

Superstatistics in the context of relativity

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The statistical properties of space and astrophysical environments often exhibit heavy-tailed distributions, reflecting deviations from thermal equilibrium. These distributions can often be represented as a superposition of statistics, i.e., superstatistics. However, the heavy-tailed nature of these distributions can lead to significant nonphysical contributions from unrealistic superluminal particles, thereby biasing thermodynamic properties and the overall dynamics. In this work, we first quantify these unphysical effects for key thermodynamic quantities, like pressure and entropy, within the three universality classes of superstatistics, and we discuss a regularized formulation that resolves the issue without requiring a fully relativistic treatment. We then develop a relativistic extension of superstatistics by generalizing the Maxwell-Jüttner distribution to nonequilibrium settings. We derive the thermodynamic properties of the resulting distributions and analyze their asymptotic behavior in the weakly relativistic and ultrarelativistic limits. Finally, we apply our model to observational data of solar energetic protons and show that it provides an excellent fit, demonstrating its relevance for describing the statistical properties of high-energy astrophysical systems.

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I. INTRODUCTION

Space and astrophysical environments are complex systems that provide an ideal natural laboratory for testing the principles and methods of statistical mechanics. The observed statistical properties of these systems encode valuable information about their internal dynamics. Despite their complexity, a common feature emerges: the observed distributions typically exhibit heavy tails, decaying more slowly than the equilibrium Maxwell-Boltzmann distribution. This behavior has been recognized since the late 1960s in space plasmas, where the kappa distribution [1–5]—along with a zoo of other empirical models [6–11]—has become a standard tool for fitting the data. Comparable heavy-tailed distributions have also been observed in gravitationally bound systems, such as stellar clusters and galaxy clusters [12–17].

These cases reflect a breakdown of standard Boltzmann statistics and are commonly interpreted as signatures of nonequilibrium effects. But what is the most effective description we can construct for such a nonequilibrium system, given only a few additional parameters beyond those required to describe the same system in equilibrium? Many different approaches have been put forward, each introducing its own set of concepts. The theoretical landscape ranges from proposals that generalize the Boltzmann-Gibbs entropy [18,19], or modify the constraints of the maximum entropy principle [20], to more physically grounded perspectives based on kinetic

theory, such as the Boltzmann equation [21] or the Fokker-Planck equation [22,23].

Arguably, the most effective strategy for describing such nonequilibrium systems is to decompose their dynamics into two (or more) distinct scales. One can then view the system as a collection of smaller subsystems (or cells) that remain infinitesimally close to equilibrium. The description can then be naturally separated into microscopic and macroscopic levels: at the microscopic level, each subsystem rapidly relaxes toward thermodynamic equilibrium and follows a standard Boltzmann distribution characterized by a well-defined local inverse temperature β . At the macroscopic scale, however, the temperature has its own dynamics, fluctuating from cell to cell according to a distribution $f(\beta)$.

In the statistical mechanics literature, this methodology is known as superstatistics [24], as it involves a superposition of different statistics (a procedure known in probability theory as *compounding* [25]). It has proven highly effective in accounting for the non-Maxwellian distributions often observed in plasmas [26–30] and gravity-dominated systems [31,32]. Its applicability extends well beyond these domains, with notable success in modeling a broad range of nonequilibrium physical systems and processes, including turbulence [33–35], ion traps [36,37], solar flares [38], spin systems [39], collisional experiments [40], and others [41–44]. Beyond physics, superstatistics has also been successfully used to describe complex systems such as traffic flow [45], financial markets [46], power grid fluctuations [47,48], DNA architecture [49], air pollution [50], and cognitive processes [51]. These empirical findings collectively point to the emergence of three distinct universality classes of superstatistics that occur frequently in the most relevant situations [52,53].

Of course, the empirical success of these distributions in fitting real-world data does not come without its share of challenges—or at least subtleties. One important issue is

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that the enhanced high-velocity tails of these non-Maxwellian distributions can lead to significant, unphysical contributions from superluminal particles, thereby biasing the resulting thermodynamic observables and the overall dynamics of the system. Thus, clarifying when such unphysical artifacts become important and extending the formalism into the relativistic domain are necessary steps.

The generalization of thermodynamics and statistical mechanics to relativistic settings has a long history, beginning with the early works of Planck [54] and Einstein [55] on the transformation of temperature. A major milestone was Jüttner's derivation of the relativistic Maxwell-Boltzmann distribution in 1911 [56], which laid the foundation for relativistic kinetic theory. In subsequent decades, substantial efforts clarified the transformation laws of thermodynamic quantities and refined the concept of equilibrium in moving frames [57,58]. Further theoretical advances deepened the understanding of relativistic ensembles and dissipative processes [59,60], forming a basis for modern formulations and motivating recent extensions within generalized statistical mechanics [19].

In the case of the Maxwellian distribution, such unphysical contributions can be safely disregarded as they are exponentially suppressed, becoming significant only at extreme temperatures (around $T \sim 10^9$ K for electrons), where relativistic corrections are required. For non-Maxwellian distributions, however, the situation is more subtle: even at moderate mean temperatures, the heavy-tailed nature of these distributions can lead to the unphysical artifact of a non-negligible fraction of particles exceeding the speed of light. This introduces a significant, unphysical contribution to thermodynamic quantities such as pressure and entropy, which are essential for assessing the state and stability of the system. This issue has been recently discussed in the context of kappa distributions [61], but it remains largely overlooked in the fitting of observational data. The aim of this work is to examine the emergence of these unphysical effects, quantify their impact for the three universality classes of superstatistics, and discuss a relativistic extension that remains valid in these cases.

The structure of the paper is as follows. To set the stage, Sec. II provides an overview of superstatistics, with a focus on the three universality classes that have received strong empirical support. In Sec. III, we quantify the unphysical contribution of superluminal particles to thermodynamic quantities such as pressure and entropy, comparing the magnitude of this effect across the three classes. We also present a practical regularization procedure to address this issue without resorting to a fully relativistic treatment. In Sec. IV, we introduce a relativistic generalization of superstatistics, extending the Maxwell-Jüttner distribution, and explore its properties in both weakly relativistic and ultrarelativistic regimes. In Sec. V, we apply our model to observational data of energetic solar protons, demonstrating excellent agreement, and elaborate on the potential physical mechanisms underlying the emergence of such distributions in solar energetic particles. Finally, in Sec. VI, we conclude by summarizing our findings and suggesting possible directions for future research. Two brief Appendixes supplement the paper: the first provides technical details on the generalized hypergeometric functions

used in the derivations, while the second presents lengthy expressions omitted from the main text.

II. SUPERSTATISTICS

Consider a system that is globally out of equilibrium but exhibits local equilibrium. We may virtually divide this system into small cells that are infinitesimally close to equilibrium. These cells represent regions from which thermalization gradually spreads across the entire system. Since the relaxation time of an individual cell, τ_0 , is typically much shorter than the global relaxation time τ , there exists an intermediate timescale, $\tau_0 \ll t \ll \tau$, during which the system remains locally equilibrated while lacking global equilibrium—a regime that can persist over a physically relevant timescale.

In practice, such a situation may occur in a plasma that has not yet equilibrated, as collisions—responsible for driving the system toward equilibrium—typically take place over a much longer timescale than the evolution of the mean electric and magnetic fields generated by the plasma. A similar scenario applies to self-gravitating systems, where the relaxation time τ grows roughly linearly with the number of particles and diverges in the thermodynamic limit [62].

In such a situation, the whole machinery of equilibrium statistical mechanics remains valid at the local level. In particular, the probability of finding the system in a state with energy E , given a local inverse temperature $\beta \equiv 1/k_B T$, is described by the Boltzmann factor, i.e., $\mathcal{P}(E|\beta) \propto e^{-\beta E}$. Globally, however, the inverse temperature β varies from one cell to another. When these fluctuations occur on a timescale much longer than the typical local relaxation time τ_0 , they can be effectively characterized by a distribution $f(\beta)$. As a result, the global statistical properties of the system are governed by a superposition of local statistics. That is [63],

$$\mathcal{P}(E) = \int f(\beta) \mathcal{P}(E|\beta) d\beta. \quad (1)$$

In the formulation given by Eq. (1), both $f(\beta)$ and the conditional distribution $\mathcal{P}(E|\beta)$ are normalized. This formulation corresponds to type-B superstatistics. A slightly different formulation, known as type-A superstatistics, leaves $\mathcal{P}(E|\beta)$ unnormalized and instead normalizes the final marginal distribution (see, e.g., Ref. [24]). The two approaches are equivalent up to a rescaling of $f(\beta)$.

