

From near-integrable to far-from-integrable: A unified picture of thermalization and heat transport

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Whether and how a system approaches equilibrium is a central issue in nonequilibrium statistical physics, underpinning our understanding of both thermalization and heat transport. Bogoliubov's three-stage (initial, kinetic, and hydrodynamic) evolution hypothesis provides a qualitative framework, but quantitative advances have mostly been restricted to near-integrable systems such as dilute gases. In this work, we investigate the relaxation dynamics of a one-dimensional diatomic hard-point gas and present a phase diagram that comprehensively characterizes relaxation behavior across the full parameter space, from near-integrable to far-from-integrable regimes. We first analyze thermalization (local energy relaxation in nonequilibrium states) and identify three universal dynamical regimes: (1) In the near-integrable regime, kinetic processes dominate (initial + kinetic phase), local energy relaxation decays exponentially, and the thermalization time τ scales with perturbation strength δ as $\tau \propto \delta^{-2}$. (2) In the far-from-integrable regime, hydrodynamic effects dominate (initial + hydrodynamic phase), local energy relaxation follows power-law decay, and thermalization time scales linearly with system size N . (3) In the intermediate regime, Bogoliubov phase emerges, marked by the transition from kinetic to hydrodynamic relaxation. The phase diagram further reveals that hydrodynamic behavior can emerge even in small systems when the system is sufficiently far from the integrable regime, challenging the conventional view that such effects are only significant in large systems. In the thermodynamic limit, the system's relaxation behavior depends on the order in which the limits ($N \rightarrow \infty$ or $\delta \rightarrow 0$) are taken. We then turn to heat transport (decay of heat-current fluctuations in equilibrium) and demonstrate that its behavior is fully consistent with the thermalization process, leading to the first unified theoretical description of thermalization and heat transport. Our results establish a comprehensive view of relaxation in classical many-body systems and suggest a route for studying similar dynamics in quantum systems, highlighting universal scaling laws and the interplay of kinetic and hydrodynamic processes across integrability regimes.

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I. INTRODUCTION

Uncovering the microscopic origins of macroscopic relaxation remains a central objective of statistical physics [1]. At the heart of this investigation lies the concept of *thermalization*, the process by which isolated systems evolve toward thermal equilibrium in their observable properties, as well as the dynamical mechanisms underlying the emergence of statistical typicality [2–4].

The first numerical study of thermalization in classical many-body systems dates to the 1950s, with the seminal

Fermi-Pasta-Ulam-Tsingou (FPUT) simulations revealing the now well-known recurrence phenomenon instead of the anticipated approach to equipartition [5–7]. Research over subsequent decades has provided substantial insight into thermalization dynamics [7–33]. In particular, a series of works by Benettin and collaborators elucidated the role of Toda integrability in FPUT-like chains [14–19]. More recent work has identified a universal scaling relation in large systems. In the near-integrable regime, the dynamics is governed by kinetic processes, and the thermalization time τ scales as $\tau \propto \delta^{-2}$, where δ quantifies the strength of nonintegrability [29,34–41]. This inverse-square scaling was further shown to be independent of spatial dimensionality [42]. Yet deviations from this behavior have been reported in small or disordered systems, where thermalization may become threshold-dependent or follow different scaling laws [43–46].

In the quantum domain, thermalization has been explored in parallel [47–59]. Landmark experiments such as the “quantum Newton cradle” [60] have demonstrated the suppression of thermalization in ultracold atomic gases.

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Theoretical frameworks, including the kinetic theory [61,62] and Fermi's golden rule [63–67], have revealed a same scaling law $\tau \propto \delta^{-2}$. However, there are exceptions where certain perturbations impede thermalization [68–71], such as integrability-breaking Kolmogorov-Arnold-Moser-like perturbations [71], and cases where anomalous scaling occurs [72,73], underscoring the need for system-specific studies. The striking similarities in thermalization dynamics between classical and quantum systems highlight their fundamental connections and suggest that advances in one domain may inform the other, offering valuable insights for the development of a unified theoretical framework of thermalization.

The universally observed inverse-square law of thermalization provides a clear understanding of thermalization mechanisms in near-integrable systems, both in classical [29,34–42] and quantum systems [61–66]. However, thermalization dynamics in far-from-integrable regimes remain underexplored. Developing a unified framework that describes thermalization across the full parameter space, from near-integrable to far-from-integrable regimes, remains a central challenge in nonequilibrium physics. The present work makes progress toward this goal.

On the other hand, the fluctuation-dissipation theorem relates the relaxation dynamics of spontaneous equilibrium fluctuations, particularly in the linear-response regime, to transport properties [74]. Progress over the past five decades has shown that thermal transport is governed by both model-specific features, such as momentum conservation and the symmetry of the interparticle interaction potential (IIP), and by spatial dimensionality through hydrodynamic effects [75–77]. The hydrodynamic theory predicts that in one-dimensional (1D) momentum-conserving systems the heat conductivity κ scales with the system size N as $\kappa \propto N^\alpha$ with $0 < \alpha < 1$, implying anomalous heat conduction and a violation of Fourier's law [78–81]. For asymmetric IIPs, the exponent is $\alpha = 1/3$ [78–84], whereas for symmetric IIPs it remains under debate whether $\alpha = 1/2$ or $\alpha = 2/5$ [85–90]. The scaling $\kappa \sim N^{1/3}$ has also been observed experimentally with recent advances in material fabrication and measurement techniques [91]. Further studies have emphasized the importance of kinetic effects in finite systems [92–98]. In models with asymmetric IIPs and weak nonintegrability, prolonged kinetic regimes lead to an apparent size-independent conductivity and an effective restoration of Fourier's law [99–101]. Despite these advances, a microscopic theory of heat transport valid across the full parameter space remains elusive. Establishing quantitative connections between microscopic dynamics and macroscopic transport behavior, and exploring their links to thermalization dynamics, are therefore additional core motivations for this study.

In this work, we take the one-dimensional diatomic hard-point (DHP) gas model [102] as an illustrating example to study thermalization dynamics and heat transport across regimes from near-integrable to strongly nonintegrable. Specifically, we investigate thermalization, through the relaxation of local energy in nonequilibrium states, and heat transport, via the decay of equilibrium heat-current fluctuations. The DHP gas usually serves as an ideal test bed for exploring fundamental questions in nonequilibrium statistical physics [99–114].

Through analytical arguments and high-precision numerical simulations, we obtain a complete phase diagram that characterizes the relaxation dynamics of the DHP model as a function of its nonintegrability strength. This diagram reveals three distinct relaxation patterns for a given system size N : (1) In the near-integrable regime, relaxation is dominated by kinetic processes. For thermalization, local energy decays exponentially with a characteristic timescale $\tau \propto \delta^{-2}$, independent of N . This behavior aligns with previous findings in classical lattice systems [29,34–42] and 1D Bose gases [61–66]. For heat conduction, the heat-current autocorrelation function (HCAF) also decays exponentially with the same scaling, resulting in normal (i.e., N -independent) heat conduction. (2) In the far-from-integrable regime, hydrodynamic processes dominate. Local energy exhibits a power-law decay, with the thermalization time scaling proportionally with N . The HCAF shows a similar power-law decay, leading to anomalous (i.e., N -dependent) heat conduction. (3) In the intermediate regime, the system enters the Bogoliubov's three-stage evolution phase, characterized by a crossover from exponential to power-law decay in both local energy relaxation and the HCAF, as the relaxation timescale increases.

Moreover, the phase diagram clearly indicates that in the far-from-integrable regime, hydrodynamic effects emerge prominently even in small systems, without requiring the large- N limit. In addition, in the thermodynamic limit, the order in which the limits $N \rightarrow \infty$ and $\delta \rightarrow 0$ are taken leads to markedly different relaxation dynamics: Taking $N \rightarrow \infty$ first results in kinetically dominated behavior, whereas taking $\delta \rightarrow 0$ first yields hydrodynamic relaxation. These results establish a unified framework for understanding relaxation dynamics in the DHP model across the entire spectrum of nonintegrability strengths. They resolve long-standing ambiguities in the DHP model, including the parameter-dependent divergence exponents of the heat conductivity κ [112], and establish a quantitative connection between integrability, thermalization, and heat transport. The resulting framework is broadly applicable to low-dimensional systems and provides insight into related phenomena in quantum settings [115].

