

Dual energy-differentiated topological transition in artificial red phosphorus chains

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We investigate the spectral and topological properties of an atomic chain of red phosphorus. We reveal the separation of flat-band states from the rest of the system and calculate its energy bands analytically. The topological properties of the system are established through the evaluation of the Berry (Zak) phase of the energy bands, revealing nontrivial topology. The Berry phase depends on the relative strength of the hopping parameters and exhibits dual energy-dependent topological phase transitions. Remarkably, the emergence of inert band edges provides a direct spectral signature of these transitions, acting as energy-resolved indicators of the redistribution of topological charge between bands. The existence of the associated edge states is proved numerically for finite lattices. The theoretical predictions, particularly the band structure and the existence of edge states, are further confirmed by numerical simulations of red phosphorus through a phononic lattice in the form of a highly structured aluminum plate.

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I. INTRODUCTION

The study of topological features in one-dimensional chains has been of enormous interest in research due to the potential for technological applications, including topological qubits in quantum computing [1–4]. From the theoretical point of view, one-dimensional chains are the simplest systems to understand difficult concepts in topological insulators. Moreover, artificial crystals as acoustic [5–15], photonic [16–28], topoelectric circuits [29], or phononic metamaterials [30,31] are ideal platforms for testing phenomena predicted from tight-binding models. The main advantage arises from the versatility to manipulate straightforward evanescent couplings and geometry in resonator arrays.

There are many important examples of one-dimensional chains with experimental realizations. The most famous is the Su-Schrieffer-Heeger (SSH) model, which embodies the topological phase transition in *trans*-polyacetylene [32–37]. This model predicts a topological phase transition when one of the hopping parameters is increased [38]. Such a topological phase has been confirmed in multiple experimental realizations, such as semiconductor quantum dots [39], topoelectrical circuits [29,40,41], superconducting circuits [42], cold atoms [43], Rydberg atoms [44], acoustic chains [7,10], polariton

lattices [45], water waves, elastic strings and metamaterials [46–48], photonic lattices, and microwave resonators [25,49]. Based on this model, similar and extended SSH models have been proposed, for instance, non-Hermitian SSH chains [50–53], two-dimensional SSH lattices [26,54–57], bearded SSH chains [25,58–60], and a twofold topological phase transition in *cis*-polyacetylene [16,30]. Each unusual version of the SSH model offers exotic phenomena and represents a significant advance in the development and understanding of topological insulator theory.

In this paper, an artificial crystalline version inspired by the red phosphorus molecule is proposed. We present a tight-binding model for both finite and infinite periodic chains to describe their electronic properties and topological features. Two nontrivial phases are found occurring at different regions of the energy spectrum. Calculation of Berry phase and edge states for the finite chain confirms the emergence of these topological phases. An artificial chain based on a coupled-resonator phononic metamaterial is proposed. The elastic wave equation is solved numerically using the finite-element method. In this system, the aluminium octagon resonators play the role of meta-atoms, while waveguides couple them evanescently. The comparison of tight-binding energy bands and phononic diagram shows an excellent agreement. Moreover, the prediction of edge states from the model is confirmed by the normal vibration modes of the finite artificial chain.

The article is organized as follows. In the first and second section, a finite red phosphorus chain is examined within the tight-binding framework. Its spectrum is analyzed, identifying two families of topologically protected edge states. The third section addresses the decoupling of the flat-band dynamics and demonstrates the equivalence of the system to an SSH-like trimer chain. In the subsequent sections, analytical expressions for the Bloch Hamiltonian, band structure, and Berry phase are derived to provide a topological characterization of

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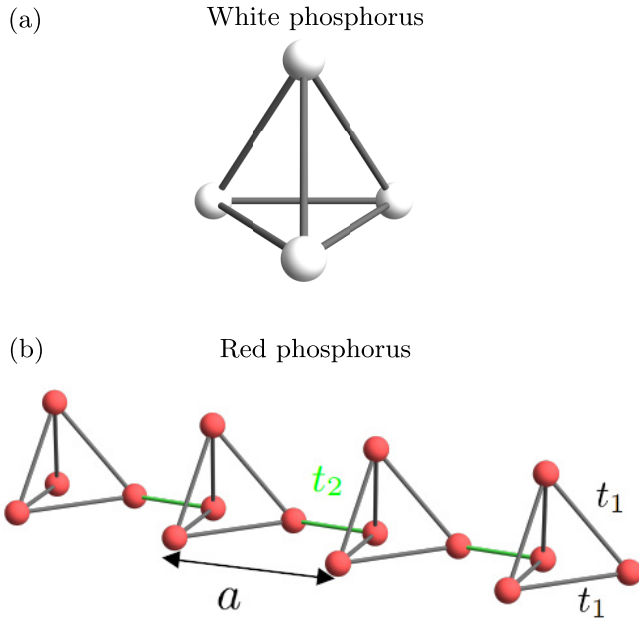


FIG. 1. (a) Pictorial representation of the P_4 molecule, also known as white phosphorus. (b) Red phosphorus lattice formed by a series of open-bond P_4 molecules, where t_1 and t_2 are the intra and interhopping parameters, respectively.

the chain. Finally, numerical simulations of an elastic experimental platform confirm the theoretical predictions regarding the band structure and the existence of topologically protected edge states.

