

Quantum computation with hybrid parafermion-spin qubits

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We propose a universal set of single- and two-qubit quantum gates acting on a hybrid qubit formed by coupling a quantum dot spin qubit to a $\mathbb{Z}_{2m}$ parafermion qubit with arbitrary integer m . The special case $m = 1$ reproduces the results previously derived for Majorana qubits. Our formalism utilizes Fock parafermions, facilitating a transparent treatment of hybrid parafermion-spin systems. Furthermore, we highlight the previously overlooked importance of particle-hole symmetry in these systems. We give concrete examples how the hybrid qubit system could be realized experimentally for $\mathbb{Z}_4$ and $\mathbb{Z}_6$ parafermions. In addition, we discuss a simple readout scheme for the fractional parafermion charge via the measurement of the spin-qubit resonant frequency.

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I. INTRODUCTION

Parafermions (PFs), considered to be a generalization of Majorana fermions [1,2], emerged initially in theoretical models of $\mathbb{Z}_d$ -symmetric spin chains [3] and field theories [4]. These quasiparticles have since been recognized for their significant potential in topological quantum computation [5–7]. Specifically, N pairs of $\mathbb{Z}_d$ PFs can create a ground-state manifold with d^N -fold degeneracy, allowing for topologically protected state manipulations through PF braiding. The case of $d = 2$ represents Majorana fermions, while PFs with $d \geq 3$ offer more complex braiding statistics, enabling a larger set of topologically protected gates.

Over the last decade, there have been a large number of proposals aiming at realizing PFs in various strongly correlated condensed matter systems based on, e.g., fractional quantum Hall states (FQHS) [8–15], topological insulators [16–19], or semiconductor/superconductor hybrid nanowires [20–25]. It was also recently demonstrated that $\mathbb{Z}_4$ parafermion edge states can be realized in a simpler setup that only requires a semiconductor nanowire (NW) with spin-orbit and electron-electron interactions as well as weak external magnetic fields [26]. This scheme does not require any complex ingredients such as superconductivity or exotic FQHS, which makes it particularly promising for the experimental realization of PFs.

However, the set of gates that is obtained by braiding $\mathbb{Z}_d$ PFs is not universal but contains at most the Clifford group

[27]. As such, additional methods to manipulate quantum information encoded in a set of PFs are needed to achieve universality. For Majorana fermions, several proposals on how to achieve a universal set of quantum gates have been made [28–36]. One such method is to couple two NWs with Majorana edge states to a singly occupied quantum dot, thus forming a hybrid Majorana-spin qubit [33].

In this work, we extend this proposal to arbitrary $\mathbb{Z}_{2m}$ PFs. Our approach leverages *Fock parafermions* that allow for transparent treatment of hybrid PF-spin systems. Notably, $\mathbb{Z}_{2m}$ Fock parafermions facilitate the separation of integer and fractional charges, significantly simplifying the description of various physical processes. Additionally, we highlight the previously overlooked importance of particle-hole (PH) symmetry in these systems, demonstrating the constraints it imposes. Deriving an effective model, we show that a hybrid PF-spin qubit enables universal quantum computation, with the special case of $m = 1$ reproducing the results for Majoranas derived in Ref. [33]. We furthermore demonstrate that this setup can be used to read out the parafermion fractional charge by measuring the spin-qubit resonant frequency given by the Zeeman splitting.

The rest of the paper is organized as follows. First, in Sec. II, we briefly review $\mathbb{Z}_{2m}$ parafermions and some of their properties, e.g., the relation between parafermions, Fock parafermions, and impenetrable anyons via nonlinear basis transformations. In Sec. III, we consider $\mathbb{Z}_{2m}$ parafermion edge states and demonstrate that the underlying PH symmetry imposes significant constraints on the permissible terms in the low-energy Hamiltonian, which describes the hybridization of these edge states in a finite system. Additionally, we discuss the generalization of PH symmetry to arbitrary systems that incorporate $\mathbb{Z}_{2m}$ PFs. In Sec. IV, we describe a model setup that consists of a single-level quantum dot tunnel coupled to a pair of NWs, each hosting two $\mathbb{Z}_{2m}$ PF edge states. In Sec. V, we derive an effective exchange Hamiltonian that acts on the

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quantum dot spin qubit and the parafermion qubit. In Sec. VI, we discuss two concrete physical systems in which our proposal can be realized: (i) a quantum dot coupled to semiconductor nanowires with spin-orbit and electron-electron interactions that host $\mathbb{Z}_4$ parafermion edge states, and (ii) a quantum dot coupled to a FQHS in the $\nu = \frac{1}{3}$ Laughlin state and superconducting nanowires hosting $\mathbb{Z}_6$ parafermion edge states. Then, in Sec. VII A, we describe the readout scheme for the fractional parafermion charges. In Sec. VII B we discuss the universal set of quantum gates that can be implemented using the hybrid setup formed by coupling a quantum dot spin qubit and a $\mathbb{Z}_{2m}$ parafermion qubit. Finally, in Sec. VIII we discuss our results and conclude.

II. $\mathbb{Z}_{2m}$ (FOCK) PARAFERMIONS

In this section, we provide a brief overview of $\mathbb{Z}_{2m}$ parafermions, Fock parafermions, and some of their properties that we will need later. For a more detailed discussion of Fock parafermions we refer, e.g., to Refs. [37,38]. We then demonstrate a simple relation between the parafermions and impenetrable (hard-core) anyons [39].

A. Parafermion and Fock parafermion operators

First, we discuss the usual parafermions. A set of ordered PF operators $\{\hat{\alpha}_j\}$ satisfy

$$\hat{\alpha}_j^{2m} = 1, \quad \hat{\alpha}_j^\dagger = \hat{\alpha}_j^{2m-1}, \quad (1)$$

and the generalized Clifford algebra commutation relations

$$\hat{\alpha}_j^\dagger \hat{\alpha}_k = e^{\frac{i\pi}{m} \text{sgn}(j-k)} \hat{\alpha}_k \hat{\alpha}_j^\dagger, \quad (2)$$

$$\hat{\alpha}_j \hat{\alpha}_k = e^{-\frac{i\pi}{m} \text{sgn}(j-k)} \hat{\alpha}_k \hat{\alpha}_j, \quad (3)$$

where $\text{sgn}(x)$ is the sign function and $\text{sgn}(0) = 0$. For later convenience we relabel the PF operators as

$$\hat{\alpha}_{j,1} \equiv \hat{\alpha}_{2j-1}, \quad \hat{\alpha}_{j,2} \equiv \hat{\alpha}_{2j}. \quad (4)$$

Clearly, for $m = 1$, the PF operators $\hat{\alpha}_{j,a}$ reduce to ordinary Majorana fermion operators.

Just like a pair of Majorana fermions can be related to a complex spinless fermion by a (linear) transformation, one can similarly relate a pair of $\mathbb{Z}_{2m}$ parafermions to a single particle, the so-called *Fock* parafermion (FPF). However, in the latter case the corresponding transformation is nonlinear and it reads as

$$\hat{\alpha}_{j,1} = \hat{\mathcal{F}}_j + (\hat{\mathcal{F}}_j^\dagger)^{2m-1}, \quad \hat{\alpha}_{j,2} = -\hat{\alpha}_{j,1} e^{\frac{i\pi}{m}(\hat{\mathcal{N}}_j - \frac{1}{2})}, \quad (5)$$

where $\hat{\mathcal{F}}_j$ ($\hat{\mathcal{F}}_j^\dagger$) is the FPF annihilation (creation) operator and

$$\hat{\mathcal{N}}_j = \sum_{l=1}^{2m-1} (\hat{\mathcal{F}}_j^\dagger)^l \hat{\mathcal{F}}_j^l \quad (6)$$

is the FPF number operator. The FPF operators satisfy the generalized Pauli principle

$$(\hat{\mathcal{F}}_j^\dagger)^{2m} = \hat{\mathcal{F}}_j^{2m} = 0, \quad (7)$$

which means that one can have up to $2m - 1$ identical Fock parafermions in the same quantum state. For the operators

acting on the same site one has

$$\hat{\mathcal{F}}_j^p (\hat{\mathcal{F}}_j^\dagger)^p + (\hat{\mathcal{F}}_j^\dagger)^{2m-p} \hat{\mathcal{F}}_j^{2m-p} = 1, \quad (8)$$

with $p = 1, \dots, 2m - 1$. The FPF operators acting on different sites satisfy the commutation relations similar to those in Eq. (2):

$$\hat{\mathcal{F}}_j^\dagger \hat{\mathcal{F}}_k = e^{\frac{i\pi}{m} \text{sgn}(j-k)} \hat{\mathcal{F}}_k \hat{\mathcal{F}}_j^\dagger, \quad (9)$$

$$\hat{\mathcal{F}}_j \hat{\mathcal{F}}_k = e^{-\frac{i\pi}{m} \text{sgn}(j-k)} \hat{\mathcal{F}}_k \hat{\mathcal{F}}_j. \quad (10)$$

