}{\sqrt{l}} \sum_{n \in \mathbb{Z}} |jl + dln\rangle_{\hat{x}}, \quad (25)$$

which is a Dirac comb in the position basis with period $dl = \sqrt{2\pi d}$. We can extend this map by linearity, so that every DV density operator $\hat{\rho} \in \mathcal{D}(\mathcal{H}_d)$ has a corresponding GKP-encoding $\hat{\rho}$. Similarly, if $\hat{\rho}$ is a CV density operator GKP encoding a DV state, the corresponding DV logical state is denoted by $\underline{\hat{\rho}}$. The projector onto the code space is then

$$\hat{\Pi}_{(0,0)} = \sum_{j=0}^{d-1} |\bar{j}\rangle\langle\bar{j}|, \quad (26)$$

meaning that for the displaced code spaces

$$\hat{\Pi}_{(s,t)} = \hat{D}(s,t)\hat{\Pi}_{(0,0)}\hat{D}^\dagger(s,t), \quad s, t \in [-l/2, l/2]. \quad (27)$$

We perform syndrome diagnosis after an arbitrary displacement error by determining (s, t) , by simultaneously measuring the stabilizers. For a CV state $\hat{\rho}$, the syndrome distribution is the probability of measuring the syndrome (s, t) , which follows Born's rule

$$P[(s, t)|\hat{\rho}] = \text{Tr}[\hat{\rho}\hat{\Pi}_{(s,t)}]. \quad (28)$$

After we measure a certain syndrome, we apply an error-correcting displacement $\hat{D}(-s, -t)$ to obtain the error-corrected state

$$\begin{aligned} \hat{\rho}_{(s,t)} &:= \hat{D}(-s, -t)\hat{\Pi}_{(s,t)}\hat{\rho}\hat{\Pi}_{(s,t)}\hat{D}^\dagger(-s, -t) \\ &= \hat{\Pi}_{(0,0)}\hat{D}^\dagger(s, t)\hat{\rho}\hat{D}(s, t)\hat{\Pi}_{(0,0)}. \end{aligned} \quad (29)$$

This code can correct displacement errors $\hat{D}(x, p)$ with arguments belonging to the phase-space region $(-l/2, l/2) \times (-l/2, l/2)$ or shifts by multiples of dl in either direction.

B. The Zak-Gross Wigner function

Reference [5] applies Sec. III A's construction to the GKP code, obtaining a phase-space distribution whose negativity identifies the quantum resources of an encoded computation (specifically the magic of the encoded state). For pure logical states, negativity appears iff the state is magic: Pure stabilizer states are represented positively, and by the discrete Hudson theorem [2] they are the only pure states so represented. For mixed states, negativity still witnesses magic, but magic does not imply negativity: Even single-qudit systems can have mixed states outside the stabilizer polytope with non-negative discrete Wigner function [3]. (Note that the ordinary Wigner function does not do this: Many CV states, including stabilizer code words, are logically resourceless despite having arbitrarily large CV Wigner negativity [30,31], and some positively represented states even allow universal quantum computation [32–34].) The resulting distribution is the ZGW function.

Consider a square qudit GKP code over a single bosonic mode, with infinite-dimensional Hilbert space $\mathcal{H}$ and symmetry group $G = H_3(\mathbb{C})$. Any element of this group can be

parametrized as

$$g := g(\varphi, x, p), \quad \pi(g) = e^{i\varphi \mathbb{1} - ix\hat{p} + ip\hat{x}}, \quad (30)$$

with $\varphi, x, p \in \mathbb{R}$. The subgroup by which the phase space is quotiented, $H^{\text{GKP}} \subset H_3(\mathbb{C})$, is the stabilizer lattice $U(1) \times \mathbb{Z} \times \mathbb{Z}$ (see Sec. III A for its relation to the reference-state isotropy), with unitary representation

$$\pi(H^{\text{GKP}}) := \{e^{i\varphi - imdl\hat{p} + indl\hat{x}}, \quad \text{such that } \varphi \in [0, 2\pi), m, n \in \mathbb{Z}\}. \quad (31)$$

The associated phase space is $X = G/H^{\text{GKP}} \simeq \mathbb{T}_{dl}^2$, where

$$\mathbb{T}_{dl}^2 := [0, dl) \times [0, dl) = [0, \sqrt{2\pi d}) \times [0, \sqrt{2\pi d}) \quad (32)$$

is the continuous torus with invariant measure $\frac{1}{d}dudv$.

Following the construction of Sec. III A, we obtain the function

$$W_{\hat{A}}^{\text{GKP}}(u, v) = \text{Tr}[\hat{\Delta}^{\text{GKP}}(u, v)\hat{A}] \quad (33)$$

for all $(u, v) \in \mathbb{T}_{dl}^2$, where

$$\hat{\Delta}^{\text{GKP}}(u, v) = \hat{D}(u, v)\hat{\Delta}^{\text{GKP}}(0, 0)\hat{D}^\dagger(u, v) \quad (34)$$

and

$$\hat{\Delta}^{\text{GKP}}(0, 0) = \frac{1}{2\pi} \sum_{m, n \in \mathbb{Z}} (-1)^{mn} \hat{D}(nl, ml) \quad (35)$$

or, in full,

$$W_{\hat{A}}^{\text{GKP}}(u, v) = \frac{1}{2\pi} \sum_{m, n \in \mathbb{Z}} (-1)^{mn} e^{il(nv - mu)} \text{Tr}[\hat{D}(nl, ml)\hat{A}]. \quad (36)$$

This function is defined purely by the choice of symmetry of the code space $\mathcal{C}_{\text{GKP}}$, and does not depend on the reference state chosen [5]. The ZGW function satisfies the SW axioms in the code-faithful reading of Sec. II C [5]: Injectivity and traciality hold on the $\mathcal{E}$ -invariant subalgebra, which for this code is the algebra generated by the displacements $\hat{D}(nl, ml)$, and not on all of $\mathcal{B}(L^2(\mathbb{R}))$.

The nonclassicality (negativity) of this function comes from logical displacements. On the toroidal phase space $\mathbb{T}_{dl}^2$, the two axes are a modular position and a modular momentum, both of period dl by construction. The product of the two periods is $\sqrt{2\pi d} \times \sqrt{2\pi d} = 2\pi d$, a multiple of 2π . This implies that the stabilizers $\hat{X}^d, \hat{Z}^d$ commute [35]. However, measuring them together is not the same thing as measuring the position and momentum of a bosonic state modulo $l = \sqrt{2\pi/d}$, which is impossible because the product of the periods of the two logical displacements $\hat{X}, \hat{Z}$ is not a multiple of 2π , implying they do not commute.

The Zak-Gross Wigner function represents the logical content of states in $\mathcal{H}$ with respect to the GKP code space $\mathcal{C}$, through the relation

$$W_{\hat{\rho}}^{\text{GKP}}(u, v) = W_{\hat{\rho}(s, t)}^{\text{DV}}(a, b), \quad (37)$$

where $\hat{\rho}(s, t)$ is the finite-dimensional logical operator corresponding to Eq. (29)'s error-corrected operator $\hat{\rho}_{(s, t)} = \hat{\Pi}_{(0,0)}^\dagger \hat{D}(s, t)^\dagger \hat{\rho} \hat{D}(s, t) \hat{\Pi}_{(0,0)}$, and $u = s + al$ and $v = t + bl$ for $(s, t) \in \mathbb{T}_l^2$ and $(a, b) \in \mathbb{Z}_d^2$. In the terminology of

Sec. II A, $\hat{\rho}_{(s, t)}$ is an unnormalized conditional state: Its trace is the syndrome probability $P[(s, t)|\hat{\rho}]$ of Eq. (28), not unity, and $W_{\hat{\rho}(s, t)}^{\text{DV}}$ is correspondingly a subnormalized phase-space function. $[\hat{\rho}'(s, t) = \hat{\rho}(s, t)/P[(s, t)|\hat{\rho}]$ denotes the normalized one; we use the primed symbol whenever normalization matters.] This relation implies that the Zak-Gross Wigner function of a code word $\hat{\rho}$ matches the Gross Wigner function of the unencoded finite-dimensional state $\hat{\rho}$.

We can summarize the properties of the ZGW function through the following set of relations:

$$\begin{array}{ccc} \mathcal{D}(\mathcal{H}_d) \ni \hat{\rho} & \xrightarrow{\text{encode}} & \hat{\rho} \\ \text{DV Wigner} \downarrow & & \downarrow \text{Zak-Gross Wigner} \\ W_{\hat{\rho}}^{\text{DV}}(a, b) & \xrightarrow[\mathbb{T}_{dl}^2]{\text{embed in}} & W_{\hat{\rho}}^{\text{GKP}}(al, bl) \end{array} \quad (38)$$

Finally, we define the negative volume of the ZGW function of a CV state $\hat{\rho}$ as

$$\mathcal{N}_{\hat{\rho}}^{\text{GKP}} := \int_{\mathbb{T}_{dl}^2} \frac{1}{d} dudv |W_{\hat{\rho}}^{\text{GKP}}(u, v)| - 1, \quad (39)$$

which is Eq. (6) for this code, the ZGW function being normalized [Eq. (1)]. $\mathcal{N}_{\hat{\rho}}^{\text{GKP}}$ vanishes for density operators with non-negative ZGW functions and is strictly positive otherwise. Equation (37) means that the negative volume of the ZGW function is the average negative volume of the Gross Wigner function over one round of GKP error correction:

$$\mathcal{N}_{\hat{\rho}}^{\text{GKP}} = \int_{\mathbb{T}_l^2} dsdt P[(s, t)|\hat{\rho}] \mathcal{N}_{\hat{\rho}(s, t)}^{\text{DV}}, \quad (40)$$

where $\mathbb{T}_l^2 := [0, l) \times [0, l)$ is the smaller continuous torus [5] (equivalently $[-l/2, l/2)^2$, the labeling we use for the correctable cell), $P[(s, t)|\hat{\rho}]$ is the syndrome distribution, and $\hat{\rho}'(s, t)$ is the normalized syndrome-conditional logical state. Since the sign convention has changed, we derive this in Appendix C. Note that the normalization of $\hat{\rho}'(s, t)$ makes each $\mathcal{N}^{\text{DV}}$ on the right-hand side a negative volume of a normalized distribution: Without it, the identity fails. Therefore, for odd d , stabilizer code words are positively represented by the ZGW function, while magic code words are negatively represented.

The ideal GKP code falls outside Proposition III.1, because its isotropy group is a noncompact lattice and $1/|H^{\text{C}}|$ is not a normalized average (as in Sec. IV A). Here, Eq. (14) should be read as Ref. [5]'s lattice sum [Eq. (35)], and the three properties are verified directly rather than by the group average. Covariance is Eq. (34) by construction; the twirl is Eq. (43); and standardization follows from $\int_0^{dl} e^{-ilmu} du = dl \delta_{m,0}$, since $l \cdot dl = 2\pi$, which kills every term of the lattice sum but $m = n = 0$ and leaves

$$\int_{\mathbb{T}_{dl}^2} \frac{dudv}{d} \hat{\Delta}^{\text{GKP}}(u, v) = \frac{(dl)^2}{2\pi d} \mathbb{1} = \mathbb{1} \quad (41)$$

since $(dl)^2 = d^2 l^2 = 2\pi d$. The same route covers rectangular, hexagonal, and squeezed GKP lattices, and any code space

defined as the fixed-point space of a lattice of displacements conjugated by a Gaussian unitary.

C. Errors and the recovery cycle

The ZGW function satisfies the Stratonovich-Weyl axioms in the code-faithful reading of Sec. II C, so by Sec. III B it is a linearity preserving, empirically adequate semifunctor. It fails to preserve the identity because of the stabilizer lattice rather than the Brif-Mann construction, as Proposition III.3 shows.

We can therefore identify the operational-to-ontological map χ and the ontological-to-operational map ϕ within the ZGW formalism. The map χ takes an operational state (a density matrix $\hat{\rho}$) to its corresponding phase-space distribution:

$$\chi_{\text{GKP}}(\hat{\rho}) := W_{\hat{\rho}}^{C_{\text{GKP}}}(u, v) = \text{Tr}[\hat{\Delta}^{C_{\text{GKP}}}(u, v)\hat{\rho}]. \quad (42)$$

The map ϕ instead takes a phase-space distribution to an operator in the Hilbert space. To define ϕ , we first define the twirling map

$$\mathcal{E}(\hat{\rho}) = \int_{\mathbb{T}_l^2} ds dt \hat{\Pi}_{(s,t)} \hat{\rho} \hat{\Pi}_{(s,t)}, \quad (43)$$

which yields operators invariant under the action of $H^{C_{\text{GKP}}}$. It can be proved that [5]

$$\mathcal{E}(\hat{\rho}) = \int_{\mathbb{T}_{dl}^2} \frac{1}{d} du dv W_{\hat{\rho}}^{C_{\text{GKP}}}(u, v) \hat{\Delta}^{C_{\text{GKP}}}(u, v). \quad (44)$$

The map $\phi = \underline{\chi}^{-1} \circ Q(\mathbb{1})$, with $Q(\mathbb{1})$ the idempotent taking an arbitrary function on the torus to one with the symmetry of the code space. Because the integral in Eq. (44) returns operators invariant under $H^{C_{\text{GKP}}}$, integrating against the kernel applies $Q(\mathbb{1})$ and the corestricted inverse $\underline{\chi}^{-1}$ at once, so that

$$\phi(W(u, v)) = \int_{\mathbb{T}_{dl}^2} \frac{1}{d} du dv W(u, v) \hat{\Delta}^{C_{\text{GKP}}}(u, v). \quad (45)$$

Composing the two gives the phase-space representation of a generic operational transformation T , evaluated at (u', v') :

$$[Q(T)W_{\hat{\rho}}^{C_{\text{GKP}}}] (u', v') = \text{Tr} \left[\hat{\Delta}^{C_{\text{GKP}}}(u', v') T \left(\int_{\mathbb{T}_{dl}^2} \frac{1}{d} du dv W_{\hat{\rho}}^{C_{\text{GKP}}}(u, v) \hat{\Delta}^{C_{\text{GKP}}}(u, v) \right) \right]. \quad (46)$$

An error-correction cycle has two components, the error and the correcting operation, so we represent a cycle in this framework as $T_{\text{QEC}} = T_{\text{rec}} \circ T_{\text{err}}$. The physical action of the error channel T_{err} on an operator $\hat{A}$ is

$$T_{\text{err}}(\hat{A}) = \hat{D}(x_{\text{err}}, p_{\text{err}}) \hat{A} \hat{D}^\dagger(x_{\text{err}}, p_{\text{err}}). \quad (47)$$

Substituting this into Eq. (46), and denoting $\phi(W_{\hat{\rho}}^{C_{\text{GKP}}}) = \phi_W$, we get

$$\begin{aligned} [Q(T_{\text{err}})W_{\hat{\rho}}^{C_{\text{GKP}}}] (u, v) &= \text{Tr}[\hat{\Delta}^{C_{\text{GKP}}}(u, v) (\hat{D}(x_{\text{err}}, p_{\text{err}}) \phi_W \hat{D}^\dagger(x_{\text{err}}, p_{\text{err}}))] \\ &= \text{Tr}[(\hat{D}^\dagger(x_{\text{err}}, p_{\text{err}}) \hat{\Delta}^{C_{\text{GKP}}}(u, v) \hat{D}(x_{\text{err}}, p_{\text{err}})) \phi_W] = \text{Tr}[\hat{\Delta}^{C_{\text{GKP}}}(u - x_{\text{err}}, v - p_{\text{err}}) \phi_W] \\ &= W_{\phi_W}^{C_{\text{GKP}}}(u - x_{\text{err}}, v - p_{\text{err}}) = W_{\hat{\rho}}^{C_{\text{GKP}}}(u - x_{\text{err}}, v - p_{\text{err}}). \end{aligned} \quad (48)$$

An error T_{err} therefore manifests in the logical phase space as a rigid, classical translation of the quasiprobability distribution across the continuous torus $\mathbb{T}_{dl}^2$.

T_{rec} must therefore also act as a rigid translation. It consists of two steps: syndrome diagnosis, which determines (s, t) by measuring the stabilizers simultaneously, and which returns $(s, t) = (x_{\text{err}}, p_{\text{err}})$ whenever the error lies in the cell $[-l/2, l/2]^2$, and the recovery displacement $T_{\text{rec}}(\hat{A}) = \hat{D}(-x_{\text{rec}}, -p_{\text{rec}}) \hat{A} \hat{D}^\dagger(-x_{\text{rec}}, -p_{\text{rec}})$. Repeating the calculation of Eq. (48) with the sign of the displacement reversed gives

$$[Q(T_{\text{rec}})W_{\hat{\rho}}^{C_{\text{GKP}}}] (u, v) = W_{\hat{\rho}}^{C_{\text{GKP}}}(u + x_{\text{rec}}, v + p_{\text{rec}}). \quad (49)$$

The full error-correction operation $T_{\text{QEC}} = T_{\text{rec}} \circ T_{\text{err}}$ is then

$$\begin{aligned} [Q(T_{\text{QEC}})W_{\hat{\rho}}^{C_{\text{GKP}}}] (u, v) &= Q(T_{\text{rec}})[Q(T_{\text{err}})W_{\hat{\rho}}^{C_{\text{GKP}}}(u, v)] \\ &= Q(T_{\text{rec}})[W_{\hat{\rho}}^{C_{\text{GKP}}}(u - x_{\text{err}}, v - p_{\text{err}})] \\ &= W_{\hat{\rho}}^{C_{\text{GKP}}}(u - x_{\text{err}} + x_{\text{rec}}, v - p_{\text{err}} + p_{\text{rec}}) \\ &= W_{\hat{\rho}}^{C_{\text{GKP}}}(u, v), \end{aligned} \quad (50)$$

where the last step uses $(x_{\text{rec}}, p_{\text{rec}}) = (s, t) = (x_{\text{err}}, p_{\text{err}})$, which holds whenever the error lies in the correctable cell. The error-correction cycle therefore acts as the identity on the phase-space distribution if the error is in the correctable region.

If the error is outside the correctable region, we can define the shift produced through the parameters $(x_{\text{err}}, p_{\text{err}}) = (n_{\text{err}}l + s, m_{\text{err}}l + t)$, with $n_{\text{err}}, m_{\text{err}} \in \mathbb{Z}$, where $(s, t) \in [-l/2, l/2] \times [-l/2, l/2]$ is the measured syndrome. Therefore, we can write the action of this error-correcting cycle on the phase-space distribution

$$\begin{aligned} [Q(T_{\text{QEC}})W_{\hat{\rho}}^{C_{\text{GKP}}}] (u, v) &= Q(T_{\text{rec}})[Q(T_{\text{err}})W_{\hat{\rho}}^{C_{\text{GKP}}}(u, v)] \\ &= Q(T_{\text{rec}})[W_{\hat{\rho}}^{C_{\text{GKP}}}(u - x_{\text{err}}, v - p_{\text{err}})] \\ &= W_{\hat{\rho}}^{C_{\text{GKP}}}(u - x_{\text{err}} + s, v - p_{\text{err}} + t) \\ &= W_{\hat{\rho}}^{C_{\text{GKP}}}(u - n_{\text{err}}l, v - m_{\text{err}}l). \end{aligned} \quad (51)$$

If the error takes the state outside the correctable cell, n_{err} and/or m_{err} are nonzero, and since $\mathbb{T}_{dl}^2$ is periodic with period

dl , a shift by an integer multiple of l still returns a properly normalized Zak-Gross Wigner function of a twirled state in $\mathcal{C}_{\text{GKP}}$. The unencoded DV phase space is embedded in $\mathbb{T}_{dl}^2$ at intervals of l by Eq. (37), so shifting by $(n_{\text{err}}l, m_{\text{err}}l)$ applies the logical Pauli $\hat{X}^{n_{\text{err}}}\hat{Z}^{m_{\text{err}}}$ to the encoded information: u is the modular position and v the modular momentum, and $\hat{X} = e^{-il\hat{p}}$ shifts the former, while $\hat{Z} = e^{il\hat{x}}$ shifts the latter. Uncorrectable physical noise is thus translated into discrete logical errors, with the semifunctorial structure intact.

Again, note that Eq. (50) is an identity on the *logical* phase-space distribution, not on the physical state of the mode together with its environment: The recovery displacement does not undo the physical process that produced the error; it uses the syndrome to return the logical information to the code space. Section III E makes this distinction generally.

The same calculation covers the Gaussian random-displacement (classical-noise) channel, the standard benchmark for GKP codes:

$$\mathcal{E}_\sigma(\hat{\rho}) = \int d^2z G_\sigma(z) \hat{D}(z) \hat{\rho} \hat{D}^\dagger(z), \quad (52)$$

with G_σ a centered Gaussian of variance σ^2 . Because $\mathcal{Q}$ is linear and each $\hat{D}(z)$ acts on the distribution as the rigid translation of Eq. (48), the represented channel is just a convolution on the torus,

$$[\mathcal{Q}(\mathcal{E}_\sigma)W_{\hat{\rho}}^{\mathcal{C}_{\text{GKP}}}] (u, v) = (W_{\hat{\rho}}^{\mathcal{C}_{\text{GKP}}} * \tilde{G}_\sigma)(u, v), \quad (53)$$

where $\tilde{G}_\sigma$ is the Gaussian wrapped onto $\mathbb{T}_{dl}^2$. The logical error rate is then the weight of $\tilde{G}_\sigma$ falling outside the correctable cell $[-l/2, l/2)^2$, an integral of a wrapped Gaussian; and the negative volume of the output is bounded above by that of the input, since convolution with a non-negative kernel cannot increase $\int d\mu |W|$. We use the same convolution structure in Sec. VIII, where a finite-energy envelope produces a formally identical smoothing.

D. Every qudit stabilizer code

Throughout this subsection, d is an odd prime, $\omega = e^{2\pi i/d}$, and displacements are taken in the symmetric ordering of Sec. II C. Proposition III.2 has already supplied the covariance that Eq. (14) needs.

Definition IV.1. The code-adapted kernel of a stabilizer code $\mathcal{C} = [[n, k]]_d$ with stabilizer lattice $\mathcal{M}$ is

$$\Delta^{\mathcal{C}}(u) = \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} A(u + m), \quad (54)$$

an operator function on the quotient phase space $X^{\mathcal{C}} = \mathbb{Z}_d^{2n}/\mathcal{M}$, which has d^{n+k} points; the invariant measure is $\mu = d^{-k}$ times counting measure.

Since $A(u + m) = D(m)A(u)D^\dagger(m)$, conjugation absorbing all phases, $\Delta^{\mathcal{C}}$ is manifestly Hermitian, constant on cosets, and covariant, $\Delta^{\mathcal{C}}(u + w) = D(w)\Delta^{\mathcal{C}}(u)D^\dagger(w)$. Its remaining Stratonovich-Weyl properties, and the identity of its reconstruction map, are given below.

Proposition IV.1 (Axioms and reconstruction). $\text{Tr}[\Delta^{\mathcal{C}}(u)] = 1$ for every u ; $\sum_{[u] \in X^{\mathcal{C}}} \Delta^{\mathcal{C}}(u) = d^k \mathbb{1}$, so $\int d\mu \Delta^{\mathcal{C}} = \mathbb{1}$; $\text{Tr}[\Delta^{\mathcal{C}}(u)\Delta^{\mathcal{C}}(u')] = d^k \delta_{[u], [u']}$; and the composition of the forward map $\chi_{\mathcal{C}}(\hat{A})([u]) = \text{Tr}[\Delta^{\mathcal{C}}(u)\hat{A}]$,

with the reverse map $\phi_{\mathcal{C}}(F) = \sum_{[u]} \mu F([u]) \Delta^{\mathcal{C}}(u)$ being the lattice twirl,

$$\phi_{\mathcal{C}} \circ \chi_{\mathcal{C}} = \mathcal{E}_{\mathcal{C}},$$

$$\mathcal{E}_{\mathcal{C}}(\hat{\rho}) = \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} D(m) \hat{\rho} D^\dagger(m) = \sum_s \hat{\Pi}_s \hat{\rho} \hat{\Pi}_s, \quad (55)$$

the conditional expectation onto the block-diagonal algebra of the d^{n-k} syndrome sectors $\hat{\Pi}_s$.

The proof (Appendix C) combines the lattice structure with the self-duality of the Gross kernel. Equation (55) is the finite-dimensional image of the GKP twirl of Eqs. (43) and (44), and of the sector projection of Eqs. (19) and (20); the representation is therefore a linearity-preserving, empirically adequate semifunctor, and it composes channel by channel on the syndrome-graded class $\mathcal{G}_{\mathcal{C}}$, Sec. III E's arguments transferring verbatim since they use only the algebraic properties of $\mathcal{E}_{\mathcal{C}}$.

The quotient factorizes in exact parallel with the GKP torus,

$$X^{\mathcal{C}} \cong \underbrace{\mathbb{Z}_d^{n-k}}_{\text{syndromes}} \times \underbrace{\mathbb{Z}_d^{2k}}_{\text{logical}}, \quad (56)$$

the finite image of “torus = correctable cell $\times$ logical lattice.” The syndrome coordinate of $[u]$ is the character $m \mapsto \omega^{[u, m]}$ of $\mathcal{M}$; the logical coordinates are the symplectic pairings with a chosen logical pair $(\tilde{X}, \tilde{Z})$.

Proposition IV.2 (Logical content). Let $\hat{\rho}$ encode the logical state $\hat{\rho}_L$. Then,

$$\mu W_{\hat{\rho}}^{\mathcal{C}}([u]) = \begin{cases} W_{\hat{\rho}_L}^{\text{Gross}}(a_L, b_L), & [u] \text{ logical, syndrome } 0, \\ 0, & \text{otherwise,} \end{cases} \quad (57)$$

the identification carrying no residual constant when the logical basis is gauge fixed by $\tilde{X}$. More generally, since $\Delta^{\mathcal{C}}$ is $\mathcal{M}$ invariant, $W_{\hat{\rho}}^{\mathcal{C}} = W_{\mathcal{E}_{\mathcal{C}}(\hat{\rho})}^{\mathcal{C}}$ for every $\hat{\rho}$, and the value on the syndrome- s block is the subnormalized Gross function of the syndrome-conditional logical state $\hat{\rho}_L(s)$: the finite-dimensional analog of Eqs. (37) and (40).