Although the superstatistical approach in principle allows for an arbitrary choice of the probability density function (PDF) $f(\beta)$, empirical studies consistently reveal a limited set of distributions that arise in practice. In particular, three main universality classes tend to appear across a wide range of physical contexts and are supported by strong observational evidence. These classes can be traced back to generic phenomenological mechanisms that plausibly underlie the stochastic nature of β [52], and they can also be derived from a maximum entropy perspective under suitable constraints [64]. The three classes are as follows:

(a) χ^2 superstatistics: In this case, the inverse temperature β is modeled by a χ^2 distribution with degree n ,

$$f_1(\beta) = \frac{1}{\Gamma(\frac{n}{2})} \left(\frac{n}{2\beta_0} \right)^{n/2} \beta^{n/2-1} \exp\left(-\frac{n\beta}{2\beta_0}\right), \quad (2)$$

where $\beta_0 = \langle \beta \rangle$ denotes the average value of β . Assuming local Maxwellian distribution (in d dimensions), the overall velocity distribution, obtained via Eq. (1), becomes

$$\begin{aligned} \mathcal{P}(v) &= \int_0^\infty d\beta f(\beta) \left(\frac{m}{2\pi\beta} \right)^{d/2} \exp\left(-\frac{\beta m v^2}{2}\right) \\ &= \frac{\beta_0 m}{\pi^{d/2} \Gamma(\frac{n}{2})} \frac{\Gamma(\frac{n+d}{2})}{(1 + \frac{\beta_0 m v^2}{n})^{(n+d)/2}}. \end{aligned} \quad (3)$$

The large-velocity asymptotics of Eq. (3) exhibit power-law decay, making this distribution especially well suited for modeling systems with suprathermal tails, such as those commonly observed in plasmas [5,65], gravitational systems [12,16], and various other physical contexts [66–70]. It is formally equivalent to the q -Gaussian form characteristic of nonextensive statistical mechanics [18]. This correspondence becomes explicit by identifying the parameters as follows [29]:

$$\tilde{q} \equiv 1 + \frac{2}{n+d} \quad \text{and} \quad \tilde{\beta} \equiv \left(\frac{n+d}{n} \right) \beta_0, \quad (4)$$

which allows Eq. (3) to be rewritten as a q -Gaussian distribution. That is [71],

$$\mathcal{P}(v) \propto \left(1 + (\tilde{q} - 1) \frac{\tilde{\beta} m v^2}{2} \right)^{1/(1-\tilde{q})}. \quad (5)$$

Note that the parameter identification in Eq. (4) is necessary to obtain an exact mapping to the q -Gaussian distribution because the analysis is carried out within type-B superstatistics. If one omits the normalization constant of the Gaussian distribution, as done in type-A superstatistics, the q -Gaussian form would follow without requiring this reparameterization (see, e.g., Refs. [24,73]). Also note that the exponent of the distribution (3) depends on the dimensionality of the system—a feature that also manifests in the context of kappa distributions [74].

Equation (3) also matches the form of the kappa distribution commonly used in plasma physics, particularly in the version proposed by Leubner [75] and subsequently adopted by several authors [5,22], although it slightly differs from the original formulation introduced by Vasyliunas in the late 1960s [2]. In statistical literature, distributions of the form given in Eq. (3) are known as Student's t distributions. They can be viewed as a specific instance of the broader Burr Type III family [76].

(b) Inverse- χ^2 superstatistics: In this case, rather than the inverse temperature β , it is the temperature itself, β^{-1} , that is modeled by a χ^2 distribution. This leads to an inverse- χ^2 distribution for β . That is,

$$f_2(\beta) = \frac{\beta_0}{\Gamma(\frac{n}{2})} \left(\frac{n\beta_0}{2} \right)^{n/2} \beta^{-(n/2+2)} \exp\left(-\frac{n\beta_0}{2\beta}\right). \quad (6)$$

Using Eq. (1), one obtains the following expression for the velocity distribution:

$$\begin{aligned} \mathcal{P}(v) &= \frac{2\beta_0}{\Gamma(\frac{n}{2})} \left(\frac{m}{2\pi} \right)^{d/2} \left(\frac{n}{2\beta_0} \right)^{n/2} \left(\frac{m v^2}{\beta_0 n} \right)^{(n-d)/4} \\ &\times \mathcal{K}_{\frac{2-d+n}{2}}(\sqrt{nm\beta_0}|v|), \end{aligned} \quad (7)$$

with $\mathcal{K}_\alpha(x)$ representing the modified Bessel function of the second kind. Notably, for large $|v|$, this distribution exhibits exponential decay in the tails. Such exponential behavior has been documented in a range of nonequilibrium systems, including vortex glasses and liquids [77], fusion plasmas [26], and harmonic oscillators coupled to solvent environments [78]. Analogous distributions have even been observed outside physics, for instance, in cancer-specific mortality rate statistics [79].

(c) Log-normal superstatistics: In this case, the inverse temperature β is assumed to be log-normally distributed, described by

$$f_3(\beta) = \frac{1}{\sqrt{2\pi}s\beta} \exp\left(-\frac{(\ln \beta - \mu)^2}{2s^2}\right), \quad (8)$$

where the mean value of β is given by $\beta_0 = \mu e^{s^2/2}$. Unlike the previous cases, the resulting velocity distribution $\mathcal{P}(v)$ lacks a closed-form solution but can be accurately evaluated through numerical methods. Log-normal superstatistics is supported by empirical data from systems where cascading and multiplicative processes dominate, such as turbulence [34,80]. It also successfully models empirical distributions observed in space plasma environments [29] and gravitational systems [32], and extends to diverse phenomena beyond the realm of physics [47,81,82].

Figure 1 illustrates representative (one-dimensional) velocity distributions generated by the three fundamental universality classes of superstatistics: the χ^2 class [Eq. (3)], the inverse- χ^2 class [Eq. (7)], and the log-normal class (evaluated numerically). In order to compare these classes on a common footing, we introduce the parameter $q := \langle \beta^2 \rangle / \beta_0^2$ to characterize the distributions. For each universality class, $q \geq 1$ can be explicitly related to the parameters of $f(\beta)$ as follows:

$$\begin{aligned} q &= 1 + \frac{2}{n} \quad (\chi^2 \text{ class}), \\ q &= \frac{n}{n-2} \quad (\text{inverse-}\chi^2 \text{ class}), \\ q &= e^{s^2} \quad (\text{log-normal class}). \end{aligned} \quad (9)$$

Adopting this parameterization enables a direct comparison of fluctuation levels across different classes. In fact, regardless of the universality class, larger values of q correspond to stronger temperature fluctuations. Conversely, as q approaches unity ($q \rightarrow 1$), the underlying distribution $f(\beta)$ collapses into a Dirac delta $\delta(\beta - \beta_0)$, causing $\mathcal{P}(v)$ to converge to the standard Maxwellian form.

It is worth noting that, even when the full superstatistical distribution is not available in closed form—as is the case with log-normal superstatistics—the velocity moments $\langle v^\ell \rangle$ can still be derived analytically. In fact, they can be expressed as

$$\langle v^\ell \rangle_i \equiv \int v^\ell \mathcal{P}(v) d^d v = \langle \langle v^\ell \rangle_{\text{MB}} \rangle_{f_i(\beta)}, \quad (10)$$

where $\langle \bullet \rangle_{\text{MB}}$ denotes the average over the Maxwellian distribution and $\langle \bullet \rangle_{f_i(\beta)}$ represents the average with respect to the temperature distribution $f_i(\beta)$, where $i = 1, 2, 3$ correspond to χ^2 , inverse- χ^2 , and log-normal superstatistics, respectively.

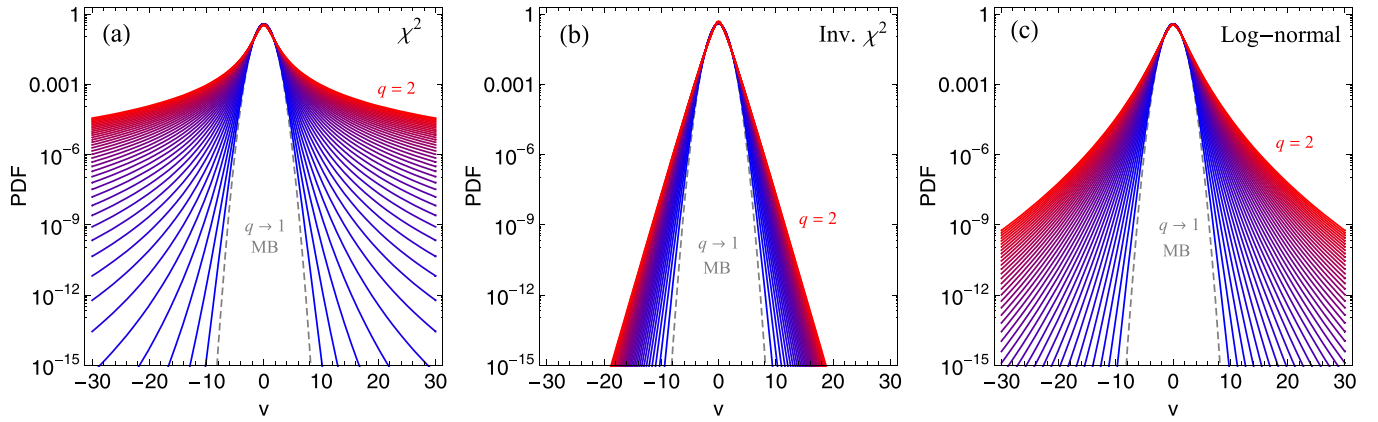


FIG. 1. One-dimensional superstatistical velocity PDFs $\mathcal{P}(v)$ for (a) χ^2 , (b) inverse- χ^2 , and (c) log-normal superstatistics, shown for a range of values of $q \equiv \langle \beta^2 \rangle / \beta_0^2$. The Maxwellian distribution (zero mean, unit variance $\beta_0 m = 1$) is recovered for $q \rightarrow 1$. Note the overpopulation at high velocities in the superstatistical case compared to the Maxwellian distribution.

The velocity moments corresponding to the Maxwellian distribution read as

$$\langle v^\ell \rangle = \left(\frac{2}{\beta m} \right)^{\ell/2} \cdot \frac{\Gamma(\frac{d+\ell}{2})}{\Gamma(\frac{d}{2})}, \quad (11)$$

while the moments of the three universality classes of $f_i(\beta)$ are given by

$$\begin{aligned} \langle \beta^l \rangle_{f_1(\beta)} &= \frac{\Gamma(\frac{n}{2} + l)}{\Gamma(\frac{n}{2})} \left(\frac{2}{n} \right)^l \beta_0^l, \\ \langle \beta^l \rangle_{f_2(\beta)} &= \frac{\Gamma(\frac{n}{2} + 1 - l)}{\Gamma(\frac{n}{2})} \left(\frac{n}{2} \right)^{l-1} \beta_0^l, \\ \langle \beta^l \rangle_{f_3(\beta)} &= e^{l(l-1)s^2/2} \beta_0^l. \end{aligned} \quad (12)$$

Using Eqs. (9), (11), and (12), the full set of superstatistical velocity moments can be compactly written as

$$\langle v^\ell \rangle_i = \xi_i(\ell, q) \frac{\Gamma(\frac{d+\ell}{2})}{\Gamma(\frac{d}{2})} \left(\frac{2}{\beta_0 m} \right)^{\ell/2}, \quad (13)$$

where we have defined the auxiliary functions

$$\begin{aligned} \xi_1(\ell, q) &= \frac{\Gamma(\frac{1}{q-1} - \frac{\ell}{2})}{\Gamma(\frac{1}{q-1})} (q-1)^{-\ell/2}, \\ \xi_2(\ell, q) &= \frac{\Gamma(\frac{q}{q-1} + \frac{\ell}{2} + 1)}{\Gamma(\frac{q}{q-1})} \left(\frac{q}{q-1} \right)^{-1-\ell/2}, \\ \xi_3(\ell, q) &= q^{\frac{\ell}{4}(\frac{\ell}{2}+1)}, \end{aligned} \quad (14)$$

which will be useful in future derivations.