II. TIGHT BINDING OF A POLYMER RED PHOSPHORUS MOLECULE—FINITE CHAIN

Red phosphorus exhibits several allotropes [61–63], typically found in amorphous form in nature. In this work, however, we consider a hypothetical crystalline one-dimensional structure constructed from P_4 molecule units. The P_4 molecules, also known as white phosphorus, have a pyramidal geometry with phosphorus atoms located at the vertices, as shown in Fig. 1(a). Our proposal consists of a periodic chain to explore the topological properties through artificial systems. In Fig. 1(b), it is shown that red phosphorus has a unit cell with an open bond P_4 molecule. The chain is formed by connecting neighboring P_4 units through an open bond, such that one phosphorus atom of each molecule binds to the adjacent unit.

Within the tight-binding approach, the three-dimensional geometry of red phosphorus is not essential. Thus, only the projected positions of the atoms along the periodic direction are taken into account. When the Bloch theorem is applied, the dimensionality is indicated through the direction in which the system is periodic. In Fig. 1, the hopping parameters t_1 and t_2 represent the probability amplitude of an electron to hop within and outside the P_4 molecule, respectively.

If we consider a finite red phosphorus chain, the tight-binding Hamiltonian is depicted by the following

block-diagonal submatrix:

$$h_{\text{rp}}(t_1) = \begin{pmatrix} E_0 & t_1 & t_1 & 0 \\ t_1 & E_0 & t_1 & t_1 \\ t_1 & t_1 & E_0 & t_1 \\ 0 & t_1 & t_1 & E_0 \end{pmatrix}, \quad (1)$$

which corresponds to the Hamiltonian of the P_4 molecule, and the nondiagonal submatrix, which is given by

$$C(t_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ t_2 & 0 & 0 & 0 \end{pmatrix}, \quad (2)$$

couples all the neighbor P_4 molecules. The full Hamiltonian for the polymer molecule of red phosphorus, in Toeplitz matrix form, is

$$H_{\text{fc}}(t, t') = \begin{pmatrix} h_{\text{rp}}(t_1) & C(t_2) & 0 & \cdots & 0 \\ C^\dagger(t_2) & h_{\text{rp}}(t_1) & C(t_2) & 0 & \cdot \\ 0 & C^\dagger(t_2) & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & C(t_2) \\ 0 & 0 & 0 & C^\dagger(t_2) & h_{\text{rp}}(t_1) \end{pmatrix}. \quad (3)$$

The numerical diagonalization of this Hamiltonian yields the energy spectrum, as shown in Fig. 2(a). For a finite chain composed of 30 unit cells, the spectrum consists of 120 eigenenergies as a function of the hopping parameter ratio $\omega \equiv t_1/t_2$. The analysis of the corresponding eigenstates reveals the existence of two pairs of topological edge states. Interestingly, and rather unconventionally, each pair is associated with a different energy. As ω increases, a pair of topological edge states appears at energy $E = 2t_1$ within the range $t_1 < \frac{t_2}{2}$, i.e., $\omega < 1/2$. A second pair emerges at energy $E = -t_1$ for $t_1 < t_2$, i.e., $\omega < 1$, corresponding to the orange lines in Fig. 2(a). The eigenstates obtained from the Hamiltonian in Eq. (3) evidence the localization of electronic states in the edges for the energies associated with the topological phases, as illustrated in Fig. 2(b). In contrast, Figs. 2(c) and 2(d) display bulk states. It is also important to note the presence of a constant energy line at $E = -t_1$ over the entire range of t_1 . This corresponds to a set of nondispersive bulk states, as seen in Fig. 2(d).

III. INFINITE LATTICE—EQUIVALENCE TO SSH-TYPE TRIMER CHAIN

From the finite red phosphorus Hamiltonian in Eq. (3), we can derive the Bloch Hamiltonian of the infinite chain by applying the Bloch theorem. In this way, the wave function in matrix representation is given by

$$\Psi(k) = (\dots, A_1 e^{ika}, A_2 e^{ika}, A_3 e^{ika}, A_4 e^{ika}, A_1 e^{2ika}, A_2 e^{2ika}, A_3 e^{2ika}, A_4 e^{2ika}, \dots), \quad (4)$$

where A_j , with $j = 1, 2, 3$, and 4, are the amplitudes for the four atomic orbitals in the unit cell and a is the unit cell length. From the Schrödinger equation $H_{\text{fc}}(t_1, t_2)\Psi(k) = E\Psi(k)$, using the Hamiltonian in Eq. (3) and wave function (4), the