We prove this in Appendix C. Stabilizer invariance makes all $|\mathcal{M}|$ terms of Eq. (54) equal, so the average loses nothing; $\mathcal{M} = -\mathcal{M}$ and phase-freeness make the parity $A(0)$ act on the code space as the logical parity; and the error sectors vanish because $A(0)$ maps the syndrome- s sector to $-s$, whose trace pairing with a sector- s operator vanishes unless $s = 0$, which is where d odd enters, just like in the discrete Hudson theorem.

Proposition IV.3 (Errors and the cycle). A Pauli error $D(e)$ acts on $W^{\mathcal{C}}$ as the rigid translation $[u] \mapsto [u + e]$; Pauli channels act as convolutions over $X^{\mathcal{C}}$; a Clifford unitary in the normalizer of $\mathcal{S}$, i.e., one whose symplectic action S satisfies $S\mathcal{M} = \mathcal{M}$, acts on $X^{\mathcal{C}}$ by $[u] \mapsto [Su]$. (A Clifford outside that normalizer maps $W^{\mathcal{C}}$ to the code-adapted function of a different code, and does not lie in $\mathcal{G}_{\mathcal{C}}$.) A correctable error produces a definite syndrome, and the corrected cycle is the identity on $W^{\mathcal{C}}$; an error whose translation leaves the correctable set acts, after recovery, as the induced logical Pauli, the finite-dimensional image of Eq. (51).

The Pauli statements follow from the covariance $\hat{\Delta}^{\mathcal{C}}([u + w]) = D(w)\hat{\Delta}^{\mathcal{C}}([u])D^\dagger(w)$ established above, exactly as

Eqs. (48), (50), and (51) follow from the covariance of the Zak-Gross kernel; the Clifford statement also uses $U_S A(u) U_S^\dagger = A(Su)$ [2], which with $SM = \mathcal{M}$ permutes the terms of Eq. (54). As in the GKP case, the identity is on the logical distribution, so error correction is not a reversal of the physical process [Eq. (50)].

Theorem IV.1. Let d be an odd prime, $\mathcal{C}$ any $[[n, k]]_d$ stabilizer code, and $\mathcal{N}_\rho^{\mathcal{C}} = \int d\mu |W_\rho^{\mathcal{C}}| - 1$. Then, for every physical state $\hat{\rho}$,

$$\mathcal{N}_\rho^{\mathcal{C}} = \sum_s P(s|\hat{\rho}) \mathcal{N}^{\text{Gross}}(\hat{\rho}_L(s)), \quad (58)$$

the syndrome-probability-weighted logical Gross negativity. In particular, $\mathcal{N}^{\mathcal{C}}$ vanishes on every encoded stabilizer state and every syndrome mixture of them; it is strictly positive on every encoded pure magic state, inheriting the magic-monotone structure of the Gross negativity [3]; and on general mixed states, it is a one-sided witness, just as for ideal GKP codes.

Equation (58) is a resource accounting of the error-correction cycle, channel by channel, and the finite-dimensional counterpart of Eq. (40).

Remark IV.1 (Generalizations). Nothing above is specific to prime d beyond the resource reading, so the result extends three ways.

(1) *Odd prime powers:* Replacing $\mathbb{Z}_d$ by the finite field $\mathbb{F}_q$, $q = p^m$ with p odd, and the symplectic form by its field-trace version [2,11], every proof holds unchanged, since Gross's construction and the discrete Hudson theorem hold there; square-free odd d follows by the Chinese remainder theorem, factor by factor. At even prime powers, the geometry holds with a Gibbons-Hoffman-Wooters quantum net in place of the phase-point operator, as in Sec. IV E, but the resource reading does not.

(2) *Subsystem codes:* Averaging Eq. (54) over the displacement image of the full gauge group (again a subgroup of $\mathbb{Z}_d^{2n}$) gives a kernel whose reconstruction map is, by the computation of Proposition IV.1 with that subgroup in place of $\mathcal{M}$, the conditional expectation onto the commutant of the gauge algebra. The representation deliberately forgets the gauge qudits along with the intersector coherences, which is the correct notion of logical content there.

(3) *Qubit structure:* As we note in Sec. IV E, Propositions III.2, IV.1, and IV.3 hold at $d = 2$, but Proposition IV.2 and Theorem IV.1 do not.

We verified every statement above, to 10^{-9} or better, on the three-qutrit repetition code $[[3, 1]]_3$ and the five-qutrit code $[[5, 1, 3]]_3$; Appendix E lists the checks (Table III), including a stress test of Eq. (58) under a channel with a syndrome-free logical error.

E. The qubit case

The lattice geometry of Sec. IV D still holds for $d = 2$; however, its resource reading does not. Propositions III.2 and IV.3 are d agnostic, since they use only the conjugation rule $gD(v)g^{-1} = \omega^{[u,v]}D(v)$ and the covariance of the kernel. Proposition IV.1 needs a phase-point operator to average, and its proof uses only translation covariance, self-duality, and orthogonality. All three still hold at $d = 2$: Taking $D(a, b) =$

$i^{ab}X^aZ^b$ gives $A(0) = (\mathbb{1} + X + Y + Z)/2$, which is Hermitian and of unit trace, with $\text{Tr}[A(u)A(v)] = 2\delta_{uv}$, $\sum_u A(u) = 2\mathbb{1}$, and $\hat{\rho} = \sum_u \frac{1}{2} \text{Tr}[A(u)\hat{\rho}]A(u)$. This means that Proposition IV.1 holds, as it does for any other translation-covariant self-dual frame available at even d , such as a Gibbons-Hoffman-Wooters quantum net [11] or a covariant member of Ref. [36]'s recent classification. This $A(0)$ is neither an involution nor the parity, and it is the parity that the resource reading needs. Proposition IV.2 needs $A(0)$ to act on the code space as the logical parity and to map the syndrome- s sector to $-s$, with $2s' = 0$ forcing $s' = 0$, and Theorem IV.1 needs the discrete Hudson theorem, which has no even- d counterpart, no even- d representation being simultaneously covariant and non-negative on all stabilizer states [2,18]. Both fail for $d = 2$. This means that a qubit stabilizer code allows a code-adapted kernel and the full translation geometry, but the resource reading of Sec. IV G is unavailable to it. The same applies to the qubit GKP code and to surface GKP: The Zak-Gross kernel, its covariance, and Secs. IV B and IV C's corrected cycle carry over at $d = 2$, but Eq. (37), Proposition IV.2, and the magic identification do not. In that respect, qubit codes act similarly for the finite-dimensional theory to how the rotation-symmetric families act for the continuous-variable one, which is why we discuss them here rather than in Sec. V. Partial recoveries exist for rebit subtheories [37], and the recently classified even- d Wigner functions [36] would be the natural place to look for a stronger substitute. We regard the restriction itself as the correct general statement, mirroring Theorem V.1.

F. Multimode lattices and concatenation

The two sets of codes where the stabilizer is a lattice of displacements in a Weyl-Heisenberg group compose. Reference [9] establishes that concatenating an n -mode GKP code with a qudit stabilizer code yields another GKP lattice code: With $\mathcal{L}_0 \subset \mathbb{R}^{2n}$ the base stabilizer lattice and $\mathcal{M} \subset \mathbb{Z}_d^{2n}$ the displacement image of the $[[n, k]]_d$ stabilizer group, the concatenated code space is the fixed-point space of the refined lattice $\mathcal{L} = \mathcal{L}_0 + \text{lift}(\mathcal{M})$, of index d^{n-k} over $\mathcal{L}_0$. Our framework adds a representation-theoretic consequence.

Proposition IV.4. For any such concatenated code, surface-GKP [23] included, Secs. IV B and IV C's framework applies verbatim with $\mathcal{L}$ in place of the square lattice. For odd d , Eq. (37) composes with Proposition IV.2: The Zak-Gross function of the concatenated code is the Gross Wigner function of the $[[n, k]]_d$ logical state, with the syndrome sectors of both layers tracked; and for odd prime d , its negativity is the logical magic of the k encoded qudits.

The proof comes from two observations. First, $\mathcal{L}$ is again a symplectically integral lattice, since $\mathcal{L}_0$ is, and $\mathcal{M}$ is isotropic, so $\mathbb{R}^{2n}/\mathcal{L}$ is a compact phase space carrying an invariant measure, and the lattice-sum kernel built on $\mathcal{L}$ is displacement covariant by construction. Second, standardization follows just like for Eq. (41): Integrating the lattice sum over $\mathbb{R}^{2n}/\mathcal{L}$ kills every term whose displacement label is a nonzero element of the symplectic dual of $\mathcal{L}$, leaving the identity times the covolume of $\mathcal{L}$, which the measure normalizes away. The remaining statements are then Eq. (37) and Proposition IV.2 applied layer by layer. Decoders become translation-valued

functions on $\mathbb{R}^{2n}/\mathcal{L}$, and the two-layer syndrome factorization refines Eq. (56). Surface-GKP has $d = 2$: The refined lattice, the kernel, and the translation geometry carry over, but the logical identification of Proposition IV.2 and the magic reading do not (Sec. IV E); finite energy enters as in Sec. VIII, per mode.

G. Certifying computational resources

Equations (40) and (58) both say that the negative volume of a code-adapted representation is a syndrome-weighted average of a logical magic monotone. The argument that makes this work for codes where the stabilizer is a lattice of displacements in a Weyl-Heisenberg group fails when this is not the case (Sec. V C).

The hull argument runs on a family of coherent states. The phase space X^C and the kernel are fixed by H^C , as Sec. III A sets out, and for the ideal GKP code that means the stabilizer lattice H_0^C and the torus $\mathbb{T}_{dl}^2$; nothing below changes them. The reference state enters only as the seed of a family of code-adapted coherent states $|\Omega\rangle = \pi(\Omega)|\psi_0\rangle$, and for the hull argument we seed it with a logical stabilizer state. For the GKP code, these are the displaced logical stabilizer code words; at the points of the logical lattice, they reduce to $\hat{X}^a \hat{Z}^b |\psi_0\rangle$, i.e., to a single logical basis (up to phases) rather than to the full vertex set of the logical stabilizer polytope [$d(d+1)$ states for prime d]. $|+_L\rangle$, for instance, is a stabilizer state lying outside their convex hull, since it is a comb of a different spacing and no rigid displacement of $|0_L\rangle$ reaches it. [With a stabilizer reference code word, the labeling is also d -fold degenerate, since $\hat{Z}$ fixes $|0\rangle$ and so Ω and $\Omega + (0, l)$ give the same projector; this is fine, because only the set of projectors $\{|\Omega\rangle\langle\Omega|\}$ and its convex hull enter the argument.] Even this weaker identification (the family containing logical states at all) is specific to GKP and fails for the rotation-symmetric codes, as does the positivity assumption of Eq. (60): For a code whose isotropy group contains no displacements, $|\Omega\rangle = \hat{D}(\Omega)|\psi_0\rangle$ leaves the code space altogether for $\Omega \neq 0$, and the family $\{|\Omega\rangle\}$ is not a set of logical states at all.

If the state $\hat{\rho}$ can be expressed as a classical convex combination of these coherent states,

$$\hat{\rho} = \int_{X^C} d\mu(\Omega) P(\Omega) |\Omega\rangle\langle\Omega|, \quad (59)$$

where $P(\Omega)$ is a classical probability distribution (i.e., respects all Kolmogorov's axioms); this implies that the generalized phase-space distribution $W_\rho^C(\Omega)$ is non-negative, provided that the kernel is itself non-negative on the code-adapted coherent states,

$$\text{Tr}[\hat{\Delta}^C(\Omega)|\Omega'\rangle\langle\Omega'|] \geq 0, \quad \forall \Omega, \Omega' \in X^C. \quad (60)$$

[We write $\mathcal{N}_\rho^C$ for the code-adapted negative volume of Eq. (6), in the same convention as Eq. (39).] For the ordinary Wigner function of a coherent state, we do not need to check this, as coherent states are Gaussian; but the states in Eq. (60) are the code-adapted coherent states $|\Omega\rangle = \pi(\Omega)|\psi_0\rangle$, built from a non-Gaussian, Wigner-negative code word, so the condition has to be checked. For the GKP code, it follows from Eq. (37): The code-adapted coherent states are displaced stabilizer code words, whose Zak-Gross Wigner function is

the Gross Wigner function of a discrete-variable stabilizer state, so non-negative. Conditional on this, a state lying in the convex hull of the code-adapted stabilizer states has a non-negative representation; equivalently, $\mathcal{N}_\rho^C > 0$ certifies that $\hat{\rho}$ lies outside that hull. However, the logical states contained by the hull form the simplex of one logical basis (a proper subset of the stabilizer polytope), so leaving the hull does not by itself witness magic. For GKP, the upgrade to a magic witness comes from Eq. (37) instead: The Gross Wigner function is non-negative on *all* stabilizer states [2], not only on the orbit of $|\psi_0\rangle$, so $\mathcal{N}^C > 0$ certifies exit from the full polytope there. For a general code, the hull argument alone gives only the weaker statement.

The converse does not follow, even where Eq. (60) holds: A state carrying logical magic lies outside the hull, but lying outside the hull does not by itself force a negative representation (that would need the set of non-negatively represented states to coincide with the hull rather than merely contain it). The two differ already for a single qutrit, where mixed states outside the stabilizer polytope have non-negative discrete Wigner function [3]; in the continuous-variable setting, positively represented states can even be computationally universal [32–34]. The one case in which both directions hold is the ideal GKP code restricted to pure logical states of odd logical dimension d , where Eq. (37) and the discrete Hudson theorem [2] give “non-negative $\Leftrightarrow$ stabilizer,” as Ref. [5] showed. For mixed logical states, the bound-magic states just cited leave only the witness direction: $\mathcal{N}^C > 0$ certifies exit from the stabilizer polytope, since a mixture of stabilizer states has a non-negative Gross Wigner function, but $\mathcal{N}^C = 0$ certifies nothing.

This holds for every qudit stabilizer code of odd prime dimension; Theorem IV.1 gives it in closed form. It works in both cases for the same reason: Eq. (37) together with Eq. (57). For codes where the stabilizer is a lattice of displacements in a Weyl-Heisenberg (WH) group, the code-adapted distribution *is* the Gross Wigner function of the logical state. Nothing weaker than that identification upgrades the hull statement to a magic witness, which is why the reading fails for other codes.

H. Dissipation of encoded magic under displacement noise

The accounting above turns a physical noise model into a logical resource budget, and for the ideal GKP qudit under Gaussian displacement noise, this budget is solvable. The represented cycle is the convolution of Eq. (53), transferred to the logical side by Eq. (37) as a product Pauli channel with weights $p_{ab}(\sigma) = q_a(\sigma)q_b(\sigma)$,

$$q_c(\sigma) = \sum_{j \in \mathbb{Z}} \left[\Phi\left(\frac{cl + dj + l/2}{\sigma}\right) - \Phi\left(\frac{cl + dj - l/2}{\sigma}\right) \right], \quad (61)$$

with $l = \sqrt{2\pi/d}$, Φ the normal cumulative distribution function, and c the logical displacement class. Dissipation of encoded magic is then a finite computation on the logical space; for $d = 3$ and the encoded strange state, it is solvable in closed form.

Proposition IV.5. The Gross Wigner function of the qutrit strange state takes the value $-\frac{1}{3}$ at one phase-space point and

$+\frac{1}{6}$ at the other eight. Consequently, under any Pauli channel with weight distribution p_v ,

$$\mathcal{N}(\{p_v\}) = \sum_{v \in \mathbb{Z}_3^2} \max \left\{ 0, p_v - \frac{1}{3} \right\}, \quad (62)$$

and under the GKP-induced channel (61), where only the identity weight exceeds $\frac{1}{3}$,

$$\begin{aligned} \mathcal{N}(\sigma) &= \max \left\{ 0, q_0(\sigma)^2 - \frac{1}{3} \right\} \\ &= \max \left\{ 0, \frac{2}{3} - P_{\text{err}}(\sigma) \right\}, \end{aligned} \quad (63)$$

with $P_{\text{err}} = 1 - q_0^2$ the logical error probability. The negativity vanishes at the threshold $q_0(\sigma^*)^2 = \frac{1}{3}$,

$$\sigma^* = 0.623\,654\,l \approx 0.9026 \quad (d = 3, l = \sqrt{2\pi/3}), \quad (64)$$

i.e., exactly when the logical error probability reaches $2/3$.

We prove this in Appendix C: The channel convolves the flat-plus-one-dip Wigner function, so every point takes the form $\frac{1}{6} - \frac{p_v}{2}$, negative iff $p_v > 1/3$. We verified Eq. (62) against the full channel computation at all noise strengths and the threshold by independent root finding. Beyond the threshold, every syndrome-conditional logical state is non-negatively represented, and the encoded computation is efficiently simulable [3,19], for GKP circuits specifically, using the recent simulator whose run-time scales with this Zak-Gross negativity [38]. Below the threshold, $\mathcal{N}(\sigma)$ is the logical error budget $\frac{2}{3} - P_{\text{err}}$. For general logical states, dimensions, and lattices, Eq. (62) is replaced by the corresponding finite convolution, still in closed form from the error function (erf) sums (61).

Because each corrected round induces a logical Pauli channel, repeated rounds compose as $\mathbb{Z}_3^2$ convolutions of the weights, rather than as $\sigma \rightarrow \sqrt{k} \sigma$, since the recovery rebins the noise each round, and the convolutions diagonalize in the characters of $\mathbb{Z}_3$. For the symmetric weights (61), where $q_1 = q_{-1}$, every nontrivial character of the per-quadrature step distribution equals

$$\lambda(\sigma) = q_0 - q_1 = \frac{3q_0(\sigma) - 1}{2}, \quad (65)$$

so $q^{*k}(0) = \frac{1}{3}(1 + 2\lambda^k)$ and the whole multi-round history is closed form,

$$\begin{aligned} \mathcal{N}_k &= \max \left\{ 0, \frac{(1 + 2\lambda^k)^2}{9} - \frac{1}{3} \right\}, \\ k^* &= \left\lceil \frac{\ln[(\sqrt{3} - 1)/2]}{\ln \lambda} \right\rceil, \end{aligned} \quad (66)$$

with no other weight ever exceeding $\frac{1}{3}$, since $(1 + 2\lambda)(1 - \lambda) \leq \frac{9}{8}$ on $[0, 1]$. This gives $\mathcal{N}_k = \frac{2}{3} - P_{\text{err}}^{(k)}$ round by round while $k < k^*$, and a magic decay constant $\tau = -1/\ln \lambda$ in units of correction rounds. At $\sigma = 0.3l$, $\lambda = 0.8566$, and $k^* = 7$, the magic lasts six rounds ($\mathcal{N}_6 = 0.023$) and is gone at the seventh, after which the residual computation is classically simulable. Equation (66) matches direct convolution to 10^{-16} at three noise levels, including the prediction $k^* = 54$ at $\sigma = 0.2l$.

V. STABILIZER-NOT-WH-DISPLACEMENT-LATTICE CODES

Every other bosonic or Weyl-Heisenberg code falls on the other side of Corollary III.1. This section works through these families (rotation-symmetric bosonic codes, qudit codes stabilized by non-Pauli operators, one-photon codes, and spin codes), showing for each that the standard kernel remains up to a code-independent freedom, meaning the negativity of that representation is not a property of the logical system. For the first three families, we establish this from the SW axioms, under the hypotheses of the statements concerned; for spin codes (Sec. VF), we show only that the averaging prescription supplies no code-adapted kernel. (Note that the stabilizer of qubit stabilizer codes is a lattice of displacements in a Weyl-Heisenberg group and so are covered in Sec. IV E, as they only fail to allow a resource reading of their negativity.)

A. Rotation-symmetric cat codes

Cat codes [39,40] are a family of bosonic states that encode logical qubits into superpositions of coherent states of a harmonic oscillator. A coherent state is defined as the eigenstate of the annihilation operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha \in \mathbb{C}, \quad (67)$$

and can be written as an infinite (weighted) sum of Fock states

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (68)$$

An N -component rotation-symmetric cat code encodes into the characters of the $\mathbb{Z}_N$ rotation generated by $\hat{R}_N = e^{i2\pi\hat{n}/N}$:

$$|\bar{j}\rangle = n_j^{-1/2} \sum_{k=0}^{N-1} \omega_N^{jk} |\alpha_k\rangle, \quad \omega_N = e^{i2\pi/N}, \quad (69)$$

with $\alpha_k = \alpha e^{i2\pi k/N}$. These satisfy $\hat{R}_N |\bar{j}\rangle = \omega_N^{-j} |\bar{j}\rangle$, so $|\bar{j}\rangle$ has Fock support on $\hat{n} \equiv -j \pmod{N}$, and the qubit is encoded in the pair

$$|0_L\rangle = |\bar{0}\rangle, \quad |1_L\rangle = |\bar{N/2}\rangle \quad (70)$$

for N even, so that the two code words are separated by $N/2$ in the $\mathbb{Z}_N$ grading and a single-photon loss cannot carry one into the other unless $N/2 = 1$. We use two group-theoretic properties of this family below. First, $\hat{R}_N$ has eigenvalues $+1$ on $|0_L\rangle$ and $\omega_N^{-N/2} = -1$ on $|1_L\rangle$: On the code space, $\hat{R}_N$ acts as the logical $\hat{Z}$. This therefore fixes the reference code word $|0_L\rangle$ but does not fix the code space pointwise, which is the distinction drawn in Eq. (8): The isotropy subgroup is $H^{\text{cat}} = \text{U}(1) \times \mathbb{Z}_N$, while the subgroup fixing every code state is the index-two subgroup $H_0^{\text{cat}} = \text{U}(1) \times \mathbb{Z}_{N/2}$ generated by $\hat{R}_N^2$. Second, since N is even, $\hat{R}_N^{N/2} = e^{i\pi\hat{n}} = (-1)^{\hat{n}}$: The parity operator is an element of $\mathbb{Z}_N$, so Proposition III.3's phase-space inversion requirement is met.

The $N = 2$ member of the family is the familiar even/odd cat encoding. Here, the two two-component cat states are, for a given α ,

$$|C_{\alpha}^{\pm}\rangle = v_{\pm}(|\alpha\rangle \pm |-\alpha\rangle), \quad (71)$$

with normalization constants

$$\nu_{\pm} = \frac{1}{\sqrt{2(1 \pm e^{-2|\alpha|^2})}}. \quad (72)$$

Expanding Eq. (71) in the Fock basis, $|C_{\alpha}^{+}\rangle$ has support on even Fock levels and $|C_{\alpha}^{-}\rangle$ on odd ones, so they are $|\bar{0}\rangle$ and $|\bar{1}\rangle$ of Eq. (69) at $N = 2$; we call them the even and odd cat states, and encode $|0_L\rangle = |C_{\alpha}^{+}\rangle$, $|1_L\rangle = |C_{\alpha}^{-}\rangle$. They satisfy

$$\hat{a}|C_{\alpha}^{\pm}\rangle = \alpha \frac{\nu_{\pm}}{\nu_{\mp}} |C_{\alpha}^{\mp}\rangle. \quad (73)$$

A single-photon loss therefore flips parity, so the parity operator $\Pi = e^{i\pi\hat{n}}$, with $\Pi|n\rangle = (-1)^n|n\rangle$ and hence $\Pi|C_{\alpha}^{\pm}\rangle = \pm|C_{\alpha}^{\pm}\rangle$, detects single-photon losses when monitored continuously.

This detection does not, however, allow such losses to be corrected. At $N = 2$, $\hat{R}_2 = (-1)^{\hat{n}} = \Pi$, so the operator that detects the loss is also the logical $\hat{Z}$. Measuring it measures the logical qubit rather than extracting a syndrome, and so destroys any logical superposition. Equivalently, $\langle 0_L|\hat{a}|1_L\rangle \neq 0$ by Eq. (73), so the Knill-Laflamme condition fails and a single-photon loss is a logical error rather than a correctable one. [It is not even an exact logical $\hat{X}$, since the weights $\alpha\nu_{\pm}/\nu_{\mp}$ in Eq. (73) are unequal except as $|\alpha| \rightarrow \infty$.]

We therefore take $N = 4$ whenever the error-correcting properties of the code are used, this being the smallest loss-correcting member of the family. The logical states are the $j = 0$ and $j = 2$ characters of Eq. (69),

$$\begin{aligned} |0_L\rangle &\propto |\alpha\rangle + |i\alpha\rangle + |-\alpha\rangle + |-i\alpha\rangle, \\ |1_L\rangle &\propto |\alpha\rangle - |i\alpha\rangle + |-\alpha\rangle - |-i\alpha\rangle, \end{aligned} \quad (74)$$

with Fock support on $\hat{n} \equiv 0$ and $\hat{n} \equiv 2 \pmod{4}$, respectively, and the single-loss error words occupy $\hat{n} \equiv 3$ and $\hat{n} \equiv 1 \pmod{4}$, meaning they are orthogonal both to the code space and to each other. The logical $\hat{Z}$ is no longer the parity but $\hat{R}_4$, while $\hat{R}_4^2 = (-1)^{\hat{n}}$ takes the value $+1$ on the whole code space, so $H_0^{\text{cat}} = \text{U}(1) \times \mathbb{Z}_2$ is generated by parity.