Note that, in the case of χ^2 superstatistics, the requirement that moments remain finite imposes constraints on the allowable values of q . Specifically, the distribution (3) asymptotically behaves as $v^{-d-2/(q-1)}$. In the three-dimensional case, for the ℓ th moment to be finite, the quantity $v^{\ell+2}\mathcal{P}(v)$ must decay at least as fast as v^{-1} . This condition leads to the bound $q < 1 + 2/\ell$. For instance, ensuring finiteness of the second-order moment (related to temperature or pressure) requires that $q < 2$.

III. THERMODYNAMIC QUANTITIES

In this section, we examine the impact of the heavy-tailed nature of superstatistical distributions on thermodynamic quantities, with particular attention to the unphysical contributions arising from superluminal particle populations. In our notation, while $\mathcal{P}(v)$ denotes a probability distribution normalized to unity, $f(v)$ represents the corresponding distribution normalized to the total number of particles, N . That is,

$$\iint f(v) d^3r d^3v = \iint n\mathcal{P}(v) d^3r d^3v = N. \quad (15)$$

In the nonrelativistic regime, all moments of the superstatistical distributions can be evaluated analytically. In particular, the lowest-order moments are given by

$$\begin{aligned} \mathcal{M}^0 &= \int f(\mathbf{r}, \mathbf{v}; t) d^3v, \\ \mathcal{M}^1 &= \frac{1}{n(\mathbf{r}, t)} \int \mathbf{v} f(\mathbf{r}, \mathbf{v}, t) d^3v = \mathbf{0}, \\ \vec{\mathcal{M}}^2 &= \int \mathbf{v} \otimes \mathbf{v} f(\mathbf{r}, \mathbf{v}; t) d^3v, \end{aligned} \quad (16)$$

which correspond, respectively, to the particle density, the mean velocity (which vanishes due to symmetry), and the stress tensor per unit mass.

A. Pressure

Assuming that the stress tensor is governed by isotropic pressure, the dyadic product $\mathbf{v} \otimes \mathbf{v}$ in Eq. (16) can be contracted to the scalar form v^2 . Under this assumption, the second-order moment yields the isotropic pressure. That is,

$$P = \frac{4\pi m}{3} \int_0^\infty v^4 f(v) dv. \quad (17)$$

For the three universality classes, this quantity can be directly written in terms of the auxiliary functions introduced in Eq. (14) as

$$P = 3n_0 \xi_i(2, q) \frac{k_B T_0}{m} \quad (i = 1, 2, 3), \quad (18)$$

where the auxiliary functions $\xi_i(2, q) \geq 1$ are given explicitly below

$$\begin{aligned}\xi_1(2, q) &= \frac{1}{2-q} \quad (1 < q < 2), \\ \xi_2(2, q) &= \frac{2q-1}{q}, \\ \xi_3(2, q) &= q,\end{aligned}\quad (19)$$

with $i = 1, 2$, and 3 corresponding, respectively, to χ^2 , inverse- χ^2 , and log-normal superstatistics.

For comparison, we also evaluate the “partial” pressure by introducing a cutoff velocity w . That is,

$$\bar{P}(w) = \frac{4\pi m}{3} \int_0^w v^4 f(v) dv. \quad (20)$$

To evaluate the contribution to the pressure from particles traveling faster than w , we define the relative contribution as the ratio

$$\frac{\delta P}{P} = \frac{P - \bar{P}(w)}{P} = 1 - \frac{\bar{P}(w)}{P}. \quad (21)$$

For the model to be physically realistic, this contribution must vanish as w approaches c . In the Maxwellian case, the partial pressure $\bar{P}(w)$ can be expressed analytically, with the relative deviation taking the form

$$\frac{\delta P}{P} = 1 - \operatorname{erf}(\hat{w}) + \frac{1}{3\sqrt{\pi}} ([4\hat{w}^3 + 6\hat{w}]e^{-\hat{w}^2}), \quad (22)$$

where

$$\operatorname{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \quad (23)$$

is the error function and $\hat{w} \equiv w/v_{\text{th}}$ (with $v_{\text{th}} = \sqrt{2k_B T_0/m}$ being the thermal speed). One may see that the expression decreases exponentially with increasing $\hat{w}$. In this case, the contribution becomes negligibly small well below c under typical conditions, rendering the unphysical influence of superluminal particles insignificant. However, as the ratio c/v_{th} approaches unity—indicative of a particle population at relativistic temperatures—the expression in Eq. (22) departs substantially from zero. This marks a breakdown in the validity of the nonrelativistic Maxwellian distribution. In this regime, the relativistic Maxwell-Jüttner distribution must be used.

Let us now evaluate this contribution for the three types of superstatistics. In the case of χ^2 superstatistics, the ratio (21) admits the following closed-form expression:

$$\begin{aligned}\frac{\delta P}{P} &= 1 - \frac{8(q-1)^{5/2} \hat{w}^5 \Gamma(\frac{3}{2} + \frac{1}{q-1})}{15\sqrt{\pi} \Gamma(\frac{1}{q-1} - 1)} \\ &\times {}_2F_1\left(\frac{5}{2}, \frac{3}{2} + \frac{1}{q-1}; \frac{7}{2}; (1-q)\hat{w}^2\right),\end{aligned}\quad (24)$$

where ${}_2F_1(a, b; c; z)$ denotes the ordinary (Gauss) hypergeometric function (see Appendix A). Note that, for large $\hat{w}$, it decays to zero following a power law. This behavior arises because the hypergeometric function asymptotically scales as $\hat{w}^{2/(1-q)}$, which causes the ratio to scale as $\hat{w}^{5-2/(q-1)}$.

For the inverse- χ^2 case, the ratio can also be expressed in a closed form, although the resulting expression is somewhat cumbersome. That is,

$$\frac{\delta P}{P} = 1 - \mathcal{A}(\hat{w}, q) - \mathcal{B}(\hat{w}, q), \quad (25)$$

with

$$\begin{aligned}\mathcal{A}(\hat{w}, q) &= \frac{8q^{3/2} \Gamma(\frac{q+1}{2q-2}) \hat{w}^5}{15(2q-1)\sqrt{\pi}(q-1)\Gamma(\frac{q}{q-1})} \\ &\times {}_1F_2\left(\frac{5}{2}; \frac{7}{2}, \frac{q-3}{2q-2}; \frac{q\hat{w}^2}{q-1}\right), \\ \mathcal{B}(\hat{w}, q) &= \frac{4(\frac{q}{q-1})^{3+\frac{1}{q-1}} \Gamma(\frac{1+q}{2-2q}) \hat{w}^{6+\frac{2}{q-1}}}{3\sqrt{\pi} \Gamma(\frac{4q-3}{q-1})} \\ &\times {}_1F_2\left(\frac{3q-2}{q-1}; \frac{13q-1}{2q-2}, \frac{4q-3}{q-1}; \frac{q\hat{w}^2}{q-1}\right),\end{aligned}\quad (26)$$

where ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$ denotes the generalized hypergeometric function (see Appendix A). Despite its closed form, this expression is cumbersome and difficult to work with, so we opt for numerical integration in the subsequent analysis. Likewise, for log-normal superstatistics, since no closed form exists, we also evaluate the ratio numerically.

The results are presented in Fig. 2, which shows the ratio $\delta P/P$ for the three universality classes, for different values of q . The ratio is plotted as a function of the normalized cutoff velocity w/v_{th} . To see how these results translate in a concrete example, take solar wind electrons at 1 A.U., where the thermal speed is approximately $v_{\text{th}} \approx 1000 \text{ km s}^{-1}$. In this context, ensuring that particles with superluminal speeds do not significantly contribute to the pressure requires the ratio to vanish below $c/v_{\text{th}} \approx 300$. This condition is generally met for inverse- χ^2 and log-normal superstatistics. However, for χ^2 superstatistics, the contribution from superluminal particles can become unacceptably large even at such moderate thermal speeds. Specifically, the ratio reaches approximately 1.4% for $q = 1.7$, rises to about 8% for $q = 1.8$, and climbs to 33% for $q = 1.9$. Under these conditions, the unphysical influence of particles exceeding the speed of light on the pressure becomes excessively high.