The paper is organized as follows. Section II introduces the DHP model, its key parameters, and the necessary definitions. Section III presents the theoretical analysis of thermalization dynamics, supported by numerical simulations. Section IV examines heat transport within the same framework, combining theoretical insights and numerical results. Finally, Sec. V provides a summary and discussion of the main findings.

II. THE DHP MODEL

The 1D DHP system consists of N particles, labeled sequentially, with alternating masses given by $m_1 = 1 - \delta/2$ and $m_2 = 1 + \delta/2$, while maintaining unit mass density. Here, $\delta \in [0, 2)$ represents the mass difference, and the system becomes integrable when $\delta = 0$, making δ the sole intrinsic parameter—referred to as the *nonintegrability strength*. For a fixed mean number density of unity, N simultaneously determines the system length. In this setup, the sound speed $c_s = \sqrt{3k_B T}$ is derived, where k_B is the Boltzmann constant (set to unity throughout) and T is the absolute temperature [116]. The unit of time in this model is $T^{-1/2}$ [117], and we set

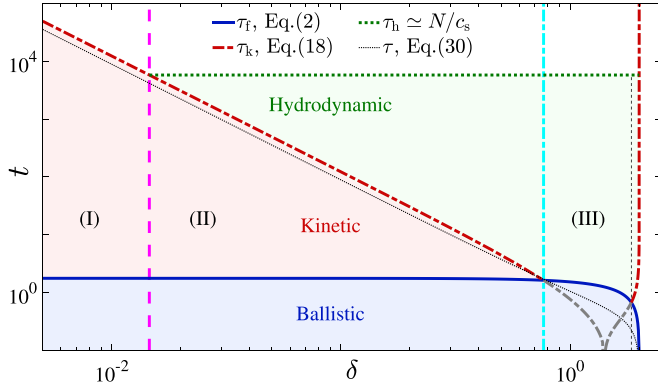


FIG. 1. Diagram of characteristic timescales versus δ in a log-log scale, with $N = 10^4$ for demonstration. The curves corresponding to Eqs. (18) and (30) are shown for reference. An appropriate coefficient is introduced so that the three curves, τ_f , τ_k , and τ , intersect at the point $\delta = \sqrt{2} - \sqrt{2}$, corresponding to $|\lambda| = g$. The left magenta vertical dashed line marks the intersection of τ_f and τ_h , while the right cyan vertical line marks the intersection of τ_f and τ_k . For specific details, see the discussion in the main text.

$T = 1$ throughout the analysis. Periodic boundary conditions are applied in all calculations.

Particles exhibit free motion except for elastic collisions with nearest neighbors. When a collision occurs between the i th and $(i + 1)$ th particles, their velocities are updated according to the following expression:

$$\begin{bmatrix} \tilde{v}_i \\ \tilde{v}_{i+1} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} m_i - m_{i+1} & 2m_{i+1} \\ 2m_i & m_{i+1} - m_i \end{bmatrix} \begin{bmatrix} v_i \\ v_{i+1} \end{bmatrix}, \quad (1)$$

where the tilde denotes the postcollision velocities.

For the system with mean particle spacing unity, adjacent particles collide *if and only if* their relative velocity $v = v_i - v_{i+1} > 0$. The mean free time between collisions, τ_f , is given by

$$\tau_f = \frac{1}{\int_0^\infty v f(v) dv} = \sqrt{\frac{\pi}{T} \left(1 - \frac{\delta^2}{4}\right)}, \quad (2)$$

where $f(v) = \frac{1}{\sqrt{2\pi T/\mu}} e^{-\frac{\mu v^2}{2T}}$ is the distribution function and $\mu = \frac{m_i m_{i+1}}{m_i + m_{i+1}} = \frac{1}{2}(1 - \frac{\delta^2}{4})$ is the reduced mass of the colliding pair.

From Eq. (2), we observe that τ_f approaches $\sqrt{\pi/T}$ (a constant) as $\delta \rightarrow 0$, while $\tau_f \rightarrow 0$ as $\delta \rightarrow 2$ (i.e., when $m_1 \rightarrow 0$ and $m_2 \rightarrow 2$), as shown by the blue curve in Fig. 1. In equilibrium, the equipartition theorem gives $\langle m_1 v_1^2 \rangle = \langle m_2 v_2^2 \rangle = k_B T$, which leads to $\langle v_1^2 \rangle \rightarrow \infty$ and $\langle v_2^2 \rangle \rightarrow T/2$ as $\delta \rightarrow 2$. This implies that light particles become ultrafast and rebound with divergent velocities, causing $\tau_f \rightarrow 0$. Yet physical speeds cannot exceed the speed of light; the velocities near $\delta \rightarrow 2$ would require relativistic corrections. This extreme regime introduces significant complexity in the analysis of relaxation dynamics and transport properties, as relativistic effects become non-negligible. A more detailed exploration of these relativistic aspects remains to be undertaken.

Inspired by the Bogoliubov hypothesis [118–121], we consider the energy relaxation process in gases, which exhibits hierarchical behaviors—ballistic, kinetic, and hydrodynamic, as illustrated in Fig. 1, specifically in region (II). For times $0 < t < \tau_f$, particles primarily exhibit *inertial motion*, leading to *ballistic behavior*. After τ_f , *local two-body collisions* result in *kinetic relaxation*. As the time progresses toward the macroscopic relaxation time $\tau_h \sim N/c_s$, *hydrodynamic effects*, characterized by the *collective motion* of particles, dominate the relaxation behavior. Therefore, τ_f marks the onset of the kinetic regime, while the characteristic time τ_k for the relaxation of a specific physical quantity (such as the fluctuation of local energy or heat current), which defines the end of the kinetic regime, requires further analysis.

To provide a unified framework for describing thermalization and heat transport, we categorize the following three cases:

Case (i): $\tau_k \gg \tau_f$ and $\tau_k \rightarrow \tau_h$ (or even $\tau_k > \tau_h$), corresponding to region (I) to the left of the magenta vertical dashed line in Fig. 1. Here, kinetic effects dominate, while hydrodynamic behavior is suppressed.

Case (ii): $\tau_f \ll \tau_k \ll \tau_h$, corresponding to region (II) between the magenta and cyan vertical lines in Fig. 1. This intermediate regime is referred to as the Bogoliubov phase, where ballistic, kinetic, and hydrodynamic effects coexist.

Case (iii): $\tau_h \gg \tau_k$ and $\tau_k \rightarrow \tau_f$, corresponding to region (III) to the right of the cyan vertical dashed line in Fig. 1. In this regime, the kinetic contribution becomes negligible, and the system exhibits pronounced hydrodynamic behavior.

These three regimes will be referenced repeatedly in subsequent discussions. Physically, the characteristic time τ_k is governed by the system's intrinsic nonintegrability and arises from *local collisions*, with negligible dependence on system size N . In contrast, $\tau_h \sim N/c_s$ scales linearly with N , making it strongly size-dependent. We now proceed to investigate the behavior of τ_k during thermalization and transport dynamics, thus elucidating their intrinsic connection.