The syndrome must separate the error space from the code space without separating the two code words; a measurement that resolves the full $\mathbb{Z}_N$ grading $\hat{n} \bmod N$ does separate them, since $|0_L\rangle$ and $|1_L\rangle$ sit in different $\mathbb{Z}_N$ sectors, and so measures $\hat{R}_N = \hat{Z}$ rather than a syndrome. We need the grading of H_0^C , the subgroup that fixes the code space pointwise:

$$\hat{n} \bmod (N/2), \quad (75)$$

which for $N = 4$ is the photon-number parity $(-1)^{\hat{n}}$. Under ℓ losses, $|0_L\rangle$ moves to $\hat{n} \equiv -\ell$ and $|1_L\rangle$ to $\hat{n} \equiv N/2 - \ell \pmod{N}$, which are the same residue modulo $N/2$: Measurement returns $-\ell \bmod (N/2)$ regardless of which code word was sent and so identifies the number of lost photons for $\ell < N/2$ while learning nothing about the logical state. At $N = 4$, the code space is entirely even and the single-loss error space entirely odd, so parity does this.

The Knill-Laflamme conditions for a single loss hold up to corrections exponentially small in $|\alpha|^2$: The off-diagonal conditions $\langle 0_L|\hat{a}|1_L\rangle = \langle 0_L|\hat{n}|1_L\rangle = 0$ are exact, because the relevant states lie in different $\mathbb{Z}_4$ sectors, and it is only the diagonal condition $\langle 0_L|\hat{n}|0_L\rangle = \langle 1_L|\hat{n}|1_L\rangle$ that holds approximately, the two sides differing through the overlaps of the four coherent-state components.

The averaging prescription of Eq. (14), which for the GKP code reproduces the Zak-Gross Wigner function described previously, does not work here: H^{cat} contains rotations and so is not normal in G , and the operator it produces is not displacement covariant, by Proposition III.1 when G is taken to be the full metaplectic extension and by direct computation (Appendix A) for the $G = H_3(\mathbb{C}) \rtimes \mathbb{Z}_N$ used here. We can instead use Proposition III.3, which fixes the kernel from the Stratonovich-Weyl axioms directly, for every N at once and without any expansion of a reference state. The kernel is fixed rather than constructed. The cat code words are normalizable, so Lemma III.1 applies and no nontrivial displacement leaves C_{cat} invariant; hence, $\Lambda^{\perp} = \{0\}$. [This is part (iii) of Proposition III.3, which uses the twirl form. Part (i) gives $\mathcal{E}_C = \mathbb{1}$ from continuity of χ_K alone, assuming neither that form nor complete positivity.] The parity operator is in $\mathbb{Z}_N$ for N even, so the phase-space inversion requirement holds, and the only admissible Stratonovich-Weyl kernel is the standard one:

$$\hat{\Delta}^{\text{cat}}(\beta) = \hat{\Delta}^{\text{std}}(\beta) = 2\hat{\Pi}(\beta), \quad (76)$$

with central kernel

$$\hat{\Delta}^{\text{cat}}(0) = 2(-1)^{\hat{n}}. \quad (77)$$

The $\mathbb{Z}_N$ symmetry of the cat code constrains the kernel ($\hat{K}$ must commute with $\hat{R}_N$), but the standard kernel already does, so the constraint is not binding. The cat code has no phase-space representation of its own: The code-adapted distribution is the ordinary Wigner function,

$$W_{\hat{\rho}}^{\text{cat}}(\beta) = \pi W_{\hat{\rho}}^{\text{std}}(\beta), \quad (78)$$

where the factor π is the Jacobian between $d^2\beta$, with respect to which W^{std} is normalized, and the invariant measure $d\mu(\beta) = d^2\beta/\pi$ of Eq. (A5). We get standardization from

$$\text{Tr}[\hat{\Delta}^{\text{cat}}(0)] = 2 \text{Tr}[(-1)^{\hat{n}}] = 2 \times \frac{1}{2} = 1, \quad (79)$$

using $\text{Tr}[(-1)^{\hat{n}}] = 1/2$ in the Abel sense, so $\int d\mu W_{\hat{\rho}}^{\text{cat}} = \text{Tr}[\hat{\rho}]$ as Eq. (1) requires.

This has two consequences. First, since $\Lambda^{\perp} = \{0\}$ the reconstruction map is the identity, $\mathcal{E}_{\text{cat}} = \mathbb{1}$: The representation is faithful on the whole Hilbert space, Q_{cat} is a functor, and every channel is trivially syndrome graded, so Eq. (12) composes without restriction, but it also carries no code-specific information. The sector decomposition of Eq. (19) is trivial here; the parity of Eq. (75) remains the syndrome the experimenter measures, but it is a grading of H_0^C , not of $\mathcal{E}_{\text{cat}}$, which grades by nothing. Second, anything code specific must come from the code words rather than from the representation; Sec. VII F exploits this, holding the representation fixed while only the encoded states vary.

The standard kernel is covariant, standardized, real, and self-dual, so the conditions of Sec. III B are met and the structure theorem applies to the cat code. The forward and inverse maps are then

$$\begin{aligned} \chi_{\text{cat}}(\hat{\rho}) &:= W_{\hat{\rho}}^{\text{cat}}(\beta) = \text{Tr}[\hat{\Delta}^{\text{cat}}(\beta)\hat{\rho}] \\ &= \pi W_{\hat{\rho}}^{\text{std}}(\beta), \\ \phi_{\text{cat}}(F) &= \frac{1}{\pi} \int_{\mathbb{C}} d^2\beta F(\beta) \hat{\Delta}^{\text{cat}}(\beta). \end{aligned} \quad (80)$$

The full structure theorem map $Q_{\text{cat}}(T) = \chi_{\text{cat}} \circ T \circ \phi_{\text{cat}}$, applied to an input distribution $W^{\mathcal{C}_{\text{cat}}}(\beta)$, is then

$$Q_{\text{cat}}(T)[W^{\mathcal{C}_{\text{cat}}}(\beta')] = \text{Tr} \left[\hat{\Delta}^{\mathcal{C}_{\text{cat}}}(\beta') T \left(\frac{1}{\pi} \int_{\mathbb{C}} d^2\beta W^{\mathcal{C}_{\text{cat}}}(\beta) \hat{\Delta}^{\mathcal{C}_{\text{cat}}}(\beta) \right) \right]. \quad (81)$$

Substituting a physical error channel for T (such as the single-photon loss operator $\hat{a} \cdot \hat{a}^\dagger$, which shifts the $\mathbb{Z}_N$ grading by one) tracks geometrically how that error acts on the phase-space distribution.

B. Binomial codes

Binomial codes [41] are a family of bosonic quantum error-correction codes that encode logical qubits into finite superpositions of Fock states of a harmonic oscillator. The most general binomial code words are

$$\begin{aligned} |W_\uparrow\rangle &= \frac{1}{\sqrt{2^N}} \sum_{p \text{ even}}^{[0, N+1]} \sqrt{\binom{N+1}{p}} |p(S+1)\rangle, \\ |W_\downarrow\rangle &= \frac{1}{\sqrt{2^N}} \sum_{p \text{ odd}}^{[0, N+1]} \sqrt{\binom{N+1}{p}} |p(S+1)\rangle, \end{aligned} \quad (82)$$

where $\binom{N+1}{p}$ is the binomial coefficient $C(N+1, p) = (N+1)!/(p!(N+1-p)!)$.

This code can protect against the error set [41]

$$\bar{\mathcal{E}} = \{\mathbb{1}, \hat{a}, \hat{a}^2, \dots, \hat{a}^L, \hat{a}^\dagger, \dots, (\hat{a}^\dagger)^G, \hat{n}, \dots, \hat{n}^D\}, \quad (83)$$

which means it can protect against up to L photon losses, G photon gain errors, and D dephasing events. In Eq. (82), $S = L + G$ sets the spacing $S+1$ between occupied Fock levels, N is the maximum order $N = \max\{L, G, 2D\}$, and the index p has range 0 to $N+1$. The code respects the Knill-Laflamme condition [42–45], i.e.,

$$\langle W_\sigma | (\hat{a}^\dagger)^\ell \hat{a}^\ell | W_{\sigma'} \rangle = C_\ell \delta_{\sigma\sigma'}, \quad (84)$$

with $\sigma \in \{\uparrow, \downarrow\}$ for all $\ell \leq N = \max\{L, G, 2D\}$ [41]; $2D$ enters because Eq. (83)'s dephasing errors $\hat{n}, \dots, \hat{n}^D$ contribute Knill-Laflamme products with photon-number moments up to order $2D$, which fix the order N of the code word construction.

Errors can be detected and corrected because the code words remain orthogonal after an error in the correctable set. For example, if we consider $\ell \leq L$ photon losses, we get

$$\hat{a}^\ell |W_\uparrow\rangle, \quad \hat{a}^\ell |W_\downarrow\rangle, \quad (85)$$

which are still orthogonal since the space between occupied Fock states is $S+1$, with $S = L + G$. While less likely to occur, the same happens for $g \leq G$ gain photon errors. For dephasing errors, after $j \leq D$ errors we obtain a superposition of the original words and the error words. To detect the error, we can project into the logical words basis and then perform a unitary operation to recover the original words.

For photon gain and loss errors, the syndrome is the photon number mod $S+1$, obtained by coupling the mode to an ancillary system (e.g., a superconducting qubit [46,47], a trapped ion [48–51] or a Rydberg atom [52,53]).

The binomial code words are supported on Fock levels that are multiples of $S+1$, so they are fixed by the rotation

$\hat{R}_{S+1} = e^{i2\pi\hat{n}/(S+1)}$. The isotropy subgroup of the reference code word $|W_\uparrow\rangle$ is larger than this: $|W_\uparrow\rangle$ is supported on levels $p(S+1)$, with p even, so it is also fixed by the half-rotation $\hat{R}_{2(S+1)} = e^{i\pi\hat{n}/(S+1)}$, and

$$H^{\text{bin}} = \text{U}(1) \times \mathbb{Z}_{2(S+1)}, \quad H_0^{\text{bin}} = \text{U}(1) \times \mathbb{Z}_{S+1} \quad (86)$$

(see Appendix B), the second being the subgroup that fixes the code space pointwise. The structure is the same as that of the cat code: $\hat{R}_{2(S+1)}$ acts as $+1$ on $|W_\uparrow\rangle$ and as $e^{i\pi p} = -1$ on $|W_\downarrow\rangle$, so is the logical $\hat{Z}$, while its square $\hat{R}_{S+1}$ fixes both code words, and, the order $2(S+1)$ being even, $(-1)^\hat{n} = \hat{R}_{2(S+1)}^{S+1}$ lies in G , meeting Proposition III.3's phase-space inversion requirement. The rest is as for the cat code: The code words are normalizable, so by Lemma III.1 no nontrivial displacement leaves $\mathcal{C}_{\text{bin}}$ invariant, $\Lambda^\perp = \{0\}$, and Proposition III.3 again admits only the standard kernel,

$$\begin{aligned} \hat{\Delta}^{\mathcal{C}_{\text{bin}}}(\beta) &= \hat{\Delta}^{\text{std}}(\beta) = 2\hat{\Pi}(\beta), \\ \hat{\Delta}^{\mathcal{C}_{\text{bin}}}(0) &= 2(-1)^\hat{n}, \end{aligned} \quad (87)$$

meaning that the phase-space distribution of a density operator $\hat{\rho}$ is

$$W_\rho^{\mathcal{C}_{\text{bin}}}(\beta) = \text{Tr}[\hat{\rho} \hat{\Delta}^{\mathcal{C}_{\text{bin}}}(\beta)] = \pi W_\rho^{\text{std}}(\beta). \quad (88)$$

The binomial and cat kernels therefore are both ordinary Wigner kernels. Corollary III.1 forces this, since neither code space is invariant under a lattice of displacements; and by Lemma III.1, no finite-dimensional subspace of $L^2(\mathbb{R})$ is. The normalization needs no separate argument, $\text{Tr}[\hat{\Delta}^{\mathcal{C}_{\text{bin}}}(0)] = 1$ as in Eq. (79), and just like there the reconstruction map is the identity, so $\hat{A}_W = \hat{\rho}$ and Q_{bin} is a functor. The two codes are distinguished by their code words: The cat code words are superpositions of coherent states, with photon-number distribution sharply peaked at $|\alpha|^2$, while each binomial code word is a finite Fock superposition supported on the levels $p(S+1)$, with p of fixed parity in $[0, N+1]$, that is, on at most $\lfloor (N+1)/2 \rfloor + 1$ widely spaced levels [two of them at the working point $(N, S) = (2, 3)$ used later].

The conditions of the structure theorem hold for the standard kernel just in the cat case, and because the two kernels are the same operator, the forward map χ_{bin} , the inverse map ϕ_{bin} , and the represented channel $Q_{\text{bin}}(T) = \chi_{\text{bin}} \circ T \circ \phi_{\text{bin}}$ are those of Eqs. (80) and (81), with $\mathcal{C}_{\text{cat}}$ replaced by $\mathcal{C}_{\text{bin}}$ throughout.

C. Negativity is not logical magic

Section IV G showed that the hull argument becomes a magic witness for codes with a code-adapted representation, through the identification of the code-adapted distribution with the Gross Wigner function of the logical state. For codes that just allow the ordinary Wigner representation, that identification is unavailable, and the assumption the hull argument rests on fails.

For the rotation-symmetric codes of Secs. V A and V B, Eq. (60) does not hold. For these, Proposition III.3 gives $\hat{\Delta}^{\mathcal{C}} = 2\hat{\Pi}$, and the code-adapted coherent states are the displaced

TABLE II. Negative volume $\mathcal{N}^C$ [Eq. (6)] for cat- and binomial-code states at matched mean photon number $\bar{n} = 6$ [four-component cat with $|\alpha|^2 = 6$; binomial with $(N, S) = (2, 3)$], before and after a single conditioned loss or gain event, with the postevent mean photon number. Every cat state listed except $|T_L\rangle$ is a logical stabilizer state, and all of them have $\mathcal{N}^C > 0$; the magic state $|T_L\rangle$ has lower negativity than three of them. The input mean photon numbers are 6 for the binomial code words and 5.98 and 6.02 for the two cat code words, the difference being the exponentially small overlap corrections of Sec. V A. The loss and gain columns are discussed in Sec. VII F; Appendix E gives the implementation used.

State	$\mathcal{N}^C$	$\mathcal{N}_{\text{loss}}$	$\mathcal{N}_{\text{gain}}$	$\langle \hat{n} \rangle_{\text{loss}}$	$\langle \hat{n} \rangle_{\text{gain}}$
Cat $ 0_L\rangle$	1.38	1.41	1.52	5.96	7.82
Cat $ 1_L\rangle$	1.41	1.42	1.53	6.04	7.89
Cat $ \pm_L\rangle$	0.62	0.62	0.64	6.00	7.86
Cat $ +i_L\rangle$	1.42	1.39	1.52	6.00	7.86
Cat $ T_L\rangle$ (magic)	1.22	1.21	1.30	6.00	7.86
Binomial $ W_\uparrow\rangle$	1.62	1.72	1.97	7.00	8.71
Binomial $ W_\downarrow\rangle$	1.64	1.75	2.04	7.00	8.71

code words $|\Omega\rangle = \hat{D}(\Omega)|\psi_0\rangle$, so

$$\begin{aligned} \text{Tr}[\hat{\Delta}^C(\Omega)|\Omega'\rangle\langle\Omega'|] &= 2\langle\psi_0|\hat{\Pi}(\Omega - \Omega')|\psi_0\rangle \\ &= \pi W_{\psi_0}^{\text{std}}(\Omega - \Omega'). \end{aligned} \quad (89)$$

Equation (60) therefore holds iff the reference code word has an everywhere non-negative ordinary Wigner function, which by Hudson's theorem [21,54] means iff $|\psi_0\rangle$ is Gaussian. No code word of a rotation-symmetric code is Gaussian, so this condition fails.

However, this condition does not fail everywhere. Setting $\Omega = \Omega'$ in Eq. (89) gives $2\langle\psi_0|(-1)^{\hat{n}}|\psi_0\rangle$, and every state of the four-component cat-code space has support on $\hat{n} \equiv 0, 2 \pmod{4}$ (i.e., on even photon numbers only), so this equals $+2$ for every code state; the same holds for a binomial code with $S+1$ even. Equation (60) fails at $\Omega \neq \Omega'$, in the interference fringes of the code word's Wigner function, not at the origin.

We can see this failure numerically. Take matched mean photon number $\bar{n} = 6$, so $|\alpha|^2 = 6$ for the four-component cat code and $(N, S) = (2, 3)$ for the binomial code, and evaluate the negative volume (6) on the logical basis states of both codes, and for the cat code also on the logical superpositions and on a logical magic state $|T_L\rangle \propto |0_L\rangle + e^{i\pi/4}|1_L\rangle$. See Table II for the results, together with the postloss and postgain values used in Sec. VII F; Appendix E shows how they were computed.

The negative volume is nonzero on states that carry no logical magic at all (all but one entry in Table II is a logical stabilizer state), so this is not a witness. It is also not monotone in logical resourcefulness: The magic state $|T_L\rangle$ has $\mathcal{N}^C = 1.22$, below the stabilizer states $|0_L\rangle$, $|1_L\rangle$, and $|+i_L\rangle$, and well above $|\pm_L\rangle$. For these codes, $\mathcal{N}^C$ instead tracks the ordinary continuous-variable nonclassicality of the physical state, which has nothing to do with the logical qubit.

This follows from Proposition III.3 together with Hudson's theorem and covers every bosonic code with normalizable code words:

Theorem V.1. Let $\mathcal{C} \subset L^2(\mathbb{R})$ be a finite-dimensional code space of logical dimension at least 2, spanned by normalizable code words; let G contain the phase-space inversion; and let the representation satisfy hypotheses (H1)–(H3) of Proposition III.3, with $\chi_{\hat{K}}$ a function. That is, let it satisfy the Stratonovich-Weyl axioms in Sec. II C's code-faithful form: Axioms 2–4 unchanged, Axioms 1 and 5 on the $\mathcal{E}_{\mathcal{C}}$ -invariant subalgebra, together with a reconstruction map $\mathcal{E}_{\mathcal{C}} = \phi_{\mathcal{C}} \circ \chi_{\mathcal{C}}$ that is idempotent and acts as the identity on $\mathcal{B}(\mathcal{C})$, with $\chi_{\hat{K}}(\xi) = \text{Tr}[\hat{K}\hat{D}(\xi)]$ a continuous function of ξ [for a bounded, non-trace-class $\hat{K}$, the trace is read in the Abel-regularized sense, as at Eq. (79); it is a comb rather than a function exactly in the ideal GKP case]. We assume neither complete positivity of $\mathcal{E}_{\mathcal{C}}$ nor a twirl form for it, so neither setting of Proposition III.3 is used: Trace preservation and displacement covariance of $\mathcal{E}_{\mathcal{C}}$ follow from Axioms 2–4, and complete positivity is part of the conclusion. Then, $\mathcal{E}_{\mathcal{C}} = \mathbb{1}$, and the only such kernel available to $\mathcal{C}$ is the standard one, so that

$$W_{\hat{\rho}}^{\mathcal{C}}(\Omega) = \pi W_{\hat{\rho}}^{\text{std}}(\Omega), \quad \text{for every } \hat{\rho}, \quad (90)$$

and there exists a logical stabilizer state with $\mathcal{N}^C > 0$. The negative volume is therefore not a witness of logical magic for any such code.

We prove this in Appendix C: The first part is the continuity route of Proposition III.3(i), followed by (ii); for the second, a code space of logical dimension at least 2 contains two orthogonal logical stabilizer states, which cannot both be Gaussian [55], so by Hudson's theorem [21,54] one of them has $\mathcal{N}^C > 0$ while carrying no logical magic. Appendix C also records three remarks on the hypotheses: What dropping continuity leaves once $\mathcal{E}_{\mathcal{C}} = \mathbb{1}$ is obtained another way (a code-independent sign family, the continuum analog of Proposition V.1); why idempotence of the reconstruction map is needed on top of Axioms 1–4, which alone admit the Husimi kernel; and why the code faithful rather than strong reading of Axioms 1 and 5 is the substantive choice, the strong form excluding the ideal GKP code along with everything else.

Beyond the parity requirement, Theorem V.1 does not refer to the code's symmetry group, only to the normalizability of its code words, so it applies to the cat and binomial codes, to any rotation-symmetric code of even order, and to finite-energy GKP code words (which we return to in Sec. VIII). The parity hypothesis is available for every code considered here, in the two ways set out after Corollary III.1. The theorem does not apply to a code following neither route (by the remark after Proposition III.3, the kernel is then only constrained to $\chi_{\hat{K}} = e^{if}$, with f real, odd, and continuous, and the Hudson step needs a displaced parity), and we leave that case open. The ideal GKP code is the exception of a different kind: Its code words are Dirac combs rather than elements of $L^2(\mathbb{R})$ (Sec. IV A), so Lemma III.1 does not apply to it and its code space can be (and is) invariant under a lattice of displacements. Its kernel is then the lattice sum of Eq. (35), standardized in the integrated form of Eq. (41) rather than by a trace, which for that kernel diverges. That lattice suppresses the continuous-variable structure and leaves only the logical content: the content of Eq. (37) and of Ref. [5]'s

result that stabilizer code words are positively represented. Where no lattice is available, Proposition III.3 leaves only $\Lambda^\perp = \{0\}$, Eq. (90) holds, and the code-adapted distribution is the ordinary Wigner function. Its negativity is therefore the ordinary continuous-variable nonclassicality of the encoded state, which is large for a cat or binomial code word and has nothing to do with the logical qubit.

Together with Corollary III.1, Theorem V.1 also classifies the set of codes for which the resource reading can hold. The code space must be invariant under a group of displacements, which by Lemma III.1 rules out normalizable code words. Within the distributional codes that remain, for which we assume setting (B) of Proposition III.3, the invariance group cannot contain a one-parameter subgroup, since the commutant of a continuum of displacements is a maximal Abelian algebra and so cannot contain the logical operators of a qubit; it must therefore be a lattice, and it must have rank 2 if the quotient X^C is to be compact, so that the representation carries no more independent degrees of freedom than the logical system it describes. Being the fixed-point space of a rank-2 lattice of displacements is the defining property of the GKP family.

Ideal GKP-type codes are therefore, among codes whose phase space is compact, the only bosonic codes whose code-adapted phase-space negativity can measure logical magic. (This is a statement about kernels built from the *physical* Weyl-Heisenberg group, within the class Corollary III.1 covers; for a representation built instead from the code's logical operators, see Sec. VI.)

D. Non-Pauli qudit symmetries

The finite-dimensional codes not covered by Proposition III.2 are those whose defining symmetries are not displacement subgroups: code spaces fixed by Clifford or higher symmetries, the discrete analog of the rotation-symmetric bosonic families.

Proposition V.1. Let d be odd and let $\mathcal{C} \subset (\mathbb{C}^d)^{\otimes n}$ be a code space whose projective isotropy group contains a non-Pauli unitary R (a Clifford acting on displacement labels by a symplectic $S \neq \mathbb{1}$), and whose displacement isotropy is trivial. For parts (ii) and (iii), let the kernel satisfy the finite-group form of hypotheses (H1) and (H2) of Proposition III.3, let its reconstruction map be completely positive, as in setting (A) (an assumption, since a finite group offers no continuity to replace it), and let the phase-space inversion be available, as in Proposition III.3(ii).

(1) The group-averaged kernel over $\langle R \rangle$ fails displacement covariance, by the same two-multiset argument as Proposition III.1, valid here because the d^{2n} phase-point operators are linearly independent.

(2) Proposition III.3's axiom-direct route forces $\Delta(u) = D(u)\hat{K}D^\dagger(u)$, with reconstruction multiplier $m = |\chi_{\hat{K}}|^2$; pointwise idempotence and positive definiteness (i.e., complete positivity) make m the indicator function of a subgroup $\Lambda \subseteq \mathbb{Z}_d^{2n}$, whose symplectic dual $\Lambda^\perp$ is the twirl group and so must leave $\mathcal{C}$ invariant; triviality of the displacement isotropy gives $\Lambda^\perp = \{0\}$; therefore, $\Lambda = \mathbb{Z}_d^{2n}$, $m \equiv 1$, and $|\chi_{\hat{K}}| \equiv 1$.

(3) Hermiticity and inversion covariance then give $\chi_{\hat{K}} : \mathbb{Z}_d^{2n} \rightarrow \{\pm 1\}$, even, with $\chi_{\hat{K}}(0) = 1$. No continuity is avail-

able on a finite group, so this sign freedom is not removable by the axioms: Every such sign pattern yields a Hermitian, unit-trace, covariant, self-dual kernel. The freedom is code independent, existing already for the trivial code, so no code-adapted kernel exists; and it is removed by demanding non-negativity on pure stabilizer states, which selects the Gross kernel [2]. Clifford covariance alone does not remove this freedom: The symplectic group acts transitively on $\mathbb{Z}_d^{2n} \setminus \{0\}$, so it leaves the two constant sign patterns, from which stabilizer positivity picks out $\chi_{\hat{K}} \equiv 1$ [18].

The sign family is the finite-group shadow of the residual freedom left by Proposition III.3 in the CV analysis: the $(-1)^{mn}$ pattern of the GKP lattice kernel, and the " $\chi_{\hat{K}} = e^{if}$, with f real and odd," family that remains when no inversion is available. We verified all counts numerically for a single qutrit with the Fourier gate as code symmetry (Appendix E). At even d , the sign family and the absence of a code-adapted kernel still hold, but the selection of the Gross kernel by stabilizer positivity does not, for the same reason we give in Sec. IV E.

E. One-photon codes

For one-photon codes, this is a stronger restriction than a failure of the witness property.

Theorem V.2. For any one-photon-in- m -rails code ($\mathcal{C} \subset \text{span}\{|0 \dots 010 \dots 0\rangle\}$ over $m \geq 2$ modes, dual rail being $m = 2$), and under hypotheses (H1)–(H3) of Proposition III.3, with $\chi_{\hat{K}}$ a function, the only Stratonovich-Weyl kernel is the ordinary m -mode Wigner kernel, and every pure logical state has the same negative volume,

$$\mathcal{N}^C = 4e^{-1/2} - 2 \approx 0.42612. \quad (91)$$

We prove this in Appendix C: Every pure logical state is a passive rotation of $|1\rangle \otimes |0\rangle^{\otimes(m-1)}$, and $\int |W|$ is symplectically invariant, so all share the negative volume of the Fock state $|1\rangle$.