B. Entropy

Similarly, we can evaluate the contribution of superluminal particles to the entropy. A general expression for the entropy S of a gas, first introduced by Gibbs [83] and widely applied in plasma physics [84,85], is given by

$$S = -k_B \iint f [\ln(f) - 1] d^3r d^3v - k_B N \ln\left(\frac{h^3}{m^3}\right). \quad (27)$$

The above definition remains valid even under nonequilibrium conditions. Additionally, it incorporates the quantum mechanical minimum phase space volume per particle and properly accounts for particle indistinguishability, thereby avoiding the Gibbs paradox. Assuming homogeneity, i.e., a constant particle density n , the entropy expression simplifies

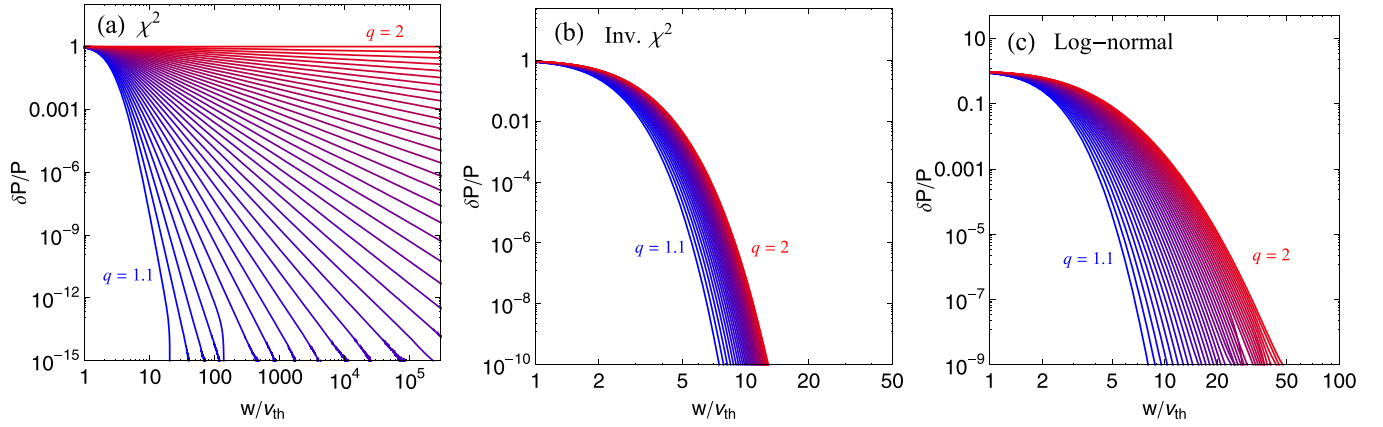


FIG. 2. The relative ratio $\delta P/P$ as a function of the normalized speed w/v_{th} , for various values of $q := \langle \beta^2 \rangle / \beta_0^2$. (a) χ^2 distribution, (b) inverse- χ^2 distribution, and (c) log-normal distribution.

to

$$\begin{aligned} \frac{S}{k_B} = & -\ln(n) \iint f(v) d^3 r d^3 v \\ & - \iint f(v) \ln[\mathcal{P}(v)] d^3 r d^3 v \\ & + \iint f(v) d^3 r d^3 v - N \ln\left(\frac{h^3}{m^3}\right), \end{aligned} \quad (28)$$

which can be rearranged into the compact form

$$S = \left[1 + \ln\left(\frac{m^3}{nh^3}\right) + \Sigma \right] k_B N, \quad (29)$$

where

$$\Sigma := - \int \mathcal{P}(v) \ln \mathcal{P}(v) d^3 v \quad (30)$$

is the “scaled entropy.” In what follows, we evaluate the contribution of superluminal particles on the scaled entropy using the relative ratio

$$\frac{\delta \Sigma}{\Sigma} = \frac{\Sigma - \bar{\Sigma}(w)}{\Sigma} = 1 - \frac{\bar{\Sigma}(w)}{\Sigma}, \quad (31)$$

where

$$\begin{aligned} \Sigma &:= -4\pi \int_0^\infty \mathcal{P}(v) \ln[\mathcal{P}(v)] v^2 dv, \\ \bar{\Sigma}(w) &:= -4\pi \int_0^w \mathcal{P}(v) \ln[\mathcal{P}(v)] v^2 dv. \end{aligned} \quad (32)$$

For a Maxwellian distribution, the (complete) scaled entropy Σ reads as

$$\Sigma = \frac{3}{2} \ln\left(\frac{2\pi k_B T_0}{m}\right) + \frac{5}{2}. \quad (33)$$

In this case, the total entropy (29) reduces to the well-known Sackur-Tetrode equation, which describes an ideal, monoatomic, classical, and nonrelativistic gas at equilibrium. For a Maxwellian distribution, the partial entropy $\bar{\Sigma}$ can be calculated analytically, and the corresponding ratio (31) can

be evaluated explicitly as

$$\frac{\delta \Sigma}{\Sigma} = \text{erfc}(\hat{w}) + \frac{2\hat{w}e^{-\hat{w}^2}(5 + 3 \ln(\pi v_{th}^2) + 2\hat{w}^2)}{\sqrt{\pi}(5 + 3 \ln(\pi v_{th}^2))}, \quad (34)$$

where $\text{erfc}(z) = 1 - \text{erf}(z)$ denotes the complementary error function. Note that, for large $\hat{w}$, $\text{erfc}(\hat{w}) \sim e^{-\hat{w}^2}/\hat{w}$ and the ratio (34) decays exponentially. As a result, under typical conditions, the contribution of particles with speeds greater than w is negligibly small, well below c .

For superstatistical distributions, it is not possible to express the entropy in a closed form, so we employ numerical integration. The results for the three universality classes are illustrated in Fig. 3. Similar trends to those observed for the pressure are evident here. In the case of χ^2 superstatistics, there is a significant unphysical contribution to the entropy arising from superluminal particles, while this effect is less pronounced in the other two classes.

C. Regularization scheme

As discussed previously, the heavy tails of superstatistical distributions—particularly within the χ^2 universality class—can result in a non-negligible contribution from superluminal particles to thermodynamic quantities. In such cases, *strictu sensu*, a relativistic treatment becomes necessary to fully capture the dynamics. However, due to the complexity of working with relativistic distributions, a practical alternative is to introduce a cutoff to eliminate the contribution of superluminal particles, as is commonly done with the regularized kappa distribution in plasma physics [86]. This artificial cutoff can be highly effective in many practical situations.

One example is Langmuir turbulence, where the Langmuir spectrum diverges at infinite wavelength—a behavior intimately connected to the heavy tails of the distribution [87]. Introducing a suitable cutoff in this scenario can prevent the so-called infrared catastrophe. Another relevant example occurs when the particle population lies at the threshold between nonrelativistic and relativistic regimes. For instance, solar wind electron distributions can extend beyond 100 keV, especially in the so-called superhalo [88], placing them near this transitional energy range.

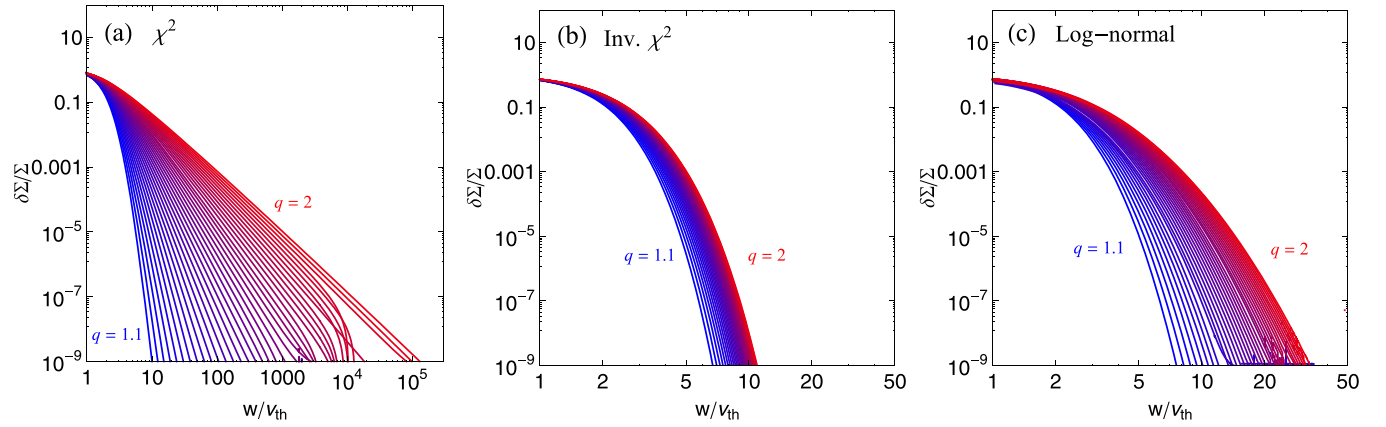


FIG. 3. The relative ratio $\delta\Sigma/\Sigma$ as a function of the normalized speed w/v_{th} , for various values of $q := \langle\beta^2\rangle/\beta_0^2$. (a) χ^2 distribution, (b) inverse- χ^2 distribution, and (c) log-normal distribution.

In our context, this prescription translates to modifying the χ^2 superstatistical distribution (3) as

$$\mathcal{P}(v) = \mathcal{C} \left(1 + \frac{(q-1)\beta_0 m v^2}{2} \right)^{\left(-\frac{3}{2} - \frac{1}{q-1}\right)} e^{-\frac{\alpha^2 \beta_0 m v^2}{2}}, \quad (35)$$

where we have assumed a three-dimensional velocity space. Here, $\alpha \in [0, 1]$ serves as a cutoff parameter, and the normalization constant $\mathcal{C}$ is given by

$$\mathcal{C} = \Gamma\left(\frac{3q-1}{2q-2}\right) \frac{(m\beta_0(q-1))^{\frac{3q+1}{2q-2}}}{2\sqrt{2}(\Theta_1 + \Theta_2)} (\alpha^2 m^2 \beta_0^2 (q-1))^{\frac{1}{1-q}}, \quad (36)$$

with

$$\begin{aligned} \Theta_1 &\equiv \pi^{3/2} \left(\frac{\alpha^2}{q-1} \right)^{\frac{1}{1-q}} \Gamma\left(\frac{1}{q-1}\right) \\ &\quad \times {}_1F_1\left(\frac{3}{2}; \frac{q-2}{q-1}; \frac{\alpha^2}{q-1}\right), \\ \Theta_2 &\equiv 2\pi \Gamma\left(\frac{3q-1}{2q-2}\right) \Gamma\left(\frac{1}{1-q}\right) \\ &\quad \times {}_1F_1\left(\frac{3q-1}{2q-2}; \frac{q}{q-1}; \frac{\alpha^2}{q-1}\right). \end{aligned} \quad (37)$$

The relevance of this regularized χ^2 superstatistical distribution lies in the fact that, when analyzing a dataset exhibiting power-law behavior, it produces results that are indistinguishable from the nonregularized form—provided the cutoff parameter α is chosen sufficiently small so that the cutoff occurs well beyond the range of the observed data. This condition is typically met in practice, as most instruments record measurements over a finite velocity range. Moreover, it extends the range of admissible values of q , which are otherwise constrained by the requirement of finite moments. For completeness, the expressions for the moments of the distribution are provided in Appendix B.