III. THERMALIZATION

In this section, we investigate the rate at which the system reaches thermal equilibrium and the underlying mechanisms. From Eq. (1), it can be easily derived that the energy of colliding particles i and $(i + 1)$ evolves as

$$\begin{bmatrix} \tilde{\mathcal{E}}_i \\ \tilde{\mathcal{E}}_{i+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathcal{E}_i \\ \mathcal{E}_{i+1} \end{bmatrix} + \mathbf{G} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad (3)$$

where $\mathcal{E}_i = \frac{1}{2} m_i v_i^2$ is the kinetic energy of particle i and

$$\mathbf{A} = \frac{1}{4} \begin{bmatrix} \delta^2 & 4 - \delta^2 \\ 4 - \delta^2 & \delta^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{\lambda}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (4)$$

is the transition matrix, with eigenvalues

$$1 \quad \text{and} \quad \lambda = \frac{\delta^2}{2} - 1 \in [-1, 1), \quad (5)$$

and

$$G = \frac{1}{2}(m_i - m_{i+1})m_i m_{i+1} v_i v_{i+1} = \pm g \sqrt{\mathcal{E}_i \mathcal{E}_{i+1}} \quad (6)$$

describes the *correlation of velocities* (or the nonlinear coupling of energies) before the collision. Here,

$$g = \delta \sqrt{1 - \frac{\delta^2}{4}} \in [0, 1] \quad (7)$$

characterizes the strength of velocity correlation between colliding particles. The sign “ $\pm$ ” depends on $\text{sgn}(m_i - m_{i+1})\text{sgn}(v_i v_{i+1})$, which indicates the type of collision. For example, a positive mass difference ($m_i - m_{i+1} > 0$) corresponds to a left-heavy/right-light particle pair, while a negative mass difference ($m_i - m_{i+1} < 0$) corresponds to a left-light/right-heavy configuration. Additionally, $v_i v_{i+1} > 0$ (or < 0) indicates a collision between particles moving in the same (or opposite) direction. In equilibrium, we have

$$\langle G \rangle = 0 \quad \text{and} \quad \langle G^2 \rangle = \frac{g^2 T^2}{4}. \quad (8)$$

From Eq. (3), it is evident that the collision redistributes the energy of the particles through a combination of weighted algebraic mean (the first term on the right-hand side, linear) and geometric mean (the second term, nonlinear). To illustrate the role of these two terms, we now consider two special cases. First, in the case of $\delta = 0$, i.e., the integrable case, $\lambda = -1$ and $g = 0$, so $G = 0$. In this case, Eq. (3) simplifies to

$$\tilde{\mathcal{E}}_i = \mathcal{E}_{i+1} \quad \text{and} \quad \tilde{\mathcal{E}}_{i+1} = \mathcal{E}_i, \quad (9)$$

which shows that collisions merely exchange energy between particles without redistributing it. Thus, the integrable system does not spontaneously reach a thermalized state. Second, in the case of $\delta = \sqrt{2}$, we have $\lambda = 0$ and $g = 1$, so $G = \pm \sqrt{\mathcal{E}_i \mathcal{E}_{i+1}}$, and Eq. (3) becomes

$$\tilde{\mathcal{E}}_i = \bar{\mathcal{E}} + G \quad \text{and} \quad \tilde{\mathcal{E}}_{i+1} = \bar{\mathcal{E}} - G, \quad (10)$$

where $\bar{\mathcal{E}} = \frac{1}{2}(\mathcal{E}_i + \mathcal{E}_{i+1})$ is the mean energy. The system can reach an equipartition state in a single collision when only the algebraic mean is considered. However, the geometric mean (G) prevents the system from staying in this state. Clearly, for $\delta \in (0, 2)$, we have $0 < |\lambda| < 1$ and $0 < |G| \leq \sqrt{\mathcal{E}_i \mathcal{E}_{i+1}}$; thus, the weighted algebraic mean drives the system toward local equilibrium, while the geometric mean drives the system away from local equilibrium. Intuitively, $|\lambda|$ and g roughly characterize the strength of these two competing effects, as illustrated in Fig. 2. Let $|\lambda| = g$, we obtain

$$\delta_c = \sqrt{2 \mp \sqrt{2}}. \quad (11)$$

A. Estimation of timescale for thermalization

In the kinetic theory of gases [122–125], the molecular chaos hypothesis, also known as the *Stosszahlansatz*, asserts that the velocities of colliding particles are uncorrelated and independent of position, i.e., $\langle G \rangle = 0$. This implies that the collisions in the system are statistically independent, making each snapshot of collision statistically identical. Consequently, the system’s evolution can be described by the dynamics of a pair of colliding particles. This leads to an

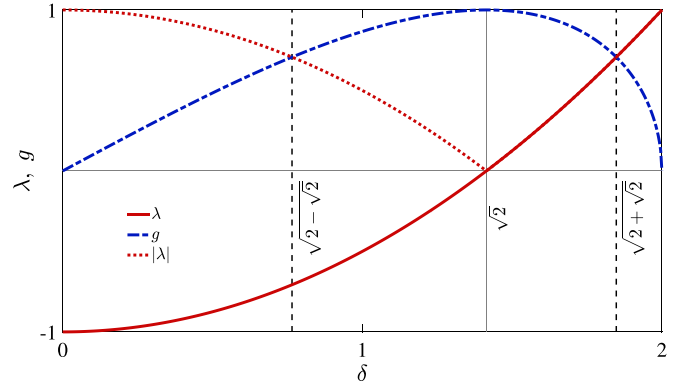


FIG. 2. Dependence of λ and g on δ . The vertical dashed lines correspond to $\delta_c = \sqrt{2} \mp \sqrt{2}$, where $|\lambda| = g$.

evolution equation for the mean energy

$$\begin{bmatrix} \tilde{E}_1 \\ \tilde{E}_2 \end{bmatrix} = A \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \Rightarrow E(1) = A E(0), \quad (12)$$

where $E_1 = \langle \mathcal{E}_i \rangle$ and $E_2 = \langle \mathcal{E}_{i+1} \rangle$ represent the ensemble averages over the initial state. Thus, the energy at time t is given by

$$E(t) = A^{t/\tau_f} E(0), \quad (13)$$

where

$$A^{t/\tau_f} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{\lambda^{t/\tau_f}}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad (14)$$

Since $|\lambda| < 1$ when $\delta \neq 0$, we have

$$\lim_{t \rightarrow \infty} \lambda^{t/\tau_f} \rightarrow 0 \Rightarrow \lim_{t \rightarrow \infty} A^{t/\tau_f} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad (15)$$

which results in

$$E_1(t) = E_2(t) = \frac{1}{2}[E_1(0) + E_2(0)], \quad (16)$$

indicating that the system reaches an equipartition state as $t \rightarrow \infty$. The rate at which λ^{t/τ_f} approaches zero reflects the speed at which the system enters this state. To characterize the relaxation behavior more clearly, we define

$$|\lambda|^{t/\tau_f} = \exp(-t/\tau_k), \quad (17)$$

where τ_k is the characteristic time of relaxation. From Eq. (17), it follows that

$$\tau_k = -\frac{\tau_f}{\ln(|\lambda|)} \propto \begin{cases} \delta^{-2}, & \delta \rightarrow 0, \\ (2 - \delta)^{-1/2}, & \delta \rightarrow 2, \end{cases} \quad (18)$$

providing an estimate for the timescale of thermalization, particularly during the stage dominated by local collisions. Equations (17) and (18) represent the main theoretical results of this work, demonstrating that interparticle collisions drive the gas toward local equilibrium in an exponential manner.

From Eq. (18), we observe that τ_k reaches its minimum at $\delta = \sqrt{2}$, where $\lambda = 0$, meaning the system achieves equipartition after a single collision. However, as shown in Fig. 2, $g = 1$ peaks at $\delta = \sqrt{2}$, suggesting that the velocity correlation is strongest, and the effect of G , which drives the system away from local equilibrium, becomes significant, as seen in Eq. (10).