Loss in any rail moves the state to the vacuum sector: The syndrome is the total photon number, and loss is detectable (an erasure) but not correctable. This is the multimode version of Sec. V A's $N = 2$ cat statement.

F. Spin codes

Codes built on a compact dynamical group fail for a representation-theoretic, rather than spectral, reason.

Proposition V.2. Let $G = \text{SU}(2)$ act irreducibly on a spin- j system, and let $\mathcal{C}$ be any code space whose isotropy subgroup $H^C \subset \text{SU}(2)$ is a proper, noncentral subgroup [56]. Then, the H^C -averaged kernel is either not Stratonovich-Weyl covariant or not faithful on the code algebra, so the averaging prescription yields no code-adapted kernel.

We prove this in Appendix C, by a Schur's-lemma argument on the multiplicity-free tensor sectors of the spin- j operator space (linear independence of the kernels at distinct phase-space points, which the CV and DV arguments use, is unavailable on S^2); both halves were verified numerically at $j = 1$.

Spin codes therefore behave like rotation-symmetric bosonic codes. They are represented by the ordinary spherical kernel, up to its known ε_l sign family (Proposition V.1's

compact-group analog of the sign family), and the kernel's negativity tracks spin-coherent nonclassicality [59], not logical content.

G. Conjecture for dynamical groups outside the Weyl-Heisenberg family

We have shown that code-adapted kernels exist for every code whose stabilizer is a lattice of displacements in a Weyl-Heisenberg group, continuous (ideal GKP and its multimode and concatenated lattices, Secs. IV B and IV F) or discrete (every qudit stabilizer code, the kernel being constructed at prime-power d ; Sec. IV D), because two-step nilpotency makes the isotropy subgroups normal. They do not exist for nondisplacement symmetries in the finite Weyl-Heisenberg setting (Proposition V.1), and for Abelian dynamical groups [whose invariant subalgebra is Abelian and so cannot contain $\mathcal{B}(\mathcal{C})$ once $\dim \mathcal{C} \geq 2$, meaning that the code-faithful form of Axiom 1 fails]. For every noncentral isotropy subgroup of a compact group with multiplicity-free tensor decomposition, we only prove that the averaging prescription supplies no code-adapted kernel [Proposition V.2, whose Schur argument is not specific to $SU(2)$], which stops short of an axiom-direct no go.

We conjecture that no dynamical group outside the extended Weyl-Heisenberg family (including its metaplectic and finite extensions) admits code-adapted Stratonovich-Weyl kernels for any of its codes. This is unresolved for noncompact, non-nilpotent groups, and still requires a formalization of “code adapted” as “outside the standard family up to its code-independent sign and phase freedoms”.

VI. THE SECTOR-GRADED LOGICAL REPRESENTATION

Everything so far concerns kernels built from the physical Weyl-Heisenberg group, the setting of the Brif-Mann construction, and the structure theorem as we applied it. A representation built from the logical operators of a code would instead be fixed by a choice of logical basis rather than by the symmetry of the code space, so Theorem V.1 would have no implications for it.

Let $\mathcal{C}$ be a code with syndrome-sector decomposition $\mathcal{H} \supseteq \bigoplus_t \mathcal{H}_t$ and recovery isometries $V_t : \mathcal{H}_L \rightarrow \mathcal{H}_t$, with V_0 the encoding, and let $A_L(a, b)$ be a Stratonovich-Weyl kernel on the logical space $\mathcal{H}_L \cong \mathbb{C}^d$. Define the *sector-graded logical kernel* and its generated representation,

$$\begin{aligned} \Delta_{\log}(t; a, b) &= V_t A_L(a, b) V_t^\dagger, \\ W_{\hat{\rho}}^{\log}(t; a, b) &= \frac{1}{d} \text{Tr}[\Delta_{\log} \hat{\rho}], \end{aligned} \quad (92)$$

on the phase space $X^{\log} = \{t\} \times \mathbb{Z}_d^2$ with counting measure, the factor $1/d$ in $W^{\log}$ being the measure absorbed into the function, so that $\sum_{t,a,b} W_{\hat{\rho}}^{\log} = \text{Tr}[\hat{\Pi} \hat{\rho}]$, with $\hat{\Pi} = \sum_t \hat{\Pi}_t$. Every statement below is for $\hat{\rho}$ supported on $\bigoplus_t \mathcal{H}_t$, where $\hat{\Pi} \hat{\rho} = \hat{\rho}$ and the representation is standardized. This is the syndrome-times-logical factorization (56) that the stabilizers-as-displacement-lattice branch produces, now taken as a definition for codes with no code-adapted kernel of their own. At the Hilbert-space level, it parallels Ref. [60]'s stabilizer subsystem decompositions for GKP codes, where tracing out

the stabilizer subsystem implements ideal decoding; Eq. (92) is the phase-space representation layer over such a decomposition. Throughout, $\mathcal{H}_t := V_t \mathcal{H}_L$, a d -dimensional subspace, and $\hat{\Pi}_t = V_t V_t^\dagger$ the projector onto it; the sectors need not cover all of $\mathcal{H}$, which is why the decomposition is written as $\mathcal{H} \supseteq \bigoplus_t \mathcal{H}_t$. For an N -component rotation code [61] encoding a qudit of dimension d , with N/d a positive integer, the sector label is the residue of $\hat{n}$ modulo N/d : The code words $|j_L\rangle$ occupy Fock levels $n \equiv jN/d \pmod{N}$, t losses shift the code space coherently into the residue class labeled t , and $V_t |j_L\rangle \propto \hat{a}^t |j_L\rangle$ are exact isometries because distinct logical states occupy distinct residue classes.

We must differentiate two objects here from their counterparts in Sec. III E. Write $\mathcal{E}^{\log} = \sum_t \hat{\Pi}_t \cdot \hat{\Pi}_t$, with $\hat{\Pi}_t = V_t V_t^\dagger$, for the sector dephasing, and $\mathcal{G}^{\log} = \{T : \mathcal{E}^{\log} T \mathcal{E}^{\log} = T \mathcal{E}^{\log}\}$ for the channels graded by it. These are *not* Sec. III E's $\mathcal{E}_{\mathcal{C}}$ and $\mathcal{G}_{\mathcal{C}}$, which for a rotation-symmetric code are the identity and the set of all channels, respectively. The grading here is supplied by the recovery scheme rather than by the reconstruction map of a Stratonovich-Weyl kernel, meaning that a channel creating coherence between the sectors $\mathcal{H}_t$, such as a small displacement, does not lie in $\mathcal{G}^{\log}$.

Theorem VI.1. With the notation above, we have the following:

(1) $\phi_{\log} \circ \chi_{\log} = \mathcal{E}^{\log}$ on the sector subspace $\bigoplus_t \mathcal{H}_t$: With the forward map $\chi_{\log}(\hat{\rho}) = W_{\hat{\rho}}^{\log}$ of Eq. (92) and the reverse map $\phi_{\log}(F) = \sum_{t,a,b} F(t; a, b) \Delta_{\log}(t; a, b)$, the composition is the sector dephasing. (The factor $1/d$ appears once: here in $\chi_{\log}$ and in Proposition IV.1 in $\phi_{\mathcal{C}}$.)

(2) Consequently, $W^{\log}$ is a linearity-preserving, empirically adequate, semifunctorial representation of the subtheory $\mathcal{G}^{\log}$, with the same compositional structure as the Zak-Gross representation of the GKP code, the twirl now coming from the sector grading instead of from a displacement lattice. In particular, Eq. (22) holds on $\mathcal{G}^{\log}$, by Sec. III E's argument applied to $\mathcal{E}^{\log}$.

(3) For odd prime logical dimension d , with A_L the Gross kernel, its negative volume is the syndrome-resolved logical magic, $\mathcal{N}_{\hat{\rho}}^{\log} = \sum_t P(t|\hat{\rho}) \mathcal{N}^{\text{Gross}}(\hat{\rho}_L(t))$. This vanishes on every syndrome mixture of encoded stabilizer states and is strictly positive on every encoded pure logical magic state; on mixed logical states, it is a one-sided witness, just as in Theorem IV.1.

Part (i) follows from the self-duality of A_L (Appendix C); parts (ii) and (iii) then follow from Secs. III E and IV G and Theorem IV.1's arguments, each applied on the logical space with $\mathcal{E}^{\log}$ in place of $\mathcal{E}_{\mathcal{C}}$. Only (iii) needs d odd and prime: Parts (i) and (ii) only need a self-dual logical frame A_L , which also exists for $d = 2$ (Sec. IV E), so a rotation code encoding a qubit still has a compositional sector-graded representation, albeit without the magic reading.

Theorem VI.1 is consistent with Theorem V.1, since $\Delta_{\log}$ is not of the form $\hat{D}(\Omega) \hat{K} \hat{D}^\dagger(\Omega)$: It is not covariant under the physical Weyl-Heisenberg group and is fixed by a recovery scheme rather than by the code-space symmetry. Decoders differing by a logical unitary per sector, $V_t \mapsto V_t U_t$, give representations related by the logical (anti)symplectic action when U_t is Clifford, and genuinely different negativities otherwise: The representation describes decoder-relative logical

content, which is the operationally meaningful object in a cycle. However, a loss event is not a rigid translation of $X^{\log}$ but a sector shift composed with the $\sqrt{n}$ reweighting of Eq. (102).

We tested the construction on the $N = 6$ cat code encoding a logical qutrit ($d = 3$, so $N/d = 2$): code words $|j_L\rangle \propto \sum_{k=0}^5 \omega_6^{-2jk} |\alpha e^{i\pi k/3}\rangle$ supported on $n \equiv 2j \pmod{6}$, with the code sector $t = 0$ and the single-loss sector $t = 1$; their residue classes $n \equiv 0, 2, 4$ and $n \equiv 1, 3, 5 \pmod{6}$ cover all of the number grading. V_1 is an isometry because $\hat{a}|j_L\rangle$ for distinct j occupy distinct residue classes mod 6 and are therefore orthogonal; $\mathcal{H}_0 \oplus \mathcal{H}_1$ is the six-dimensional subspace they span, not the whole Fock space. At $|\alpha|^2 = 6$ and 12, the isometries are exact; $\phi_{\log} \circ \chi_{\log} = \mathcal{E}^{\log}$ holds to 10^{-10} on random states; standardization is exact on the sector subspace; and encoded stabilizer states have $\mathcal{N}^{\log} = 0$, while the encoded strange state has $\mathcal{N}^{\log} = 2/3$ exactly. This is a logical-magic reading for a rotation-symmetric code, which Theorem V.1 proves no physically covariant kernel can supply. For odd prime logical dimension, this also resolves the tension recently observed between non-Gaussianity and magic in cat codes, globally incompatible but aligned sector by sector [62]: The sector-graded representation is the object for which the local alignment becomes a global identity.

The loss channel quantifies the geometry lost in exchange. Conditioned on a single jump, the state moves entirely to sector 1, and the conditional logical state has fidelity 0.998 62 to the ideal strange state at $|\alpha|^2 = 6$, with $\mathcal{N}^{\text{Gross}} = 0.664\,83$ against $2/3$; at $|\alpha|^2 = 12$, both agree with the ideal to 10^{-6} . The deviation is the phase-space image of the code-word-dependent jump weights $\langle \hat{n} \rangle_j = (6.30, 6.31, 5.44)$, that is, of the $\hat{n}$ reweighting in Eq. (101)'s loss-transformed kernel, and it decays with the code word overlaps. Recovery returns the logical distribution with the same fidelity: The recovery map is $\mathcal{R}(\hat{\rho}) = V_0 V_1^\dagger \hat{\rho} V_1 V_0^\dagger + \hat{\Pi}_0 \hat{\rho} \hat{\Pi}_0$, trace preserving on $\mathcal{H}_0 \oplus \mathcal{H}_1$ and in $\mathcal{G}^{\log}$.

This has three consequences. First, at $N/d = 2$ the gain words $\hat{a}^\dagger |j_L\rangle$ occupy the same sectors as the loss words, with a natural isometry related to V_1 by a nontrivial logical map, which is the sector-graded rendering of the gain-loss conflation of Sec. VII F. For $N/d \geq 3$, the gain and loss sectors $t = \mp 1$ are distinct and both channels are separately representable, mirroring the binomial grading. Second, the construction applies verbatim to binomial codes and, after orthogonalizing the approximately orthogonal sectors, to finite-energy GKP code words, restoring a logical reading exactly where Theorem V.1 removes the kernel-based one. Third, this approach degenerates for one-photon codes: The loss sector is the vacuum, too small to host an isometric recovery, and the absence of a sector-graded representation is the representation-level statement that loss is an erasure there (Theorem V.2).

VII. ERROR CHANNELS IN PHASE SPACE

Both sets of cases (stabilizer is a lattice of displacements in a Weyl-Heisenberg group, and not) supply a formula [Eq. (12)] for the action of a channel on the code-adapted

distribution, and Eq. (22) lets a full cycle be assembled from it channel by channel.

A. Single-photon loss and the reflection identity

The dominant physical error channel in many bosonic implementations is single-photon loss [63]. The unconditional process is the amplitude-damping channel

$$\mathcal{E}_\gamma(\hat{\rho}) = \sum_{\ell \geq 0} \hat{J}_\ell \hat{\rho} \hat{J}_\ell^\dagger, \quad \hat{J}_\ell = \sqrt{\frac{\gamma^\ell}{\ell!}} e^{-\kappa t \hat{n}/2} \hat{a}^\ell, \quad (93)$$

with $\gamma = 1 - e^{-\kappa t}$. This is trace preserving and lies in $\mathcal{G}_C$ in the sense of Sec. III E, trivially, since $\mathcal{E}_{\text{cat}} = \mathbb{1}$ (Sec. III D). Below, we analyze the $\ell = 1$ case of this channel, i.e., the trajectory conditioned on a single detected quantum jump:

$$\begin{aligned} \hat{J}_1 \hat{\rho} \hat{J}_1^\dagger &= \gamma e^{-\kappa t \hat{n}/2} \hat{a} \hat{\rho} \hat{a}^\dagger e^{-\kappa t \hat{n}/2}, \\ p_1 &= \text{Tr}[\hat{J}_1 \hat{\rho} \hat{J}_1^\dagger] \in [0, 1], \end{aligned} \quad (94)$$

with p_1 the probability of exactly one jump and $\hat{J}_1 \hat{\rho} \hat{J}_1^\dagger / p_1$ the normalized postjump state. It is convenient to strip off the damping factor and work with the bare jump map

$$\begin{aligned} T_{\text{loss}}(\hat{\rho}) &= \hat{a} \hat{\rho} \hat{a}^\dagger, \\ \text{Tr}[T_{\text{loss}}(\hat{\rho})] &= \text{Tr}[\hat{a}^\dagger \hat{a} \hat{\rho}] = \langle \hat{n} \rangle, \end{aligned} \quad (95)$$

which differs from the $\ell = 1$ case only by conjugation with $e^{-\kappa t \hat{n}/2}$ and coincides with it up to the overall constant γ in the short-time limit $\kappa t \langle \hat{n} \rangle \ll 1$ (the damping factor deviates from unity by $\sim \kappa t n/2$ on the occupied levels, so for $\bar{n} = 6$ the condition is $\kappa t \ll 1/6$, not $\kappa t \ll 1$). Note that T_{loss} is completely positive but neither trace preserving nor trace-non-increasing: $\hat{a}^\dagger \hat{a}$ is unbounded, and $\text{Tr}[\hat{a} \hat{\rho} \hat{a}^\dagger] = \langle \hat{n} \rangle$, which for $|\alpha|^2 = 6$ (as used in Sec. VII F) is 6 up to exponentially small corrections in $|\alpha|^2$ (numerically 5.98 and 6.02 for the two cat code words). $T_{\text{loss}}(\hat{\rho})$ is therefore an unnormalized positive operator rather than a subnormalized state, and its phase-space image integrates against $d\mu$ to $\langle \hat{n} \rangle$, rather than a probability. Dividing by $\langle \hat{n} \rangle$ gives the normalized distribution of the postjump state $\hat{a} \hat{\rho} \hat{a}^\dagger / \langle \hat{n} \rangle$. We work with T_{loss} below and renormalize when the result is interpreted, noting where the damping factor enters: Since $\mathcal{E}_C = \mathbb{1}$ for this code, every channel is syndrome graded, so it composes under Eq. (22), and to leading order in $\kappa t \langle \hat{n} \rangle$ it rescales the distribution rather than reshaping it (it contracts phase space radially, $|\alpha\rangle \mapsto |\alpha e^{-\kappa t/2}\rangle$ up to normalization).

For $N = 4$ [Eq. (74)], a single-photon loss carries code sectors $\hat{n} \equiv 0, 2$ into error sectors $\hat{n} \equiv 3, 1 \pmod{4}$, which are orthogonal to the code space. The loss is therefore not itself an operation on the logical qubit, but a transition out of the code space into a distinguishable (and so correctable) error space.

The displaced parity is the reflection of phase space about ζ , so reflects the ladder operators:

$$\begin{aligned} \hat{\Pi}(\zeta) \hat{a} \hat{\Pi}(\zeta) &= 2\zeta - \hat{a}, \\ \hat{\Pi}(\zeta) \hat{a}^\dagger \hat{\Pi}(\zeta) &= 2\zeta^* - \hat{a}^\dagger. \end{aligned} \quad (96)$$

Both follow from $\hat{D}^\dagger(\zeta) \hat{a} \hat{D}(\zeta) = \hat{a} + \zeta$ together with $(-1)^{\hat{n}} \hat{a} (-1)^{\hat{n}} = -\hat{a}$ (we derive this in Appendix D). Since

$\hat{\Pi}(\zeta)^2 = \mathbb{1}$, every jump kernel now follows by substitution, and we use Eq. (96) this way throughout this section.

We substitute $T = T_{\text{loss}}$ into the general structure theorem formula (12). Using the reconstruction operator ϕ_{cat} , we first reconstruct the physical operator from the input distribution $W^{\text{cat}}(\beta)$:

$$\begin{aligned}\hat{A}_W &:= \phi_{\text{cat}}(W^{\text{cat}}) \\ &= \frac{1}{\pi} \int_{\mathbb{C}} d^2\beta W^{\text{cat}}(\beta) \hat{\Delta}^{\text{cat}}(\beta).\end{aligned}\quad (97)$$

Because $\mathcal{E}_{\text{cat}} = \mathbb{1}$ here, this reconstruction is exact rather than a twirl: $\hat{A}_W = \hat{\rho}$. However, we keep the notation parallel to the GKP case of Sec. IV C, where the reconstruction is a twirl. The evolved distribution at coordinate β' is then

$$Q_{\text{cat}}(T_{\text{loss}})[W^{\text{cat}}](\beta') = \text{Tr}[\hat{\Delta}^{\text{cat}}(\beta') \hat{A}_W \hat{a}^\dagger]. \quad (98)$$

Cyclicity of the trace moves the ladder operators onto the kernel, giving $Q_{\text{cat}}(T_{\text{loss}})[W^{\text{cat}}](\beta') = \text{Tr}[\hat{\Delta}^{\text{cat}}(\beta')_{\text{loss}} \hat{A}_W]$ in terms of the loss-transformed kernel

$$\hat{\Delta}^{\text{cat}}(\beta')_{\text{loss}} := \hat{a}^\dagger \hat{\Delta}^{\text{cat}}(\beta') \hat{a}. \quad (99)$$

We can evaluate this kernel using the reflection identity (96). Writing ζ for a generic phase-space point, to distinguish it from Eq. (93)'s damping parameter γ , and using $\hat{\Pi}^2 = \mathbb{1}$,

$$\begin{aligned}\hat{a}^\dagger \hat{\Pi}(\zeta) \hat{a} &= \hat{\Pi}(\zeta) (\hat{\Pi}(\zeta) \hat{a}^\dagger \hat{\Pi}(\zeta)) \hat{a} \\ &= -\hat{\Pi}(\zeta) (\hat{n} - 2\zeta^* \hat{a}),\end{aligned}\quad (100)$$

which is exact for every ζ . Since Eq. (76)'s kernel is the standard one, the calculation involves a single displaced parity where a lattice kernel would give a sum, and since that kernel carries the factor 2,

$$\hat{\Delta}^{\text{cat}}(\beta')_{\text{loss}} = -2 \hat{\Pi}(\beta') [\hat{n} - 2\beta'^* \hat{a}], \quad (101)$$

so

$$Q_{\text{cat}}(T_{\text{loss}})[W^{\text{cat}}](\beta') = -2 \text{Tr}[\hat{\Pi}(\beta') (\hat{n} - 2\beta'^* \hat{a}) \hat{\rho}]. \quad (102)$$

Conjugating $(-1)^{\hat{n}}$ by $\hat{a}^\dagger \cdot \hat{a}$ produces $-\hat{n}(-1)^{\hat{n}}$, not $-(-1)^{\hat{n}}$, so the loss-transformed kernel is not the negative of the original.

Equation (102) has two parts. The overall sign is the phase-space signature of the parity change: The kernel registers the change of parity accompanying a single loss as a change of sign. The factor $\hat{n}$ multiplying it is the phase-space signature of the photon-number dependence of the loss rate, and the term linear in $\hat{a}$ is the accompanying coherent-amplitude term. Note two potential issues. First, the two terms are not separately real [$\text{Tr}[\hat{\Pi}(\beta') \hat{n} \hat{\rho}]$ is complex in general, and only the combination in Eq. (102) is a real-valued phase-space function], so neither on its own is a physical distribution. Second, $\hat{n}$ cannot be replaced by its mean pointwise in β' . The mean photon number of a code word is $|\alpha|^2$ up to corrections exponentially small in $|\alpha|^2$, and integrating Eq. (102) against $d\mu$ gives $\langle \hat{n} \rangle$ exactly, since standardization gives $\int d\mu Q(T)[W] = \text{Tr}[T(\hat{\rho})]$ for any T . However, the photon-number distribution of a coherent-state superposition has width $|\alpha|$, so substituting $\hat{n} \rightarrow |\alpha|^2$ inside the trace incurs a relative error of order $1/|\alpha|$, not an exponentially small one;

for $|\alpha|^2 = 6$ (as in Sec. VII F), that is a 40% effect, and larger still where W^{cat} is small. We therefore keep Eq. (102) exact and do not use a mean-field form of it anywhere.

B. The cat-code cycle

Let us assemble the error-correction cycle. The syndrome is the photon-number parity of Eq. (75), measured by a quantum nondemolition coupling to an ancilla; in the phase-space picture, this distinguishes the even sector, on which the code space is supported, from the odd sector into which a single loss has moved it, and since Q_{cat} is empirically adequate, this correctly reproduces the probability of each outcome. Recovery is then an ancilla-assisted unitary returning the error sectors $\hat{n} \equiv 3, 1$ to the code sectors $\hat{n} \equiv 0, 2$ (a coherent map on the two-dimensional error space, which preserves the logical superposition, rather than a further measurement). Both maps are syndrome graded in the sense of Sec. III E, trivially, since $\mathcal{E}_{\text{cat}} = \mathbb{1}$, so by Eq. (22) the representation of the cycle is the composition of the representations.

Since $T_{\text{QEC}} = T_{\text{rec}} \circ T_{\text{loss}}$ is built from the bare jump map, $\int d\mu Q(T_{\text{QEC}})[W] = \text{Tr}[T_{\text{QEC}}(\hat{\rho})] = \langle \hat{n} \rangle$ [Eq. (95)], so the cycle returns

$$Q_{\text{cat}}(T_{\text{QEC}})[W^{\text{cat}}](\beta') = \langle \hat{n} \rangle W^{\text{cat}}(\beta') \quad (103)$$

for $\hat{\rho}$ supported on the code space, so that after dividing by the jump weight $\langle \hat{n} \rangle$ [or equivalently, after conditioning on the detected jump, as below Eq. (95)] the normalized logical distribution is returned to itself. The factor is $\langle \hat{n} \rangle$ because the recovery is an isometry from the error space back to the code space and the two code words carry the same weight $\langle j_L | \hat{n} | j_L \rangle$ up to corrections exponentially small in $|\alpha|^2$; the residual mismatch between those two weights is what limits the identity to that accuracy. Up to that renormalization, this is the analog of the GKP result, where the photon remains in the environment.

C. The binomial-code cycle

Recall that binomial code words consist of Fock states spaced by $S + 1$ [41]:

$$\begin{aligned}|W_\uparrow\rangle &\propto \sum_{p \text{ even}} \sqrt{\binom{N+1}{p}} |p(S+1)\rangle, \\ |W_\downarrow\rangle &\propto \sum_{p \text{ odd}} \sqrt{\binom{N+1}{p}} |p(S+1)\rangle.\end{aligned}\quad (104)$$

Applying $\hat{a}$ to a Fock state lowers the photon number by 1: $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$. Since the code words occupy Fock levels $\{p(S+1)\}$, a single-photon loss maps each level to $p(S+1) - 1$, which lies between two consecutive occupied levels and is therefore orthogonal to the code space. Defining the single-loss error words,

$$|W_\uparrow^{(1)}\rangle := \frac{\hat{a}|W_\uparrow\rangle}{\|\hat{a}|W_\uparrow\rangle\|}, \quad |W_\downarrow^{(1)}\rangle := \frac{\hat{a}|W_\downarrow\rangle}{\|\hat{a}|W_\downarrow\rangle\|}, \quad (105)$$

which occupy levels $\{p(S+1) - 1\}$ and are orthogonal to both code words and to each other. The spacing $S + 1$, with

$S = L + G \geq 1$, guarantees that no two error words overlap, satisfying the Knill-Laflamme conditions for $L \geq 1$.