Figure 4 displays the ratios of pressure ($\delta P/P$) and entropy ($\delta\Sigma/\Sigma$) for the regularized χ^2 superstatistical distribution, with parameters $\alpha = 0.1$ and $q = 1.8$. One can see that the contribution from particles with velocities above w is strongly reduced. For reference, we recall that the unregularized χ^2

distribution with $q = 1.8$ yields a pressure contribution of roughly 8% from superluminal particles.

IV. SUPERSTATISTICS IN THE RELATIVISTIC REGIME: THE MAXWELL-JÜTTNER DISTRIBUTION

The Maxwell-Jüttner distribution is the relativistic generalization of the classical Maxwell-Boltzmann distribution to the relativistic regime. It describes the equilibrium momentum (or energy) distribution of particles in a relativistic ideal gas, where thermal energies are comparable to or exceed the rest mass energy of the particles. In d dimensions, it reads as [56,89]

$$\mathcal{P}(\mathbf{p}|\beta_J) d^d p = \frac{\exp[-\beta_J \gamma(\mathbf{p})]}{Z_J} d^d p, \quad (38)$$

which is expressed here in momentum space rather than velocity space. Here, m_0 denotes the rest mass of a gas par-

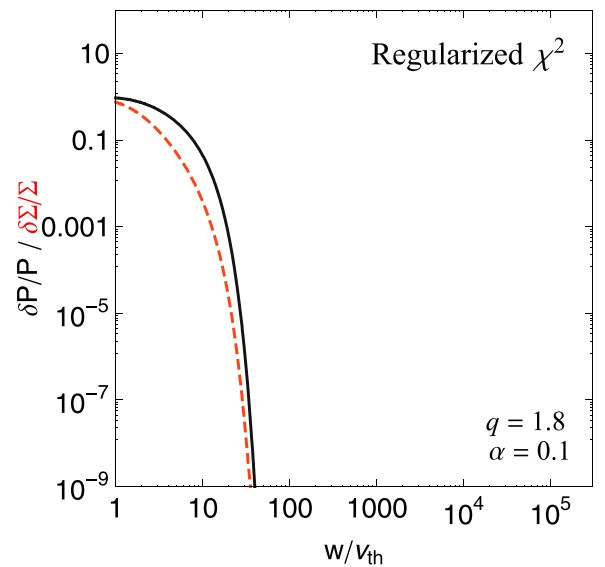


FIG. 4. The relative ratios $\delta P/P$ (solid line) and $\delta\Sigma/\Sigma$ (dashed line) for the regularized χ^2 distribution (35), with $q = 1.8$ and $\alpha = 0.1$.

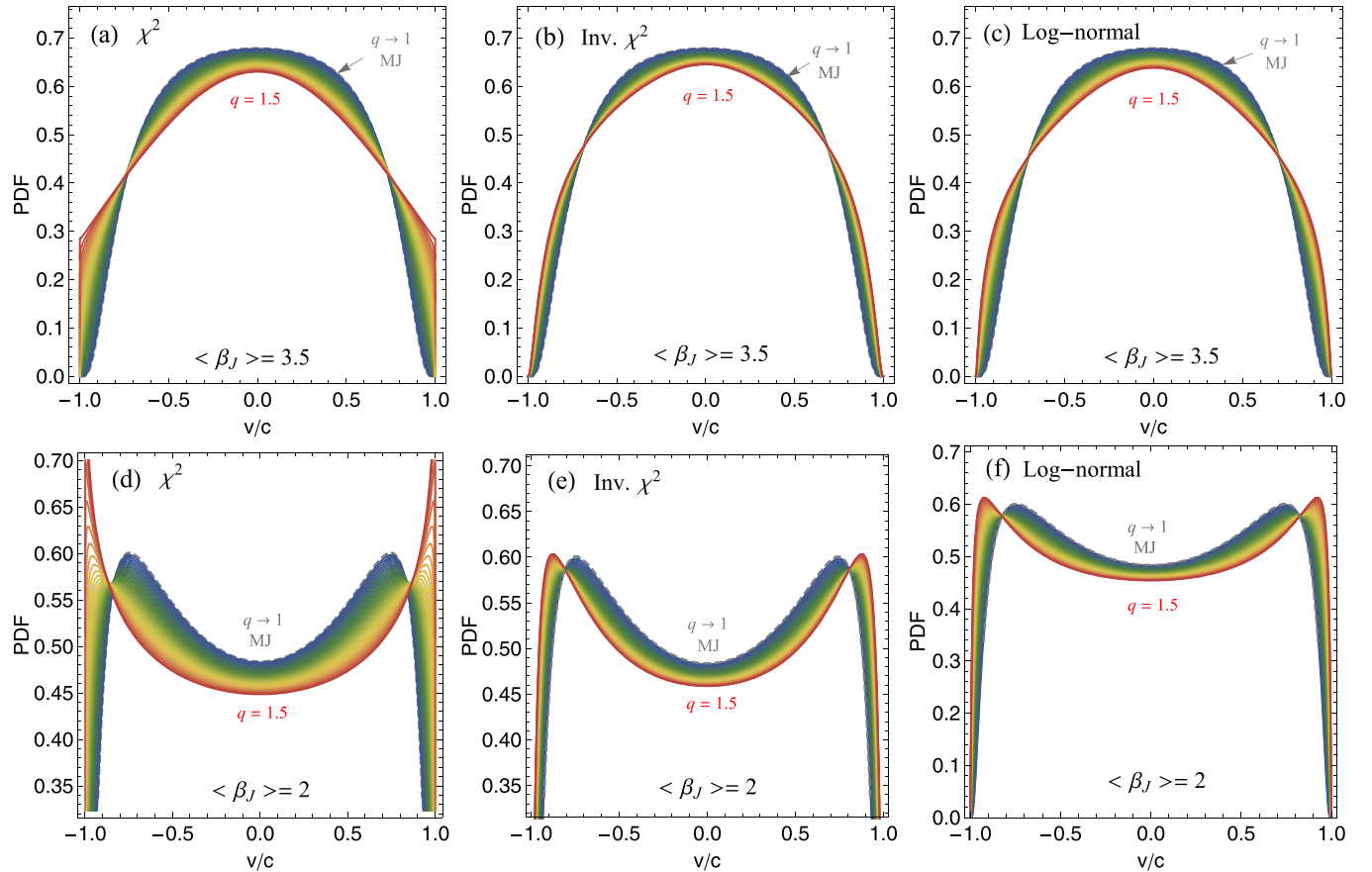


FIG. 5. One-dimensional superstatistical extensions of the Maxwell-Jüttner distribution for the three universality classes of superstatistics, shown for various values of $q := \langle \beta_J^2 \rangle / \langle \beta_J \rangle^2$. The distributions exhibit a transition from a unimodal to a bimodal form at $\langle \beta_J \rangle = 3$. The upper panel corresponds to $\langle \beta_J \rangle = 3.5$, while the lower panel represents $\langle \beta_J \rangle = 2$.

ticle, $\beta_J \equiv m_0 c^2 / k_B T$ is the inverse normalized temperature (sometimes referred to as the relativistic coldness; see, e.g., Ref. [93]), and $\gamma(p) = \sqrt{1 + (p^2 / m_0^2 c^2)}$ is the Lorentz factor. The normalization constant reads as

$$Z_J = \pi^{\frac{d-1}{2}} 2^{\frac{d+1}{2}} (m_0 c)^d \beta_J^{\frac{1-d}{2}} \mathcal{K}_{\frac{d+1}{2}}(\beta_J). \quad (39)$$

Here, quantities like the particle number density, the temperature, and the relativistic spatial momentum should be understood as measured by an observer comoving with the gas. In the nonrelativistic regime $|\mathbf{p}| \ll m_0 c$, $m_0 c^2 \ll k_B T$, the Maxwell-Jüttner distribution (38) approaches the Maxwellian distribution.

Many generalizations of the Maxwell-Jüttner distribution have been proposed in the literature to account for heavy-tailed behavior observed in high-energy systems. For instance, a special relativistic version of the kappa distribution was first introduced by Naito *et al.* [94], followed by several alternative formulations [95–98]. These generalizations are designed to be Lorentz invariant and to retain a natural cutoff at $v = c$, just like the Maxwell-Jüttner distribution, to which they reduce in an appropriate limit.

Here, we consider a superstatistical extension of the Maxwell-Jüttner distribution, i.e., distributions of the form of

$$\mathcal{P}(\mathbf{p}; \langle \beta_J \rangle, q) = \int_0^\infty \mathcal{P}(\mathbf{p} | \beta_J) f_i(\beta_J) d\beta_J. \quad (40)$$

The underlying idea is that the relativistic gas remains locally in equilibrium, with its local statistical properties described by the Maxwell-Jüttner distribution, while the temperature, or equivalently $\beta_J \equiv m_0 c^2 / k_B T$, fluctuates according to a distribution $f(\beta_J)$ that reflects its underlying dynamics. One difficulty in this context is that such distributions generally require numerical treatment, as we are not in a position to derive a closed-form expression for Eq. (40). Nonetheless, their properties can still be analyzed analytically in the weakly relativistic and ultrarelativistic limits. Moreover, as will be shown in Sec. V, these distributions can provide excellent fits to observational data.

Figure 5 displays (one-dimensional) superstatistical relativistic distributions of the form given in Eq. (40), corresponding to the three universality classes of $f(\beta_J)$, plotted against v/c . These superstatistical extensions feature heavy tails, making them potentially well suited for fitting observational data. Notably, they preserve a characteristic property of the original Maxwell-Jüttner distribution: a transition from a unimodal to a bimodal shape occurs at $\beta_{J0} := \langle \beta_J \rangle = d + 2$.