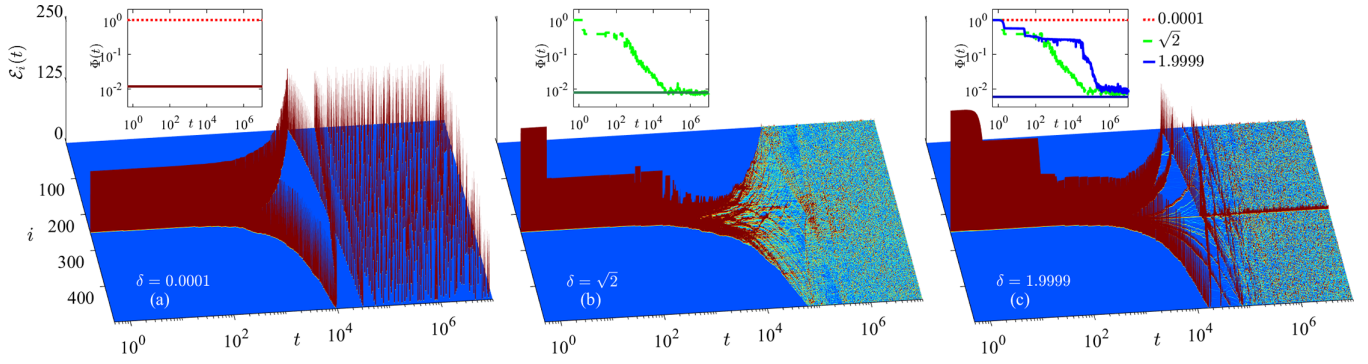


FIG. 3. Evolution of energy $\mathcal{E}_i(t)$ at the fixed $N = 500$. Panels (a)–(c) correspond to $\delta = 10^{-4}$, $\sqrt{2}$, and 1.9999, respectively. Insets: Evolution of the IPR obtained from the data in the main panel. The horizontal lines in insets represent the theoretical values for the thermalized state [see Eq. (21)].

Physically, the analysis presented above applies primarily when $g \ll |\lambda|$. When $g > |\lambda|$, i.e., $\sqrt{2} - \sqrt{2} < \delta < \sqrt{2} + \sqrt{2}$, collective motion driven by spatial correlations becomes significant, leading to pronounced hydrodynamic behavior characterized by a long-time tail. Moreover, it is evident that the diagonal elements of the transition matrix A are smaller than the off-diagonal elements, i.e., $\delta^2 < (4 - \delta^2)$ for $\delta \in [0, \sqrt{2})$. This implies that after a collision, particles transfer more energy to each other than they retain, resulting in a *weak memory effect*. Conversely, for $\delta \in (\sqrt{2}, 2)$, we have $\delta^2 > (4 - \delta^2)$, meaning that most of the energy stays with the colliding particles, resulting in a *stronger memory effect*, which leads to more pronounced hydrodynamic behavior. These predictions will be validated through numerical simulations.

B. Numerical verification of thermalization

In this subsection, we perform molecular dynamics simulations to validate the theoretical analysis, particularly the estimation of the thermalization rate, as provided by Eq. (18). To this end, we utilize the inverse participation ratio (IPR), $\Phi(t)$, defined as

$$\Phi(t) = \frac{\sum_{i=1}^N \mathcal{E}_i(t)^2}{\sum_{i=1}^N \mathcal{E}_i(0)^2}, \quad (19)$$

which serves as a measure [126] of the evolution of local energy fluctuations over time [127]. For $\delta = 0$, the system is integrable, and the IPR remains constant, signifying no thermalization, consistent with ballistic motion. However, for $\delta \neq 0$, the IPR changes, indicating the onset of thermalization. The ensemble average $\langle \Phi(t) \rangle$ is expected to stabilize at a minimum value in the thermalized state. The system is evolved using the event-driven algorithm [110].

In the numerical simulations, the particle spacing is initialized randomly, and then two particles, i and j , are randomly selected, setting the total energy of the system while ensuring momentum conservation, i.e.,

$$m_i v_i^2 + m_j v_j^2 = NT \quad \text{and} \quad m_i v_i + m_j v_j = 0, \quad (20)$$

while the remaining particles are kept at rest. Under these conditions, it is straightforward to derive that

$$\langle \Phi(\mathcal{T}_{\text{eq}}) \rangle = \frac{3(8 + \delta^2)}{N(4 + \delta^2)} \in \left(\frac{9}{2N}, \frac{6}{N} \right) \quad (21)$$

in the thermalized state, where $\mathcal{T}_{\text{eq}}$ represents the thermalization time, i.e., when $\langle \Phi(t) \rangle$ stabilizes to the corresponding value of Eq. (21), signaling that the system has reached thermal equilibrium.

To visualize the system's dynamics, Fig. 3(a) shows the evolution of the particle energy for $\delta = 10^{-4}$ and $N = 500$, where the initial total energy is assigned to the 249th and 250th particles. The energy propagates almost ballistically, and the corresponding $\Phi(t)$ remains nearly constant within the computed time range (red dotted line in the inset). Specifically, $\Phi(t) \approx 1 \gg \langle \Phi(\mathcal{T}_{\text{eq}}) \rangle = 0.012$, indicating that the system is still far from equilibrium, as it remains in the near-integrable regime, making thermalization difficult.

Figure 3(b) presents the results for $\delta = \sqrt{2}$, where energy rapidly transfers to adjacent particles, and $\Phi(t)$ decreases quickly (green dotted line in the inset), signifying the system's approach to equilibrium. Around $t \sim 10^6$, $\Phi(t)$ stabilizes at $\langle \Phi(\mathcal{T}_{\text{eq}}) \rangle = 0.008$ (horizontal line in the inset), suggesting that the system has reached thermal equilibrium.

Figure 3(c) shows the results for $\delta = 1.9999$. After a brief period (approximately $t \sim 10^4$), several well-defined energy wave packets form and propagate outward. After multiple collisions, these wave packets gradually dissipate, but a localized wave packet remains at the center. The system continues to evolve toward equilibrium, with $\Phi(t)$ slowly approaching the theoretical value $\langle \Phi(\mathcal{T}_{\text{eq}}) \rangle = 0.006$, as indicated by the blue line in the inset, with the other two lines shown for comparison.

To gain further insight into the system's thermalization, Fig. 4(a) shows $\langle \Phi(t) \rangle$ for various values of δ at the fixed $N = 10^4$, averaged over 10^4 trajectories with random initialization to minimize fluctuations. The rate at which $\langle \Phi(t) \rangle$ approaches a stable value as δ increases is nonmonotonic, in qualitative agreement with the results in Fig. 3. The curve for $\delta = \sqrt{2}$ decays the fastest, as shown by the black dashed line.

Figure 4(b) reveals a collapse for small δ when t is rescaled by δ^2 . The curve approximates exponential decay (see the