It is easy to see that the number operator satisfies

$$[\hat{\mathcal{N}}_j, \hat{\mathcal{F}}_k^l] = -l \delta_{jk} \hat{\mathcal{F}}_j^l, \quad [\hat{\mathcal{N}}_j, (\hat{\mathcal{F}}_k^\dagger)^l] = l \delta_{jk} (\hat{\mathcal{F}}_j^\dagger)^l, \quad (11)$$

for $l = 1, \dots, 2m - 1$. One can easily see that Eqs. (6)–(10) reduce to the canonical fermion relations for $m = 1$.

For completeness, we note that by using the Fradkin-Kadanoff transformation [40], one can write the $\mathbb{Z}_{2m}$ FPF operator $\hat{\mathcal{F}}_j$ as

$$\hat{\mathcal{F}}_j = \left(\prod_{k < j} \hat{Z}_k \right) \left[\frac{2m-1}{2m} \hat{X}_j - \frac{1}{2m} \hat{X}_j \sum_{l=1}^{2m-1} \hat{Z}_j^l \right], \quad (12)$$

where X_j and Z_j are the generalized Pauli matrices. They satisfy the relations

$$\hat{X}_j^{2m} = \hat{Z}_j^{2m} = 1, \quad \hat{X}_j^{2m-1} = \hat{X}_j^\dagger, \quad \hat{Z}_j^{2m-1} = \hat{Z}_j^\dagger, \quad (13)$$

$$\hat{X}_j \hat{Z}_k = e^{i\frac{\pi}{m} \delta_{jk}} \hat{Z}_k \hat{X}_j, \quad (14)$$

with δ_{jk} being the Kronecker delta. The relation (12) generalizes the well-known Jordan-Wigner transformation and relates Fock parafermions to spinlike (commonly referred to as “clock”) variables.

B. From Fock parafermions to impenetrable anyons

Let us now consider the operator

$$\hat{a}_j \equiv \hat{\mathcal{F}}_j^m. \quad (15)$$

Then, from Eq. (7) we immediately see that $\hat{a}_j$ satisfies the Pauli principle

$$\hat{a}_j^2 = (\hat{a}_j^\dagger)^2 = 0, \quad (16)$$

and the relations (8)–(10) yield

$$\hat{a}_j^\dagger \hat{a}_k + e^{i(m-1)\pi \text{sgn}(j-k)} \hat{a}_k \hat{a}_j^\dagger = \delta_{jk}, \quad (17)$$

$$\hat{a}_j \hat{a}_k + e^{-i(m-1)\pi \text{sgn}(j-k)} \hat{a}_k \hat{a}_j = 0. \quad (18)$$

The operator $\hat{a}_j$ satisfying Eqs. (16), (17), and (18) is nothing else than a hard-core (impenetrable) anyon with the statistical angle $(m-1)\pi$. It is well known that it can be represented in the following way [39]

$$\hat{a}_j = e^{i(m-1)\pi \sum_{k=1}^{j-1} \hat{f}_k^\dagger \hat{f}_k} \hat{f}_j, \quad (19)$$

where $\hat{f}_j$ is the spinless fermion satisfying the canonical relations

$$\hat{f}_j^2 = 0, \quad \{\hat{f}_j, \hat{f}_k^\dagger\} = \delta_{jk}, \quad \{\hat{f}_j, \hat{f}_k\} = 0. \quad (20)$$

In general, the quantum statistical angle of an impenetrable anyon can be any real number. However, in our case the

statistical angle is an integer multiple of π . Therefore, we immediately see that for odd values of m the exponential string factor in Eq. (19) disappears. On the contrary, for even values of m , Eq. (19) simply yields the Jordan-Wigner transformation from fermions to hard-core bosons (equivalently, spin- $\frac{1}{2}$ operators). Thus, we obtain

$$\hat{a}_j = \hat{f}_j \text{ (} m \text{ odd)}, \quad \hat{a}_j = \hat{b}_j \text{ (} m \text{ even)}, \quad (21)$$

where $\hat{b}_j$ is the hard-core boson annihilation operator that satisfies $[\hat{b}_j, \hat{b}_k^\dagger] = \delta_{jk}(1 - 2\hat{b}_j^\dagger \hat{b}_j)$. Keeping in mind Eq. (15), we see that the $\mathbb{Z}_{2m}$ Fock parafermion $\hat{\mathcal{F}}_j$ can be thought of as the “ m th root” of a fermion $\hat{f}_j$ (hard-core boson $\hat{b}_j$) if m is odd (even).

We note in passing that for $m = 2$ one can easily construct an operator mapping from $\mathbb{Z}_4$ onto the spin- $\frac{1}{2}$ fermions since in both cases the local Hilbert space is four dimensional and the required anticommutation relations on different sites can be achieved with the help of Jordan-Wigner-type string operators [7,41,42]. As a side remark, $\mathbb{Z}_3$ Fock parafermions can be mapped onto spin- $\frac{1}{2}$ fermions with an infinitely strong repulsive interaction between different spin species, which effectively excludes the doubly occupied states [43].

III. PARA-FERMION EDGE STATES AND PARTICLE-HOLE SYMMETRY

Throughout this paper, we assume that all considered systems hosting parafermion edge states possess particle-hole symmetry. Indeed, this symmetry is required for the edge states to be employed in the construction of topologically protected qubits. In this section, we discuss how particle-hole symmetry manifests in effective low-energy Hamiltonians that describe parafermion edge states. A related discussion, involving systems that contain both parafermions and fermions, can be found in Appendix B.

A. Effective Hamiltonian for PF edge states

Consider a finite system hosting $\mathbb{Z}_{2m}$ PF edge states $\hat{\alpha}_1$ and $\hat{\alpha}_2$. Let $\hat{H}_{\text{edge}}$ be the low-energy effective Hamiltonian describing the hybridization of the edge states. For instance, for Majorana ($m = 1$) edge states $\hat{\alpha}_i \equiv \hat{\gamma}_i$, with $i \in \{1, 2\}$, one has $\hat{H}_{\text{edge}} = i\eta\hat{\gamma}_1\hat{\gamma}_2$, with the hybridization amplitude η being exponentially small in the system size [44]. For $\mathbb{Z}_{2m}$ PFs with $m > 1$ the low-energy Hamiltonian $\hat{H}_{\text{edge}}$ is more complicated, as it can include terms involving higher powers of PF operators. Requiring the conservation of $\mathbb{Z}_{2m}$ parity, which is generated by the operator

$$\hat{P} = e^{i\pi/m} \hat{\alpha}_1^\dagger \hat{\alpha}_2, \quad (22)$$

one finds the following form of the effective Hamiltonian [10,45]:

$$\hat{H}_{\text{edge}} = \frac{1}{2} \sum_{k=1}^m \eta_k (e^{i\theta_k} (\hat{\alpha}_1^\dagger)^k \hat{\alpha}_2^k + \text{H.c.}), \quad (23)$$

where the factor $\frac{1}{2}$ is introduced for convenience and the parameters η_k and θ_k are real.