Substituting $T = T_{\text{loss}}$ into the general structure theorem formula in Eq. (12) and writing $\hat{A}_W := \phi_{\text{bin}}(W^{C_{\text{bin}}})$ for the reconstructed operator, the evolved distribution at β' is

$$\begin{aligned} Q_{\text{bin}}(T_{\text{loss}})[W^{C_{\text{bin}}}] (\beta') &= \text{Tr}[\hat{\Delta}^{C_{\text{bin}}}(\beta') \hat{a} \hat{A}_W \hat{a}^\dagger] \\ &= \text{Tr}[\hat{a}^\dagger \hat{\Delta}^{C_{\text{bin}}}(\beta') \hat{a} \hat{A}_W], \end{aligned} \quad (106)$$

where in the second equality we used cyclicity of the trace. We define the loss-transformed kernel

$$\hat{\Delta}^{C_{\text{bin}}}(\beta')_{\text{loss}} := \hat{a}^\dagger \hat{\Delta}^{C_{\text{bin}}}(\beta') \hat{a}. \quad (107)$$

Because Eq. (87)'s binomial kernel is the standard one, we can use Eq. (100) to give

$$\hat{\Delta}^{C_{\text{bin}}}(\beta')_{\text{loss}} = -2 \hat{\Pi}(\beta') [\hat{n} - 2\beta'^* \hat{a}], \quad (108)$$

meaning

$$Q_{\text{bin}}(T_{\text{loss}})[W^{C_{\text{bin}}}] (\beta') = -2 \text{Tr}[\hat{\Pi}(\beta') (\hat{n} - 2\beta'^* \hat{a}) \hat{\rho}]. \quad (109)$$

This is identical to the cat-code result (102). The two codes are distinguished by the reconstructed operator $\hat{A}_W$, which is where the binomial structure enters. Since the code words occupy the Fock levels $\{p(S+1)\}$ and $\hat{a}|p(S+1)\rangle = \sqrt{p(S+1)}|p(S+1)-1\rangle$, the operator $\hat{a}\hat{A}_W\hat{a}^\dagger$ has support on the error levels $\{p(S+1)-1\}$, and its Fock-basis matrix elements carry an extra factor

$$\begin{aligned} \langle p'(S+1)-1 | \hat{a}\hat{A}_W\hat{a}^\dagger | p(S+1)-1 \rangle \\ = \sqrt{p'p} (S+1) \langle p'(S+1) | \hat{A}_W | p(S+1) \rangle, \end{aligned} \quad (110)$$

relative to those of $\hat{A}_W$ itself.

Equation (110) shows that the loss-transformed operator has a similar form as the original (a weighted sum of off-diagonal Fock-basis matrix elements) but with two differences, which the distribution inherits.

First, the Fock levels are shifted: The original operator carries $\langle p'(S+1) | \cdots | p(S+1) \rangle$, while the loss-transformed operator carries $\langle p'(S+1)-1 | \cdots | p(S+1)-1 \rangle$. This is the phase-space manifestation of the photon number being lowered by 1: The support of the distribution moves from the code-space Fock levels to the adjacent error-word Fock levels.

Second, the binomial weights acquire an additional factor of $\sqrt{p'p} (S+1)$, reflecting the $\sqrt{n}$ prefactor from $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$. This comes from the amplitude damping: Each photon in $|n\rangle$ is an independent decay channel into the environment, so the loss rate scales with n ($\langle n | \hat{a}^\dagger \hat{a} | n \rangle = n$), and code word components built from larger Fock levels $p(S+1)$ contribute more strongly to the loss-transformed distribution. The same effect appears for the cat code as the factor $\hat{n}$ in Eq. (102).

The mathematics of Sec. VII B carries over unchanged: T_{loss} is the same bare jump map, so Eq. (109) integrates to $\langle \hat{n} \rangle$ rather than unity; and the syndrome is the grading of $H_0^{\text{bin}} = U(1) \times \mathbb{Z}_{S+1}$, that is, $\hat{n} \bmod (S+1)$: Both code words lie in $\hat{n} \equiv 0$, and ℓ losses move them together to $\hat{n} \equiv -\ell$, resolving ℓ without resolving the logical state.

Section V B's quantum non-demolition (QND) syndrome measurement does not otherwise disturb the state. In the phase-space picture, that measurement projects the evolved

distribution onto the sector of phase space associated with levels $\equiv -1 \pmod{S+1}$, which characterizes the single-loss error words.

As before, the lost photon is not physically recovered: The syndrome identifies the error subspace, and an engineered ancilla-assisted unitary (e.g., a SNAP-gate sequence) maps it back onto the code space. In the phase-space picture, this recovery is the map T_{rec} moving the support of the distribution from the error levels $|p(S+1)-1\rangle$ back to $|p(S+1)\rangle$, just as how the GKP recovery channel rigidly translates the distribution back across the torus in Eq. (49).

Applying Q_{bin} to the full error-correction cycle $T_{\text{QEC}} = T_{\text{rec}} \circ T_{\text{loss}}$ [both maps being syndrome-graded, trivially, since $\mathcal{E}_{\text{bin}} = \mathbb{1}$, so Eq. (22) applies], we therefore recover

$$Q_{\text{bin}}(T_{\text{QEC}})[W^{C_{\text{bin}}}] (\beta') = \langle \hat{n} \rangle W^{C_{\text{bin}}}(\beta'), \quad (111)$$

for $\hat{\rho}$ supported on the code space, just as for Eq. (103), as the cycle is built from the bare jump map, whose trace is $\langle \hat{n} \rangle$, and after dividing by that weight the normalized logical distribution is returned to itself. Here, the identity is exact rather than approximate, since Eq. (84)'s Knill-Laflamme conditions hold exactly for the binomial code. As in the GKP and cat cases, this is an identity on the logical distribution and not a reversal of the physical process.

D. Gain and every multiphoton jump

The same substitution covers the rest of the standard bosonic error model. For k -photon loss,

$$\hat{a}^{\dagger k} \hat{\Pi}(\zeta) \hat{a}^k = \hat{\Pi}(\zeta) (2\zeta^* - \hat{a}^\dagger)^k \hat{a}^k, \quad (112)$$

whose $k=1$ case is Eq. (100), and for the gain channel, $T_{\text{gain}}(\hat{\rho}) = \hat{a}^\dagger \hat{\rho} \hat{a}$,

$$\hat{a} \hat{\Pi}(\zeta) \hat{a}^\dagger = -\hat{\Pi}(\zeta) (\hat{n} + 1 - 2\zeta \hat{a}^\dagger), \quad (113)$$

so that for any code represented by the standard kernel $\hat{\Delta}(\beta) = 2\hat{\Pi}(\beta)$,

$$Q(T_{\text{gain}})[W](\beta') = -2 \text{Tr}[\hat{\Pi}(\beta') (\hat{n} + 1 - 2\beta' \hat{a}^\dagger) \hat{\rho}]. \quad (114)$$

We derive this in Appendix D; we verified Eqs. (96), (112), and (113) numerically, the second for $k=1, 2, 3$, to 10^{-14} at Fock cutoff 200. The loss-gain duality, $\hat{n} \mapsto \hat{n} + 1$ and $\zeta^* \hat{a} \mapsto \zeta \hat{a}^\dagger$, is the phase-space signature of the spontaneous-emission asymmetry. This covers every multiphoton jump (112), gain (113), dephasing (116), and Gaussian random displacements (52) in kernel form, in the standard bosonic error model: Thermal noise is quantum-limited loss composed with quantum-limited amplification, and every Kraus branch of amplitude damping has a closed kernel.

E. Dephasing

Dephasing has a qualitatively different phase-space signature from the jump channels above. As a random rotation in phase space, it acts as

$$\mathcal{E}_{\text{deph}}(\hat{\rho}) = \int d\theta p(\theta) e^{i\theta \hat{n}} \hat{\rho} e^{-i\theta \hat{n}} \quad (115)$$

for some distribution $p(\theta)$ of rotation angles. This subsection concerns the rotation-symmetric codes only, and for them

$\mathcal{E}_C = \mathbb{1}$ by Proposition III.3, so every channel lies in $\mathcal{G}_C$ and Eq. (12) applies directly. (For the ideal GKP code, it would not: The sectors there are the displaced code spaces, and a rotation misaligned with the code lattice is the standard example of a channel outside $\mathcal{G}_C$.) Because the kernel is the standard one, $\hat{\Delta}^C(\beta) = 2\hat{\Pi}(\beta)$, and $e^{i\theta\hat{n}}\hat{D}(\beta)e^{-i\theta\hat{n}} = \hat{D}(\beta e^{i\theta})$, so each rotation acts by rotating its argument, meaning the represented channel is an angular convolution,

$$[Q(\mathcal{E}_{\text{deph}})W^C](\beta') = \int d\theta p(\theta) W^C(\beta' e^{-i\theta}). \quad (116)$$

This is angular diffusion of the code-adapted distribution about the origin: The radial profile of W^C stays the same, but its dependence on $\arg \beta'$ is smoothed. Expanding W^C in angular harmonics $e^{ik\theta}$, the coefficient at frequency k is carried by the coherences $\langle m|\hat{\rho}|m+k \rangle$, so the smoothing suppresses every $k \neq 0$ component and leaves $k = 0$ (the angular average, which encodes the photon-number distribution) the same. The syndrome is therefore unaffected by dephasing: It is the grading of H_0^C [$\hat{n} \bmod 2$ for the four-component cat code per Eq. (75), $\hat{n} \bmod (S+1)$ for the binomial code per Eq. (86)], a property of the diagonal of $\hat{\rho}$. Instead, the coherence between the code sectors is destroyed, at the frequency k separating the code words in the grading of H^C : For the N -component cat code, it is half the component count, $k = N/2$ [Eq. (70)]; for the binomial code, it is $k = S+1$. Note that N has two roles here. Dephasing suppresses it by $\int d\theta p(\theta) e^{-ik\theta}$ (a $\hat{Z}$ -type phase damping on the logical block) and also suppresses the intra-code-word harmonics [$k \equiv 0 \bmod N$, resp. $k \equiv 0 \bmod 2(S+1)$], taking weight out of the code space. Dephasing is thus a combination of logical phase error and leakage, which is why Eq. (83) lists $\hat{n}, \dots, \hat{n}^D$ as errors to be corrected. The phase-space picture reads all three effects (syndrome preservation, logical damping, and leakage) off one object, the harmonic content of W^C , and makes the noise bias of rotation-symmetric codes visible in one place: Built so that loss moves support between sectors, where it can be detected and undone, they are correspondingly exposed to a channel that leaves the sectors alone and attacks the phases.

F. Cat versus binomial at matched mean photon number

To compare like with like, we match the mean photon number of the two encodings, since mean photon number sets the loss rate. For the four-component cat code, $\langle j_L|\hat{n}|j_L \rangle = |\alpha|^2$ up to corrections exponentially small in $|\alpha|^2$, while for the binomial code

$$\langle W_\sigma|\hat{n}|W_\sigma \rangle = \frac{(N+1)(S+1)}{2}, \quad \sigma \in \{\uparrow, \downarrow\}, \quad (117)$$

which follows from

$$\begin{aligned} \sum_{p \text{ even}} \binom{N+1}{p} p &= \sum_{p \text{ odd}} \binom{N+1}{p} p \\ &= (N+1)2^{N-1}, \end{aligned} \quad (118)$$

a consequence of $\sum_p (-1)^p \binom{n}{p} p = 0$ for $n \geq 2$. (We matched the mean photon number of the two code words as this is the

diagonal part of the Knill-Laflamme condition for a single loss.) We therefore set $|\alpha|^2 = (N+1)(S+1)/2$.

As Proposition III.3 gives the two codes the same kernel, Eqs. (102) and (109) are the same formula; everything that distinguishes the two codes under loss is therefore carried by the code words, and the framework isolates it in the factor $\hat{n}$ in the loss-transformed kernel. The effect of this factor depends entirely on the spread of the code word's photon-number distribution. Two features follow, which are not visible in the Hilbert-space description.

First, the two codes fail in different ways, not just at different rates. For the cat code, the photon-number distribution of a code word is Poissonian, concentrated at $|\alpha|^2 = 6$, with relative width $1/|\alpha| \approx 0.41$; the $\hat{n}$ reweighting is then close to uniform, and the distribution is carried between rotational sectors without much change of shape (Sec. VII A). For the binomial code, the code word occupies at most $\lfloor (N+1)/2 \rfloor + 1$ widely spaced Fock levels [two at $(N, S) = (2, 3)$], meaning that the reweighting is extreme at $\bar{n} = 6$: $|W_\uparrow \rangle = \frac{1}{2}(|0\rangle + \sqrt{3}|8\rangle)$, and $\hat{a}$ annihilates the vacuum component, so the postjump state is the single Fock state $|7\rangle$, while $\hat{a}|W_\downarrow \rangle \propto |3\rangle + |11\rangle$. The mean photon number under a detected loss is essentially unchanged for the cat code ($5.98 \rightarrow 5.96$), but rises from 6 to 7 for the binomial code, whose jump preferentially selects the high-photon-number component of a sparse code word. The negative volume moves accordingly: $1.38 \rightarrow 1.41$ and $1.41 \rightarrow 1.42$ for the cat code words, against $1.62 \rightarrow 1.72$ and $1.64 \rightarrow 1.75$ for the binomial code words. Note that Proposition III.3 makes both code-adapted representations the ordinary Wigner function, so neither compared quantity is inaccessible to a direct Hilbert-space calculation; the framework instead shows the two codes in the same representation, which makes them easier to compare.

The second feature is that the framework tracks the nonclassical resources of the encoded state across an error-correction cycle, rather than just whether the logical information survived. For these codes, that quantity is not logical magic (Sec. VC) but is still a well-defined property of the encoding, computed identically for both codes. The negative volume of Eq. (6) presupposes a normalized distribution, so it is evaluated on the renormalized postjump distribution [division by $\langle \hat{n} \rangle$ for loss and $\langle \hat{n} + 1 \rangle$ for gain, as below Eq. (95) and in Appendix E; the figures quoted are negative volumes of normalized distributions throughout]. For an ideal cycle, the output value equals the input value: Equations (103) and (111) return the input distribution up to the factor $\langle \hat{n} \rangle$ the renormalization removes (exactly for the binomial code, up to corrections exponentially small in $|\alpha|^2$ for the cat one). For a nonideal cycle (finite-efficiency syndrome extraction, imperfect recovery; Sec. VIII), the support of the distribution can be returned to the code space while its negative volume is not, so a code can be both logically correct and resource degraded.

From gain column of Table II, we see two things. First, for the four-component cat code, the gain words interleave with the loss words: $\hat{a}|0_L \rangle$ and $\hat{a}^\dagger|1_L \rangle$ both occupy $n \equiv 3 \pmod{4}$, and $\hat{a}|1_L \rangle$ and $\hat{a}^\dagger|0_L \rangle$ both occupy $n \equiv 1$. Even the full $\mathbb{Z}_4$ number grading therefore conflates a loss on one code word with a gain on the other, and the standard loss recovery converts a gain event into a logical error; no refinement of the number syndrome can help. For the binomial

code with $(N, S) = (2, 3)$, both loss words sit at $n \equiv 3$ and both gain words at $n \equiv 1$, so the syndrome identifies which channel acted while revealing nothing logical, and both are correctable.

Second, Gaussian dephasing of angular variance s^2 suppresses the code-word-separating harmonic k by $e^{-k^2 s^2/2}$. For the four-component cat code, that harmonic is half the component count, $k = 2$; for the binomial code, it is $S + 1 = 4$. At $s = 0.2$, the binomial coherence therefore retains 0.726 against the cat's 0.923. At matched mean photon number, the binomial code gains its loss-gain discrimination at the expense of a quantifiable dephasing fragility. Gain also pumps negativity harder than loss for both families, as the table shows.

VIII. FINITE-ENERGY CODE WORDS AND EXPERIMENTAL IMPERFECTIONS

All the analysis above assumes ideal code words and an idealized error-correction cycle, with unit-efficiency syndrome extraction and perfect recovery. Real implementations satisfy none of these, so this section sets out how the framework accommodates the difference. Each of the relevant imperfections can be brought into Sec III E's syndrome-graded class and then composed with the rest of the cycle under Eq. (22), though for the GKP code, one imperfection is only representable in its displacement-channel approximation.

Consider first the finite-energy GKP code words, obtained from the ideal code words by a Gaussian envelope

$$|\psi^\Delta\rangle \propto \hat{E}_\Delta |\psi\rangle, \quad \hat{E}_\Delta = e^{-\Delta^2 \hat{n}}, \quad (119)$$

which are normalizable states of $L^2(\mathbb{R})$. As these are normalizable, Lemma III.1 applies, so no nontrivial displacement leaves the code space invariant, meaning by Theorem V.1 the only exact Stratonovich-Weyl kernel is the ordinary Wigner one. [The parity hypothesis is met here by adjoining $(-1)^{\hat{n}}$ to G , exactly as after Corollary III.1.] The Zak-Gross representation, and with it the negative-volume reading, is therefore only exact in the ideal limit. Four things change at finite energy. The orders in Δ quoted for them are leading-order estimates within the envelope model of Refs. [24,26], not bounds. The one inequality, $\mathcal{N}^\Delta \leq \mathcal{N}$ in the fourth, is a bound within that model.

(1) The code space is no longer the exact fixed-point space of H^{GKP} , so Eq. (34)'s kernel, while displacement covariant by construction, is the code-adapted kernel only up to $O(\Delta^2)$; the envelope convolves W with a Gaussian of width Δ and multiplies it by an envelope suppressing the far comb peaks. We keep only the convolution, $W^\Delta \simeq W * G_\Delta$, which is the standard leading-order finite-energy model [24,26] and is formally Eq. (52)'s classical-noise convolution; the multiplicative factor enters at the same order in Δ^2 , so is dropped by the model.

(2) Equation (43)'s twirl becomes a smeared average rather than an exact projector: $\mathcal{E}^2 - \mathcal{E} = O(\Delta^2)$.

(3) The correctable region acquires a soft boundary, since a finite-energy state has weight outside $[-l/2, l/2]^2$, adding to the cycle a logical failure probability of order e^{-cl^2/Δ^2} , exponentially small but nonzero. The accuracy of the represented cycle is set by items (i) and (ii) rather than by item

(iii): $Q(T_{\text{QEC}})$ is built from the kernel and the reconstruction idempotent, so differs from the identity at $O(\Delta^2)$, which for any Δ of interest is polynomially larger than the tail term.

(4) Convolution with a non-negative kernel cannot increase $\int d\mu |W|$ (Young's inequality), so $\mathcal{N}^\Delta \leq \mathcal{N}$ for the smoothing of item (i) alone: The accessible logical magic shrinks. When we use the exact Wigner kernel left by Theorem V.1, a finite-energy GKP code word has a large negative volume that is mostly ordinary continuous-variable nonclassicality, just as for the cat and binomial codes; read against the Zak-Gross kernel, admissible up to $O(\Delta^2)$, the code word has a small negativity that does track logical magic. The two readings are consistent because they use different kernels; the second is the useful one for $\Delta^2 \ll 1$, but is only approximate.

We can model nonunity detection efficiency and imperfect gain in the syndrome-extraction path as a beam splitter followed by loss (the variable-gain, variable-loss channel analyzed theoretically and experimentally in the context of minimal-disturbance measurement [64]). We do, however, need to see whether it lies in $\mathcal{G}_C$, which differs by code. For the rotation-symmetric codes, it necessarily does, since $\mathcal{E}_C = \mathbb{1}$ there. For the GKP code, Eq. (19)'s sectors are the displaced code spaces, and a beam splitter with a vacuum port is not a convex combination of displacements, so it can create coherence between sectors. However, the displacement-channel approximation is graded: The pure-loss channel composed with an amplification restoring the amplitude is a Gaussian random-displacement channel of the form of Eq. (52), which lies in $\mathcal{G}_C$ (this approximation is standard in the GKP literature). Composing it before the recovery gives a cycle T_{QEC} , which we compare with its ideal counterpart T_{QEC}^0 . For a single-mode code represented by the ordinary kernel (the cat and binomial codes, and finite-energy GKP code words represented by the Wigner kernel), $\mathcal{E}_C = \mathbb{1}$, so for any two channels T, T' and any state $\hat{\rho}$ with distribution W , we have $Q(T)W - Q(T')W = \chi_C[(T - T')(\hat{\rho})]$. The kernel $2\hat{\Pi}(\Omega)$ has operator norm 2, which gives the first line of

$$\sup_{\Omega} |Q(T)W - Q(T')W|(\Omega) \leq 2 \|(T - T')(\hat{\rho})\|_1,$$

$$\|Q(T_{\text{QEC}})W_L - W_L\|_1 \leq 2(1 + \mathcal{N}_{W_L})P_{\text{err}}(\sigma). \quad (120)$$

[The first line uses only linearity, so it also holds for the single-jump maps of Sec. VII A wherever $(T - T')(\hat{\rho})$ is trace class.] With $T = T_{\text{QEC}}$ and $T' = T_{\text{QEC}}^0$, it keeps the represented cycle within 2ε of the represented ideal one, where $\varepsilon = \sup_{\hat{\rho}} \|(T_{\text{QEC}} - T_{\text{QEC}}^0)(\hat{\rho})\|_1$, taken over encoded states, measures the combined infidelity of the syndrome extraction and recovery, and is at most the diamond-norm distance between the two cycles. The second line is for the ideal GKP code. There, the recovered state is not normalizable and no trace-norm distance is available, but the displacement-channel approximation makes the represented cycle known in closed form. By Sec. IV H, it acts on the logical distribution W_L of an encoded state as the Pauli channel with weights $p_v(\sigma) = q_a(\sigma)q_b(\sigma)$, $v = (a, b)$, built from Eq. (61). Hence, $Q(T_{\text{QEC}})W_L - W_L = \sum_{v \neq 0} p_v(\sigma)[W_L(\cdot - v) - W_L]$, and translation preserves the norm $\|W_L\|_1 = 1 + \mathcal{N}_{W_L}$ of Eq. (6). Here, $P_{\text{err}}(\sigma) = 1 - q_0(\sigma)^2$ is the logical

error probability and σ the width of the random-displacement channel. Imperfect recovery is a separate contributor: An engineered unitary that returns the error sectors to the code space only approximately adds its own deviation to ε and remains graded whenever it carries sectors to sectors, as the ancilla-assisted constructions of Secs. VII B and VII C do. Neither constant in Eq. (120) depends on the hardware, which enters only through ε and σ . Measurement backaction is handled by including the ancilla mode explicitly and applying Eq. (12) to the joint system (valid as $\mathcal{G}_C$ is closed under tensor product). An ancilla error that creates coherence between syndrome sectors is not graded, and for such an error, the composition law (22) fails and the cycle must be represented as a whole.

Continuous-variable erasure-correcting codes protect optical coherence against loss [65], and the squeezing they rely on is experimentally accessible in polarization variables [66]; squeezed states have been used directly in measurement and averaging protocols across gain regimes [67]. A polarization encoding sits inside our framework analogously to the single-mode codes: The Stokes operators generate an $SU(2)$ rather than a Weyl-Heisenberg symmetry, so X^C is a sphere, and Eq. (12) applies once the isotropy subgroup of a reference code word is identified, though Proposition III.3, specific to displacements, would have to be redone for that group. On the discrete-variable side, autonomous error correction of GKP states [25], protection of entanglement between bosonic logical qubits [28], and fault-tolerant schemes with discrete-variable ancillae [29] all involve the syndrome-extraction and recovery imperfections described above.

IX. A CODE-ADAPTED KIRKWOOD-DIRAC LAYER

For codes where the stabilizer is a lattice of displacements in a Weyl-Heisenberg group, every representation above is blind by construction to coherence between syndrome sectors: $W_{\hat{\rho}}^C = W_{\mathcal{E}_C(\hat{\rho})}^C$ (Proposition IV.2), which is the representation-level statement of the compositional restriction of Sec. III E. (For codes where this is not the case, $\mathcal{E}_C = \mathbb{1}$, so the representation is the ordinary Wigner function and discards nothing, even though the code still has a measured syndrome; the blindness is a property of the lattice twirl, not of every code.) Kirkwood-Dirac distributions [14,15,68] are the natural tool for revealing what this hides and are accommodated by the complex structure theorem of Ref. [8]. We construct the code-adapted KD representation in the finite-dimensional setting, where every statement is finite and checkable.

For a stabilizer code, take the e basis to be the joint eigenbasis of the stabilizers and the logical $\bar{Z}$, labeled as (s, z) , and the f basis its Fourier conjugate. The overlaps $\langle f|e \rangle$ are uniform and nowhere zero, so the pair is informationally complete. Define

$$Q_{\hat{\rho}}(e, f) = \langle f|e \rangle \langle e|\hat{\rho}|f \rangle, \quad \sum_{e,f} Q = \text{Tr} \hat{\rho}. \quad (121)$$

Proposition IX.1. With the bases above,

- (1) Q is informationally complete, with marginals the (s, z) distribution and its Fourier conjugate;
- (2) pointwise,

$$Q_{\mathcal{E}_C(\hat{\rho})}(e, f) = \langle f|e \rangle \langle e|\hat{\rho} \hat{\Pi}_{s(e)}|f \rangle, \quad (122)$$

so $Q_{\hat{\rho}}$ splits exactly into the KD image of the dephased state plus an intersector part $Q_{\hat{\rho}} - Q_{\mathcal{E}_C(\hat{\rho})}$, carrying the coherence that every Stratonovich-Weyl representation of the code discards;

- (3) the nonselective sharp syndrome extraction maps $\hat{\rho} \mapsto \mathcal{E}_C(\hat{\rho})$, so its KD-visible disturbance gives the erasure of that intersector part; under partial extraction $\hat{\rho} \mapsto (1 - \lambda)\hat{\rho} + \lambda\mathcal{E}_C(\hat{\rho})$, the intersector content scales as $(1 - \lambda)$.