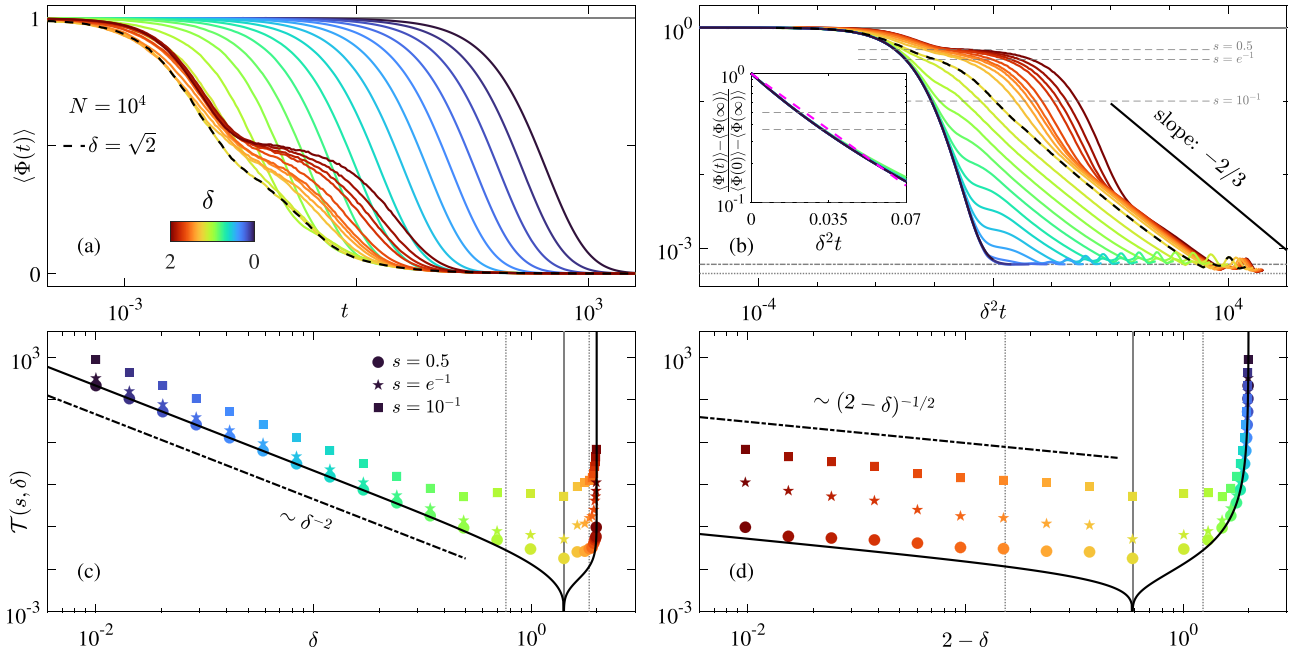


FIG. 4. (a) IPR $\langle \Phi(t) \rangle$ for various δ . (b) Same as panel (a), but with the horizontal axis rescaled by δ^2 in a log-log scale. The two horizontal reference lines at the bottom correspond to the values of $6/N$ and $9/(2N)$, respectively. Inset: The ordinates are rescaled and plotted in a semilogarithmic scale, with the magenta line representing the exponential decay function for reference. (c) $\mathcal{T}(s, \delta)$ as a function of δ for various s ; see the horizontal gray reference lines in panel (b). (d) Same as the panel (c), but the horizontal axis replotted as $2 - \delta$.

inset in semilogarithmic scale) toward its minimum value $6/N$, indicating that $\mathcal{T}_{\text{eq}} \propto \tau_k \propto \delta^{-2} \gg \tau_h$ as $\delta \rightarrow 0$. This suggests that the system remains in region (I) (see Fig. 1), where kinetics dominate.

As δ increases, a crossover occurs in $\langle \Phi(t) \rangle$ from exponential decay (typical of the kinetic process) to power-law decay (characteristic of the hydrodynamic process), signaling a transition from region (I) to region (II). Specifically, for larger δ (e.g., $\delta \rightarrow \sqrt{2}$), $\langle \Phi(t) \rangle \sim t^{-2/3}$ before stabilizing, indicating that the system enters region (III).

For a large δ , particularly as $\delta \rightarrow 2$, $\langle \Phi(t) \rangle$ exhibits complex decay behavior: It rapidly decays to a small plateau, then decays again, and finally follows a power-law decay of $\langle \Phi(t) \rangle \sim t^{-2/3}$, approaching its stable value. This behavior suggests that for very large δ , the thermalization process is dominated by hydrodynamic effects, as $\delta^2 \gg (4 - \delta^2)$ when $\delta \rightarrow 2$, reflecting a stronger memory effect, as discussed in the theoretical analysis subsection.

To detect the scaling behavior of $\mathcal{T}_{\text{eq}}$ with respect to δ , we introduce $\mathcal{T}(s, \delta)$, defined by the condition $\langle \Phi(\mathcal{T}, \delta) \rangle = s$, where s serves as a threshold. Figure 4(c) shows the non-monotonic relationship between $\mathcal{T}$ and δ for different values of s , corresponding to the three horizontal lines in Fig. 4(b), from top to bottom. For a small δ , the scaling behavior is independent of s , indicating that $\mathcal{T}_{\text{eq}} \propto \delta^{-2}$, consistent with the universal thermalization law observed in lattice systems, driven by kinetic effects [29,34–41]. Notably, $\mathcal{T}$ reaches its minimum at $\delta = \sqrt{2}$ for short times, as indicated by the black dashed line in Fig. 4(a). Additionally, Fig. 4(d) demonstrates that $\mathcal{T}_{\text{eq}} \propto (2 - \delta)^{-1/2}$ as $\delta \rightarrow 2$. These numerical results strongly support the prediction given by Eq. (18).

Figure 5(a) shows the evolution of $\langle \Phi(t) \rangle$ for different system sizes N at the fixed $\delta = 0.05$, exploring the influence

of finite-size effects on thermalization behavior. The time is rescaled by $\sqrt{N}$, as it follows from Eq. (20) that changing the number of particles at a fixed temperature or changing the temperature at a fixed particle number is equivalent in the initial stage. Therefore, the change in system size introduces a timescale factor similar to the temperature unit. It is evident that $\langle \Phi(t) \rangle$ eventually converges to its theoretical equilibrium value, as indicated by the horizontal dashed lines corresponding to Eq. (21), signifying thermal equilibrium. For small system sizes, the decay follows an exponential form (see the inset), which is nearly independent of system size and rapidly approaches the theoretical value, meaning the system remains in region (I) of Fig. 1.

When the system size exceeds a critical threshold N_c , a transition occurs from exponential decay to power-law decay, signaling that the system enters region (II). Moreover, multiple oscillations are observed before the system reaches its theoretical value, with peaks occurring around $t \sim N/c_s$, marked by the gray vertical line in Fig. 5(b). These oscillations are characteristic of hydrodynamic effects, arising from the collision of acoustic modes under periodic boundary conditions [84].

Figure 5(c) shows the results for $\delta = \sqrt{2}$. It is observed that $\langle \Phi(t) \rangle$ exhibits qualitatively identical behavior across all system sizes, decaying in a power-law manner (i.e., $\langle \Phi(t) \rangle \sim t^{-2/3}$) and displaying oscillatory behavior before stabilizing. Furthermore, the onset of oscillations depends on N , but becomes independent of δ for sufficiently large N . For example, the curve undergoing power-law decay at $\delta = 0.05$ (indicated by the gray dashed line) closely matches the result obtained at the same size with $\delta = \sqrt{2}$. Additionally, Fig. 5(d) shows nearly complete overlap between all oscillating segments of the curves, suggesting that $\mathcal{T}_{\text{eq}} \sim \tau_h \simeq N/c_s$ (see the gray