It is convenient to rewrite the Hamiltonian $\hat{H}_{\text{edge}}$ defined in Eq. (23) in terms of the FPF operators. Using Eq. (5), we find

$$(\hat{\alpha}_1^\dagger)^k \hat{\alpha}_2^k = (-1)^k e^{\frac{i\pi k}{m} (\hat{\mathcal{N}} - \frac{k}{2})}, \quad (24)$$

where $\hat{\mathcal{N}}$ is the FPF number operator defined in Eq. (6). Consequently, the Hamiltonian $\hat{H}_{\text{edge}}$ expressed in terms of FPFs becomes

$$\hat{H}_{\text{edge}} = \sum_{k=1}^m (-1)^k \eta_k \cos \left[\frac{\pi k}{m} \left(\hat{\mathcal{N}} - \frac{k}{2} \right) + \theta_k \right]. \quad (25)$$

The generator of $\mathbb{Z}_{2m}$ parity, when expressed in terms of FPF operators, is given by $\hat{P} = e^{\frac{i\pi}{m} \hat{\mathcal{N}}}$, which obviously commutes with $\hat{H}_{\text{edge}}$.

For Majorana edge states, the Hamiltonian $\hat{H}_{\text{edge}}$ reduces to

$$\hat{H}_{\text{edge}} = \eta(2\hat{c}^\dagger \hat{c} - 1) \quad (m = 1), \quad (26)$$

where we denoted $\eta = \eta_1 \sin \theta_1$ and introduced the fermionic operator $\hat{c} = (\hat{\gamma}_1 - i\hat{\gamma}_2)/2$ that satisfies the canonical anticommutation relations $\{\hat{c}, \hat{c}^\dagger\} = 1$. Obviously, the Hamiltonian (26) is particle-hole symmetric. This symmetry is reflected in its energy spectrum, which consists of $\pm\eta$ and is symmetric around zero energy.

Interestingly, for the general case of $\mathbb{Z}_{2m}$ PFs with arbitrary m , the spectrum of $\hat{H}_{\text{edge}}$, defined in Eq. (25), is symmetric around zero energy *if and only if* the hybridization amplitudes satisfy $\eta_k = 0$ for all even values of k . Only under these conditions does the Hamiltonian $\hat{H}_{\text{edge}}$ exhibit particle-hole symmetry. In the remainder of this section, we will provide an analytic proof of this observation.

B. Particle-hole symmetry for $\mathbb{Z}_{2m}$ FPFs

In the presence of particle-hole symmetry, the energy levels of $\hat{H}_{\text{edge}}$ are necessarily symmetric around zero energy. This symmetry implies the existence of an operator $\hat{\mathcal{C}}$ that acts on the low-energy Hamiltonian as

$$\hat{\mathcal{C}} \hat{H}_{\text{edge}} \hat{\mathcal{C}}^{-1} = -\hat{H}_{\text{edge}}. \quad (27)$$

The operator $\hat{\mathcal{C}}$ corresponds to the particle-hole transformation, which must be antiunitary and satisfy $\hat{\mathcal{C}}^2 = 1$. One can always write the particle-hole transformation as

$$\hat{\mathcal{C}} = \hat{U} \hat{\mathcal{K}}, \quad (28)$$

where $\hat{U}$ is unitary and $\hat{\mathcal{K}}$ is the complex conjugation, whose action is $\hat{\mathcal{K}} i \hat{\mathcal{K}}^{-1} = -i$.

In order to understand the meaning of particle-hole transformation for $\mathbb{Z}_{2m}$ PFs and construct an explicit form of $\hat{\mathcal{C}}$, let us first briefly do so for ordinary fermions. In the case of Majorana edge states the particle hole transformation $\hat{\mathcal{C}}$ that anticommutes with the edge state Hamiltonian (26) is simply given by

$$\hat{\mathcal{C}} = (\hat{c} + \hat{c}^\dagger) \hat{\mathcal{K}} \quad (m = 1). \quad (29)$$

Indeed, one can always choose a suitable basis in which $\hat{\mathcal{K}} \hat{c} \hat{\mathcal{K}}^{-1} = \hat{c}$, so that $\hat{\mathcal{C}} \hat{c} \hat{\mathcal{C}}^{-1} = \hat{c}^\dagger$ and $\hat{\mathcal{C}} \hat{c}^\dagger \hat{\mathcal{C}}^{-1} = 1 - \hat{c}^\dagger \hat{c}$. Thus, the edge-state Hamiltonian $\hat{H}_{\text{edge}}$ satisfies Eq. (27). Clearly, the operator $\hat{\mathcal{C}}$ is not unique as it is defined up to a

unitary transformation that commutes with the Hamiltonian $\hat{H}_{\text{edge}}$.

Generalization to the case of $\mathbb{Z}_{2m}$ FPFs with arbitrary m is straightforward. Since the particle-hole transformation acts on integer charges, the natural candidate for the generator of the particle-hole transformation is

$$\hat{C} = (\hat{F}^m + (\hat{F}^\dagger)^m) \hat{K} \quad (\text{any } m), \quad (30)$$

where $\hat{F}$ is the FPF annihilation operator. Note that $\hat{C}$ does not simply exchange the $\mathbb{Z}_{2m}$ FPF annihilation and creation operators. The reason is that the action of particle-hole transformation on a fractional charge is not well defined. However, for an operator $\hat{F}^m$, which annihilates an integer charge, the behavior under the particle-hole transformation is as expected and one has $\hat{C} \hat{F}^m \hat{C}^{-1} = (\hat{F}^m)^\dagger$. Similarly, using Eqs. (7) and (11), one can easily show that the action of $\hat{C}$ on the FPF number operator $\hat{N}$, given by Eq. (6), reads as

$$\hat{C} \hat{N} \hat{C}^{-1} = m - 2m(\hat{F}^m)^\dagger \hat{F}^m + \hat{N}. \quad (31)$$

Note that the term $(\hat{F}^m)^\dagger \hat{F}^m$ is diagonal in the Fock space basis and has integer eigenvalues $[\mathcal{N}/m]$, where $\mathcal{N} \in \{0, 1, \dots, 2m-1\}$ is the eigenvalue of the $\mathbb{Z}_{2m}$ FPF number operator $\hat{N}$.

It is now easy to check whether the PF edge state Hamiltonian $\hat{H}_{\text{edge}}$ in Eq. (25) possesses particle-hole symmetry. Using Eq. (31), we see that under the transformation $\hat{C} \hat{H}_{\text{edge}} \hat{C}^{-1}$ the cosine in Eq. (25) acquires a phase shift πk . Therefore, in order to satisfy the particle-hole symmetry condition $\hat{C} \hat{H}_{\text{edge}} \hat{C}^{-1} = -\hat{H}_{\text{edge}}$ [equivalently, Eq. (27)], the sum over k in Eq. (25) must contain only terms with k odd, i.e., we have $\eta_k \equiv 0$ for k even.

Before we finish this section, let us comment on the particle-hole symmetry in the general case of a many-body $\mathbb{Z}_{2m}$ FPF system. There, the particle-hole transformation needs to be slightly modified and it reads as

$$\hat{C} = \prod_j e^{i\pi m \sum_{k < j} (\hat{F}_k^\dagger)^m \hat{F}_k^m} [\hat{F}_j^m + (\hat{F}_j^\dagger)^m], \quad (32)$$

where the Jordan-Wigner-type string $e^{i\pi m \sum_{k < j} (\hat{F}_k^\dagger)^m \hat{F}_k^m}$ ensures that $\hat{C}^2 = 1$. Then, one has

$$\hat{C} \hat{F}_j^m \hat{C}^{-1} = (-1)^{m(j-1)} (\hat{F}_j^m)^\dagger, \quad (33)$$

whereas for $\hat{C} \hat{N}_j \hat{C}^{-1}$ one obtains the relation equivalent to Eq. (31).