Superstatistical distributions of the form of Eq. (40), being simply local Maxwell-Jüttner distributions averaged over $f(\beta_J)$, allow one to make use of the known moments of the standard Maxwell-Jüttner distribution (see, e.g., Ref. [99]), in combination with the moments of the three universality classes of $f(\beta_J)$ [viz., Eq. (12)]. For example, the average

velocity can be written as

$$\langle v \rangle_i = \left\langle \frac{2c}{\beta_J} \left(\frac{1}{\beta_J} + 1 \right) \frac{e^{-\beta_J}}{\mathcal{K}_2(\beta_J)} \right\rangle_{f_i(\beta_J)}, \quad (41)$$

where $\langle \bullet \rangle_{f_i}$ denotes the average with respect to the distribution $f_i(\beta_J)$ and the expression inside the angle brackets corresponds to the known average velocity for the Maxwell-Jüttner distribution.

Let us examine two limiting cases. First, consider a distribution $f(\beta_J)$ sharply peaked around $\langle \beta_J \rangle \gg 1$. In this scenario, the dominant contribution to Eq. (41) comes from large values of β_J , corresponding to a superstatistical gas in the weakly relativistic regime. Expanding Eq. (41) in a series and employing the moments (12), one obtains

$$\langle v \rangle_i \approx \xi_i(1, q) \sqrt{\frac{8}{\pi} \frac{k_B T_0}{m_0}} - \frac{7\xi_i(3, q)}{2\sqrt{2\pi}} \frac{1}{c^2} \sqrt{\frac{k_B T_0}{m_0}}, \quad (42)$$

where the auxiliary functions read explicitly as

$$\begin{aligned} \xi_1(1, q) &\equiv \frac{\Gamma[\frac{1}{q-1} - \frac{1}{2}]}{\Gamma[\frac{1}{q-1}] \sqrt{q-1}}, \\ \xi_2(1, q) &\equiv \left(\frac{q-1}{q} \right)^{3/2} \frac{\Gamma[\frac{1}{q-1} + \frac{5}{2}]}{\Gamma[\frac{q}{q-1}]}, \\ \xi_3(1, q) &\equiv q^{3/8}, \end{aligned} \quad (43)$$

and

$$\begin{aligned} \xi_1(3, q) &\equiv \frac{\Gamma[\frac{1}{q-1} - \frac{3}{2}]}{\Gamma[\frac{1}{q-1}] (q-1)^{3/2}}, \\ \xi_2(3, q) &\equiv \left(\frac{q-1}{q} \right)^{5/2} \frac{\Gamma[\frac{1}{q-1} + \frac{7}{2}]}{\Gamma[\frac{q}{q-1}]}, \\ \xi_3(3, q) &\equiv q^{15/8}, \end{aligned} \quad (44)$$

with $i = 1, 2$, and 3 corresponding, respectively, to χ^2 , inverse- χ^2 , and log-normal superstatistics.

In the opposite limit, where the distribution $f(\beta_J)$ is sharply peaked around $\langle \beta_J \rangle \ll 1$, corresponding to the ultrarelativistic regime, the mean velocity exhibits a universal behavior; that is,

$$\langle v \rangle_i \approx c - q \frac{m_0^2 c^5}{4k_B^2 T_0^2}. \quad (45)$$

A. Pressure

In special relativity, the pressure in the local rest frame is represented by the spatial diagonal components of the energy-momentum tensor

$$T^{\mu\nu} = c \int \frac{d^3 p}{p^0} p^\mu p^\nu f(p), \quad (46)$$

where $p^0 = \sqrt{\mathbf{p}^2 + (m_0 c)^2}$ and $d^3 p/p^0$ denotes the Lorentz-invariant volume element in phase space (see, e.g., Refs. [100,101]).

In the isotropic case, this reduces to a scalar pressure

$$P = \frac{c}{3} \int \frac{d^3 p}{p^0} p^2 f(p). \quad (47)$$

For the standard Maxwell-Jüttner distribution, this yields

$$P = \frac{1}{\beta_J} n m_0 c^2, \quad (48)$$

which coincides with the pressure of a nonrelativistic gas in equilibrium. In fact, the ideal gas law $P = nk_B T$ holds in both the nonrelativistic and relativistic regimes. For the superstatistical distributions, combining Eq. (48) with the moments of the three universality classes [viz., Eq. (12)], one recovers the pressure as

$$P = \xi_i(2, q) n k_B T_0 \quad (i = 1, 2, 3), \quad (49)$$

where the auxiliary functions $\xi_i(2, q)$ are explicitly given in Eq. (19).

The pressure (48) is identical to that of a nonrelativistic superstatistical gas [29]. This is expected given that superstatistical distributions are formed by averaging equilibrium distributions over $f(\beta)$. As a result, the correction depends exclusively on the underlying temperature dynamics, which is encoded in $f(\beta)$, and does not vary between relativistic and nonrelativistic regimes. Note that this contrasts with empirical distributions, such as the kappa distribution commonly used in plasma physics, in both its nonrelativistic and relativistic formulations [97]. In that case, the correction to the pressure behaves differently in the relativistic and nonrelativistic regimes; that is, the multiplicative factor in Eq. (49) differs between the two regimes.

B. Internal Energy

We now examine the effect of superstatistics on the internal energy U of a relativistic gas. The internal energy, normalized by the rest energy $E_0 \equiv m_0 c^2$, can be expressed as

$$\begin{aligned} \frac{U}{E_0} &= \langle \gamma \rangle_i = \int_1^\infty \mathcal{P}(\gamma, \langle \beta_J \rangle, q) \gamma d\gamma \\ &= \left\langle \left(3\beta_J + \frac{\mathcal{K}_1(1/\beta_J)}{\mathcal{K}_2(1/\beta_J)} \right) \right\rangle_{f_i(\beta_J)}, \end{aligned} \quad (50)$$

where, in the last equality, we have used the expression for the internal energy corresponding to the standard Maxwell-Jüttner distribution [99]. For the three superstatistical universality classes, a closed-form expression for the internal energy does not appear to be accessible analytically; however, it can be readily computed numerically. The results are presented in Fig. 6 for the three classes and various values of the parameter q . It can be observed that, for a fixed average temperature T_0 , the internal energy increases as q deviates from the equilibrium value $q = 1$. This behavior is a consequence of the heavy-tailed nature of the superstatistical distributions.

It is of interest to examine the behavior of the internal energy in the weakly relativistic and ultrarelativistic limits, where analytical expressions can be derived. In the weakly relativistic regime, assuming a temperature distribution $f(\beta_J)$ that is sufficiently peaked around a mean value $\langle \beta_J \rangle \gg 1$, the

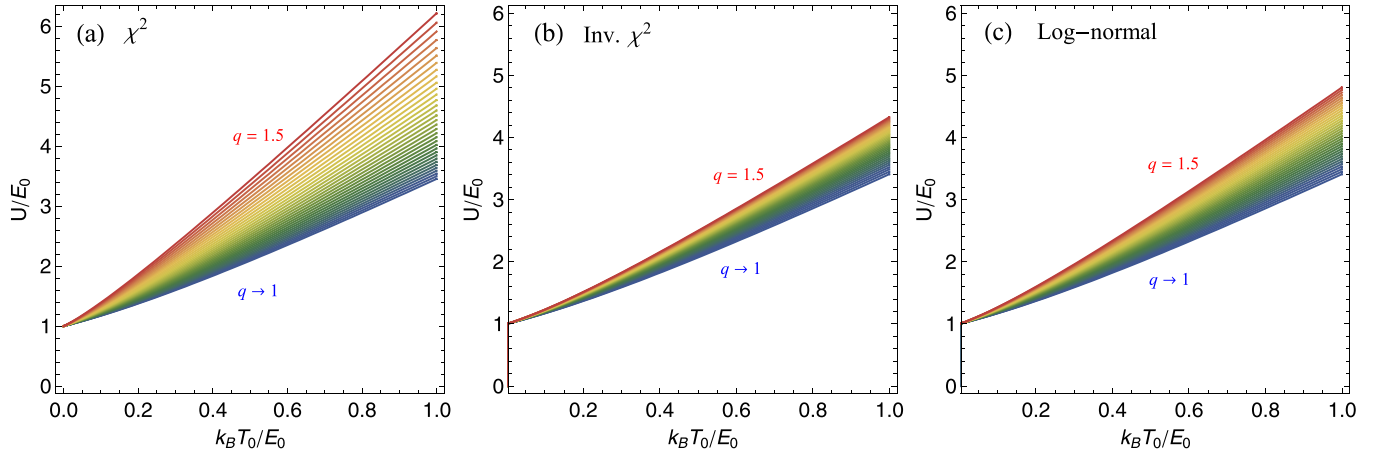


FIG. 6. Normalized internal energy U/E_0 as a function of the normalized mean temperature $k_B T_0/E_0$, plotted for various values of the parameter $q := \langle \beta_J^2 \rangle / \langle \beta_J \rangle^2$. (a) χ^2 distribution, (b) inverse- χ^2 distribution, and (c) log-normal distribution.