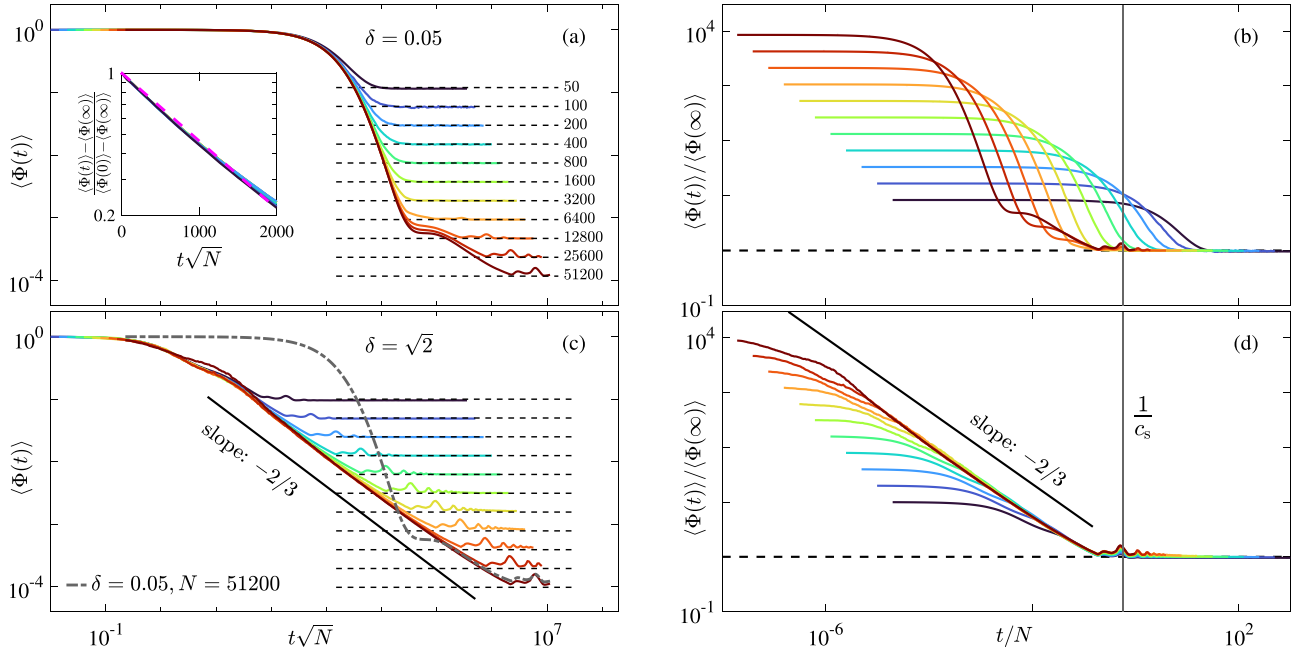


FIG. 5. IPR $\langle \Phi(t) \rangle$ for various N in a log-log scale. (a), (b) For $\delta = 0.05$ and (c), (d) for $\delta = \sqrt{2}$. The horizontal panels differ only in the rescaling of the axes. All panels share the legend. Lines with different slopes are drawn for reference.

vertical line). This indicates significant size dependence due to hydrodynamic effects, specifically the collisions of acoustic modes.

In short, for small δ , when the system size N is less than a critical size N_c (i.e., $N < N_c$), the kinetic process dominates the thermalization behavior, resulting in an exponential decay of the IPR, with $\mathcal{T}_{\text{eq}} \sim \tau_k \sim \delta^{-2}$. When $N > N_c$, a crossover from exponential to power-law decay occurs due to the predominance of hydrodynamic effects, and $\mathcal{T}_{\text{eq}} \sim \tau_h \sim N/c_s$. As shown in Fig. 1, to observe significant hydrodynamic effects, the condition $\tau_h \gg \tau_k$ must be met, which implies that $N_c \propto \delta^{-2}$, corresponding to the Boltzmann-Grad limit [128]. Clearly, there are three characteristic regions in the thermalization process of a given system, as illustrated in Fig. 1. For a very small δ , $\langle \Phi(t) \rangle$ decays exponentially, and the system stays in region (I). As δ increases, $\langle \Phi(t) \rangle$ initially decays exponentially, followed by power-law decay, placing the system in region (II). When δ approaches $\sqrt{2}$, the decay behavior of $\langle \Phi(t) \rangle$ becomes almost entirely power law, indicating that the system is in region (III). It is important to note that, in the thermodynamic limit, the order in which the limits are taken can significantly affect the conclusions regarding the thermalization dynamics. For instance, when $N \rightarrow \infty$ is taken first, followed by $\delta \rightarrow 0$ (observing behavior along the horizontal axis in Fig. 1), the integrable limit ensures that the kinetic process dominates. In contrast, if $\delta \rightarrow 0$ is taken first, followed by $N \rightarrow \infty$ (as along the vertical axis in Fig. 1 to observe the relaxation behavior), the system's thermalization dynamics are governed by hydrodynamics. This distinction is crucial in mesoscopic systems, where the order of limits can determine whether the system exhibits symmetry breaking or a cusp catastrophe [129].

Integrability is an intrinsic property of a system that governs its relaxation dynamics in both nonequilibrium and equilibrium states. According to the Onsager's regression

hypothesis [130], the relaxation patterns of nonequilibrium processes and equilibrium fluctuations are identical within the linear-response regime. In the following, we examine the relaxation behavior of heat-current fluctuations in the DHP system under equilibrium conditions. Based on the linear-response theory [131,132], this reflects the system's linear transport properties, allowing us to explore the similarities and differences in the relaxation behaviors of various physical quantities described by the same underlying dynamics in both equilibrium and nonequilibrium states.

IV. HEAT TRANSPORT

In this section, we examine the heat transport properties of the DHP system by analyzing the heat-current autocorrelation function in equilibrium.

A. Theoretical analysis

The total heat current (HC) is given by $J \equiv \sum_{i=1}^N \frac{m_i v_i^3}{2}$ [102]. According to the molecular chaos hypothesis, the HC is a *quasistationary quantity*, with its instantaneous rate of change depending solely on its current value [133]. Therefore, the HCAF is expressed as

$$C(t) = \langle J(t)J(0) \rangle = \langle J(0)^2 \rangle e^{-t/\tau} = C(0)e^{-t/\tau}, \quad (22)$$

where the Taylor expansion at $t = 0$ gives

$$C(t) \approx C(0) - tC(0)/\tau. \quad (23)$$

Here, $C(0) = \langle J(0)^2 \rangle$. After a detailed calculation, we obtain

$$C(0) = NT^3 \left(\frac{15}{4 - \delta^2} - \frac{9}{4} \right). \quad (24)$$

Next, considering a collision between particles i and $i + 1$, the change in HC due to this collision is

$$\Delta_{1,i} = J_1 - J_0 = \frac{1}{4} \left(1 - \frac{\delta^2}{4} \right) (m_i - m_{i+1}) v_i^3, \quad (25)$$

where J_0 and J_1 represent the HC before and after the collision, and $v_i = v_i - v_{i+1} > 0$ is the collision condition. The HCAF for a single collision is then

$$C(1) = \langle J_1 J_0 \rangle = \langle (J_0 + \Delta_{1,i}) J_0 \rangle = C(0) + \mathcal{S}, \quad (26)$$

where

$$\mathcal{S} \equiv \langle J_0 \Delta_{1,i} \rangle = -\frac{48T^3\delta^2}{4 - \delta^2}. \quad (27)$$

For n times collisions, the HCAF becomes

$$C(n) = \langle J_n J_0 \rangle = C(0) + \xi n \mathcal{S}, \quad (28)$$

where $\xi = \langle J_0 (\sum_{l=1}^n \Delta_{l,i_l}) \rangle / (n\mathcal{S})$ is a dimensionless coefficient and $\Delta_{l,i_l} = J_l - J_{l-1}$ denotes the change in HC during the l th collision between particles i_l and $(i_l + 1)$.

Comparing Eqs. (23) and (28), and using $t = \frac{n\tau_f}{N}$, we obtain

$$\tau = -\frac{\tau_f C(0)}{\xi N \mathcal{S}}. \quad (29)$$

Here, ξ is the sole parameter that needs to be determined. From the thermalization results, we observe that $|\lambda|$ and g can roughly describe the strength of the kinetic and hydrodynamic effects for $\delta \in (0, \sqrt{2})$, where $\delta^2 < (4 - \delta^2)$, as shown in Fig. 2. Assuming that the kinetic effect vanishes—or, more precisely, that the kinetic and hydrodynamic effects become indistinguishable—when $|\lambda| = g$, the system undergoes a transition from region (II) to region (III) (see Fig. 1), so that $\tau \simeq \tau_f$ at $\delta_c = \sqrt{2 - \sqrt{2}}$. This gives $\xi \approx 1/4$, leading to Eq. (29) concentered as

$$\tau = \left(\frac{3}{16} + \frac{1}{2\delta^2} \right) \sqrt{\frac{\pi}{T} \left(1 - \frac{\delta^2}{4} \right)}. \quad (30)$$

Note that as $\delta \rightarrow 0$, $\tau \propto \delta^{-2}$, which is the same behavior as τ_k , as shown in Eq. (18). As discussed in Sec. II, when $\delta \rightarrow 2$, the transport behavior of the system becomes complex, and we will not address it further here. We focus on the range $\delta \in [0, \sqrt{2}]$ going forward.