IV. PARA-FERMION EDGE STATES COUPLED TO QUANTUM DOT

Let us now proceed with describing our setup. We consider a pair of $\mathbb{Z}_{2m}$ parafermion edge states $\hat{\alpha}_{v,1}$ and $\hat{\alpha}_{v,2}$ separated by a finite distance L , as shown in Fig. 1. In this notation $v = l$ for the PFs in the left NW and $v = r$ for the PFs in the right NW. The hybridization of the PFs can be described by the Hamiltonian [10,45]

$$\hat{H}_{\text{PF}} = \frac{1}{2} \sum_{v=l,r} \sum_{k=1}^m \eta_{vk} (e^{i\theta_{vk}} (\hat{\alpha}_{v,1}^\dagger)^k \hat{\alpha}_{v,2}^k + \text{H.c.}), \quad (34)$$

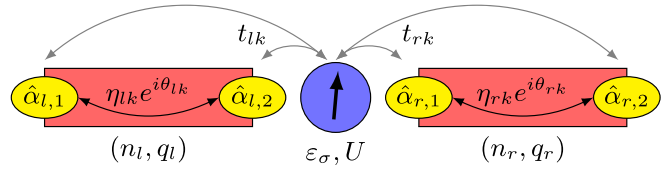


FIG. 1. Setup of two NWs (red bars) hosting a pair of $\mathbb{Z}_{2m}$ PF edge states on the left (right) NW, $\hat{\alpha}_{l,1}$ and $\hat{\alpha}_{l,2}$ ($\hat{\alpha}_{r,1}$ and $\hat{\alpha}_{r,2}$). These parafermions have quantum numbers n_v and q_v , indicating their integer and fractional charge, respectively. The PFs on each NW can overlap as described by the Hamiltonian defined in Eq. (34). The amplitude of the hybridization process $(\hat{\alpha}_{v,1}^\dagger)^k \hat{\alpha}_{v,2}^k$ is given by $\eta_{vk} e^{i\theta_{vk}}$, with k taking odd values in the interval $[1, m]$. A quantum dot (blue) is placed between the two NWs. It hosts the spin qubit with two energy levels $\varepsilon_{\sigma=\uparrow,\downarrow}$, for spin up and spin down, respectively, and includes a Coulomb repulsion U . Integer charges can tunnel between the dot and the NWs with an amplitude t_{vk} .

where the amplitudes $\eta_{vk} > 0$, the phases θ_{vk} are real, and we have assumed that there is no overlap between PFs on different nanowires. As we have shown in Sec. III, in order for the Hamiltonian (34) to exhibit particle-hole symmetry, the hybridization amplitudes must satisfy the condition $\eta_{vk} = 0$ for even values of k . A perturbative instanton calculation shows that the amplitudes η_{vk} are exponentially small in the system size L and one has $\eta_{vk} \propto e^{-\kappa_v L}$, where $\kappa_v > 0$ [45]. We note that, using the ordering $l < r$, parafermion operators from different wires follow the same commutation relations as defined in Eq. (2).

In terms of the FPFs, the Hamiltonian $\hat{H}_{\text{PF}}$ reads as

$$\hat{H}_{\text{PF}} = \sum_{v=l,r} \sum_{k=1}^m (-1)^k \eta_{vk} \cos \left[\frac{\pi k}{m} \left(\hat{N}_v - \frac{k}{2} \right) + \theta_{vk} \right]. \quad (35)$$

The eigenvalues of $\hat{N}_v$ are $\mathcal{N}_v \in \{0, 1, \dots, 2m-1\}$ and they correspond to the fractional charge $Q_v = \mathcal{N}_v e/m$ carried by the pair of parafermion states $\hat{\alpha}_{v,1}$ and $\hat{\alpha}_{v,2}$, with e being the electron charge. It is then useful to separate the integer and fractional parts of the charge and write $Q_v = n_v e + q_v e/m$, where $n_v \in \{0, 1\}$ and $q_v \in \{0, \dots, m-1\}$ (see Fig. 2). The integer part n_v can be changed in electron tunneling processes, whereas the fractional part q_v is a topologically protected quantity, which is conserved unless the system is coupled to a reservoir of fractional charges. The spectrum of the Hamiltonian $\hat{H}_{\text{PF}}$ [see Eq. (35)] is particle-hole symmetric and the particle-hole pairs are built up from the states with charge $Q_v = q_v e/m$ and $Q_v = (m + q_v) e/m$, which correspond to $n_v = 0$ and 1 , respectively. Note that this symmetry is not reflected in the schematics of Fig. 2.

Let us now consider a single-level quantum dot, hosting a spin qubit, placed between the two nanowires, as shown in Fig. 1. The Hamiltonian of the quantum dot is

$$\hat{H}_{\text{D}} = \sum_{\sigma=\uparrow,\downarrow} \varepsilon_{\sigma} \hat{N}_{\sigma} + U \hat{N}_{\uparrow} \hat{N}_{\downarrow}, \quad (36)$$

where $\hat{N}_{\sigma} = \hat{d}_{\sigma}^\dagger \hat{d}_{\sigma}$, the operator $\hat{d}_{\sigma}^\dagger$ ($\hat{d}_{\sigma}$) creates (annihilates) an electron in the spin- σ state on the quantum dot, ε_{σ} is the energy of the spin- σ level in the quantum dot, and $U > 0$ is the interaction energy, which we take to be the largest

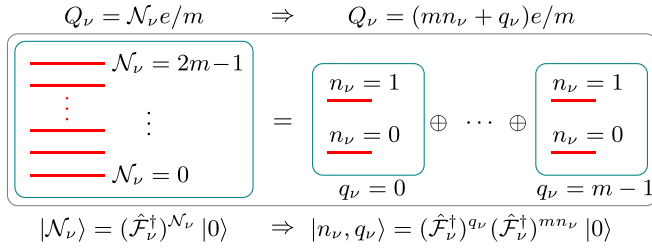


FIG. 2. Schematics of the Hilbert space structure for a $\mathbb{Z}_{2m}$ Fock parafermion in the v th nanowire. The electric charge Q_v is proportional to the number N_v of Fock parafermions and one has $N_v = mn_v + q_v$, where $n_v \in \{0, 1\}$ and $q_v \in \{0, 1, \dots, m-1\}$. Relabeling the states by the pair of quantum numbers (n_v, q_v) , the $2m$ -dimensional local Hilbert space of the Fock parafermion states can be viewed as m copies of two-dimensional Hilbert spaces, which can be associated with an effective fermionic degree of freedom. In this picture, the quantum numbers n_v and q_v are, respectively, the integer and fractional part of the electric charge Q_v .

energy scale. More precisely, we choose the chemical potential (absorbed in ε_σ) on the quantum dot such that its ground state contains one spinful electron and, moreover, that it costs less charging energy to remove the electron from the dot (e.g., in a virtual process) than to add a second electron to it (suppressed by large U). For more details we refer to Appendix C.