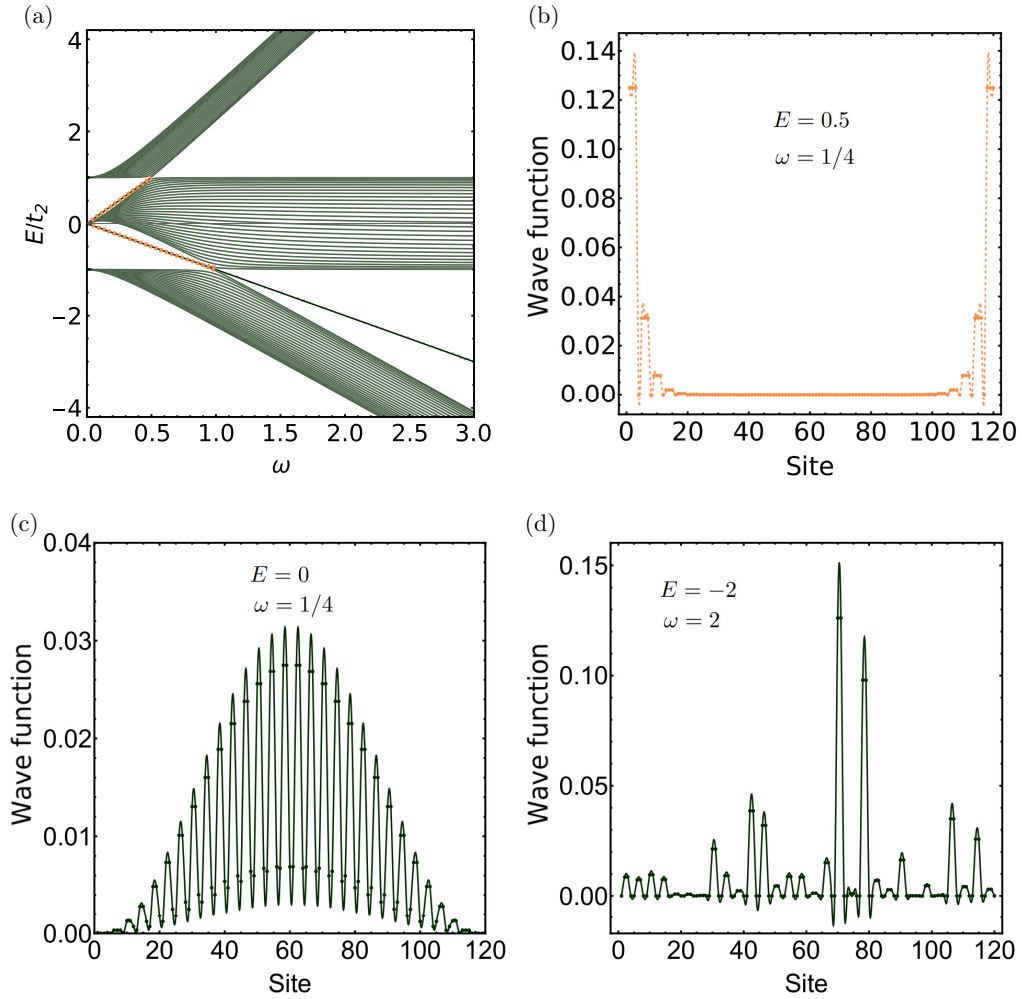


FIG. 2. (a) Energy spectrum of the finite red phosphorus chain as a function of $\omega = t_1/t_2$; we fix $t_2 = 1$. The green bands correspond to bulk states, while the dashed orange ones correspond to edge states. (b)–(d) Squared modulus of the wave function for the topological edge (dashed orange) and bulk states (green) as a function of the site number. Panel (d) shows the wave function for the unique green flat band, and panel (c) shows it for dispersive bulk states.

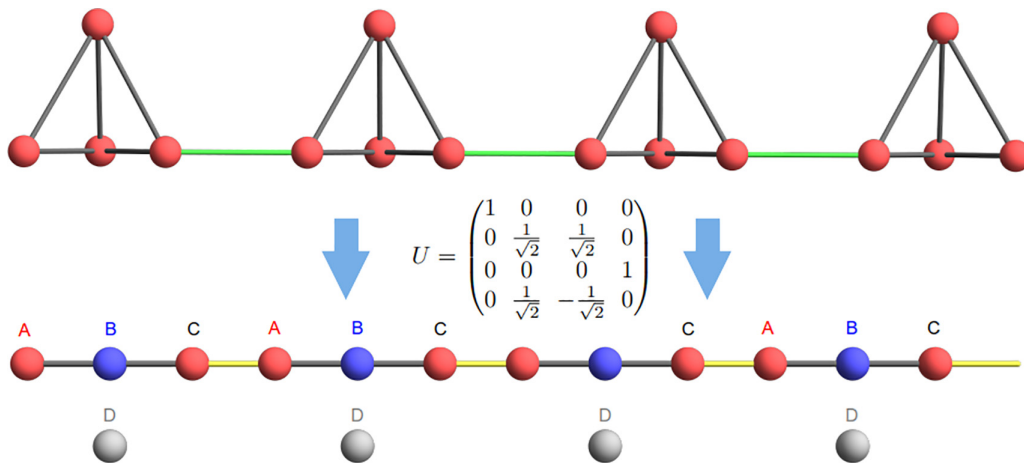


FIG. 3. Auxiliary atomic chain described by the Hamiltonian (7). Flat-band states are located on the atoms of the D chain.