integrand can be expanded using the corresponding asymptotic expressions [102]:

$$\begin{aligned} \mathcal{K}_1(\beta) &\sim e^{-\beta} \sqrt{\frac{\pi}{2\beta}} \left(1 + \frac{3}{8\beta} - \frac{15}{128\beta^2} + \dots \right), \\ \mathcal{K}_2(\beta) &\sim e^{-\beta} \sqrt{\frac{\pi}{2\beta}} \left(1 + \frac{15}{8\beta} + \frac{105}{128\beta^2} + \dots \right), \end{aligned} \quad (51)$$

to obtain

$$U \approx m_0 c^2 + \frac{3}{2} \xi_i(2, q) k_B T_0 + \frac{15}{8} \xi_i(4, q) \frac{(k_B T_0)^2}{m_0 c^2}, \quad (52)$$

where $\xi_i(2, q)$ (for $i = 1, 2, 3$) are given in Eq. (19) and $\xi_i(4, q)$ are explicitly given as

$$\begin{aligned} \xi_1(4, q) &\equiv \frac{1}{6 + (2q - 7)q}, \\ \xi_2(4, q) &\equiv 6 + \frac{2 - 7q}{q^2}, \\ \xi_3(4, q) &\equiv q^3, \end{aligned} \quad (53)$$

with $i = 1, 2$, and 3 corresponding, respectively, to χ^2 , inverse- χ^2 , and log-normal superstatistics. In the ultrarelativistic limit, assuming a temperature distribution sharply peaked around $\langle \beta_J \rangle \ll 1$ and using the expansion

$$\begin{aligned} \mathcal{K}_1(\beta) &\sim \frac{1}{\beta}, \\ \mathcal{K}_2(\beta) &\sim \frac{2}{\beta^2}, \end{aligned} \quad (54)$$

we obtain

$$U \approx 3 \xi_i(2, q) k_B T_0 + \frac{m_0^2 c^4}{2 k_B T_0}. \quad (55)$$

C. Entropy

We now focus on the entropy corresponding to the relativistic superstatistics (40), for the three universality classes

of superstatistics. The (scaled) entropy can be expressed as

$$\Sigma = -4\pi \int_0^\infty \mathcal{P}(\mathbf{p}; \langle \beta_J \rangle, q) \ln \mathcal{P}(\mathbf{p}; \langle \beta_J \rangle, q) p^2 dp. \quad (56)$$

In general, the integral (56) cannot be solved analytically and will therefore be evaluated numerically. In the limit $q \rightarrow 1$, where the superstatistical distributions (40) reduce to the standard Maxwell-Jüttner distribution, the entropy can be computed analytically and takes the form

$$\Sigma = \frac{U}{T_0} + k_B \ln Z_J, \quad (57)$$

which serves as a reference to assess the influence of temperature fluctuations on the entropy of a relativistic gas.

The results of the numerical integration are shown in Fig. 7 for the three universality classes. One may see that, for a fixed mean temperature T_0 , the entropy increases compared to that of a relativistic gas in equilibrium. This increase reflects the heavy-tailed nature of the distributions and ultimately arises from the underlying temperature fluctuations, which introduce additional uncertainty and thereby raise the entropy (see also discussion in Ref. [103]). As the parameter q deviates from its equilibrium value $q = 1$, the entropy increase is most pronounced for the χ^2 superstatistics, followed by the log-normal and then the inverse- χ^2 class, consistent with the prominence of the heavy tails of these universality classes (see Fig. 5).

V. APPLICATION: MODELING THE OBSERVED SPECTRA OF SOLAR ENERGETIC PROTONS

To validate our approach with real-world data, we compare the superstatistical model to observational data of solar energetic particles, which often exhibit a pronounced non-Maxwellian high-energy tail distribution [3,104,105]. These particles, generated by solar flares, coronal activity, and interplanetary shocks, have a significant impact on space weather conditions. Accurately characterizing their fluxes is essential, not only for advancing scientific understanding but also for enhancing practical space weather predictions. Specifically, we focus on data from relativistic protons observed during the 28 October 2003 solar event, measured by balloon flights

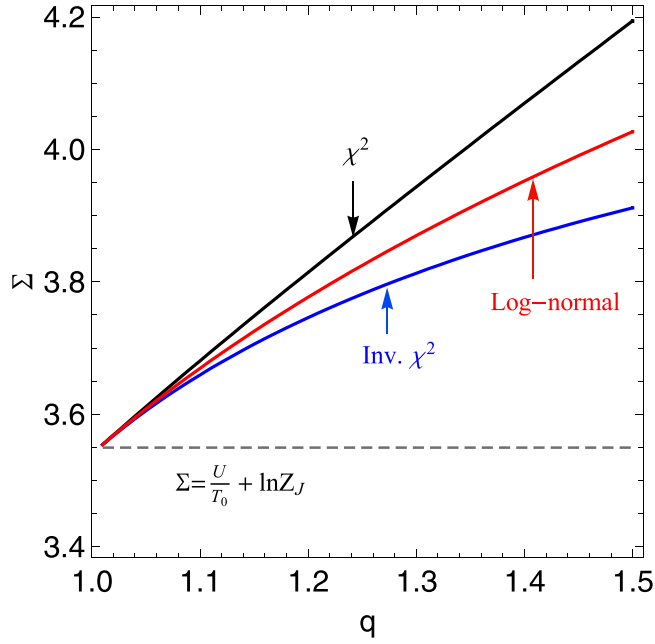


FIG. 7. The scaled entropy Σ as a function of $q := \langle \beta_J^2 \rangle / \langle \beta_J \rangle^2$ for the three universality classes of superstatistics. Parameters are set to $k_B = c = \langle \beta_J \rangle = m = 1$.

over Apatity and by the *GOES-10/11* spacecraft [106] (see also Ref. [107]).

To test our model, we first need to compute the differential flux associated with the superstatistical distributions. In space plasma physics and particle transport theory, the differential particle flux $J(E, \mathbf{r})$ provides a direct observable: it represents the number of particles of a given species arriving at position $\mathbf{r}$ per unit energy, per unit solid angle, per unit area, and per unit time. This quantity is often measured by particle detectors onboard spacecraft. The differential flux is related to the distribution function $f(\mathbf{p}; \mathbf{r})$ through the relation [108]

$$J(E; \mathbf{r}) dE d\Omega = f(\mathbf{p}; \mathbf{r}) (p/m) p^2 dp d\Omega, \quad (58)$$

where $m = \gamma m_0$ is the relativistic mass and $f(\mathbf{p}; \mathbf{r})$ is normalized to the number density. Using the relation between the kinetic energy E and the scalar momentum $(E + m_0 c^2)^2 = p^2 c^2 + m_0^2 c^4$, such that $mdE = pdp$, the flux reads as

$$J(E) = p^2 f(p), \quad (59)$$

where we assume isotropy and a constant density, such that $f(\mathbf{p}; \mathbf{r}) \rightarrow f(p)$. This relation reflects the fact that the differential flux incorporates both the density of states and the particle momentum. Expressing the momentum in terms of the kinetic energy, we have

$$p^2 = \frac{m_0 E(E + 2E_0)}{E_0}. \quad (60)$$

By introducing the dimensionless (scaled) variables

$$p_s \equiv \frac{p}{m_0 c} \quad \text{and} \quad E_s \equiv \frac{E}{E_0}, \quad (61)$$

the differential flux can be written as

$$J = \frac{c E_s (E_s + 2)}{E_0} f(p_s), \quad (62)$$

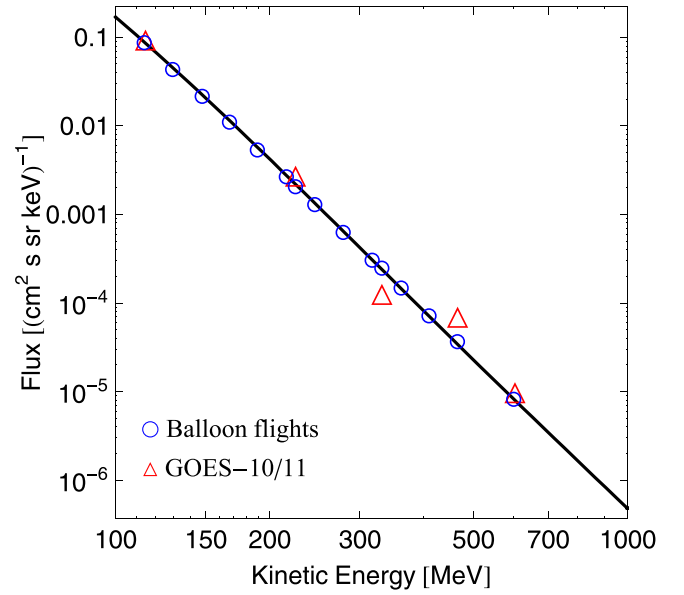


FIG. 8. Fit of the differential flux using χ^2 superstatistics for the 28 October 2003 solar event [106]. Blue circles represent data from balloon-borne measurements over Apatity, while red triangles correspond to observations from the *GOES-10/11* spacecraft.

which serves as the primary basis for modeling observational data.

We compared the fluxes (62) associated with the three universality classes of superstatistics to the observational measurements of energetic protons. Our results show that the χ^2 universality class provides an excellent fit. In Fig. 8, we present the data along with the best fit obtained using χ^2 superstatistics, where the optimal parameters for the fit are $n = 2.8 \times 10^{-5} \text{ cm}^{-3}$, $\beta_J = 540.54$ ($k_B T_0 \approx 1.74 \text{ MeV}$), and $q = 1.58$. The two other universality classes are not shown, as they fail to reproduce the data. Likewise, the standard Jüttner distribution is omitted, as it falls far below the measured fluxes in the energy range considered.

To assess the goodness of fit between the model and the observed data, we computed the sum of squared residuals (SSR) and the coefficient of determination R^2 . The SSR is defined as the sum of the squared differences between the observed values y_i and the model predictions $f(x_i)$, that is,

$$\text{SSR} = \sum_{i=1}^N (y_i - f(x_i))^2. \quad (63)$$

To further quantify the quality of the fit, we used the coefficient of determination,

$$R^2 = 1 - \frac{\text{SSR}}{\text{SST}}, \quad (64)$$

where SST is the total variance of the data with respect to its mean. In our case, with the previously stated parameters, we obtained $R^2 = 0.99$, which confirms the excellent agreement between the model and the observational data.

Beyond the quality of the fit, the choice of universality class offers insight into the nature of the underlying random processes. Specifically, the emergence of the χ^2 class of superstatistics, rather than the other two classes, points to the

presence of *additive* random processes [52]. In fact, according to the central limit theorem, this class arises when a large number of microscopic random variables contribute to β_J in an additive way. This suggests that additive processes are at play, generating the heavy tails observed in the distribution of solar energetic particles.