The quantum dot is coupled to the NWs and electrons can tunnel between the dot and edge states of the nanowires, as shown in Fig. 1. Clearly, all tunneling processes between the dot and the edge states transfer integer values of electric charge and vary it by $\pm e$. Thus, the coupling Hamiltonian can be written as

$$\hat{V} = \sum_{v=l,r} \sum_{\sigma=\uparrow,\downarrow} \hat{V}_{v\sigma}, \quad (37)$$

$$\hat{V}_{v\sigma} = \sum_{k=0}^{2m-1} (t_{vk}^* \hat{K}_v^{(m)} e^{\frac{i\pi k^2}{2m}} \hat{\alpha}_{v,1}^{m-k} \hat{\alpha}_{v,2}^k \hat{d}_\sigma + \text{H.c.}), \quad (38)$$

where the terms proportional to t_{v0} (t_{vm}) describe the tunneling between the edge states on the left (right) end of the v th NW and the quantum dot (see Fig. 1). Similarly, the terms proportional to t_{vk} with $k \neq 0, m$ correspond to the process in which an electron on the quantum dot couples simultaneously to the edge states on both ends of the v th NW. For simplicity, we assume all tunneling processes to be spin independent. The case of spin-dependent tunneling between the quantum dot and the nanowires can be treated in the spirit of Ref. [46]. In Eq. (37) we have introduced the Klein factors $\hat{K}_v^{(m)}$ that ensure that the terms corresponding to the tunneling of different spin states to different wires are independent and therefore should commute with each other:

$$[\hat{V}_{\mu\sigma}, \hat{V}_{\nu s}] = 0, \quad \mu \neq \nu, \quad \sigma \neq s. \quad (39)$$

It is easy to check that for odd m Eq. (39) is satisfied with $\hat{K}_v^{(m)} = 1$. In this form, the Hamiltonian (37) was used in Ref. [47]. However, the case of even m requires

nontrivial Klein factors,¹ which we will specify below [see Eq. (55)].

In order to see that $\hat{V}$ indeed describes the tunneling of integer charges, we note that using Eq. (5) one obtains

$$\hat{\alpha}_{v,1}^m = (\hat{F}_v^\dagger)^m + \hat{F}_v^m, \quad \hat{\alpha}_{v,2}^m = \hat{\alpha}_{v,1}^m e^{i\pi(\hat{N}_v + \frac{m}{2})}. \quad (40)$$

Thus, the PF operator $\hat{\alpha}_{v,1}^{m-k} \hat{\alpha}_{v,2}^k$ in Eq. (37) can be rewritten as $\hat{\alpha}_{v,1}^{m-k} \hat{\alpha}_{v,2}^k = (-1)^k [(\hat{F}_v^\dagger)^m + \hat{F}_v^m] e^{\frac{i\pi k}{m}(\hat{N}_v - \frac{k}{2})}$, where we used Eq. (24). Therefore, we see that the total Hamiltonian

$$\hat{H} = \hat{H}_D + \hat{H}_{PF} + \hat{V} \quad (41)$$

includes the PF operators only of the form $\hat{N}_v$, $\hat{F}_v^m$, and $(\hat{F}_v^\dagger)^m$ and does not include operators that create or annihilate a fractional charge. Note that the spectrum of $\hat{V}$ is symmetric around zero energy for any values of t_{vk} , so that the particle-hole symmetry is preserved. In Appendix B, we present an explicit construction of the particle-hole transformation $\hat{C}$ that anticommutes with both the edge state Hamiltonian H_{edge} and the tunneling Hamiltonian $\hat{V}$.

Since the Hamiltonian (41) acts on the mixed fermion-parafermion Hilbert space, let us comment on the commutation relations between the FPF and fermion operators, following Ref. [38]. Keeping in mind that the operator $\hat{F}_v^m$ satisfies the commutation relations of the hard-core anyons, from Eqs. (15) and (19), we have

$$\{\hat{F}_v^m, \hat{d}_\sigma\} = \{\hat{F}_v^m, \hat{d}_\sigma^\dagger\} = 0, \quad (42)$$

where we took into account that the operators $\hat{d}_\sigma$ and $\hat{f}_v$ [see Eq. (19)] satisfy $\{\hat{d}_\sigma, \hat{f}_v\} = [\hat{d}_\sigma, \hat{f}_v^\dagger \hat{f}_v] = 0$. Therefore, one can easily check that the relation (42) is satisfied provided that

$$\hat{F}_v \hat{d}_\sigma = e^{i\chi} \hat{d}_\sigma \hat{F}_v, \quad \hat{F}_v \hat{d}_\sigma^\dagger = e^{-i\chi} \hat{d}_\sigma^\dagger \hat{F}_v, \quad (43)$$

where the quantum statistical angle χ satisfies $e^{im\chi} = -1$. For any m the natural choice of the statistical angle in Eq. (43) is $\chi = \pm\pi/m$, although for m odd one can also simply choose $\chi = \pi$ so that $\{\hat{F}_v, \hat{d}_\sigma\} = 0$. For the sake of completeness, in Appendix A we discuss a concrete matrix representation of operators acting on the mixed fermion-parafermion Hilbert space. The results of Appendix A can be also used to easily verify parafermion algebraic relations, such as the one in Eq. (40).

Before we proceed with the derivation of an effective model, let us discuss the symmetries of the Hamiltonian $\hat{H}$ defined in Eq. (41) and the structure of the Hilbert space. Clearly, the Hamiltonian of Eq. (41) commutes with the total $\mathbb{Z}_{2m}$ parity operator that includes both the parafermion and the fermion degrees of freedom:

$$\hat{P}_{\text{tot}} = e^{\frac{i\pi}{m}(\hat{N}_l + \hat{N}_r)} (-1)^{\hat{N}_l + \hat{N}_r}. \quad (44)$$

The eigenvalues of the total parity operator (44) are $P_{\text{tot}} = \exp(\frac{i\pi}{m}Q)$, with $Q \in \{0, 1, \dots, 2m-1\}$. Therefore, the total $4 \times (2m)^2$ -dimensional Hilbert space decomposes into $2m$ sectors with definite total parity, with each sector being

¹We thank the anonymous referee for pointing out the need for introducing the Klein factors in Eq. (38).

$8m$ dimensional. Clearly, in the sector with the total parity $\exp(\frac{i\pi}{m}\mathcal{Q})$, one has $\mathcal{Q} = [\mathcal{N}_l + \mathcal{N}_r - m(N_\uparrow + N_\downarrow)] \bmod 2m$. Therefore, in this sector we can choose the following basis states:

$$|\mathcal{N}_l, \mathcal{N}_r\rangle \otimes |N_\uparrow, N_\downarrow\rangle = |\mathcal{N}, \mathcal{Q} - \mathcal{N} + m(N_\uparrow + N_\downarrow)\rangle \otimes |N_\uparrow, N_\downarrow\rangle, \quad (45)$$

where $\mathcal{N} \in \{0, 1, \dots, 2m-1\}$, and $N_\sigma \in \{0, 1\}$.

Recall that [see discussion after Eq. (35)] the number of Fock parafermions in the ν th NW can be represented as

$$\mathcal{N}_\nu = mn_\nu + q_\nu, \quad (46)$$

where the quantum numbers $n_\nu \in \{0, 1\}$ and $q_\nu \in \{0, 1, \dots, m-1\}$ can be thought of, respectively, as the integer and fractional parts of the total electric charge carried by the ν th parafermion state. By construction, the Hamiltonian defined in Eq. (41) does not couple the states with different values of q_l and q_r , as the quantum dot only allows an integer charge to tunnel through. This means that each fixed-parity sector of the Hilbert space can be further decomposed into subsectors with fixed values of q_l and q_r . Taking into account that Eq. (46) gives us $q_\nu = \mathcal{N}_\nu \bmod m$, from Eq. (45) we find

$$q_l = \mathcal{N} \bmod m, \quad q_r = (\mathcal{Q} - \mathcal{N}) \bmod m. \quad (47)$$

Thus, each $8m$ -dimensional fixed parity sector splits into m eight-dimensional subsectors with q_l and q_r given by Eq. (47). These sectors are spanned by the states

$$|n_l, n_r\rangle \otimes |N_\uparrow, N_\downarrow\rangle, \quad (48)$$

where $n_\nu = \lfloor \mathcal{N}_\nu/m \rfloor$, with $\lfloor \cdot \rfloor$ being the floor function. In what follows we restrict ourselves to the case of a singly occupied quantum dot, $N_\uparrow + N_\downarrow = 1$, and choose the parity sector with $\mathcal{Q} = m$ [i.e., $P_{\text{tot}} = -1$].