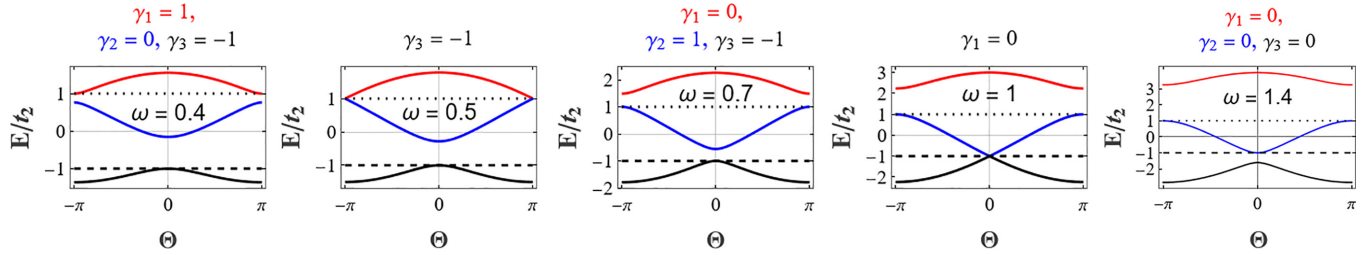


FIG. 4. Energy bands $E_{1,2,3}/t_2$ in dependence on $\Theta = ka$ for five different values of $\omega = t_1/t_2$. We fixed $a = 1$ and $t_2 = 1$. The Berry phase $\gamma_{1,2,3}$ of the bands is indicated above the figures.

4×4 Bloch Hamiltonian yields

$$H_{\text{RP}}(k) = \begin{pmatrix} 0 & t_1 & t_1 & t_2 e^{-iak} \\ t_1 & 0 & t_1 & t_1 \\ t_1 & t_1 & 0 & t_1 \\ t_2 e^{iak} & t_1 & t_1 & 0 \end{pmatrix}. \quad (5)$$

Let us perform the following unitary transformation of the Hamiltonian H_{RP} :

$$H(k) = U H_{\text{RP}}(k) U^\dagger = \begin{pmatrix} 0 & \sqrt{2}t_1 & t_2 e^{-ika} & 0 \\ \sqrt{2}t_1 & t_1 & \sqrt{2}t_1 & 0 \\ t_2 e^{ika} & \sqrt{2}t_1 & 0 & 0 \\ 0 & 0 & 0 & -t_1 \end{pmatrix},$$

$$\text{with } U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}. \quad (6)$$

The Hamiltonian acquires a block-diagonal form. It explicitly reveals the decoupling of the flat-band dynamics from the rest of the system. The states given by multiples of the vector $(0, 0, 0, 1)^T$ correspond to the flat-band energy $-t_1$.

The operator (6) can be considered the Bloch Hamiltonian of an auxiliary atomic ladder composed of two parallel chains of atoms. The first one is an SSH-type trimer chain with a staggered potential. Let us refer to the three atoms in the elementary cell of the chain as A , B , and the C atoms. The on-site energy vanishes on the atoms A and C , the B atoms have an on-site energy t_1 . The second parallel chain is composed of noninteracting D atoms (see Fig. 3). The quasiparticles located on these chains are represented by the corresponding components of the transformed Bloch wave function,

$$\Psi(k) = (\psi_A(k), \psi_B(k), \psi_C(k), \psi_D(k))^T.$$

The states located on the D atoms correspond to the flat band and do not interact with the rest of the system. The tight-binding Hamiltonian for the auxiliary SSH-type trimer associated with the Bloch operator (6) can be written as follows:

$$H = t_1 \sum_n B_n^\dagger B_n + \sum_n (\sqrt{2}t_1 A_n^\dagger B_n + \sqrt{2}t_1 B_n^\dagger C_n + t_2 A_n^\dagger C_{n+1} + \text{H.c.}) - t_1 \sum_n D_n^\dagger D_n. \quad (7)$$

In the following section, we investigate spectral and topological properties of the system described by $H(k)$ in Eq. (6).

Since it is unitarily equivalent to $H_{\text{RP}}(k)$, the results will be directly applicable to the chain of red phosphorus.

IV. ENERGY BANDS, BERRY PHASE, AND INERT BAND-EDGE STATES

The eigenvalues of $H(k)$ in Eq. (6) can be found straightforwardly. The secular equation $\det[H(k) - E] = 0$ is of fourth order in E . As noted earlier, one solution corresponds to the flat-band energy $E_{\text{fb}} = -\omega t_2 = -t_1$. The other three energy bands are solutions of a cubic equation for ϵ . The equation can be written in the following form:

$$\cos ka = \frac{1}{4\omega^2}(\epsilon - \omega)(\epsilon^2 - 1) - \epsilon, \quad \epsilon = E/t_2, \quad \omega = t_1/t_2. \quad (8)$$

The third-order equation can be solved analytically with the use of Cardano's formula; see, e.g., the Appendix in Ref. [64]. This way, we get

$$E_1 = \frac{t_2}{3} \left[\omega + 2\sqrt{3 + 13\omega^2} \cos \left(\frac{1}{3} \arccos q \right) \right],$$

$$E_2 = \frac{t_2}{3} \left[\omega + 2\sqrt{3 + 13\omega^2} \cos \left(\frac{2\pi}{3} + \frac{1}{3} \arccos q \right) \right],$$

$$E_3 = \frac{t_2}{3} \left[\omega - 2\sqrt{3 + 13\omega^2} \sin \left(\frac{1}{3} \arcsin q \right) \right],$$

$$q = \frac{\omega[-9 + 19\omega^2 + 54\omega \cos(ka)]}{(3 + 13\omega^2)^{3/2}}. \quad (9)$$