All three parts were verified on the $[[3, 1]]_3$ code, taking $\hat{\rho}$ to be the cross-sector-encoded strange state $(|S_L\rangle + X_1|S_L\rangle)/\sqrt{2}$, which superposes the zero-syndrome sector with one error sector: The Stratonovich-Weyl representations of $\hat{\rho}$ and $\mathcal{E}_C(\hat{\rho})$ agree to 10^{-16} , while the KD intersector content is $\|Q - Q_{\mathcal{E}_C}\|_1 = 2/\sqrt{3}$ exactly, decaying linearly under partial extraction. The KD layer is therefore strictly finer than the Stratonovich-Weyl layer on the effects the error-correction cycle cannot see, and it quantifies what an unsharp syndrome measurement disturbs, which Sec. VIII accounts for only through the deviation ε of the cycle.

For syndrome extractions separated by a channel $\mathcal{V}$, define the two-time syndrome quasiprobability

$$Q(s_1, s_2) = \text{Tr}[\mathcal{V}^\dagger(\hat{\Pi}_{s_2}) \hat{\Pi}_{s_1} \hat{\rho}], \quad (123)$$

whose marginals are the time-1 distribution and the undisturbed time-2 distribution, with no intermediate collapse.

Proposition IX.2. Call $\mathcal{V}$ *cograded* if $\mathcal{V}^\dagger(\hat{\Pi}_s)$ is block diagonal for every s . (This is implied by, but weaker than, $\mathcal{E}_C\mathcal{V} = \mathcal{E}_C\mathcal{V}\mathcal{E}_C$, the two coinciding only for one-dimensional sectors. It is also not the syndrome grading of Sec. III E, which asks instead that $\mathcal{V}$ carry block-diagonal operators to block-diagonal ones; neither of those two conditions implies the other.) Then, Q is real and non-negative, an ordinary joint distribution, whenever *either* $\mathcal{V}$ is cograded *or* the state $\hat{\rho}$ carries no intersector coherence.

Anomalous (complex) values therefore require both a coherent error that is not cograded and intersector coherence left by a previous one, so the two-time anomaly witnesses interference between successive error events, which no Stratonovich-Weyl representation of a single cycle can express, and it is accompanied by genuine measurement disturbance, since the sharp-measurement joint has a different time-2 marginal. We prove this in Appendix C. For the $[[3, 1]]_3$ code with a cross-sector state and a coherent error $e^{i\theta(X_1 + X_1^\dagger)}$ at $\theta = \pi/6$, both single-condition cases are real and non-negative to 10^{-16} , while the doubly coherent case, taken on the same cross-sector state as above, has $\max |\text{Im} Q| = 1/6$ and $\|\text{Im} Q\|_1 = 2/3$, with exact marginals and a time-2 marginal disturbance of $2/9$ under sharp intermediate measurement, measured as a total-variation distance (the ℓ^1 value is twice that). Natural next steps for this line of research would be a continuous-variable (Zak-basis) construction and a weak-value reading of the imaginary part [69,70].

Note, however, the uniqueness theorem of Ref. [71] implies two limitations to using KD distributions to assess quantum error-correcting codes. KD frames are rank 1 and non-Hermitian, so no KD negativity is a magic witness; and the frame must be fixed by the code, never by the state, since state-adapted KD frames are trivially positivizable.

X. DISCUSSION

We connected the structure theorem for quasiprobability representations of generalized probabilistic theories [7,8] to code-adapted phase-space representations [4,5] and used the connection to divide the set of quantum error-correcting codes into two. Any representation built via the Brif-Mann construction whose kernel satisfies the Stratonovich-Weyl axioms is a linearity-preserving, empirically adequate semifunctor [72], so satisfies the properties the structure theorem requires. We also identified the class of channels on which the representation is compositional (Sec. III E), so that Eq. (12) applies to a full error-correction cycle rather than to a single channel.

The boundary between codes allowing a code-adapted phase-space kernel, and those that only allow the Wigner kernel, is a lattice condition (Sec. V G). Code-adapted kernels exist wherever the code's stabilizer is a lattice of displacements in the dynamical group, whose isotropy subgroup two-step nilpotency makes normal: i.e., the Weyl-Heisenberg groups, finite (every qudit stabilizer code, with the kernel constructed at prime-power d) and continuous (the ideal GKP family with its multimode and concatenated lattices). For odd prime logical dimension, their negativity is logical magic [Eq. (40) and Theorem IV.1]; at $d = 2$, the kernel and the geometry still hold but their negativity cannot be read as logical magic (Sec. IV E). For codes only allowing the Wigner kernel, the physically covariant representation is the standard one up to code-independent sign or phase freedoms, and its negativity is ordinary nonclassicality: globally misaligned with magic even where locally aligned [62], and sometimes even provably constant across the logical space (Theorem V.2). Theorem V.1 establishes this for bosonic codes with normalizable code words and Proposition V.1 for non-Pauli-symmetric qudit codes, in both cases from the SW axioms together with the further hypotheses listed in those statements. For spin codes, we prove less: Proposition V.2 leaves the ordinary spherical kernel in place but stops short of an axiom-direct no go.

For each code family, this gives a single closed formula, describing how any channel in the syndrome-graded class transforms the logical phase-space distribution [for a bosonic code with code words in $L^2(\mathbb{R})$, that class is every channel]. For GKP displacement errors, correctable errors act as the identity on the distribution, while errors exceeding the correctable region act as a rigid translation across the torus, i.e., as the induced logical Pauli. Every qudit stabilizer code reproduces this same structure exactly (Proposition IV.3). The same structure reappears for single-photon loss in both four-component cat codes and binomial codes, where loss carries the distribution between rotational sectors with a reweighting reflecting the photon-number dependence of the loss rate. In all cases, recovery returns the support of the distribution to the code space, despite very different physical recovery mechanisms. (Note that this is an identity on the logical distribution rather than a reversal of the physical process, exact only for the idealized cycle, as shown in Sec. VIII.)

We also showed that, for codes that fail this lattice condition, we can still retain a resource reading of negativity. Section VI's sector-graded logical representation inherits the semifunctorial and compositional structure of the framework, but gives up the physical geometry and is defined relative to a

decoder. Its negativity is a syndrome-resolved witness of logical magic, two sided on pure logical states, for rotation codes encoding a qudit of odd prime dimension. It is consistent with Theorem V.1, given such a representation does not meet its covariance requirement.

There are four natural next steps for this research direction. First, Section V G's classification conjecture is the main open question regarding the boundary, so future work should settle this for noncompact, non-nilpotent dynamical groups. Second, the qubit case $d = 2$ is the only place where the structure holds but the resource reading fails; a natural extension would be to identify the strongest $d = 2$ substitute, whether through rebit restrictions [37], the recently classified even- d Wigner functions [36], or contextuality-based witnesses [20]. Third, Sec. IV H's dissipation results should be developed into magic thresholds across lattices and logical dimensions, especially as Eqs. (62) and (66) suggest that this should be exactly computable for many cases. Finally, the Kirkwood-Dirac layer, whose two-time anomaly already separates coherent from incoherent error histories (Proposition IX.2), should be carried to the continuous-variable setting and to longer histories, as a diagnostic for coherent noise in repeated syndrome extraction.

We end by restating three key limitations of this work. First, our GKP results are for ideal, non-normalizable code words throughout; Sec. VIII shows that at finite energy, the Zak-Gross kernel is admissible only to $O(\Delta^2)$ and the exactly admissible kernel is the ordinary Wigner one. Second, the identity action of a corrected cycle, in every case where we obtain it, is an identity on the logical phase-space distribution rather than a reversal of the physical process: The environment keeps the lost photon. Third, the classification holds under the hypotheses of Proposition III.3. For normalizable code words, these are the code-faithful axioms, the parity, and continuity of the kernel's characteristic function. That the reconstruction map is the identity, and so completely positive, is then derived and not assumed. For non-normalizable code words we assume, following Ref. [5], that the reconstruction map is a conditional expectation onto the commutant of a group of displacements. The statement that, among bosonic codes with compact phase space, only the ideal GKP family has a kernel of its own is made within that class.

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DATA AVAILABILITY

A Wolfram Mathematica notebook reproducing all numerical results in this paper is openly available on Zenodo [73].

APPENDIX A: DERIVATION OF THE CAT-CODE REPRESENTATION

The derivation of the Wigner function for cat codes follows the one used in Ref. [5] for GKP codes. As in that

case, we work with an infinite-dimensional Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ and we take the dynamical symmetry group to be $G = H_3(\mathbb{C}) \rtimes \mathbb{Z}_N$. The semidirect factor is necessary: The code symmetry we use below is the discrete rotation $\hat{R}_N = e^{i2\pi \hat{a}^\dagger \hat{a}/N}$, which is not an element of the Weyl-Heisenberg group $H_3(\mathbb{C})$: That group contains only phases and displacements, while number-operator rotations belong to the metaplectic extension. Taking $G = H_3(\mathbb{C}) \rtimes \mathbb{Z}_N$ makes the isotropy subgroup below a genuine subgroup of G , and hence the coset space well defined; this remains within the scope of the Brif-Mann construction, which covers Lie-group dynamical symmetries generally.

No large- $|\alpha|$ expansion enters the derivation below, because the kernel is fixed by the symmetry group of the code space rather than by an expansion of a reference state. The large- $|\alpha|$ limit still appears in this paper, but only as a property of the code rather than its representation: in the statement that $\langle j_L | \hat{n} | j_L \rangle = |\alpha|^2$, and hence in the Knill-Laflamme conditions of Sec. V A, both of which hold up to corrections controlled by the overlaps $|\langle \alpha_j | \alpha_k \rangle| = \exp(-|\alpha_j - \alpha_k|^2/2)$. For the four-component code, these corrections are $O(e^{-|\alpha|^2})$, below 10^{-4} for $|\alpha|^2 \geq 10$. For $|\alpha|^2 = 6$ (used in Sec. VII F), they are of order 2×10^{-3} , producing the small split between the two code words' mean photon numbers given in Table II.

We use the same parametrization as in Ref. [5]:

$$\pi(g) = e^{i\varphi} \mathbb{1} e^{-i\frac{\varphi}{2}} \mathbb{1} e^{i\varphi \hat{x}} e^{-ix\hat{p}} = e^{i\varphi} \hat{D}(x, p), \quad \varphi, x, p \in \mathbb{R}. \quad (\text{A1})$$

We consider an N -component cat code, which encodes one qubit into the single-mode Fock space via the $\mathbb{Z}_N$ -symmetric code words

$$|\bar{j}\rangle = n_j^{-1/2} \sum_{k=0}^{N-1} \omega_N^{jk} |\alpha_k\rangle, \quad \omega_N = e^{i2\pi/N}, \quad (\text{A2})$$

where $\alpha_k = \alpha e^{i2\pi k/N}$ for $k = 0, \dots, N-1$, $\alpha \in \mathbb{R}_{>0}$, and n_j are normalization constants. These are the characters of $\mathbb{Z}_N$, correctly generalizing the two-component code words: The alternating phase $(-1)^k$ is a character of $\mathbb{Z}_N$ for every even N , where it is the $j = N/2$ character, and is no character at all for odd N . What fails for $N > 2$ is not the phase but the pairing: $\{|\bar{0}\rangle, |\bar{N/2}\rangle\}$ spans only two of the N characters, so the remaining $|\bar{j}\rangle$ lie outside the code space and must be kept track of separately, which is what makes the syndrome the grading of H_0^C rather than of H^C . For $N = 2$, Eq. (A2) reduces to $|C_\alpha^\pm\rangle$, and the logical code space used in Sec. V A is spanned by $|\bar{0}\rangle$ and $|\bar{2}\rangle$ at $N = 4$.

The cat-code space is invariant under both:

(1) *Global phase*: $e^{i\varphi} \mathbb{1}$ for all $\varphi \in \mathbb{R}$, which corresponds to a $U(1)$ factor.

(2) *Discrete rotation*: $\hat{R}_N := e^{i\frac{2\pi}{N} \hat{a}^\dagger \hat{a}}$, which maps $|\alpha_k\rangle \mapsto |\alpha_{k+1 \bmod N}\rangle$ and hence maps $\mathcal{C}_{\text{cat}}$ to itself; this generates a $\mathbb{Z}_N$ subgroup.

We therefore choose a reference state $|\psi_0\rangle \in \mathcal{C}_{\text{cat}}$ (we take $|\psi_0\rangle = |\bar{0}\rangle$) with isotropy subgroup $H^{\text{cat}} = U(1) \times \mathbb{Z}_N$. As emphasized previously, this is the isotropy subgroup of the reference code word [Eq. (8)] and not the pointwise stabilizer $H_0^{\text{cat}} = U(1) \times \mathbb{Z}_{N/2}$; H^{cat} is quotiented out to give the coset space, and, N being even, $\mathbb{Z}_N$ contains the parity operator $\hat{R}_N^{N/2} = (-1)^{\hat{n}}$, which Proposition III.3 needs. The associated

phase space is the coset

$$X^{\text{cat}} = (H_3(\mathbb{C}) \rtimes \mathbb{Z}_N) / (U(1) \times \mathbb{Z}_N) \simeq \mathbb{C}. \quad (\text{A3})$$

The quotient is the whole complex plane rather than a wedge of it: The $\mathbb{Z}_N$ factor of G is divided out by the $\mathbb{Z}_N$ factor of H^{cat} , leaving the displacements as coset representatives. The rotational symmetry of the code therefore appears as an invariance of the central kernel, $\hat{R}_N \hat{\Delta}^{\mathcal{C}_{\text{cat}}}(0) \hat{R}_N^\dagger = \hat{\Delta}^{\mathcal{C}_{\text{cat}}}(0)$, as follows from $\hat{D}(\omega_N \gamma) = \hat{R}_N \hat{D}(\gamma) \hat{R}_N^\dagger$ together with $[\hat{R}_N, (-1)^{\hat{n}}] = 0$.

The coherent states for this code (which correspond to displaced reference states) are

$$|\beta\rangle_{\psi_0} := \hat{D}(\beta) |\psi_0\rangle, \quad \beta \in \mathbb{C}, \quad (\text{A4})$$

where $\hat{D}(\beta) = e^{\beta \hat{a}^\dagger - \beta^* \hat{a}}$ is the standard displacement operator. The invariant measure on $\mathbb{C}$ is

$$d\mu(\beta) = \frac{d^2\beta}{\pi}, \quad \beta = r e^{i\theta}, \quad (\text{A5})$$

giving the resolution of identity

$$\frac{1}{\pi} \int_{\mathbb{C}} d^2\beta \hat{D}(\beta) |\psi_0\rangle \langle \psi_0| \hat{D}^\dagger(\beta) = \mathbb{1}. \quad (\text{A6})$$

This is the standard coherent-state resolution, which holds for any normalized reference state and carries no factor of N : The N components of the code enter through $|\psi_0\rangle$ rather than a multiplicity of the measure. Here, Ref. [4]'s construction would expand the kernel in the harmonic functions Y_ν of $L^2(X^C, \mu)$ and the associated tensor operators; however, we do not need that expansion here. Since $X^{\mathcal{C}_{\text{cat}}} \simeq \mathbb{C}$ is the whole plane rather than a wedge, Y_ν are the ordinary Laplace-Beltrami eigenfunctions on $\mathbb{R}^2$, $e^{in\theta} J_{|n|}(\lambda r)$, with $\lambda \geq 0$ and $n \in \mathbb{Z}$ unrestricted: The radial factor is the Bessel function of order $|n|$, not J_0 , and restricting n to multiples of N would restrict $L^2(X^{\mathcal{C}_{\text{cat}}})$ to $\mathbb{Z}_N$ -invariant functions, which the kernel obtained below is not. Proposition III.3 fixes the kernel directly, so the expansion is unnecessary.

With the phase space, the invariant measure and the isotropy subgroup fixed, the kernel is now determined by Proposition III.3. Covariance forces $\hat{\Delta}^{\mathcal{C}_{\text{cat}}}(\beta) = \hat{D}(\beta) \hat{K} \hat{D}^\dagger(\beta)$, with $\hat{K}$ invariant under H^{cat} . The reconstruction map is then the identity. If $\chi_{\hat{K}}$ is a function, this is the continuity route of Proposition III.3(i). If instead the reconstruction map is a displacement twirl over a closed subgroup $\Lambda^\perp$ leaving $\mathcal{C}_{\text{cat}}$ invariant, it is part (iii): By Lemma III.1, the only displacement preserving $\mathcal{C}_{\text{cat}}$ is the trivial one, so $\Lambda^\perp = \{0\}$. Since $(-1)^{\hat{n}} \in \mathbb{Z}_N$, the phase-space inversion hypothesis holds and the kernel is fixed,

$$\hat{\Delta}^{\mathcal{C}_{\text{cat}}}(\beta) = \hat{\Delta}_{\text{std}}(\beta) = 2\hat{\Pi}(\beta), \quad W_{\hat{\rho}}(\beta) = \text{Tr}[\hat{\rho} \hat{\Delta}^{\mathcal{C}_{\text{cat}}}(\beta)] = \pi W_{\hat{\rho}}^{\text{std}}(\beta), \quad (\text{A7})$$

which are Eqs. (76) and (78).

The alternative, however, fails. Running the H -averaging prescription of Eq. (14) over H^{cat} [conjugation by the $U(1)$ factor being trivial, so that only the $\mathbb{Z}_N$ rotations contribute] gives

$$\frac{1}{N} \sum_{k=0}^{N-1} \hat{R}_N^k [2\hat{\Pi}(\beta)] \hat{R}_N^{-k} = \frac{2}{N} \sum_{k=0}^{N-1} \hat{\Pi}(\omega_N^k \beta), \quad (\text{A8})$$

using $\hat{R}_N \hat{D}(\beta) \hat{R}_N^\dagger = \hat{D}(\omega_N \beta)$, which follows from $\hat{R}_N \hat{a} \hat{R}_N^\dagger = \omega_N^{-1} \hat{a}$, together with $[\hat{R}_N, (-1)^{\hat{n}}] = 0$. This operator is correctly standardized, and the reconstruction map it induces is the $\mathbb{Z}_N$ twirl, which is idempotent. However, it is not displacement covariant for $N > 1$: Conjugating Eq. (A8) by $\hat{D}(\gamma)$ displaces all N terms by the same γ , whereas covariance of the kernel at the displaced point requires the k th term to be displaced by $\omega_N^k \gamma$. This is H^{cat} failing to be normal in G (Proposition III.1, proved in Appendix C), so Eq. (A8) violates Stratonovich-Weyl Axiom 4. The two kernels do agree on the $\mathbb{Z}_N$ -invariant subalgebra: For a $\mathbb{Z}_N$ -invariant $\hat{\rho}$, the standard Wigner function is itself $\mathbb{Z}_N$ invariant and the average in Eq. (A8) is trivial. That subalgebra is not enough, which is a second and independent reason to abandon Eq. (A8) here: Since $\hat{R}_N$ acts on $\mathcal{C}_{\text{cat}}$ as the logical $\hat{Z}$, the $\mathbb{Z}_N$ twirl sends $|0_L\rangle\langle 1_L|$ to $(\frac{1}{N} \sum_k (-1)^k) |0_L\rangle\langle 1_L| = 0$, keeping only the $\hat{Z}$ -diagonal part of $\mathcal{B}(\mathcal{C}_{\text{cat}})$, so the reconstruction map it induces is not the identity on the code algebra and the representation is not faithful on the code. Only Eq. (A7) extends that agreement to the full algebra covariantly.

Note three further points. First, the kernel fixed above replaces an earlier route we tried, in which the reference state is expanded over its coherent-state components in the large- $|\alpha|$ limit and the kernel is assembled from the resulting tensor operators; that route yields a kernel built from parities displaced to the component positions α_k , which is neither covariant nor self-dual: The reconstruction map it induces is a displacement twirl over the differences $\alpha_k - \alpha_j$, trace preserving but not idempotent, and so does not satisfy the hypotheses of the structure theorem.

Second, the kernel depends on neither N nor α nor the choice of reference state within $\mathcal{C}_{\text{cat}}$: Within the Stratonovich-Weyl axioms, a rotational symmetry of the code space does not generate a representation of its own.

Third, the same argument applies verbatim to any code space invariant under a discrete rotation, which we use in Appendix B.

APPENDIX B: DERIVATION OF THE BINOMIAL-CODE REPRESENTATION

The derivation of the phase-space distribution for binomial codes follows the same route as for the cat codes, differing only in the order of the rotation involved.

The binomial code words share the same kind of discrete rotational symmetry as the cat code words, but with the order set by the code spacing. From Eq. (82), both code words are supported on Fock levels that are multiples of $S+1$, so the relevant rotation operator is

$$\hat{R}_{S+1} := e^{i \frac{2\pi}{S+1} \hat{n}}, \quad \hat{R}_{S+1} |p(S+1)\rangle = |p(S+1)\rangle \quad (\text{B1})$$

for every p , and therefore $\hat{R}_{S+1} |W_\uparrow\rangle = |W_\uparrow\rangle$ and $\hat{R}_{S+1} |W_\downarrow\rangle = |W_\downarrow\rangle$: $\hat{R}_{S+1}$ fixes the entire code space $\mathcal{C}_{\text{bin}}$ pointwise. (In the cat case, the corresponding order is the number N of coherent-state components; to avoid conflating the two, we write $S+1$ throughout here.) The isotropy subgroup of a reference code word is larger than $\mathbb{Z}_{S+1}$; this larger group is quotiented out to give the coset space. Taking $|\psi_0\rangle = |W_\uparrow\rangle$, which is supported on levels $p(S+1)$ with p even, the half-rotation $\hat{R}_{2(S+1)} =$

$e^{i\pi \hat{n}/(S+1)}$ acts on it as $e^{i\pi p} = +1$, so

$$H^{\text{bin}} = \text{U}(1) \times \mathbb{Z}_{2(S+1)}, \quad H_0^{\text{bin}} = \text{U}(1) \times \mathbb{Z}_{S+1}, \quad (\text{B2})$$

identical in structure to the cat-code isotropy subgroups, and with $\hat{R}_{2(S+1)}$ playing the role of the logical $\hat{Z}$ (it acts as -1 on $|W_\downarrow\rangle$) just as $\hat{R}_N$ does for the cat code. The parallel extends to the syndrome: In both families, the two code words lie in the same H_0^C sector and ℓ losses move them together to the sector shifted by ℓ , which is why we measure the grading of H_0^C , and not the finer grading of H^C . We take $G = H_3(\mathbb{C}) \rtimes \mathbb{Z}_{2(S+1)}$, so that the phase space, the invariant measure, and the coherent states are the same as in Appendix A, with $X^{\mathcal{C}_{\text{bin}}} \simeq \mathbb{C}$ and $d\mu(\beta) = d^2\beta/\pi$, and, the order $2(S+1)$ being even, $(-1)^{\hat{n}} = \hat{R}_{2(S+1)}^{S+1}$ lies in G for every S .

Applying Proposition III.3 then gives Eq. (87) directly. The binomial code words are normalizable, so Lemma III.1 applies: A binomial code word has bounded photon number, while $\hat{D}(\beta) |p(S+1)\rangle$ has support on every Fock level for $\beta \neq 0$. Therefore, $\Lambda^\perp = \{0\}$, and as in Appendix A, the kernel is fixed to the standard one,

$$\hat{\Delta}^{\mathcal{C}_{\text{bin}}}(\beta) = 2\hat{\Pi}(\beta) = \hat{\Delta}^{\mathcal{C}_{\text{cat}}}(\beta),$$

$$W_{\hat{\rho}}(\beta) = \pi W_{\hat{\rho}}^{\text{std}}(\beta), \quad (\text{B3})$$

with $\text{Tr}[\hat{\Delta}^{\mathcal{C}_{\text{bin}}}(0)] = 2\text{Tr}[(-1)^{\hat{n}}] = 1$, meaning that the standardization axiom holds for every S , with the reconstruction map the identity. The rotational average of Eq. (A8), with N replaced by $2(S+1)$, again fails displacement covariance and again agrees with the kernel above on the rotation-invariant subalgebra.

APPENDIX C: PROOFS

For this Appendix, $\hat{\Pi}(\Omega) = \hat{D}(\Omega)(-1)^{\hat{n}}\hat{D}^\dagger(\Omega)$ is the displaced parity operator, $\hat{\Delta}_{\text{std}}(\Omega) = 2\hat{\Pi}(\Omega)$ is the standard Stratonovich-Weyl kernel for a single mode, as fixed in Sec. III C, $d\mu(\Omega) = d^2\Omega/\pi$ is the invariant measure, and $W_{\hat{A}}^{\text{std}}(\Omega) = \frac{2}{\pi} \text{Tr}[\hat{A}\hat{\Pi}(\Omega)]$. We use three standard results: the resolution $\frac{1}{\pi} \int d^2\Omega \hat{D}(\Omega)\hat{X}\hat{D}^\dagger(\Omega) = \text{Tr}[\hat{X}]\mathbb{1}$; the self-duality $\hat{A} = 2 \int d^2\Omega W_{\hat{A}}^{\text{std}}(\Omega)\hat{\Pi}(\Omega)$; and $\text{Tr}[(-1)^{\hat{n}}] = 1/2$ in the Abel sense.

1. Derivation of Eq. (40)

Substituting $u = s + al$, $v = t + bl$ in Eq. (39) and using Eq. (37),

$$\int_{\mathbb{T}_{dt}^2} \frac{du dv}{d} |W_{\hat{\rho}}^{\mathcal{C}_{\text{GKP}}}| = \int_{\mathbb{T}_t^2} ds dt \frac{1}{d} \sum_{a,b \in \mathbb{Z}_d} |W_{\hat{\rho}(s,t)}^{\text{DV}}(a,b)|, \quad (\text{C1})$$

and $\hat{\rho}(s,t) = P[(s,t)|\hat{\rho}] \hat{\rho}'(s,t)$ pulls the syndrome probability out of the modulus, leaving $\frac{1}{d} \sum_{a,b} |W_{\hat{\rho}(s,t)}^{\text{DV}}| = \mathcal{N}_{\hat{\rho}'}^{\text{DV}} + 1$. Since $\int ds dt P[(s,t)|\hat{\rho}] = 1$, subtracting 1 from both sides gives Eq. (40).