V. EFFECTIVE MODEL

Inside each of the subsectors with fixed q_ν , the FPF number operator can be written as $\hat{\mathcal{N}}_\nu = m\hat{n}_\nu + q_\nu$, where we introduced the operator

$$\hat{n}_\nu \equiv (\hat{\mathcal{F}}_\nu^\dagger)^m \hat{\mathcal{F}}_\nu^m. \quad (49)$$

Keeping in mind Eqs. (15) and (19), one also has $\hat{n}_\nu = \hat{a}_\nu^\dagger \hat{a}_\nu = \hat{f}_\nu^\dagger \hat{f}_\nu$. Thus, projecting $\hat{H}_{\text{PF}}$ in Eq. (35) onto the Hilbert space sector with the fixed values of q_ν , we obtain a simple expression

$$\hat{H}_{\text{PF}} = \sum_{\nu=l,r} \delta_\nu(q_\nu) (2\hat{n}_\nu - 1), \quad (50)$$

where we introduced the effective q -dependent splitting

$$\delta_\nu(q) = \sum_{k=1}^m \eta_{\nu k} \cos \left[\frac{\pi k}{m} \left(q - \frac{k}{2} \right) + \theta_{\nu k} \right], \quad (51)$$

where we used the fact that due to the particle-hole symmetry one has $\eta_{\nu k} = 0$ for k even.

Keeping in mind that the fractional part of the total $\mathbb{Z}_{2m}$ charge in each nanowire is conserved, we project $\hat{V}$ onto the subspace with the fixed values of q_l and q_r . Then, taking into account that $\hat{a}_{\nu,1}^m = \hat{a}_\nu^\dagger + \hat{a}_\nu$, as follows from Eqs. (15) and (40), we rewrite Eq. (38) in terms of hard-core anyons as

$$\hat{V}_{\nu\sigma} = it_{\nu-}^*(q_\nu) \hat{K}_\nu^{(m)} \hat{a}_\nu^\dagger \hat{d}_\sigma - it_{\nu+}^*(q_\nu) \hat{K}_\nu^{(m)} \hat{a}_\nu \hat{d}_\sigma + \text{H.c.}, \quad (52)$$

where we used $(\hat{a}_\nu^\dagger + \hat{a}_\nu)e^{i\pi k \hat{n}_\nu} = \hat{a}_\nu^\dagger + (-1)^k \hat{a}_\nu$ and introduced the effective tunneling amplitude

$$t_{\nu\pm}(q) = \mp i \sum_{k=0}^{2m-1} (\pm 1)^k t_{\nu k} e^{-i\pi k q/m}, \quad (53)$$

which depends on the fractional charge quantum numbers q_ν . In deriving Eqs. (52) and (53), we included the factors of $\pm i$ for later convenience and used the expression for $(\hat{a}_{\nu,1}^\dagger)^k \hat{a}_{\nu,2}^k$ in Eq. (24).

Then, keeping in mind the relation between the hard-core anyons $\hat{a}_\nu$ and the fermionic operators $\hat{f}_\nu$ in Eq. (19), in order to satisfy Eq. (39) we require

$$\hat{K}_\nu^{(m)} \hat{a}_\nu = \hat{f}_\nu, \quad \hat{K}_\nu^{(m)} \hat{a}_\nu^\dagger = \hat{f}_\nu^\dagger, \quad (54)$$

which can be satisfied by choosing the Klein factors as

$$\hat{K}_\nu^{(m)} = e^{i(m-1)\pi \sum_{\mu < \nu} (\hat{\mathcal{F}}_\mu^\dagger)^m \hat{\mathcal{F}}_\mu^m}. \quad (55)$$

We then rewrite the effectively coupling Hamiltonian as

$$\hat{V}_{\nu\sigma} = it_{\nu-}^*(q_\nu) \hat{f}_\nu^\dagger \hat{d}_\sigma - it_{\nu+}^*(q_\nu) \hat{f}_\nu \hat{d}_\sigma + \text{H.c.}, \quad (56)$$

which clearly satisfies Eq. (39) for any m . Note that the operators $\hat{f}_\nu$ and $\hat{d}_\sigma$ anticommute with each other.

The case of $m = 1$, which corresponds to the Majorana edge states coupled to the quantum dot, has been studied in Ref. [33]. Then, comparing our Eqs. (50) and (56) with Eqs. (2) and (4) in Ref. [33], we see that for any m the total Hamiltonian $\hat{H} = \hat{H}_D + \hat{H}_{\text{PF}} + \sum_{\nu,\sigma} \hat{V}_{\nu\sigma}$ coincides exactly with the corresponding Hamiltonian in Ref. [33], if we simply replace the coefficients $t_{\nu\pm}$ from Ref. [33] with $t_{\nu\pm}(q_\nu)$ given by Eq. (53). Therefore, *all* results obtained for $m = 1$ in Ref. [33] can be immediately extended to any value of m .

In what follows we focus on the regime of a singly occupied quantum dot, i.e., we require $\varepsilon_\sigma < 0$ (the chemical potential of the NWs is set to zero) and U being the largest energy scale. Assuming that the coupling between the nanowires and the quantum dot is weak compared to the difference in energies of the dot electrons and the PF edge states, we apply the Schrieffer-Wolff transformation to the total Hamiltonian $\hat{H} = \hat{H}_{\text{PF}} + \hat{H}_D + \sum_{\nu,\sigma} \hat{V}_{\nu\sigma}$ [48,49]. Then, projecting the resulting effective Hamiltonian onto the subspace with a singly occupied quantum dot and taking the limit $U \rightarrow +\infty$, to the second order in the tunneling amplitudes we obtain the effective Hamiltonian

$$\hat{\mathcal{H}}_{\text{eff}} = \hat{H}_D + \hat{H}_{\text{PF}} + \hat{\mathcal{H}}_T^{(s)} + \hat{\mathcal{H}}_T^{(e)} + \hat{\mathcal{H}}_T^{(o)}, \quad (57)$$

where $\hat{H}_D$ and $\hat{H}_{PF}$ are given by Eqs. (36) and (50), respectively, and we introduced the effective coupling terms (see Appendix C for more details)

$$\hat{\mathcal{H}}_T^{(s)} = \frac{1}{2} \sum_{v,\sigma} \left(\frac{|t_{v+}(q_v)|^2}{\varepsilon_\sigma + 2\delta_v(q_v)} \hat{f}_v^\dagger \hat{f}_v + \frac{|t_{v-}(q_v)|^2}{\varepsilon_\sigma - 2\delta_v(q_v)} \hat{f}_v \hat{f}_v^\dagger \right) \hat{\mathcal{B}}_\sigma, \quad (58)$$

$$\begin{aligned} \hat{\mathcal{H}}_T^{(o)} &= \frac{1}{2} \sum_{v,\sigma} \left[t_{v+}^*(q_v) t_{v+}(q_v) \left(\frac{\hat{\mathcal{A}}_\sigma^\dagger}{\varepsilon_\sigma + 2\delta_v(q_v)} + \frac{\hat{\mathcal{A}}_\sigma}{\varepsilon_\sigma + 2\delta_v(q_v)} \right) \right. \\ &\quad \times \hat{f}_v^\dagger \hat{f}_v + t_{v-}^*(q_v) t_{v-}(q_v) \left(\frac{\hat{\mathcal{A}}_\sigma^\dagger}{\varepsilon_\sigma - 2\delta_v(q_v)} \right. \\ &\quad \left. \left. + \frac{\hat{\mathcal{A}}_\sigma}{\varepsilon_\sigma - 2\delta_v(q_v)} \right) \hat{f}_v \hat{f}_v^\dagger \right], \quad (59) \end{aligned}$$

$$\begin{aligned} \hat{\mathcal{H}}_T^{(e)} &= -\frac{1}{2} \sum_{v,\sigma} \left[t_{v+}^*(q_v) t_{v-}(q_v) \left(\frac{\hat{\mathcal{A}}_\sigma^\dagger}{\varepsilon_\sigma - 2\delta_v(q_v)} \right. \right. \\ &\quad \left. \left. + \frac{\hat{\mathcal{A}}_\sigma}{\varepsilon_\sigma + 2\delta_v(q_v)} \right) \hat{f}_v \hat{f}_v + \text{H.c.} \right], \quad (60) \end{aligned}$$

with the operators $\hat{\mathcal{A}}_\sigma = \hat{N}_\sigma + \hat{d}_\sigma^\dagger \hat{d}_\sigma$ and $\hat{\mathcal{B}}_\sigma = \hat{\mathcal{A}}_\sigma + \hat{\mathcal{A}}_\sigma^\dagger$. The term $\hat{\mathcal{H}}_T^{(s)}$ involves the tunneling between the dot and a single nanowire, whereas the terms $\hat{\mathcal{H}}_T^{(o)}$ and $\hat{\mathcal{H}}_T^{(e)}$ involve the tunneling between the dot and both NWs and act on the subspace with odd and even parity of the NWs, respectively.