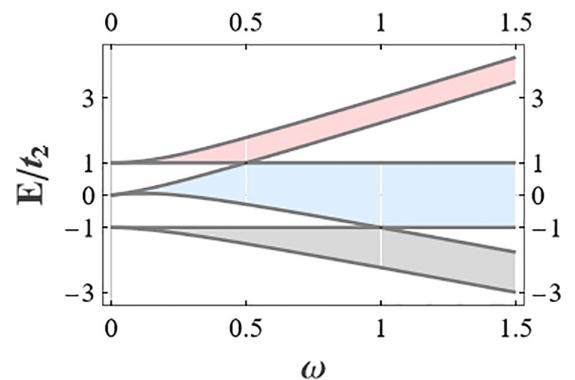


FIG. 5. Energy bands in dependence on $\omega = t_1/t_2$. The gap between the lower and middle bands gets closed for $\omega = 1$, whereas the gap between the upper and the middle band closes for $\omega = 1/2$. The edges of the energy gap are solutions of Eqs. (10) and (11).

TABLE I. The values of the Berry phase $\gamma_{1,2,3}$ for each energy band in dependence on ω .

	$\omega < \frac{1}{2}$	$\frac{1}{2} < \omega < 1$	$1 < \omega$
γ_1	1	0	0
γ_2	0	1	0
γ_3	-1	-1	0

The energy bands are plotted for different parameter values in Fig. 4.

Let us analyze the minima and maxima of the energy bands $E_{1,2,3}$ in the first Brillouin zone. The right-hand side of Eq. (8) is a third-order polynomial in ϵ that goes to $\pm\infty$ as $\epsilon \rightarrow \pm\infty$, respectively. The left-hand side can acquire values in the interval $[-1, 1]$. Therefore, the solutions of Eq. (8) are the values of ϵ where the polynomial falls into the interval $[-1, 1]$. It is straightforward to find their explicit values as they correspond to the solutions of Eq. (8) when either $\cos ka = 1$ (i.e., $k = 0$) or $\cos ka = -1$ (i.e., $k = \pm\pi/a$). When $k = 0$, Eq. (8) can be factorized as

$$(\epsilon + 1)(\epsilon^2 - \epsilon(1 + \omega) - \omega(1 - 4\omega)) = 0, \quad (10)$$

whereas for $k = \pm\pi/a$, we have

$$(\epsilon - 1)(\epsilon^2 + \epsilon(1 - \omega) - \omega(1 + 4\omega)) = 0. \quad (11)$$

It is rather remarkable that one solution in Eqs. (10) and (11) is independent of ω ,

$$\epsilon_{\pm} = \pm 1 \implies E_{\pm} = \pm t_2. \quad (12)$$

We call $E_{\pm}$ *inert band edges*. It also follows from Eqs. (10) and (11) that two energy bands close when the ratio $\omega = t_1/t_2$ is either $\omega = \frac{1}{2}$ or $\omega = 1$. The dependence of the energy bands on ω is illustrated in Fig. 5. The eigenstates of $H(0)$ and $H(\pi/a)$ corresponding to the inert band edges $E_{\pm}$ can be found in a straightforward manner. The two states coincide; they have exactly the following form:

$$\Psi_{\text{inert}} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$(H(0) + E_-)\Psi_{\text{inert}} = (H(\pi/a) - E_+)\Psi_{\text{inert}} = 0. \quad (13)$$

They are located on the sublattices A and C. As they do not depend on ω , we call them inert band-edge states.

Let us consider ω as the tunable parameter, whereas t_2 is fixed. The bandgaps close and open again as ω crosses the

critical values $\omega = 1/2$ and $\omega = 1$. The closing and opening of the energy gaps can be associated with an abrupt change of the topological properties of the energy bands. We calculate the Berry phase as the indicator of nontrivial topological characteristics.

The Berry phase can be calculated with the use of the normalized eigenstates of $H(k)$ corresponding to $E_j(k) \equiv t_2 \epsilon_j(k)$. They can be found in a straightforward manner. They satisfy

$$H(k)\Psi_j(k) = E_j(k)\Psi_j(k), \quad \langle \Psi_j(k) | \Psi_j(k) \rangle = 1, \\ j = 1, 2, 3. \quad (14)$$

We do not present their explicit form here, as the formulas are rather large. The eigenstates can be used to calculate the corresponding Berry phase γ_j associated with each $\Psi_j(k)$,

$$\gamma_j = -\frac{i}{\pi} \int_{-\pi/a}^{\pi/a} \Psi_j^\dagger(k) \partial_k \Psi_j(k) dk, \quad j = 1, 2, 3. \quad (15)$$

The numerical values of γ_j as functions of ω and t_2 are illustrated in Table I.

The results confirm that when ω crosses its critical values, the energy gap closes and the Berry phase of the two touching bands undergoes a corresponding change. In this process, the associated topological charge is transferred from one band to the other. The presence or absence of inert band edges correlates directly with the Berry phase: Bands with zero or two inert edges are topologically trivial, whereas a band hosting a single inert edge acquires a nontrivial Berry phase. This intriguing relation between topological phase and inert edge states merits a more detailed analysis elsewhere.

These results, together with the analysis of the second section, confirm the bulk-edge correspondence. The existence of topological edge states in the finite lattice is unambiguously correlated with the nontrivial Berry phase of the infinite chain.