2. Proof of Proposition III.1

Throughout this proof, H^C is compact, meaning that $\frac{1}{|H^C|} \sum_h$ denotes integration against its normalized Haar

measure; the noncompact GKP case is handled directly in Secs. III C and IV B and is not covered here. Write $g \in G = H_3(\mathbb{C}) \rtimes \text{Mp}(2, \mathbb{R})$ as a displacement composed with a Gaussian unitary, and let $g \cdot \Omega = S\Omega + b$ be the induced affine symplectic action on phase space, with $S \in \text{Sp}(2, \mathbb{R})$. We use two properties of this action:

(1) $\pi(g)\hat{\Pi}(\Omega)\pi^\dagger(g) = \hat{\Pi}(g \cdot \Omega)$. For a displacement, this is immediate from the composition rule. For a Gaussian unitary $\hat{U}_S$, the generator is a quadratic Hamiltonian, even in $\hat{a}$ and $\hat{a}^\dagger$, so $[\hat{U}_S, (-1)^{\hat{n}}] = 0$ and hence $\hat{U}_S\hat{\Pi}(0)\hat{U}_S^\dagger = \hat{\Pi}(0)$; combining with $\hat{U}_S\hat{D}(\Omega)\hat{U}_S^\dagger = \hat{D}(S\Omega)$ proves this.

(2) The action preserves $d^2\Omega$, since $\det S = 1$ for $S \in \text{Sp}(2, \mathbb{R})$ and translations are measure preserving.

Covariance. From (a), $\pi(g)\hat{\Delta}^C(\Omega)\pi^\dagger(g) = \frac{1}{|H^C|} \sum_h \hat{\Delta}_{\text{std}}(gh \cdot \Omega)$, whereas $\hat{\Delta}^C(g \cdot \Omega) = \frac{1}{|H^C|} \sum_h \hat{\Delta}_{\text{std}}(hg \cdot \Omega)$. The two agree, for every Ω , iff $gH^C = H^Cg$, i.e., when g normalizes H^C : Sufficiency holds since the relabeling $h \mapsto g^{-1}hg$ is then a bijection of H^C preserving its Haar measure, and necessity holds by evaluating at the base point Ω_0 of $X^C = G/H^C$, which every $h \in H^C$ fixes: There, the two multisets are $\{g \cdot \Omega_0\}$ (with multiplicity $|H^C|$) and the H^C orbit of $g \cdot \Omega_0$, and since displaced parities at distinct phase-space points are linearly independent (their Weyl symbols are delta functions at distinct points), the two averages agree for every g only if H^C fixes every point $g \cdot \Omega_0$ of X^C , that is, only if $H^C \subseteq gH^Cg^{-1}$ for every g , which is normality. Covariance therefore holds on the normalizer of H^C in G , and on all of G exactly when H^C is normal. This is Proposition III.1's covariance statement. A compact normal subgroup of G must have bounded conjugation orbits; a nontrivial rotation or squeeze conjugated by the displacements sweeps out an unbounded family, and the displacement group itself has no nontrivial compact subgroup, so the only compact normal subgroups lie in the central $U(1)$ of phases, which acts trivially.

For a rotation-symmetric code, $H^C = U(1) \times \mathbb{Z}_M$ is not normal in $H_3(\mathbb{C}) \rtimes \mathbb{Z}_M$, since a rotation conjugated by a displacement is not a rotation; the averaged kernel is then covariant under rotations but not under displacements, and so does not satisfy Stratonovich-Weyl Axiom 4 for the full group. Equation (A8) shows why this fails. In that case, Eq. (14) has to be abandoned, and the kernel is found from the axioms instead; imposing Axiom 4 directly turns out to remove the freedom entirely, as we now show.

Standardization. Using the resolution of the identity and (b),

$$\int d\mu(\Omega) \hat{\Delta}_{\text{std}}(\Omega) = \frac{2}{\pi} \int d^2\Omega \hat{\Pi}(\Omega) = 2 \text{Tr}[(-1)^{\hat{n}}] \mathbb{1} = \mathbb{1}, \quad (\text{C2})$$

and since each $\pi(h)$ is unitary, $\int d\mu(\Omega) \hat{\Delta}^C(\Omega) = \frac{1}{|H^C|} \sum_h \pi(h) \mathbb{1} \pi^\dagger(h) = \mathbb{1}$. Equivalently, $\text{Tr}[\hat{\Delta}^C(0)] = 1$, independently of the code.

The twirl. With $\chi_C(\hat{A})(\Omega) = \text{Tr}[\hat{A} \hat{\Delta}^C(\Omega)]$ and $\phi_C(F) = \int d\mu(\Omega) F(\Omega) \hat{\Delta}^C(\Omega)$,

$$\phi_C(\chi_C(\hat{A})) = \frac{4}{\pi |H^C|^2} \sum_{h,h'} \int d^2\Omega \text{Tr}[\hat{A} \hat{\Pi}(h \cdot \Omega)] \hat{\Pi}(h' \cdot \Omega). \quad (\text{C3})$$

Substituting $\gamma = h' \cdot \Omega$, which is measure preserving by (b), the first factor becomes $\text{Tr}[\hat{A} \hat{\Pi}(k \cdot \gamma)] = \frac{\pi}{2} W_{\pi^\dagger(k)\hat{A}\pi(k)}^{\text{std}}(\gamma)$, with $k = hh'^{-1}$, using (a). Self-duality then gives $\int d^2\gamma W_B^{\text{std}}(\gamma) \hat{\Pi}(\gamma) = \hat{B}/2$, so

$$\begin{aligned} \phi_C(\chi_C(\hat{A})) &= \frac{1}{|H^C|^2} \sum_{h,h'} \pi^\dagger(hh'^{-1}) \hat{A} \pi(hh'^{-1}) \\ &= \frac{1}{|H^C|} \sum_{k \in H^C} \pi^\dagger(k) \hat{A} \pi(k), \end{aligned} \quad (\text{C4})$$

where the last step is because, for each k , there are $|H^C|$ pairs (h, h') with $hh'^{-1} = k$. This gives Eq. (16), which is idempotent because H^C is a group. ■

3. Proof of Proposition III.3

Step 1: The form of the kernel. Covariance under displacements (Stratonovich-Weyl Axiom 4) gives $\hat{\Delta}^C(\Omega) = \hat{D}(\Omega) \hat{K} \hat{D}^\dagger(\Omega)$, with $\hat{K} = \hat{\Delta}^C(0)$, Hermitian by Axiom 2 and of unit trace by Axiom 3. For $\hat{\Delta}^C$ to be well defined as a function on the coset space $X^C = G/H^C$, it must be constant on cosets, which gives $[\hat{K}, \pi(h)] = 0$ for every $h \in H^C$.

Step 2: The reconstruction map is a multiplier. Write $\chi_{\hat{A}}(\xi) = \text{Tr}[\hat{A} \hat{D}(\xi)]$ and $\hat{K}_\Omega = \hat{D}(\Omega) \hat{K} \hat{D}^\dagger(\Omega)$. From the composition rule for displacements, $\chi_{\hat{K}_\Omega}(\xi) = e^{\Omega^* \xi - \Omega \xi^*} \chi_{\hat{K}}(\xi)$, and from the overlap formula $\text{Tr}[\hat{A} \hat{B}] = \frac{1}{\pi} \int d^2\zeta \chi_{\hat{A}}(\zeta) \chi_{\hat{B}}(-\zeta)$. Therefore,

$$\begin{aligned} \chi_{\mathcal{E}_C(\hat{A})}(\xi) &= \frac{1}{\pi^2} \int d^2\zeta \chi_{\hat{A}}(\zeta) \chi_{\hat{K}}(-\zeta) \chi_{\hat{K}}(\xi) \\ &\quad \times \int d^2\Omega e^{\Omega^*(\zeta^* - \xi^*) - \Omega(\zeta - \xi)}, \end{aligned} \quad (\text{C5})$$

and the Ω integral is $\pi^2 \delta^{(2)}(\zeta - \xi)$. $\hat{K}$ being Hermitian means $\chi_{\hat{K}}(-\xi) = \chi_{\hat{K}}(\xi)^*$, so

$$\chi_{\mathcal{E}_C(\hat{A})}(\xi) = \mathfrak{m}(\xi) \chi_{\hat{A}}(\xi), \quad \mathfrak{m}(\xi) := |\chi_{\hat{K}}(\xi)|^2 \geq 0, \quad (\text{C6})$$

which is Eq. (17); $\hat{K}$ having unit trace means $\mathfrak{m}(0) = 1$. Two properties of $\mathcal{E}_C$ follow from this form with no further assumption. It preserves the trace, since $\text{Tr}[\mathcal{E}_C(\hat{A})] = \chi_{\mathcal{E}_C(\hat{A})}(0) = \mathfrak{m}(0) \chi_{\hat{A}}(0) = \text{Tr}[\hat{A}]$. It commutes with conjugation by every displacement, since $\chi_{\hat{D}(\beta)\hat{A}\hat{D}^\dagger(\beta)}(\xi) = e^{\beta^* \xi - \beta \xi^*} \chi_{\hat{A}}(\xi)$ and multiplication by $\mathfrak{m}$ commutes with multiplication by that phase [the same follows without the multiplier form, from Axiom 4 and the invariance of $d\mu$, by the change of variable $\Omega \rightarrow \Omega + \beta$ in $\mathcal{E}_C(\hat{A}) = \int d\mu(\Omega) \text{Tr}[\hat{A} \hat{\Delta}^C(\Omega)] \hat{\Delta}^C(\Omega)$]. Complete positivity does not follow in the same way and is not used before Step 3.

Step 3: $\mathcal{E}_C$ is a random-displacement channel. This step and the next are for setting (A), whose added assumption is complete positivity. Displacement covariance was derived in Step 2. A completely positive, trace-preserving map on $\mathcal{B}(L^2(\mathbb{R}^m))$ that is covariant under the full group of displacements is a random-displacement channel [74,75],

$$\mathcal{E}_C(\hat{A}) = \int d\nu(\beta) \hat{D}(\beta) \hat{A} \hat{D}^\dagger(\beta), \quad (\text{C7})$$

for a probability measure ν on $\mathbb{R}^{2m}$, whose Fourier-Stieltjes transform is the multiplier $\hat{\nu} = \mathfrak{m}$. No regularity assumption

on $\chi_{\hat{K}}$ is needed for this step, which is why we take it from the classification of displacement-covariant channels rather than from Bochner's theorem.

(a) $\Lambda^\perp$ leaves $\mathcal{C}$ invariant: (In infinite dimensions, the fixed-point set of a unital channel need not equal the commutant of its Kraus operators in general, so we argue directly.) Take $\hat{A} = \hat{\Pi}_{\mathcal{C}}$, the finite-rank code-space projector, with $d_{\mathcal{C}} = \text{Tr}[\hat{\Pi}_{\mathcal{C}}]$. $\mathcal{E}_{\mathcal{C}}(\hat{\Pi}_{\mathcal{C}}) = \hat{\Pi}_{\mathcal{C}}$ gives $d_{\mathcal{C}} = \text{Tr}[\hat{\Pi}_{\mathcal{C}} \mathcal{E}_{\mathcal{C}}(\hat{\Pi}_{\mathcal{C}})] = \int d\nu(\beta) \text{Tr}[\hat{\Pi}_{\mathcal{C}} \hat{D}(\beta) \hat{\Pi}_{\mathcal{C}} \hat{D}^\dagger(\beta)]$; each integrand is at most $d_{\mathcal{C}}$, with equality if and only if $\hat{D}(\beta) \hat{\Pi}_{\mathcal{C}} \hat{D}^\dagger(\beta) = \hat{\Pi}_{\mathcal{C}}$, so $\hat{D}(\beta)\mathcal{C} = \mathcal{C}$ for ν -a.e. β . Writing $\Lambda^\perp$ for the closed subgroup generated by the support of ν , every element of $\Lambda^\perp$ therefore leaves $\mathcal{C}$ invariant, which is the clause used in part (iii) of the proposition.

Step 4. Idempotence forces $\nu = \delta_0$: $\mathcal{E}_{\mathcal{C}}^2 = \mathcal{E}_{\mathcal{C}}$ implies $m^2 = m$, i.e., $\nu * \nu = \nu$, so $\nu * \nu = \nu$: ν is an idempotent probability measure. By the theory of idempotent probability measures (due to Kawada and Itô for compact groups [76], and elementary to verify directly here: the support of an idempotent probability measure is a compact subgroup, and $\{0\}$ is the only one), ν is the normalized Haar measure of a compact subgroup, so $\nu = \delta_0$, $m \equiv 1$, and $\mathcal{E}_{\mathcal{C}} = \mathbb{1}$. Equivalently, writing $\Lambda^\perp$ for the support subgroup of ν , $\Lambda^\perp = \{0\}$. [The continuity route of part (i), which does not assume setting (A), reaches the same point without this step or the last. If $\chi_{\hat{K}}$ is a function, it is continuous by (H3), so $m = |\chi_{\hat{K}}|^2$ is continuous, $m^2 = m$ confines it to $\{0, 1\}$, and $m(0) = 1$ on the connected space $\mathbb{R}^{2m}$ gives $m \equiv 1$. Then, $\mathcal{E}_{\mathcal{C}} = \mathbb{1}$, which is completely positive.]

Step 5: The kernel. $m \equiv 1$ means that $|\chi_{\hat{K}}(\xi)| = 1$ for every ξ , so $\chi_{\hat{K}}(\xi) = e^{if(\xi)}$ for a real f with $f(0) = 0$; we take $\chi_{\hat{K}}$ to be continuous [hypothesis (H3)], as it is for every normalizable-code word kernel here (for the parity kernel, $\chi_{\hat{K}} \equiv 1$ in the Abel-regularized sense); f is then continuous too. Hermiticity of $\hat{K}$ gives $\chi_{\hat{K}}(-\xi) = \chi_{\hat{K}}(\xi)^*$, so f is odd modulo 2π . If G contains the phase-space inversion [the parity operator, which acts on displacements by $(-1)^{\hat{n}} \hat{D}(\xi) (-1)^{\hat{n}} = \hat{D}(-\xi)$ and on X^C by $\Omega \mapsto -\Omega$], then covariance evaluated at the fixed point $\Omega = 0$ of that action gives $(-1)^{\hat{n}} \hat{K} (-1)^{\hat{n}} = \hat{K}$, i.e., $[\hat{K}, (-1)^{\hat{n}}] = 0$. (For the codes considered here, the inversion also lies in H^C , so this also follows from Step 1.) Hence, $\chi_{\hat{K}}(\xi) = \text{Tr}[\hat{K} \hat{D}(\xi)] = \text{Tr}[\hat{K} \hat{D}(-\xi)] = \chi_{\hat{K}}(-\xi)$ and f is even. Odd and even together give $2f \in 2\pi\mathbb{Z}$, so $\chi_{\hat{K}} = \pm 1$ pointwise, and continuity with $\chi_{\hat{K}}(0) = 1$ gives $\chi_{\hat{K}} \equiv 1$. Since $\chi_{2^m(-1)^{\hat{n}_{\text{tot}}}} \equiv 1$ and the characteristic function determines the operator, $\hat{K} = 2^m(-1)^{\hat{n}_{\text{tot}}}$ and $\hat{\Delta}^C = \hat{\Delta}_{\text{std}}$. Without either route to $[\hat{K}, (-1)^{\hat{n}}] = 0$ (no inversion in G and no even-order rotation in H^C), the argument stops at $\chi_{\hat{K}} = e^{if}$, with f real, continuous, and odd, which is why parts (ii) and (iii) of the proposition are stated with the parity hypothesis rather than without it. ■

Two remarks on the scope of this proof. First, Steps 3 and 4 assume that $\mathcal{E}_{\mathcal{C}}$ is a channel on $\mathcal{B}(L^2(\mathbb{R}))$ [setting (A)]. The GKP twirl of Eq. (43) is not: It is an average over a noncompact lattice, whose “normalized Haar measure” does not exist as a probability measure, and it is defined on the code algebra of a rigged Hilbert space rather than on $\mathcal{B}(L^2(\mathbb{R}))$. It belongs to setting (B), where the twirl form is assumed and not derived. There, the clause of (a) is immediate: (H2) puts $\mathcal{B}(\mathcal{C})$ inside the fixed-point algebra of $\mathcal{E}_{\mathcal{C}}$, which is the commutant

of $\Lambda^\perp$, so every $\hat{D}(\lambda)$ with $\lambda \in \Lambda^\perp$ commutes with the code-space projector and leaves $\mathcal{C}$ invariant. This is the same caveat as in Sec. IV A and is the loophole the ideal GKP code exploits: There, $\Lambda^\perp$ is the stabilizer lattice and m is the indicator function of its symplectic dual, the logical lattice $L\mathbb{Z}^2$, read off from Eq. (43) directly, since $\mathcal{E}_{\mathcal{C}}(\hat{D}(\xi)) = \hat{D}(\xi)$ for $\xi \in L\mathbb{Z}^2$ and 0 otherwise, rather than from the formula $m = |\chi_{\hat{K}}|^2$: $\chi_{\hat{K}}$ there acts as a comb, whose square is undefined. This m takes the values $\{0, 1\}$ that idempotence demands, but Steps 3 and 4 do not apply to it, for the reason just given. Continuity of $\chi_{\hat{K}}$ also fails here, which matters separately at Step 5; one admissible $\hat{K}$ is the corresponding sum of displacement operators, which is Eq. (35), uniqueness being the caveat noted after Proposition III.3. Second, for a code whose code words are normalizable, Lemma III.1 closes the loophole directly once the twirl form is in hand, without the complete-positivity hypothesis: No nontrivial displacement can leave such a code space invariant, so $\Lambda^\perp = \{0\}$.

4. Proof of Lemma III.1

Suppose $\hat{D}(\beta)\mathcal{C} = \mathcal{C}$ with $\dim \mathcal{C} < \infty$ and $\mathcal{C} \subset L^2(\mathbb{R}^m)$. First, take $m = 1$. Then, $\hat{D}(\beta)$ restricted to $\mathcal{C}$ is a unitary on a finite-dimensional space and so has an eigenvector in $\mathcal{C}$. However, $\hat{D}(\beta) = e^{i\lambda \hat{K}_\theta}$, where $\hat{K}_\theta = \hat{x} \sin \theta - \hat{p} \cos \theta$ is a quadrature operator and $\lambda \propto |\beta|$; $\hat{K}_\theta$ is unitarily equivalent to $\hat{x}$ and so has purely absolutely continuous spectrum, from which $e^{i\lambda \hat{K}_\theta}$ has purely absolutely continuous spectrum on the unit circle and no eigenvectors in $L^2(\mathbb{R})$ for $\lambda \neq 0$. Therefore, $\beta = 0$. For $m > 1$, a passive (beam splitter) network $\hat{U}$ maps $\beta \in \mathbb{C}^m$ to $(|\beta|, 0, \dots, 0)$, and $\hat{U} \hat{D}(\beta) \hat{U}^\dagger = \hat{D}(|\beta|) \otimes \mathbb{1}^{\otimes(m-1)}$; since $\hat{U}\mathcal{C}$ is again a finite-dimensional subspace left invariant by that operator, the single-mode argument applies on the first factor and gives $|\beta| = 0$. ■

Note three things. First, the argument uses only $\dim \mathcal{C} < \infty$ and $\mathcal{C} \subset L^2(\mathbb{R}^m)$; it does not care how the code is built, so it applies to cat, binomial, rotation-symmetric and numerically optimized codes, and finite-energy GKP code words. Second, it fails for the ideal GKP code because its code words are Dirac combs and $\mathcal{C}_{\text{GKP}} \not\subset L^2(\mathbb{R})$: A Dirac comb is an eigenvector of a displacement, in the distributional sense of Sec. IV A. Third, the same spectral argument excludes a nontrivial squeeze, whose generator is the dilation operator $(\hat{x}\hat{p} + \hat{p}\hat{x})/2$, again with purely absolutely continuous spectrum. Rotations $e^{i\theta \hat{n}}$ are the exception: $\hat{n}$ has discrete spectrum and the Fock states are its eigenvectors. The isotropy group of a cat or binomial code therefore contains no displacement and no squeeze, and is the compact group $U(1) \times \mathbb{Z}_M$.

5. Proof of Proposition III.4

For $T_1, T_2 \in \mathcal{G}_{\mathcal{C}}$,

$$\mathcal{E} T_2 T_1 \mathcal{E} = \mathcal{E} T_2 \mathcal{E} T_1 \mathcal{E} = T_2 \mathcal{E} T_1 \mathcal{E} = T_2 T_1 \mathcal{E}, \quad (\text{C8})$$

where the first equality uses $T_1 \in \mathcal{G}_{\mathcal{C}}$, the second $T_2 \in \mathcal{G}_{\mathcal{C}}$, and the third $T_1 \in \mathcal{G}_{\mathcal{C}}$ again; we write $\mathcal{E}$ for $\mathcal{E}_{\mathcal{C}}$ throughout. For a convex combination, linearity gives

$$\begin{aligned} \mathcal{E}(\lambda T_1 + (1-\lambda)T_2)\mathcal{E} &= \lambda \mathcal{E} T_1 \mathcal{E} + (1-\lambda) \mathcal{E} T_2 \mathcal{E} \\ &= (\lambda T_1 + (1-\lambda)T_2)\mathcal{E}, \end{aligned} \quad (\text{C9})$$

and for a tensor product, the sector decomposition of a composite system is the product of the decompositions of its parts, so the joint twirl factorizes and the same computation applies factorwise. ■

6. Proof of Proposition III.5

Composing the representations inserts $\phi_C \circ \chi_C = \mathcal{E}_C$ between each pair of adjacent channels. By Proposition III.4, each such insertion may be absorbed into the channel applied before it, using $\mathcal{E}_C T_j \mathcal{E}_C = T_j \mathcal{E}_C$ repeatedly, together with $\mathcal{E}_C \circ \phi_C = \phi_C$; this leaves a single $\mathcal{E}_C$ at the input, as in Eq. (12). ■

7. Proof of Theorem V.1

The first part follows by the multiplier route [the continuity route of Proposition III.3(i)], which does not require considering complete positivity: Step 2 of Proposition III.3's proof, which needs only covariance and the integral formulas, shows that $\mathcal{E}_C$ acts on characteristic functions by multiplication by $m = |\chi_{\hat{K}}|^2$; idempotence gives $m^2 = m$ pointwise; continuity of $\chi_{\hat{K}}$ (hypothesized in the theorem) with $m(0) = |\text{Tr } \hat{K}|^2 = 1$ then gives $m \equiv 1$ on the connected plane, so $\mathcal{E}_C = \mathbb{1}$; and Step 5's parity argument fixes $\hat{K} = 2(-1)^{\hat{n}}$, the parity being available by hypothesis. Lemma III.1 gives an independent route to the same conclusion whenever we use Proposition III.3's twirl form, but the theorem does not need it. Equation (90) is then just the definition of W^C , which holds for every state, not just those supported on $\mathcal{C}$.

For the second part, take the logical basis states $|0_L\rangle, |1_L\rangle \in \mathcal{C}$, which are logical stabilizer states and are mutually orthogonal. Two orthogonal pure states cannot both be Gaussian, because the overlap of two Gaussian pure states is strictly positive (see, e.g., Ref. [55]). Therefore, at least one of them, say $|\psi\rangle$, is non-Gaussian, and by Hudson's theorem [21,54] its Wigner function takes negative values. Then, $\int d\mu |W_\psi^C| > |\int d\mu W_\psi^C| = 1$, i.e., $\mathcal{N}_\psi^C > 0$, while a witness of logical magic must vanish there. ■

Note two points. First, the theorem constrains only the negativity of the representation; the representation itself remains well defined, empirically adequate, and compositional on $\mathcal{G}_C$ (indeed on everything, since $\mathcal{E}_C = \mathbb{1}$), as Sec. VII F uses. Second, the proof shows that the GKP construction gets its strength not from the compactness of a group but the (un)normalizability of the code words: Because the ideal code words are Dirac combs, Lemma III.1 does not apply, the code space can be invariant under a lattice of displacements, and the resulting lattice kernel can and does represent stabilizer code words non-negatively.

Three further remarks on the hypotheses of Theorem V.1. First, continuity is used twice in the proof: to obtain $\mathcal{E}_C = \mathbb{1}$ and then to select $\chi_{\hat{K}} \equiv 1$. If it is dropped, $\mathcal{E}_C = \mathbb{1}$ has to come from elsewhere: from complete positivity [Proposition III.3(i), setting (A)] or from the twirl form with Lemma III.1 [Proposition III.3(iii)]. With $\mathcal{E}_C = \mathbb{1}$ granted in one of those two ways, dropping the continuity requirement leaves, in place of $\chi_{\hat{K}} \equiv 1$, the family of ± 1 -valued even functions on $\mathbb{R}^2$ with $\chi_{\hat{K}}(0) = 1$, the continuum analog of Proposition V.1's sign family. That family is code independent, existing already for the trivial code, none of its members being selected by any property of

$\mathcal{C}$, and each measurable member of it giving a kernel that is Hermitian, unit trace, covariant, and self-dual. Therefore, even without continuity, no code-adapted kernel is available for a code satisfying the remaining hypotheses together with one of those two; continuity instead means that the kernel is the standard one. Second, the reconstruction hypothesis separately excludes the Husimi kernel $\hat{\Delta}_Q(\beta) = |\beta\rangle\langle\beta|$, as despite being linear, Hermitian, standardized against $d^2\beta/\pi$, displacement covariant, and commuting with the parity (so satisfying Axioms 1–4 with the inversion available), its reconstruction has multiplier $e^{-|\xi|^2}$, which is not idempotent. Axioms 4 alone therefore do not fix the kernel, with or without parity; we need idempotence of the reconstruction map as well. Third, under the strong form of Axioms 1–5, injectivity, and traciality on all of $\mathcal{B}(L^2(\mathbb{R}))$, the first conclusion is immediately obvious, since either forces $\phi_C \circ \chi_C = \mathbb{1}$ directly, and the ideal GKP code would be excluded along with everything else. The content of the theorem is that the weakened, code-faithful form suffices here.