Before we proceed with discussing concrete platforms that can be used to realize the proposed hybrid parafermion-spin qubits, let us briefly comment on the physical origin of the pair creation and annihilation terms in Eq. (60). In the case of odd m , these terms originate from the underlying superconductor and can be understood simply in terms of a Cooper pair being created in one of the wires, with one of the pair electrons subsequently tunneling to another wire through the dot (see, e.g., Fig. 2 in Ref. [33] for the illustration of this process in the case of $m = 1$). The case of even m is more peculiar and the precise physical mechanism is less obvious. As we discuss in the beginning of Sec. VI, in systems with even- m $\mathbb{Z}_{2m}$ parafermions, there is no superconductivity to drive pair processes analogously to Cooper pairing. Instead, the pair creation and annihilation terms $\hat{f}_v \hat{f}_v^\dagger$ and $\hat{f}_v^\dagger \hat{f}_v$ in Eq. (60) should originate from the strong electron-electron interactions at half-filling.

VI. PHYSICAL PLATFORMS

A. $\mathbb{Z}_4$ PF edge states

The setup we propose to realize $\mathbb{Z}_4$ parafermions is based on Ref. [26]. There, a Rashba NW with strong electron-electron interaction is gated such that the chemical potential alternates between values μ_1 and μ_2 (see Fig. 3). The exact values of μ_1 and μ_2 depend on the system parameters and they are given in Ref. [26]. The Rashba spin-orbit interaction (SOI) has strength α_R , giving the SOI momentum $k_{SO} = m\alpha_R$, with m being the effective mass of the electrons in the NW. A magnetic field $\vec{B}$ is applied perpendicular to the nanowire and

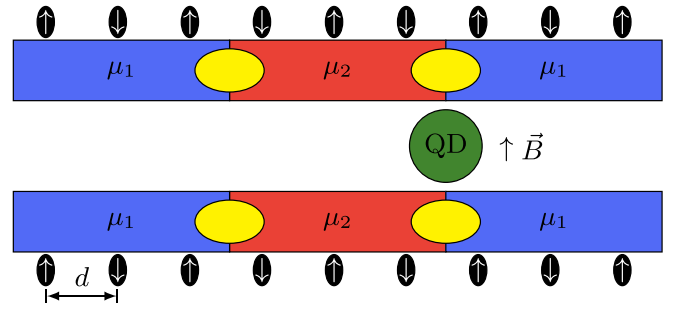


FIG. 3. Sketch of a setup to experimentally get $\mathbb{Z}_4$ parafermions. The setup is adapted from Ref. [26]. Two nanowires are each gated such that the chemical potentials alternate between μ_1 (blue regions) and μ_2 (red regions). Parafermion bound states (yellow) appear at the domain walls between the different chemical potentials. A magnetic field $\vec{B}$ is applied perpendicular to the nanowires and a quantum dot (green) placed between the two nanowires. Two chains of nanomagnets (black) with alternating magnetization (white arrows) and distance d are placed close to the nanowires.

a row of nanomagnets with alternating magnetizations and a distance $d = \pi/4k_{SO}$ between the nanomagnets is placed close to the nanowire. In this setup, parafermions appear at the domain walls between the regions with different chemical potential. In order to get two pairs of parafermions, two nanowires are placed on either side of a quantum dot, as shown in Fig. 3.

We emphasize that this realization of $\mathbb{Z}_4$ parafermions does not require superconductivity but instead relies on a strongly interacting semiconductor nanowire with spin-orbit coupling and weak magnetic fields at half-filling. The effective “particle” and “hole” sectors can be mapped onto each other by virtue of this special interplay of interaction, filling fraction, and band structure in the wire. Consequently, a robust particle-hole symmetry emerges as an effective symmetry of the low-energy sector, giving rise to $\mathbb{Z}_4$ parafermion bound states localized at interfaces [26].

For $\mathbb{Z}_4$ parafermions, the effective tunneling Hamiltonian is given by Eqs. (58)–(60) with $m = 2$. The effective splitting of the edge states $\delta_v(q)$ defined in Eq. (51) becomes

$$\delta_v(q) = \eta_{v,1} \cos \left(\frac{q\pi}{2} - \frac{\pi}{4} + \theta_{v,1} \right), \quad (61)$$

where $q \in \{0, 1\}$.

B. $\mathbb{Z}_6$ PF edge states

The paradigmatic setup for the experimental realization of $\mathbb{Z}_6$ PFs is based on Refs. [11,47,50,51], where a trench is etched into a FQHS and filled with superconducting material. A parafermion then appears at each end of the trench. In order to get $\mathbb{Z}_6$ parafermions, the filling factor of the FQHS must be $\nu = 1/3$ [1,8,10,47,51,52]. In such a setup, crossed Andreev reflection, which is essential for the existence of parafermions in this system, has been experimentally observed [50]. However, it is important to note that the presence of crossed Andreev reflection does not conclusively prove the existence of parafermions in the system [53]. Because the proposed setup requires two pairs of parafermions, two trenches must

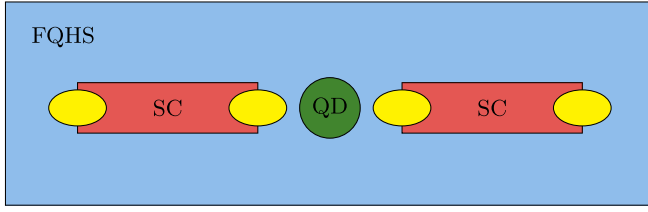


FIG. 4. Sketch of a setup to experimentally get $\mathbb{Z}_6$ parafermions. Two trenches are etched into a FQHS with filling factor $\nu = 1/3$. The trenches are filled with a superconducting material (red), which leads to parafermions (yellow) emerging at the ends of the trench. Between the trenches, a quantum dot (green) is located.

be etched into the FQHS (see Fig. 4). In-between the two superconducting trenches, there is a quantum dot, which can, e.g., be realized as an antidot [54–65].

In the case of $\mathbb{Z}_6$ parafermions, the effective tunneling Hamiltonian follows from Eqs. (58)–(60) with $m = 3$. The effective splitting of the edge states $\delta_\nu(q)$, defined in Eq. (51), becomes

$$\delta_\nu(q) = \eta_{\nu,1} \cos\left(\frac{\pi q}{3} - \frac{\pi}{6} + \theta_{\nu,1}\right) - \eta_{\nu,3} \sin(\pi q + \theta_{\nu,3}), \quad (62)$$

where $q \in \{0, 1, 2\}$.

VII. READOUT AND QUANTUM GATES

A. Readout of parafermion states

A possible application of the proposed hybrid parafermion-spin-qubit platform is the readout of the parafermion fractional charge. As we demonstrate below, this can be achieved via a simple measurement of the energy splitting of the spin qubit.