The edge states localized at the boundaries have nonzero energies. It is worth noting in this context that topological edge states with nonzero energy were discussed, e.g., in the generalized SSH model in Ref. [65]. There, the deviation of the edge-state energies from zero was caused by on-site interaction (staggered potential). This explanation appears to be true in our model too: As we have shown, the dynamics of the system is equivalent to the trimer-SSH chain with nonvanishing on-site interaction.

TABLE II. Equivalences between the condensed-matter and coupled-resonator phononic tight-binding models.

Tight-binding model	Condensed-matter variables	Coupled-resonator phononic system	Metamaterial variables
Atoms		Mechanical resonators or meta-atoms	
Molecules or materials		Metamolecules or metamaterials	
σ bond		Finite phononic crystal (FPnC)	
Energy	E	Frequency	f
Atomic orbital	ψ_0	High- Q normal mode or <i>metaorbital</i>	w_0
Site energy	E_0	Resonant frequency of the high- Q normal mode	f_0
Hopping	t	Evanescent (Bragg) overlap through the FPnCs	t

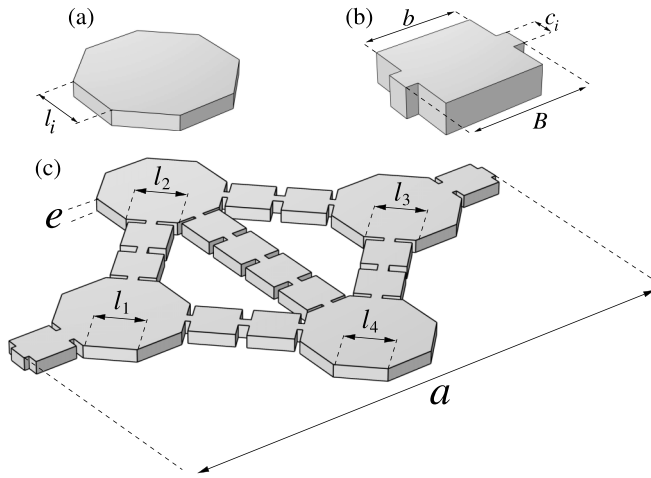


FIG. 6. (a) Artificial atomic site consists of an octagonal resonator of length l . The bond is implemented by a finite phononic crystal whose unit cell is given in panel (b); it is composed of a square plate of length b and two smaller plates with side length $B - b$ and width c_i . Here, the width c_1 controls the intracell hopping and c_2 the intercell hopping. (c) Unit cell (of length a) of the artificial red phosphorus designed on a plate of thickness e . The geometrical parameters are $l_1 = l_3 = 23$ mm and $l_2 = l_4 = 23.1$ mm, $B = 26$ mm, $b = 21$ mm, $c_1 = 7, 6.652, 6.4, 6$, and 5.5 mm, $c_2 = 8.6258$ mm, $e = 6.35$ mm, and $a = 26.01$ mm. The mechanical properties of the plates correspond to those of aluminum.

V. ARTIFICIAL RED PHOSPHORUS BASED ON COUPLED-RESONATOR PHONONIC METAMATERIAL

While a tight-binding model for red phosphorus has been developed, it still lacks experimental or numerical validation. As discussed in sec. I, several platforms are available to test the predicted results, including optical lattices, electrical circuits, and mechanical resonators. One platform we have successfully implemented is based on coupled-resonator phononic waveguides. These waveguides are composed of resonators connected to one another through finite phononic

crystals (FPnCs). When a normal-mode frequency of a resonator lies within an FPnC bandgap, the resonator modes couple via evanescent waves. This approach offers several advantages:

(1) The waveguides maintain perfect transmission even in the presence of bends.

(2) Its dynamics is governed by a tight-binding model analogous to that used in condensed-matter physics, with the correspondences listed in Table II.

(3) Finite waveguides can emulate artificial molecules, offering therefore a versatile platform for studying a wide range of condensed-matter phenomena.

(4) Both nearest-neighbor and higher-order hoppings can be systematically engineered.

The emulation of the hypothetical red phosphorus molecule is achieved by phononic crystals shown in Fig. 1(b). These artificial structures enable precise control of the hopping parameters by tuning the geometry of the waveguides that provide evanescent coupling between the resonances of the meta-atoms. Figure 6 shows the unit cell designed to emulate red phosphorus. It is composed of octagonal resonators [Fig. 6(a)] coupled via FPnCs. The FPnC unit cell [see Fig. 6(b)] consists of a square plate with two attached cuboids. As shown in Fig. 6, two types of FPnCs are employed: FPnC-A structures containing either two or four unit cells to implement intracell hopping amplitudes, and FPnC-B, with two unit cells to realize intercell hoppings. As discussed below, when a resonance of the octagonal resonators lies within the bandgaps of both FPnCs, the corresponding vibrational modes become spatially confined to the resonators. These localized modes are weakly coupled through evanescent Bloch waves that decay inside the FPnCs, giving rise to an effective tight-binding phononic regime. Two slightly different sizes of the octagonal resonators were used to compensate for small shifts in the resonance frequencies (of a few hertz) caused by the different coupler configurations.