In addition, the hydrodynamic theory predicts that the HCAF in the large-size limit exhibits a long-time tail, specifically $C(t) \sim t^{-2/3}$ [78–80,109]. Considering both kinetic and hydrodynamic effects, we decompose the complete HCAF as

$$C(t) = C(0)[\eta e^{-t/\tau} + (1 - \eta)e^{-1(t/\tau)^{-2/3}}], \quad (31)$$

where η is a weight factor. Specifically, $\eta = 1$ for $t < \tau$, and $\eta \in [0, 1]$ for $t \geq \tau$.

Following the linear-response theory, the heat conductivity can be derived using the Green-Kubo formula as

$$\kappa = \lim_{t_c \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{T^2 N} \int_0^{t_c} C(t) dt, \quad (32)$$

where $t_c = N/c_s$ is the truncation time [75]. Inserting Eq. (22) into Eq. (32), we obtain the heat conductivity induced by

kinetic effects as

$$\kappa_k = \frac{(24 + 9\delta^2)T}{16 - 4\delta^2} \tau = \frac{3(8 + 3\delta^2)^2}{128\delta^2} \sqrt{\frac{\pi T}{4 - \delta^2}} \quad (33)$$

for $t_c \gg \tau$. Combining Eq. (31) with Eq. (32), we get

$$\kappa(t) = \kappa_k [\eta(1 - e^{-t/\tau}) + (1 - \eta)h(t)], \quad (34)$$

where

$$h(t) = 1 - 4e^{-1} + 3e^{-1(t/\tau)^{1/3}} \quad (35)$$

represents the hydrodynamic contribution. From Eq. (33), we have $\kappa(\tau) = \frac{3T}{2}\tau$ for $\delta = 0$, where the system is integrable and exhibits ballistic transport behavior [134].

To address the transport properties in a unified framework, the characteristic time of the HCAF in the kinetic regime, given by Eq. (30), is shown in Fig. 1 (black dashed line). In region (I), the kinetic effects dominate, leading to exponential decay of the HCAF. In region (II), the three effects coexist, and we will observe a crossover from exponential decay to power-law decay. In region (III), the hydrodynamic effects dominate, resulting in power-law decay of the HCAF.

Furthermore, from Eq. (30), we see that $\tau/\tau_f \propto \delta^{-2}$ as $\delta \rightarrow 0$, implying that the kinetic process dominates in the integrable limit, with $\kappa \simeq \kappa_k \propto \delta^{-2}$, which is size-independent. To observe significant hydrodynamic effects ($\kappa \gg \kappa_k$) for a given δ , the system size N must exceed a critical value $N_c \propto \delta^{-2}$, similar to the thermalization process. This requirement indicates that hydrodynamic effects become less noticeable in the integrable limit. Whether hydrodynamic effects can be observed also depends on the order of taking the limits: $\delta \rightarrow 0$ first or $N \rightarrow \infty$ first, as discussed in Sec. III. For instance, if $\delta \rightarrow 0$ first, hydrodynamic transport behavior may emerge as $N \rightarrow \infty$, provided that $N > N_c \propto \delta^{-2}$, meaning that N must grow faster than $\delta \rightarrow 0$. All of the above analysis will be verified numerically below.

B. Numerical results for heat transport

Figure 6(a) shows the normalized HCAF for various values of δ at the fixed small $N = 50$. For small δ , the kinetic predictions and numerical results agree well. However, as δ increases, deviations emerge. The kinetic region rapidly shrinks, and hydrodynamic effects begin to dominate, as evidenced by the oscillations arising from the collision of acoustic modes at the boundary [84]. These results suggest that even small systems exhibit strong hydrodynamic effects when the system is in the far-from-integrable limit, challenging the common belief that hydrodynamic effects are primarily observed in large systems.

Figure 6(b) presents the normalized HCAF for various system sizes at the fixed $\delta = 0.05$. The HCAF decays exponentially, and all results collapse nearly completely, matching the kinetic predictions (magenta dashed lines). In Fig. 6(c), for $\delta = 0.2$, the HCAF of small systems still follows an exponential decay and aligns with the kinetic predictions (see the magenta dashed line). However, as the system size increases, a transition from exponential to power-law decay emerges. For larger systems, the power-law decay behavior matches the hydrodynamic prediction, $C(t) \sim t^{-2/3}$ [78–80,109].

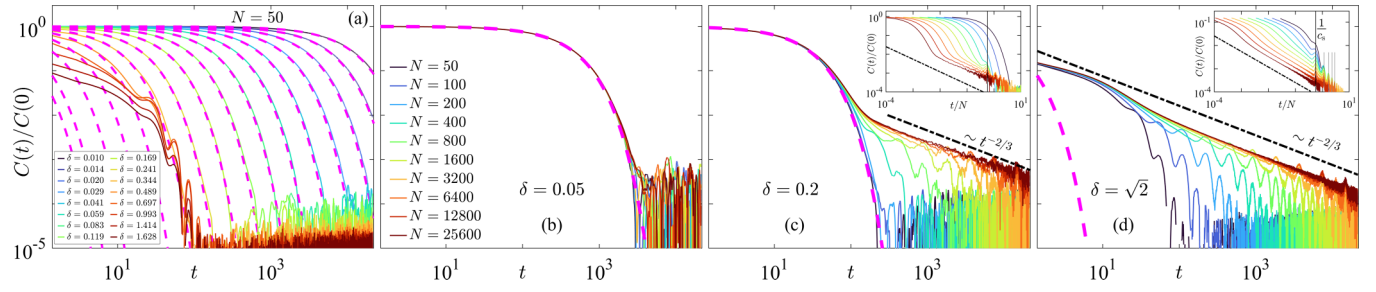


FIG. 6. (a) Normalized HCAF for various δ at $N = 50$. The magenta dashed lines represent predictions from Eq. (22), with τ given by Eq. (30). (b)–(d) Normalized HCAF for different N at $\delta = 0.05$, 0.2 , and $\sqrt{2}$, respectively. These panels share the legend. The magenta dashed lines are theoretical predictions. Inset: Same as the main panel, but with the horizontal axis rescaled. The black vertical solid lines in the insets correspond to $1/c_s$. All black lines with a slope of $-2/3$ are for reference.

Figure 6(d) shows the results for $\delta = \sqrt{2}$, where nearly all curves exhibit power-law decay and hydrodynamic behavior, characterized by oscillations due to the collision of acoustic modes under periodic boundary conditions [84]. The peaks of the oscillations occur at integer multiples of N/c_s , as indicated by the vertical lines in the inset. As N increases, the exponent of the power-law decay approaches the theoretical value of $-2/3$ in hydrodynamics. These phenomena are fully consistent with the framework outlined in Fig. 1.

Figure 7(a) shows the HCAF for different values of δ at a large system size, $N = 10^4$. For all values of δ , over short timescales ($t < \tau_f$), the HCAF remains constant, corresponding to a linear increase in heat conductivity, as shown in Fig. 7(b), indicating ballistic transport. For $\delta = 0.05$, kinetic processes dominate, and the HCAF decays exponentially, as illustrated by the blue solid line, corresponding to $\kappa \simeq \kappa_k$, which indicates normal heat conduction. For $\delta = 0.2$, the exponential decay phase narrows while the power-law decay phase expands, as seen with the green dashed-dotted line. At $\delta = \sqrt{2} - \sqrt{2} \approx 0.77$, $\tau/\tau_f \approx 1$, indicating that the exponential decay phase (kinetic regime) nearly disappears, leading to a direct transition from ballistic to hydrodynamic behavior (power-law decay), as shown by the red dotted line in Fig. 7(a). This transition is marked by a crossover from $\kappa \sim t$ to $\kappa \sim t^{1/3}$ in Fig. 7(b). Additionally, oscillations due to acoustic mode collisions are observed, with a characteristic

time at $t_c \approx 5773.5$ in Fig. 7(a). As δ increases further, $\tau < \tau_f$, indicating that light particles become localized by heavy particles, which leads to a nonmonotonic decay of the HCAF for $\delta = \sqrt{2}$ (refer to the orange dotted line), with hydrodynamic processes continuing to dominate heat transport.