Specifically, let X_i ($i = 1, \dots, n$) be n independent Gaussian random variables, each with zero mean and identical variance. Defining

$$\beta_J := \sum_{i=1}^n X_i^2, \quad (65)$$

the random variable β_J then follows a χ^2 distribution (2) with n degrees of freedom. Here, squaring the microscopic random variables ensures that β_J is positive. These microscopic random variables are the underlying source of the fluctuations in β_J . In a general scenario, temperature fluctuations can be linked either to classical mechanisms (e.g., heat exchange) or to quantum fluctuations, reflecting fundamental uncertainty in temperature measurements.

Within this picture, the number of effective degrees of freedom, n , determines the shape of the distribution expected to appear in the data. Observe that in the limit of infinitely many degrees of freedom, $n \rightarrow \infty$, the χ^2 distribution (2) collapses to a Dirac delta function, and equilibrium statistical mechanics (the standard Maxwell-Jüttner distribution in our context) is recovered. The appearance of non-Maxwellian profiles is thus a signature of a finite number of effective degrees of freedom.

In our scenario involving energetic solar particles, the fluctuations of β_J are expected to be primarily driven by heat exchange processes. Given the intrinsically three-dimensional nature of the system, the minimal number of relevant degrees of freedom is $n_{\min} = 3$, corresponding to the three spatial directions through which heat can flow. Equation (9) thus implies the following upper bound for q :

$$q_{\max} = 1 + \frac{2}{3} = 1.666\,67, \quad (66)$$

which is consistent with the value obtained from our fit to this specific dataset, namely, $q = 1.58$. A value of q smaller than the asymptotic value (66) indicates the presence of more effective degrees of freedom contributing to β_J . These may include degrees of freedom associated with temporal evolution (see Ref. [37] for an experimental realization involving explicit time dependence of temperature fluctuations). One may also envisage the presence of secondary competing mechanisms involving alternative microscopic scenarios, such as multiplicative random processes, which are ubiquitous in turbulent systems [33,34] (notably, turbulence pervades both the solar corona and the solar wind, where energetic solar protons are generated and propagate [109]).

Note also that the lower bound on n (or, equivalently, the upper bound on q) imposes a limit on the magnitude of temperature fluctuations. For a χ^2 distribution (2), the relative variance of β_J is given by

$$\frac{\langle \beta_J^2 \rangle - \langle \beta_J \rangle^2}{\langle \beta_J \rangle^2} = \frac{2}{n}. \quad (67)$$

For $n_{\min} = 3$, this yields a maximum relative temperature uncertainty of approximately $\delta T/T_0 \approx 0.81$.

Since the general arguments underlying the bound given in Eq. (66) do not rely on any assumptions specific to the relativistic regime, this bound is expected to hold in the non-relativistic regime as well. It appears to be satisfied by all available measurements in space and astrophysical plasmas. Although only a limited number of datasets have so far been analyzed using the χ^2 superstatistical approach as formulated here, a useful comparison can be made with the kappa distribution, which has been more extensively employed to fit a wide range of observational data over the past decades.

Asymptotically, both the kappa distribution and the χ^2 superstatistical distribution exhibit power-law behavior of the form $E^{-\alpha}$, with $\alpha = \kappa + 3/2$ for the kappa distribution and $\alpha = q/(q-1)$ for χ^2 superstatistics. Observed values of κ typically range from $\kappa \approx 100 \pm 50$ in planetary nebulae to $\kappa \approx 1.5 \pm 0.10$ in the ambient solar wind (see Table 1.1 in Ref. [110] and references therein). When translated into our formalism, these values correspond to $n \in [3.8, 301]$, which are fully compatible with the lower bound $n_{\min} = 3$.

VI. CONCLUSION

In this work, we have examined the thermodynamic and physical implications of heavy-tailed distributions commonly observed in space and astrophysical environments, particularly through the lens of superstatistics. While superstatistical models have proven remarkably successful in capturing the behavior of a wide range of nonequilibrium systems, their heavy-tailed nature may lead to nonphysical artifacts implying superluminal particle populations. We first quantified the magnitude of these unphysical effects on pressure and entropy for the three universality classes of superstatistics that have strong empirical support. We showed that, in particular within the χ^2 class of superstatistics, these contributions are non-negligible and can substantially distort the thermodynamic description of the system if left uncorrected. To address this, we discussed a regularization procedure that can be used in practice and effectively suppresses superluminal contributions without invoking a fully relativistic framework.

To go beyond this regularization prescription, we introduced a superstatistical generalization of the Maxwell-Jüttner distribution, applicable to nonequilibrium settings and retaining a natural cutoff for velocities near the speed of light. We analyzed the resulting distributions and their thermodynamic properties, examining their behavior in both the weakly relativistic and ultrarelativistic limits. Finally, we applied our relativistic superstatistical model to observational data of solar energetic protons [106]. The model showed excellent agreement with the data, without the need for *ad hoc* assumptions. Beyond its potential as a predictive tool, the model provides insight into the mechanisms responsible for the emergence of such distributions by relying on the microscopic foundations of superstatistics. In our case, the non-Maxwellian profiles were interpreted as arising from additive random processes involving heat exchange, leading to an upper bound on the deviation from the Maxwellian form. Interestingly, this bound appears to be satisfied by other reports that successfully fit

data using distributions with similar asymptotic power-law behavior [110].

The aspects discussed here offer a pathway toward a more physically grounded statistical mechanics for space and astrophysical environments and naturally open the door to further investigations. For instance, the regularized χ^2 superstatistical model may help prevent the so-called infrared catastrophe in Langmuir turbulence, where the spectrum diverges in the long-wavelength limit due to the heavy tails of the distribution [87]. It may also prove valuable in scenarios where the particle population lies near the transition between nonrelativistic and relativistic regimes. Moreover, applying the relativistic superstatistical fluxes discussed here to a broader set of observational data—ideally including larger datasets—can help assess their validity in modeling solar energetic particles and, more broadly, space and astrophysical systems. In particular, fitting data that span a wider energy range may help distinguish the superstatistical model presented in this work from existing empirical models that also exhibit power-law tails [94–98], due to differences in their behavior at lower energies. Given that some of these systems, such as the solar wind, are known to be turbulent [109], the log-normal class is expected to come into play, as it naturally emerges in the context of cascading dynamics [33,34]. With this in mind, it would be worthwhile to investigate whether this class leaves observable signatures in such turbulent astrophysical environments.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: GENERALIZED HYPERGEOMETRIC FUNCTIONS

For reference, we provide the definition of the generalized hypergeometric series, ${}_pF_q$, and two particular cases used in the main text.

The generalized hypergeometric series is defined as

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{z^n}{n!}, \quad (\text{A1})$$

where $a_1, \dots, a_p, b_1, \dots, b_q \in \mathbb{C}$, and $(a)_n$ denotes the Pochhammer symbol (or rising factorial), defined by

$$(a)_0 = 1, \\ (a)_n = a(a+1)(a+2) \cdots (a+n-1), \quad \text{for } n \in \mathbb{N}. \quad (\text{A2})$$

The two special cases used in the main text are

$${}_1F_2(a; b_1, b_2; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b_1)_n (b_2)_n} \cdot \frac{z^n}{n!}, \quad (\text{A3})$$

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}. \quad (\text{A4})$$

The function ${}_2F_1$ is known as the *Gauss* hypergeometric function, and ${}_1F_2$ is a special case of the generalized hypergeometric function ${}_pF_q$. These functions are analytic in a neighborhood of $z = 0$, and can often be extended analytically beyond their radius of convergence.

For further properties and identities involving these functions, see, e.g., Refs. [111,112].

APPENDIX B: MOMENTS OF THE REGULARIZED χ^2 SUPERSTATISTICAL DISTRIBUTION

This Appendix provides the expressions for the moments of the regularized χ^2 superstatistical distribution (35). Assuming a three-dimensional isotropic system, the ℓ th moment takes the form

$$\langle v^\ell \rangle \equiv 4\pi \int_0^\infty v^{2+\ell} \mathcal{P}(v) dv \equiv \mathcal{C} \mathcal{I}(\beta_0, \ell, q, \alpha), \quad (\text{B1})$$

where $\mathcal{C}$ is the normalization constant given in Eq. (36) and $\mathcal{I}(\beta_0, \ell, q, \alpha)$ reads as

$$\mathcal{I} = \mathcal{A}_1 \left[\mathcal{A}_2 \times {}_1F_1\left(\frac{3+\ell}{2}, \frac{\ell q + 2q - \ell - 4}{2(q-1)}, \frac{\alpha^2}{q-1}\right) + \mathcal{A}_3 \times {}_1F_1\left(\frac{3q-1}{2(q-1)}, \frac{\ell q - \ell - 2q}{2(q-1)}, \frac{\alpha^2}{q-1}\right) \right], \quad (\text{B2})$$

where we have defined

$$\begin{aligned} \mathcal{A}_1 &\equiv \frac{2^{\frac{5+\ell}{2}} \alpha^3 \pi}{(q-1)^{3/2} \Gamma\left(\frac{3q-1}{2(q-1)}\right)} (\alpha^2 \beta_0 m)^{\frac{1-(3+2\ell)q}{2(q-1)}} \\ &\quad \times (\beta_0 m (q-1))^{\frac{2+\ell q}{2(1-q)}} (\alpha^2 \beta_0^2 m^2 (q-1))^{\frac{2+\ell+\ell q}{2(q-1)}}, \\ \mathcal{A}_2 &\equiv (\alpha^2 \beta_0 m)^{\ell/2} (\beta_0 m (q-1))^{\frac{\ell q}{2(1-q)}} \\ &\quad \times \Gamma\left(\frac{3+\ell}{2}\right) \Gamma\left(\frac{2+\ell-\ell q}{2(q-1)}\right), \\ \mathcal{A}_3 &\equiv (\alpha^2 \beta_0 m)^{\frac{1}{1-q}} (\beta_0 m (q-1))^{\frac{2+\ell}{2(1-q)}} \\ &\quad \times \Gamma\left(\frac{3q-1}{2(q-1)}\right) \Gamma\left(\frac{\ell q - \ell - 2}{2(q-1)}\right). \end{aligned} \quad (\text{B3})$$

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