All numerical results presented below were calculated using the finite-element method implemented in COMSOL Multiphysics. In Fig. 7(a), the band structures of the

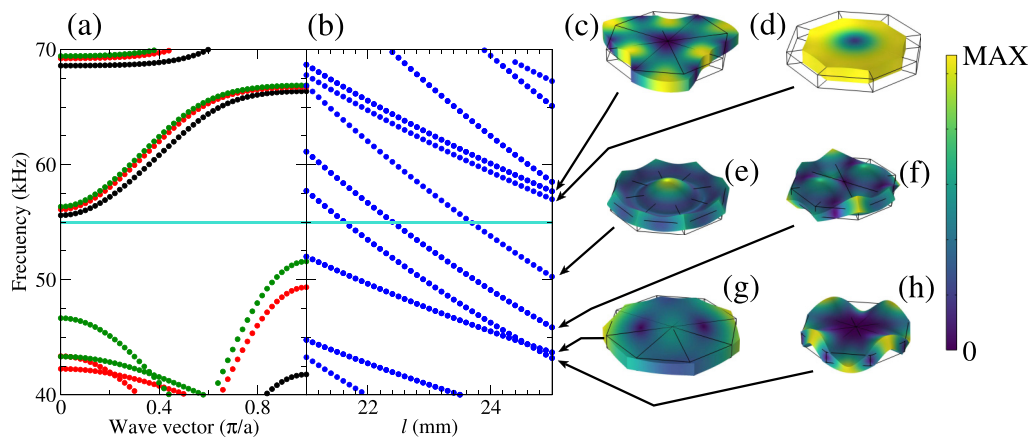


FIG. 7. (a) Band structure of a crystal whose unit cell is given in Fig. 6(b). The green, red, and black dots correspond to different values of c_2 , namely, 3.5, 5.4, and 6 mm, respectively. (b) Frequency spectrum of an octagonal resonator as a function of its side length l . The horizontal (cyan) line indicates 55 kHz. (c) Normal-mode wave amplitudes of the resonator for $l = 25$ mm and frequencies above 40 kHz in order of increasing frequency. (d)–(h) Other normal-mode wave amplitudes of the resonator near the bandgap in panel (a).

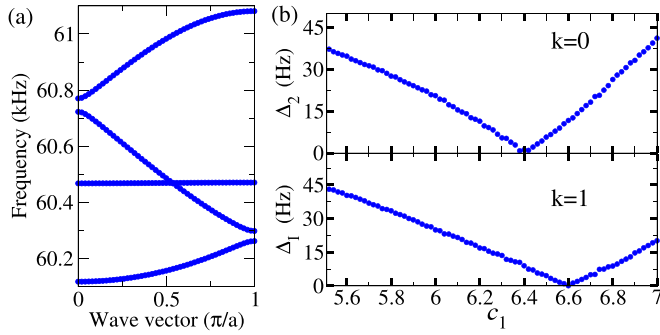


FIG. 8. (a) Band structure of the artificial phononic red phosphorus one-dimensional chain for $c_2 = 8.625$ mm and $c_1 = 7$ mm. (b) Bandgap widths defined as $\Delta_2 = f_3 - f_2$ for $k = 0$ (upper panel) and $\Delta_1 = f_2 - f_1$ for $k = \pi/a$ (lower panel).

finite phononic crystals, used to emulate different hopping amplitudes, are shown; they were calculated using Floquet boundary conditions. Since the intracell hopping t_1 is emulated with FPnCs of two and four unit cells [see Fig. 6] two different values of c_1 were used to yield the same hopping parameter: $c_1 = 5.4$ mm for a coupler with two cells and $c'_1 = 5.41$ mm for the coupler with four cells. As can be seen, for the parameters used, the band structures exhibit a bandgap around 55 kHz. The resonator frequency will be located within this bandgap. In Fig. 7(b), the normal-mode frequencies of the octagonal plate are given as a function of its side length l . A frequency of the octagonal plate falls within the bandgap of the FPnCs for $l \approx 23.75$ mm. The out-of-plane normal displacement field associated with this normal mode amplitude associated with this frequency is given in Fig. 7(e) together with other wave amplitudes.

A band structure obtained for the artificial red phosphorus is given in Fig. 8(a) for fixed values of the parameters c_1 and c_2 , associated with the intracell and intercell hoppings, respectively. The geometrical parameters of the unit cell are given in the caption of Fig. 6. Since the unit cell has four resonators, four bands emerge. One flat band appears and corresponds to modes moving in the Y direction, i.e., without transport in the X direction; this flat band is then uncoupled from the other bands and can be classified as an orthogonal flat band [66]. As it can be seen, in Fig. 8(b), where the bandgaps at $k = 0, 1$ are plotted as a function of c_1 maintaining constant c_2 , the three remaining bands undergo two topological phase transitions.

When $c_1 \gtrsim 7$ mm, all bands are separated by small gaps. The middle and lower bands merge, forming a Dirac cone for $c_1 \approx 6.652$ mm; the bands then separate again, opening a small bandgap for $c_1 \approx 6.5$ mm. For $c_1 \approx 6.4$ mm, the middle and upper bands join, forming a Dirac cone. For values of $c_1 \lesssim 6.38$ mm, all bands are again separated by small gaps. This is in agreement with the theoretical results of the previous section.