It is clear that all numerical results are in excellent agreement with theoretical predictions, confirming that the transport properties are entirely determined by the location of system parameters within the phase diagram (see again Fig. 1). According to the phase diagram, the dependence of the thermal conductivity divergence exponent on the parameter δ in the DHP model [112] can be naturally understood: Different values of δ correspond to varying contributions from hydrodynamic effects, which in turn determine the system-size range required to observe the asymptotic scaling law $\kappa \sim N^{1/3}$. Conversely, for a given system-size range, the κ - N relationships corresponding to different δ values exhibit subtle variations, manifesting as distinct effective divergence exponents.

V. SUMMARY AND DISCUSSION

In summary, the relaxation dynamics of the DHP model has been systematically analyzed with emphasis on both thermalization and heat transport. By combining theoretical analysis with high-precision numerical simulations, a comprehensive phase diagram is established that unifies relaxation behavior from the near-integrable to the far-from-integrable regime.

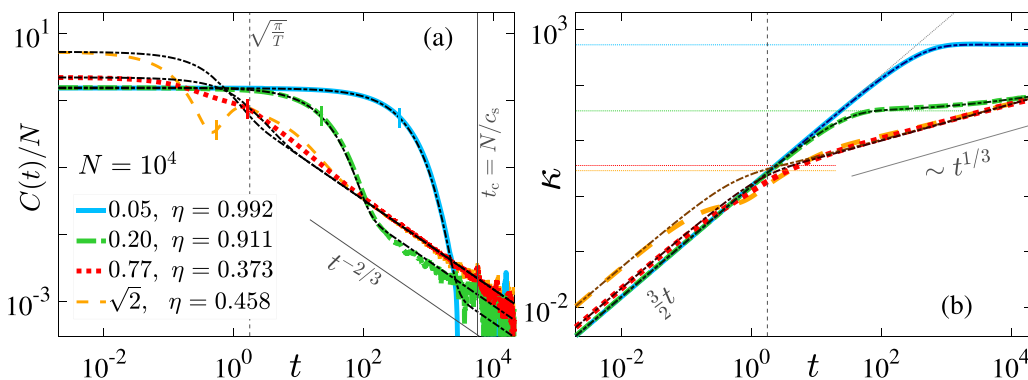


FIG. 7. (a) Normalized HCAF for various δ . The horizontal coordinate of the short vertical line on the curve indicates the value of τ from Eq. (30). The vertical gray dashed line represents the value of τ_f at $\delta = 0$ [see Eq. (2)]. The black dashed-dot lines represent the fitting curves using expression (31). (b) Heat conductivity κ obtained from Eq. (32). The horizontal lines are predicted from Eq. (33), and the dark dots represent predictions from Eq. (34). Both panels share the same legend, and all gray lines serve as reference.

A common relaxation structure is identified for thermalization and heat transport, consisting of three distinct dynamical regimes. In the near-integrable regime, relaxation is governed by kinetic processes, with both the IPR and the HCAF decaying exponentially. The characteristic relaxation time follows $\tau \propto \delta^{-2}$, and the heat conductivity is size-independent, $\kappa \approx \kappa_k$ [Eq. (33)]. In the far-from-integrable regime, hydrodynamic processes dominate, leading to power-law decay of the IPR and HCAF, $\sim t^{-2/3}$. The relaxation time scales linearly with system size, $\tau \propto N$, and the heat conductivity exhibits anomalous scaling, $\kappa \propto N^{1/3}$. The intermediate regime corresponds to a Bogoliubov phase, characterized by a crossover from kinetic to hydrodynamic behavior and a clear separation of ballistic, kinetic, and hydrodynamic timescales. Equation (34) further implies a power-law temperature dependence of the heat conductivity, $\kappa(T) \sim T^{1/2}$. Additionally, results for the relaxation dynamics of the DPH model suggest that thermalization is independent of chaos, at least in this model, since the system exhibits no local instability [109]. In particular, the Lyapunov exponent is identically zero and does not depend on δ [111].

Moreover, the phase diagram also clarifies how relaxation depends on δ and N . Hydrodynamic behavior is shown to emerge even in relatively small systems when the dynamics is sufficiently far from integrable regime, demonstrating that the large- N limit is not a prerequisite for hydrodynamic relaxation. In the thermodynamic limit, the relaxation behavior depends on the order in which the limits are taken: Taking $N \rightarrow \infty$ prior to $\delta \rightarrow 0$ yields kinetically dominated relaxation, whereas reversing the order leads to hydrodynamic dominance. This noncommutativity highlights the essential role of integrability breaking in determining relaxation pathways.

Although the DHP gas is a toy model, it captures essential physical mechanisms. The predicted scaling of the heat conductivity with system size and temperature is qualitatively consistent with experimental observations in ultrathin van der Waals NbSe₃ nanowires, where $\kappa \sim N^{1/3}$ and $\kappa(T) \sim T$ reported [91]. A central advantage of the DHP model lies in the complete decoupling of integrability breaking and temperature, which allows their effects on relaxation and transport to be isolated in a controlled manner. In contrast, in realistic nonlinear lattice systems such as NbSe₃, phonons are the dominant heat carriers and temperature plays a dual role. It not only governs the statistical distribution of phonons but also modifies the relative weight of anharmonic and harmonic interactions in the Hamiltonian, thereby affecting the effective interaction strength among carriers. As a result, temperature influences heat transport through more intricate mechanisms than in the simplified setting considered here. Continued advances in experimental techniques are expected to enable systematic tests of the unified relaxation phase diagram identified in this work.

The proposed phase diagram also provides a qualitative framework for understanding numerical results in 1D lattice models [92–98]. In such systems, temperature or energy density plays a role analogous to the perturbation strength δ . Consistent patterns emerge in comparisons with anharmonic chains. Lattices with asymmetric IIPs, including FPUT- α - β chains, Lennard-Jones or Morse chains [97], and related models [92], reproduce the DHP scenario and exhibit a clear

separation of dynamical timescales. Weak nonintegrability, corresponding to low temperatures, produces an extended kinetic regime and normal heat conduction [92–95], whereas strong nonintegrability leads to a reduced kinetic regime and size-dependent hydrodynamic transport [96,98].

The analysis further sheds light on anharmonic chains with symmetric IIPs. In systems such as the FPUT- β chain, trivial phonon number-conserving resonances dominate the kinetic equation [135] and give rise to strong phonon-hydrodynamic effects [136]. These resonances primarily induce frequency shifts and do not efficiently promote energy mixing [26,27,135,136]. Because this mechanism is controlled mainly by system size rather than nonintegrability strength, symmetric-potential models display pronounced size dependence in thermalization and heat transport over a broad parameter range [14,34,90]. This motivates extending the phase diagram of symmetric models to include an additional phonon-hydrodynamic regime [136]. By contrast, lattice models with broken translational symmetry, such as the Frenkel-Kontorova [137,138], ϕ^4 (or Klein-Gordon) [28,139,140], and rotor chains [141–145], lack momentum conservation and retain only energy and mass as conserved quantities [144,145]. These systems exhibit normal heat conduction [146], suggesting the absence of a hydrodynamic regime in their relaxation phase diagram, an expectation that remains to be examined systematically.

Overall, this work establishes a unified framework linking integrability breaking, thermalization, and heat transport in low-dimensional systems. Since the thermal diffusivity satisfies $D = \kappa/c_v$, where c_v is the specific heat [147], diffusive transport is expected to follow the same scenario. The results also provide insight into relaxation dynamics in quantum systems [115], open directions for exploring the interplay between system size, interaction symmetry, and dynamical regimes in classical and quantum transport, and highlight the potential of size-independent kinetic transport for the design of high-efficiency thermoelectric materials [148–150].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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