8. Proof of Proposition IV.1

Unit trace. $\text{Tr} A(u) = 1$ for every u , and Eq. (54) averages.

Standardization. Summing Δ^C over a transversal of X^C and over $\mathcal{M}$ covers $\mathbb{Z}_d^{2n}$ once, so $\sum_{[u]} \Delta^C = \frac{1}{|\mathcal{M}|} \sum_v A(v) = \frac{d^n}{|\mathcal{M}|} \mathbb{1} = d^k \mathbb{1}$.

Traciality. $\text{Tr}[\Delta^C(u)\Delta^C(u')] = \frac{1}{|\mathcal{M}|^2} \sum_{m,m'} d^n \delta_{u+m,u'+m'}$; the number of pairs with $m - m' = u' - u$ is $|\mathcal{M}|$ if $[u] = [u']$ and 0 otherwise, giving $d^k \delta_{[u],[u']}$.

Reconstruction. Writing $v = u + m$,

$$\begin{aligned} \phi_C(\chi_C(\hat{A})) &= \frac{d^{-k}}{|\mathcal{M}|^2} \sum_{[u]} \sum_{m,m'} \text{Tr}[A(u+m)\hat{A}] A(u+m') \\ &= \frac{d^{-k}}{|\mathcal{M}|^2} \sum_v \sum_{m'' \in \mathcal{M}} \text{Tr}[A(v)\hat{A}] A(v+m'') \\ &= \frac{d^{n-k}}{|\mathcal{M}|^2} \sum_{m'' \in \mathcal{M}} D(m'') \hat{A} D^\dagger(m'') = \mathcal{E}_C(\hat{A}), \quad (\text{C10}) \end{aligned}$$

using $A(v+m) = D(m)A(v)D^\dagger(m)$ to pull the m'' conjugation out of the inner sum, then the self-duality $\sum_v \text{Tr}[A(v)\hat{A}] A(v) = d^n \hat{A}$, and $d^{n-k}/|\mathcal{M}|^2 = 1/|\mathcal{M}|$. The $\mathcal{M}$ twirl equals the syndrome-sector dephasing because averaging over an Abelian group of unitaries is the conditional expectation onto the block diagonal of the commutant, here graded by the characters of $\mathcal{M}$; we verified the operator identity directly on the test codes. ■

9. Proof of Proposition IV.2

$\mathcal{M}$ invariance of the kernel means $W_{\hat{\rho}}^C = W_{\mathcal{E}_C(\hat{\rho})}^C$, and $\mathcal{E}_C$ -invariant states decompose over syndrome blocks, each of which is a displaced encoded state; by translation covariance, it therefore suffices to treat an encoded state $\hat{\rho}$.

(i) Since $D(m)\hat{\rho}D^\dagger(m) = \hat{\rho}$ and $A(u+m) = D(m)A(u)D^\dagger(m)$, all $|\mathcal{M}|$ terms of Eq. (54) have equal trace against $\hat{\rho}$: $\text{Tr}[\Delta^C(u)\hat{\rho}] = \text{Tr}[A(u)\hat{\rho}]$.

(ii) *Zero-syndrome sector:* The parity satisfies $A(0)D(m)A(0) = D(-m)$; as $\mathcal{M} = -\mathcal{M}$ and $\mathcal{S}$ is

phase-free, $A(0)$ preserves $\mathcal{C}$ and acts there as an involution conjugating the logical displacements by inversion, $A(0)\bar{D}(v)A(0) = \bar{D}(-v)$. Gauge fixing the logical basis by $\bar{X}$ aligns it with the logical parity $A_L(0)$. For $u = u_L$, a logical vector, $D(u_L)$ restricted to $\mathcal{C}$ is the logical displacement $D_L(a_L, b_L)$ in the same symmetric ordering, so $\text{Tr}[A(u_L)\hat{\rho}] = \text{Tr}_L[A_L(a_L, b_L)\hat{\rho}_L]$, which is Eq. (57), with no residual constant once the measures are attached.

(iii) *Error sectors*: For ϕ in the syndrome sector s , $D(m)A(0)\phi = A(0)D(-m)\phi = \omega^{-s(m)}A(0)\phi$, so $A(0)$ maps sector s to sector $-s$. Hence, for u with syndrome $s \neq 0$, $\text{Tr}[A(u)\hat{\rho}] = \text{Tr}[A(0)D^\dagger(u)\hat{\rho}D(u)]$ is the trace of an operator mapping sector s' into sector $-s' \neq s'$, which vanishes: $2s' = 0$ forces $s' = 0$ because d is odd. ■

10. Proof of Theorem IV.1

By Proposition IV.2, on the syndrome- s block of X^C the function $\mu W_{\hat{\rho}}^C$ equals $P(s|\hat{\rho})$ times the logical Gross function of $\hat{\rho}_L(s)$, translated by the block label. Summing $|\cdot|$ over each block gives $P(s)(\mathcal{N}^{\text{Gross}}(\hat{\rho}_L(s)) + 1)$; summing over s and subtracting 1 gives Eq. (58). The vanishing and positivity statements are those of the Gross negativity [2,3] applied per sector. ■

11. Proof of Theorem VI.1(i)

With $\chi_{\log}(\hat{\rho})(t; a, b) = \frac{1}{d} \text{Tr}[\Delta_{\log}(t; a, b)\hat{\rho}]$, $\phi_{\log}(F) = \sum_{t,a,b} F(t; a, b)\Delta_{\log}(t; a, b)$, and the qudit self-duality $\sum_{a,b} \frac{1}{d} \text{Tr}[A_L(a, b)\sigma]A_L(a, b) = \sigma$,

$$\begin{aligned} \phi_{\log}(\chi_{\log}(\hat{\rho})) &= \sum_t V_t \left(\sum_{a,b} \frac{1}{d} \text{Tr}[A_L V_t^\dagger \hat{\rho} V_t A_L] \right) V_t^\dagger \\ &= \sum_t V_t V_t^\dagger \hat{\rho} V_t V_t^\dagger = \sum_t \hat{\Pi}_t \hat{\rho} \hat{\Pi}_t, \end{aligned} \quad (\text{C11})$$

using $V_t^\dagger V_t = \mathbb{1}_L$ in the second line. The single factor of $\mu_L = 1/d$ sits in $\chi_{\log}$; inserting it in $\phi_{\log}$ as well gives $\frac{1}{d} \mathcal{E}^{\log}$. ■

12. Proof of Proposition V.1

(i) Conjugating the averaged kernel $\Delta_H(u) = \frac{1}{|R|} \sum_k A(S^k u)$ by $D(w)$ gives $\frac{1}{|R|} \sum_k A(S^k u + w)$, whereas covariance demands $\frac{1}{|R|} \sum_k A(S^k(u + w))$. Since the d^{2n} phase-point operators are an orthogonal, hence linearly independent, operator basis, the two agree for all u, w only if the label multisets agree, that is, only if $S = \mathbb{1}$.

(ii) Covariance under displacements forces $\Delta(u) = D(u)\hat{K}D^\dagger(u)$; the reconstruction map acts on characteristic functions $\chi_{\hat{A}}(\xi) = \text{Tr}[\hat{A}D(\xi)]$ as multiplication by $\mathfrak{m}(\xi) = |\chi_{\hat{K}}(\xi)|^2$, which is the finite-group transcription of Step 2 of Proposition III.3's proof, with all integrals now finite sums. Idempotence of $\phi \circ \chi$ gives $\mathfrak{m}^2 = \mathfrak{m}$ pointwise, so $\mathfrak{m}(\xi) \in \{0, 1\}$; complete positivity (assumed in the statement) and $\mathfrak{m}(0) = 1$ make $\mathfrak{m}$ the Fourier transform of an idempotent probability distribution on $\mathbb{Z}_d^{2n}$, that is, of the uniform distribution on a subgroup $\Lambda^\perp$; hence, $\mathfrak{m} = \mathbb{1}_\Lambda$ with $\Lambda = (\Lambda^\perp)^\perp$, and the twirl over $\Lambda^\perp$ must leave $\mathcal{C}$ invariant. Triviality of the

displacement isotropy gives $\Lambda^\perp = \{0\}$, meaning that $\Lambda = \mathbb{Z}_d^{2n}$ and $\mathfrak{m} \equiv 1$.

(iii) $|\chi_{\hat{K}}| \equiv 1$, with $\chi_{\hat{K}}(-\xi) = \chi_{\hat{K}}(\xi)^*$ (Hermiticity) and $\chi_{\hat{K}}(-\xi) = \chi_{\hat{K}}(\xi)$ [inversion covariance, which is the parity hypothesis of Proposition III.3, which holds here whenever the isotropy contains an even-order Clifford squaring to the parity, as the Fourier code does via $F^2 = A(0)$, and which is available in general by adjoining the $S = -\mathbb{1}$ Clifford to G] forces $\chi_{\hat{K}}$ real, even, and unimodular: a ± 1 -valued even function with $\chi_{\hat{K}}(0) = 1$. Each such function defines $\hat{K} = d^{-n} \sum_{\xi} \chi_{\hat{K}}(\xi) D(-\xi)$, which is Hermitian with unit trace, and $|\chi_{\hat{K}}| \equiv 1$ is exactly self-duality; so all $2^{(d^{2n}-1)/2}$ sign patterns satisfy the axioms used, and the freedom exists independently of $\mathcal{C}$. Stabilizer positivity selects $\chi_{\hat{K}} \equiv 1$ [2]; Clifford covariance leaves the two constant sign patterns, the symplectic group being transitive on $\mathbb{Z}_d^{2n} \setminus \{0\}$ [18]. The sign kernels sit inside the recent complete classification of $d \times d$ discrete Wigner functions [36]. All counts, including the 16 admissible sign kernels at $d = 3$, $n = 1$, of which one is stabilizer-non-negative and two are Clifford-covariant, were confirmed numerically (Appendix E). ■

13. Proof of Theorem V.2

$\chi_{\hat{K}}$ is a continuous function, so the continuity route of Proposition III.3(i) gives $\mathcal{E}_C = \mathbb{1}$. [Where the twirl form is available the same follows from part (iii): The code words are normalizable, so the multimode version of Lemma III.1, whose quadrature-spectrum argument does not depend on the mode count, gives $\Lambda^\perp = \{0\}$.] The code space is fixed pointwise by $\rho^{\otimes \hat{n}_{\text{tot}}}$, so the parity hypothesis holds and Proposition III.3(ii) forces the ordinary Wigner kernel. Any pure logical state is $\sum_i c_i |0 \cdots 1_i \cdots 0\rangle = \hat{b}^\dagger |0\rangle^{\otimes m}$ for the mode $\hat{b} = \sum_i c_i^* \hat{a}_i$, i.e., a passive (linear-optical) rotation of $|1\rangle \otimes |0\rangle^{\otimes (m-1)}$. Passive rotations are symplectic and $\int |W|$ is symplectically invariant, so every pure logical state shares the negative volume of the Fock state $|1\rangle$, which is $\int_0^\infty |4r^2 - 1| e^{-2r^2} 4r dr - 1 = (4 - \sqrt{e})/\sqrt{e} - 1 = 4e^{-1/2} - 2$. ■

14. Proof of Proposition V.2

The spin- j operator space decomposes multiplicity-free into irreducible tensor sectors $l = 0, \dots, 2j$, and any covariant kernel has the form $\Delta(\Omega) = \sum_{l,m} \varepsilon_l Y_{lm}^*(\Omega) T_{lm}$, with unimodular ε_l [4,77,78]. Averaging over H^C inserts, in each sector l , the projector $P_H^l = \int_H dh D^l(h)$ onto the H -invariant subspace. Covariance of the averaged kernel under all of $\text{SU}(2)$ requires each P_H^l to commute with the irreducible D^l , i.e., by Schur's lemma, to be 0 or $\mathbb{1}$. $P_H^l = \mathbb{1}$ means H acts trivially on sector l , which for a noncentral H fails for every $l \geq 1$, since $\ker D^l = \{\pm \mathbb{1}\}$ for integer l ; $P_H^l = 0$ means sector l is absent from the kernel. By Eq. (16), the reconstruction map of the averaged kernel is the H^C twirl, whose range is the H^C -invariant subalgebra $\bigoplus_l \text{Ran } P_H^l$; for those where covariance is imposed every P_H^l is 0 or $\mathbb{1}$ and this is $\bigoplus_{l: P_H^l = \mathbb{1}} \text{Ran } P_H^l$. Faithfulness on the code requires that range to contain $\mathcal{B}(\mathcal{C})$, which it does not for a noncentral H^C . For a rotation-symmetric spin code, $\text{U}(1)$ acts nontrivially on $\mathcal{C}$, whose code words are $\hat{J}_z$ eigenvectors of distinct eigenvalue,

so the twirl keeps only the diagonal part of $\mathcal{B}(\mathcal{C})$, just as for the cat code in Appendix A. For the binary polyhedral groups, the code space is fixed pointwise, but the invariant sectors are too few: For $2O$, the first invariant tensor sector above $l = 0$ is $l = 4$, so wherever $2j < 4$ only $l = 0$ remains, the twirl is depolarizing, and it annihilates every traceless element of $\mathcal{B}(\mathcal{C})$ once $\dim \mathcal{C} \geq 2$. Unlike the CV and DV cases, we cannot argue here from linear independence of the kernels at distinct phase-space points, since on S^2 they are not independent; we argue by Schur's lemma instead. We verified both halves numerically at $j = 1$: The $\mathbb{Z}_2$ -averaged Stratonovich kernel fails covariance under an x rotation, and the invariant projectors have rank 1 of 3 and rank 3 of 5 on the $l = 1, 2$ sectors, nonscalar as the argument requires. ■

15. Proof of Proposition IX.1

(i) The e and f bases are Fourier conjugate, so $\langle f|e \rangle$ is uniform and nowhere zero and the operators $|e\rangle\langle f|/\langle f|e \rangle$ form a basis of $\mathcal{B}(\mathcal{H})$; $Q_{\hat{\rho}}$ lists the coefficients of $\hat{\rho}$ in the dual basis and so determines it. The marginals follow from completeness: $\sum_f \langle f|e \rangle \langle e|\hat{\rho}|f \rangle = \langle e|\hat{\rho}|e \rangle$ and $\sum_e \langle f|e \rangle \langle e|\hat{\rho}|f \rangle = \langle f|\hat{\rho}|f \rangle$.

(ii) Each $|e\rangle$ is a joint stabilizer eigenvector, so $\langle e|\hat{\Pi}_s = \delta_{s,s(e)}\langle e|$. Hence, $\langle e|\mathcal{E}_C(\hat{\rho})|f \rangle = \sum_s \langle e|\hat{\Pi}_s \hat{\rho} \hat{\Pi}_s|f \rangle = \langle e|\hat{\rho} \hat{\Pi}_{s(e)}|f \rangle$, which is Eq. (122).

(iii) The nonselective sharp extraction is by definition $\hat{\rho} \mapsto \sum_s \hat{\Pi}_s \hat{\rho} \hat{\Pi}_s = \mathcal{E}_C(\hat{\rho})$, so its KD image is $\mathcal{Q}_{\mathcal{E}_C(\hat{\rho})}$ and the disturbance is $Q_{\hat{\rho}} - Q_{\mathcal{E}_C(\hat{\rho})}$. Q is linear in $\hat{\rho}$, so under $\hat{\rho} \mapsto (1 - \lambda)\hat{\rho} + \lambda\mathcal{E}_C(\hat{\rho})$ the intersector part is multiplied by $(1 - \lambda)$, $\mathcal{E}_C$ being idempotent. ■

16. Proof of Proposition IX.2

If $\mathcal{V}$ is cograded, $B_{s_2} := \mathcal{V}^\dagger(\hat{\Pi}_{s_2})$ is a positive block-diagonal operator, so $\text{Tr}[B_{s_2} \hat{\Pi}_{s_1} \hat{\rho}] = \text{Tr}[(B_{s_2})_{s_1} \hat{\rho}_{s_1 s_1}]$ is the trace of a product of two positive operators supported in sector s_1 : real and non-negative. If instead $\hat{\rho}$ is sector diagonal, $\hat{\Pi}_{s_1} \hat{\rho} = \hat{\Pi}_{s_1} \hat{\rho} \hat{\Pi}_{s_1}$, and $Q(s_1, s_2) = \text{Tr}[\hat{\Pi}_{s_2} \mathcal{V}(\hat{\Pi}_{s_1} \hat{\rho} \hat{\Pi}_{s_1})]$ is a probability. The marginal statements follow from $\sum_{s_1} \hat{\Pi}_{s_1} = \mathbb{1}$ and trace preservation of $\mathcal{V}$. ■

17. Proof of Proposition IV.5

The strange-state Gross function has one negative point, $-\frac{1}{3}$, and eight equal positive points, $+\frac{1}{6}$. A Pauli channel acts as convolution by p_v (Proposition IV.3 on the logical space), so every output value is $p_v(-\frac{1}{3}) + (1 - p_v)\frac{1}{6} = \frac{1}{6} - \frac{p_v}{2}$, negative iff $p_v > \frac{1}{3}$; summing $|\cdot|$ and subtracting 1 gives Eq. (62). For the weights (61), $p_0 = q_0^2$ alone exceeds $\frac{1}{3}$ until uniformization, giving $\mathcal{N} = q_0^2 - \frac{1}{3}$ and the stated threshold, located by root finding on the erf sums at working precision 10^{-20} . ■

APPENDIX D: THE REFLECTION IDENTITY AND THE JUMP KERNELS

The reflection identity (96) comes from $\hat{D}^\dagger(\zeta)\hat{a}\hat{D}(\zeta) = \hat{a} + \zeta$ and $(-1)^{\hat{n}}\hat{a}(-1)^{\hat{n}} = -\hat{a}$:

$$\begin{aligned} \hat{\Pi}(\zeta)\hat{a}\hat{\Pi}(\zeta) &= \hat{D}(\zeta)(-1)^{\hat{n}}(\hat{a} + \zeta)(-1)^{\hat{n}}\hat{D}^\dagger(\zeta) \\ &= \hat{D}(\zeta)(\zeta - \hat{a})\hat{D}^\dagger(\zeta) = 2\zeta - \hat{a}. \end{aligned} \quad (\text{D1})$$

TABLE III. Numerical verification of Sec. IV D on two different qutrit codes ($d = 3$). $|\mathbb{S}\rangle \propto |1\rangle - |2\rangle$ is the strange state.

	$[[3, 1]]$	$[[5, 1, 3]]$
Isotropy of a generic code word $= \mathcal{M}$	✓	✓
$\sum_{x^C} \Delta^C = d^k \mathbb{1}$; $\text{Tr} \Delta^C = 1$	✓	✓
Equation (57), no residual constant; error sectors 0	✓	✓
Error = rigid translation; cycle = id	✓	✓
Twirl = syndrome dephasing	✓	✓
$\mathcal{N}^C$ encoded $ 0_L\rangle, +_L\rangle$	0	0
$\mathcal{N}^C$ encoded $ \mathbb{S}\rangle$	2/3	2/3
Equation (58) under Pauli noise	✓	✓

Equation (112) follows from $\hat{\Pi}^2 = \mathbb{1}$: $\hat{a}^{\dagger k} \hat{\Pi} \hat{a}^k = \hat{\Pi}(\hat{\Pi} \hat{a}^\dagger \hat{\Pi})^k \hat{a}^k = \hat{\Pi}(2\zeta^* - \hat{a}^\dagger)^k \hat{a}^k$. The gain form (113) can be obtained the same way or derived directly: With $(\hat{a} + \zeta)(-1)^{\hat{n}} = (-1)^{\hat{n}}(-\hat{a} + \zeta)$,

$$\begin{aligned} \hat{a}\hat{\Pi}(\zeta)\hat{a}^\dagger &= \hat{D}(\zeta)(\hat{a} + \zeta)(-1)^{\hat{n}}(\hat{a}^\dagger + \zeta^*)\hat{D}^\dagger(\zeta) \\ &= -\hat{D}(\zeta)(-1)^{\hat{n}}[\hat{n} + 1 + \zeta^* \hat{a} - \zeta \hat{a}^\dagger - |\zeta|^2]\hat{D}^\dagger(\zeta) \\ &= -\hat{\Pi}(\zeta)(\hat{n} + 1 - 2\zeta \hat{a}^\dagger), \end{aligned} \quad (\text{D2})$$

using $\hat{D}(\zeta)\hat{n}\hat{D}^\dagger(\zeta) = \hat{n} - \zeta \hat{a}^\dagger - \zeta^* \hat{a} + |\zeta|^2$ and the invariance of $\zeta^* \hat{a} - \zeta \hat{a}^\dagger$ under the same conjugation. The loss identity (100) follows from the same algebra with the order of $\hat{a}$ and $\hat{a}^\dagger$ exchanged. We verified both numerically to 10^{-14} at Fock cutoff 200.

APPENDIX E: NUMERICAL METHODS

Discrete-variable codes are implemented with displacements as index permutations with phases, so kernel traces take $O(d^n)$ per phase-space point; the $[[5, 1, 3]]_3$ verification evaluates the full 729-coset quotient exactly, with cyclic stabilizers $XZZ^{-1}X^{-1}\mathbb{1}$ checked for symplectic closure, logical pairs found in the commutant modulo the lattice, and the logical basis gauge fixed by $\bar{X}$. The sector-graded cat construction of Sec. VI uses exact residue-class supports at cutoff 160.

Every statement of Sec. IV D was verified, to 10^{-9} or better, on the three-qutrit repetition code $[[3, 1]]_3$ and the five-qutrit code $[[5, 1, 3]]_3$; Table III summarizes the results. The encoded qutrit strange state $|\mathbb{S}\rangle \propto |1\rangle - |2\rangle$ has $\mathcal{N}^C = 2/3$, its logical Gross value, preserved by Eq. (58) under a Pauli channel of correctable errors. As a stress test, take instead the three-component channel with weights $\{\frac{1}{2}\mathbb{1}, \frac{1}{4}\bar{Z}, \frac{1}{4}X_1\}$, whose $\bar{Z}$ branch is a syndrome-free logical error, so that the zero-syndrome conditional $\hat{\rho}_L(0) = \frac{2}{3}|\mathbb{S}\rangle\langle\mathbb{S}| + \frac{1}{3}Z|\mathbb{S}\rangle\langle\mathbb{S}|Z^\dagger$ is a rank-2 mixture. Both sides of Eq. (58) give $\mathcal{N}^C = 5/12$, the right-hand side as $\frac{3}{4} \times \frac{1}{3} + \frac{1}{4} \times \frac{2}{3}$.

For Proposition V.1, all counts were verified for a single qutrit with the Fourier gate F as code symmetry, any two-dimensional span of F eigenvectors being F invariant: The $\langle F \rangle$ average fails displacement covariance; all $2^{(d^2-1)/2} = 16$ sign kernels pass every axiom used; exactly one representation, Gross's, is non-negative on the 12 pure stabilizer states; and F covariance cuts 16 to 4. The spin-1 checks for Proposition V.2 are described with its proof in Appendix C.

Table II's cat and binomial negativities used two independent implementations. In the first, we expanded each code word in the Fock basis and truncated at $n_{\max} = 280$, which at $|\alpha|^2 = 6$ leaves a residual weight far below machine precision; the binomial code words are exact at this truncation. Writing $\beta = re^{i\theta}$ and using $\hat{D}(\beta) = e^{i\theta\hat{n}}\hat{D}(r)e^{-i\theta\hat{n}}$, the Wigner function separates as $W(r, \theta) = \frac{2}{\pi} \sum_{m,n} \rho_{nm} e^{i(m-n)\theta} \langle m|\hat{\Pi}(r)|n\rangle$, with the real-argument displaced parity $\langle m|\hat{\Pi}(r)|n\rangle = \sum_k (-1)^k \langle m|\hat{D}(r)|k\rangle \langle n|\hat{D}(r)|k\rangle$; we obtained the real orthogonal matrix $\langle m|\hat{D}(r)|k\rangle$ by exponentiating the real antisymmetric generator $r(\hat{a}^\dagger - \hat{a})$, avoiding the overflow the closed-form Laguerre expression suffers at large r . The grid is 460 radial by 1400 angular points out to $|\beta| = 6.5$, beyond which the weight is negligible. Two checks bound the discretization error: $\int d^2\beta W^{\text{std}} = 1$ is recovered to better than 10^{-4} , and halving the grid spacing changes each tabulated value by less than 10^{-3} (why we quote two decimal places). The implementation was checked against a direct evaluation of $\langle m|\hat{D}(\gamma)|n\rangle$ in the Laguerre form and against the known Fock-state values $\mathcal{N}_{|1\rangle} = 0.426$ and $\mathcal{N}_{|2\rangle} = 0.729$. The second implementation, the one in the deposited notebook [73], uses Laguerre-harmonic quadrature

at Fock cutoff 64 on a polar grid of 500 radial points out to $|\beta| = 7$ and 480 angles, and agrees with the first to the quoted precision.

Table II lists the negative volumes of the input code words and of the normalized postevent states $\hat{a}\hat{\rho}\hat{a}^\dagger/\langle\hat{n}\rangle$ and $\hat{a}^\dagger\hat{\rho}\hat{a}/\langle\hat{n}+1\rangle$, so that every entry is the negative volume of a normalized distribution.

Every quantitative claim above was verified in Wolfram Mathematica: the single-qutrit facts, strange-state Wigner values, and sign-family counts in exact arithmetic; the full $[[3, 1]]_3$, Kirkwood-Dirac, and two-time computations, yielding the exact values $2/\sqrt{3}$, $1/6$, $2/3$, $2/9$, and $5/12$; the operator identities, thresholds, multi-round sequence, and spin-1 checks; and the sector-graded cat numbers. The $[[5, 1, 3]]_3$ negativities were rederived by an algorithmically independent characteristic-function and symplectic-finite Fourier transform (FFT) route, giving coset sums 3 and 5, and hence $\mathcal{N} = 2/3$, exactly; the deposited notebook runs every row of Table III on both codes, the $[[5, 1, 3]]_3$ rows by that route, which it checks against explicitly constructed phase-point operators.

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