A finite coupled-resonator phononic analog of phosphorus of 40 sites, i.e., ten unit cells, is now analyzed; its geometry is shown in Fig. 9. Its frequency spectrum is displayed in Fig. 10. Here, with a value of $c_1/c_2 \approx 0.5$, the topologically protected states appear at ≈ 60.1 and ≈ 60.45 kHz. The topological nature of these states is apparent in this figure since they remain spectrally isolated within the bandgaps when varying c_1/c_2 for a wide range of parameters. The wave amplitudes associated with these states are given in Figs. 9(a) and 9(b), and, as expected, they are localized at the boundaries of the structure. Both wave amplitudes correspond to symmetric states. The flat band is also captured by the phononic model and appears around 60.75 kHz for $c_1/c_2 = 0.85$. This confirms that these states do not interact with the other bands. The finite phononic design could be used to experimentally observe the dual energy-differentiated topological transition described here.

VI. CONCLUSIONS AND FINAL REMARKS

We proposed an artificial version of the hypothetical red phosphorus molecule that exhibits two topological transitions emerging at two different energies. This behavior is atypical, as most one-dimensional topological chains, such as the Su-Schrieffer-Heeger model, host edge states pinned at zero energy in the middle of the bandgap. Through a tight-binding study of both finite and infinite chains, we identified distinct topological phases emerging for the hopping parameter regimes $t_1 \leq \frac{t_2}{2}$ and $t_1 \leq t_2$.

A key result of this work is the identification of a unitary transformation that block diagonalizes the Bloch Hamiltonian, separating the dispersive bands from a trivial flat band. This reduction reveals that the red phosphorus lattice can be effectively interpreted as an SSH-type trimer chain. Within this framework, we derived exact analytical expressions for the energy spectrum and eigenstates, enabling a direct and transparent computation of the Berry phase.

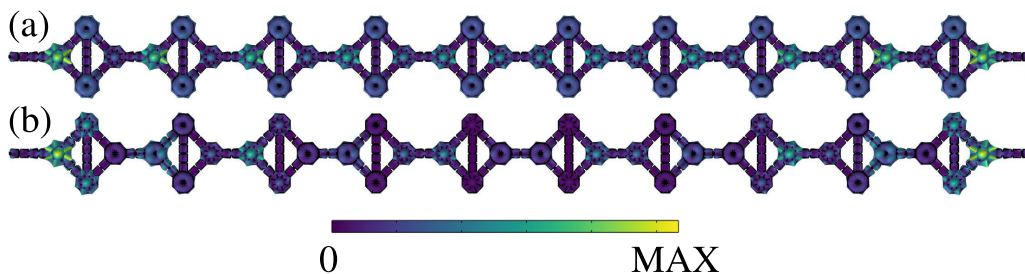


FIG. 9. Topologically protected symmetric states (solitons) of a finite artificial phononic analog of red phosphorus at different frequencies. The wave amplitude shown in panel (a) corresponds to a mode lying within the upper bandgap (label A in Fig. 10), while the one shown in panel (b) corresponds to a topological state lying within the lower bandgap (label B in Fig. 10). In both cases, the amplitudes decay evanescently toward the bulk of the chain.

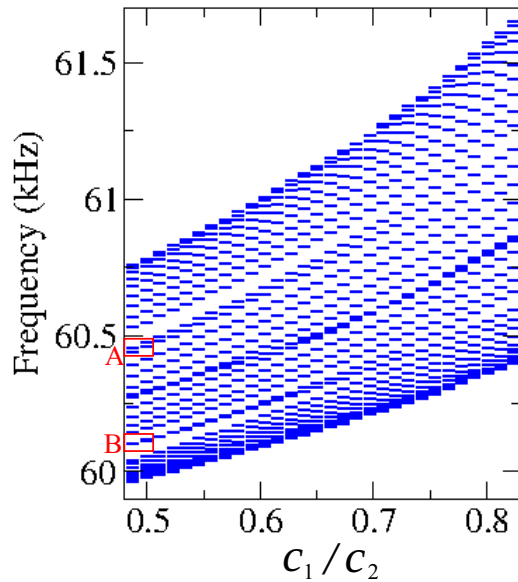


FIG. 10. Frequency spectrum evolution as a function of c_1 associated with the intracell hopping amplitude t_1 . The intercell hopping amplitude t_2 is fixed by setting $c_2 = 8.625$ mm. The occurrence of a double topological phase transition at different frequencies is observed. The frequencies labeled “A” and “B” correspond to the topological states.

The topological characterization of the system uncovers an unusual dependence of the Berry phase on the hopping parameter ratio $\omega = t_1/t_2$, with nonzero values in the topological regimes. This behavior departs from that of conventional one-dimensional topological models and highlights the energy-dependent nature of the topological transitions.

We observed a correlation between the existence of the inert band-edge states [Eq. (13)] and the topological charge of the energy bands that would be worth further analysis.

Finally, we demonstrate that phononic crystals are realistic platforms for emulating red phosphorus chains. Finite-element calculations of vibrational modes and phononic band structures fully corroborate the topological predictions of the tight-binding model. These results are experimentally accessible in phononic crystals and metamaterials such as those of Refs. [8,67,68], and open avenues for the realization of the robust topological states, charge-pumping protocols, engineering quantum states, and testing fundamental topological concepts in controllable toy-model platforms.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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