Without loss of generality, we read out the fractional charge of the left NW $q_l \equiv q$. Thus, we set all couplings to the right NW to zero, i.e., $t_{rk} = 0$. In this case, the effective tunneling terms $\hat{\mathcal{H}}_T^{(o)}$ and $\hat{\mathcal{H}}_T^{(e)}$ defined in Eqs. (59) and (60), respectively, vanish. Thus, the full effective Hamiltonian $\hat{\mathcal{H}}_{\text{eff}}$ is given by

$$\hat{\mathcal{H}}_{\text{eff}} = \hat{H}_D + \hat{H}_{\text{PF}} + \hat{\mathcal{H}}_T^{(s)}, \quad (63)$$

where $\hat{H}_D$, $\hat{H}_{\text{PF}}$, and $\hat{\mathcal{H}}_T^{(s)}$ are defined in Eqs. (36), (50), and (58), respectively.

In the absence of the tunneling to the right NW, the electric charge in the left NW is conserved (since the quantum dot is in the singly occupied regime). Therefore, the effective Hamiltonian $\hat{\mathcal{H}}_{\text{eff}}$ [see Eq. (63)] decomposes into two-dimensional blocks corresponding to $n_l = 0$ and 1, and one can write Eq. (63) as

$$\mathcal{H}_{\text{eff}}^{(1Q)}(n_l, q_l) = \frac{\varepsilon_\uparrow - \varepsilon_\downarrow + \tilde{t}_z(n_l, q_l)}{2} \sigma^z + \frac{\tilde{t}_0(n_l, q_l)}{2} \sigma^x, \quad (64)$$

where σ^α are the Pauli matrices acting on the quantum dot degrees of freedom and where we omitted a constant term. The parameters $\tilde{t}_z(n, q)$ and $\tilde{t}_0(n, q)$ in Eq. (64) are given by

$$\tilde{t}_\beta(n, q) = \sum_{\sigma=\uparrow, \downarrow} (\sigma^\beta)_{\sigma\sigma} \frac{n|t_+(q)|^2 + (1-n)|t_-(q)|^2}{\varepsilon_\sigma - 2(-1)^n \delta(q)}, \quad (65)$$

where we redefined $t_\pm \equiv t_{l\pm}$, with $t_{l\pm}$ given by Eq. (53) and $\beta \in \{0, x, y, z\}$ indicates the Pauli matrix used for the summation. Clearly, the spin-qubit energy splitting (frequency) $\Delta E_n = \sqrt{[\varepsilon_\uparrow - \varepsilon_\downarrow + \tilde{t}_z(n, q)]^2 + \tilde{t}_0^2(n, q)}$ depends on the value of the fractional charge q in the nanowire via the effective splitting $\delta(q)$, defined in Eq. (51). For simplicity, we set $t_{lk} = 0$ and $\eta_{lk} = 0$ for all $k > 1$. In this case, the splitting reads as

$$\delta(q) = \eta_1 \cos\left[\frac{\pi}{2m}(2q - 1) + \theta_1\right], \quad (66)$$

where we have defined $\eta_1 = \eta_{l1}$. Furthermore, the parameters $t_\pm$ [see Eq. (53)] take the form

$$t_\pm(q) = \mp i(t_0 \pm t_1 e^{-i\pi q/m}), \quad (67)$$

where we have defined $t_k = t_{lk}$. The absolute value of $t_\pm(q)$ is thus given by

$$|t_\pm(q)|^2 = t_0^2 + t_1^2 \pm 2t_0 t_1 \cos\left(\frac{\pi q}{m}\right). \quad (68)$$

We note that the Kramers degeneracy is broken due to the presence of a magnetic field needed to create parafermions in the first place.² However, we assume that $\varepsilon_\uparrow - \varepsilon_\downarrow \ll |t_\pm(q)|, \delta(q)$, such that to lowest order in $\varepsilon_\uparrow - \varepsilon_\downarrow$ we can approximate ΔE_n as

$$\Delta E_n(\theta_1) = \frac{2[t_0^2 + t_1^2 + 2(2n - 1)t_0 t_1 \cos(\frac{\pi q}{m})]}{\varepsilon - 2(-1)^n \eta_1 \cos[\frac{\pi}{2m}(2q - 1) + \theta_1]}, \quad (69)$$

where we have defined $\varepsilon = (\varepsilon_\uparrow + \varepsilon_\downarrow)/2$. The extrema of $\Delta E_n(\theta_1)$ are at $\frac{\pi}{m}(\frac{1}{2} - q) + \pi\mathbb{Z}$ and the height of the maxima is

$$\Delta E_{n,\text{max}} = \frac{2[t_0^2 + t_1^2 + 2(2n - 1)t_0 t_1 \cos(\frac{\pi q}{m})]}{\varepsilon - 2\eta_1}, \quad (70)$$

thus depending on both the fractional charge q and the integer charge n . Thus, varying the phase θ_1 and measuring at which values for θ_1 the spin-qubit frequency ΔE has its extrema and the height of the maxima, the fractional charge q and the integer charge n in the nanowire can be determined (see Fig. 5). To tune θ_1 , we note that the phase θ_1 depends on the product μL [47,52,66], where μ is the chemical potential and L is the length of the NW. Therefore, by tuning the chemical potential, θ_1 can be changed, which allows us to determine the parafermion fractional charge by measuring the experimentally accessible qubit frequency [67,68].

B. Two-qubit gates

Clearly, the effective Hamiltonian $\hat{\mathcal{H}}_{\text{eff}}$ [see Eq. (57)] can be mapped onto a two-qubit Hamiltonian. The mapping designates the first qubit to the spin of the quantum dot, while the second qubit (the topological qubit) arises from the parafermion edge states on the two nanowires. Furthermore, the terms $\hat{\mathcal{H}}_T^{(\alpha)}$ enable an effective anisotropic exchange interaction between these two qubits. Note that in order to

²Note that in practice the condition $\varepsilon_\uparrow = \varepsilon_\downarrow$ requires extra care, so that the magnetic field needed for creating the PFs in the first place does not cause a Zeeman splitting on the dot.

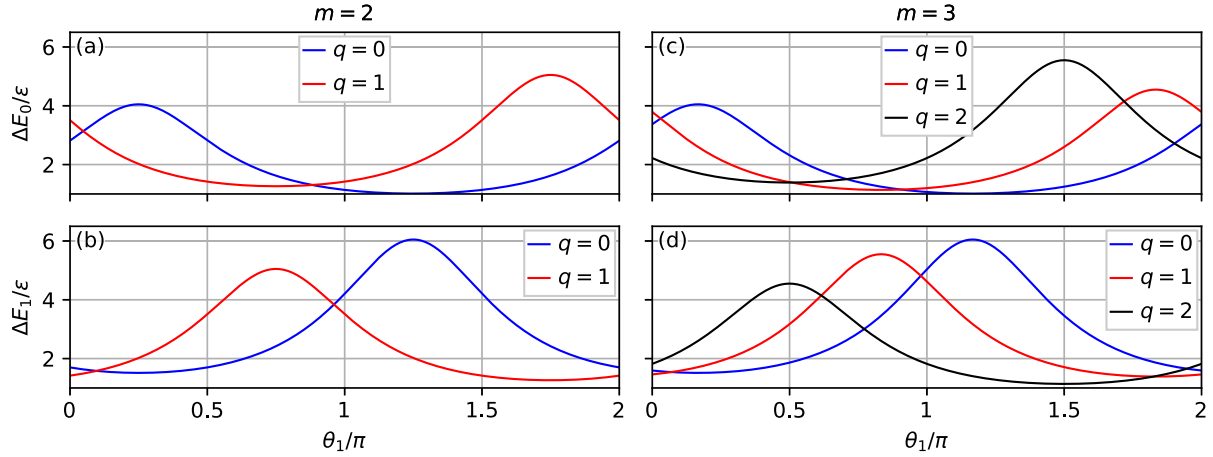


FIG. 5. The spin-qubit energy splitting ΔE_n , as defined in Eq. (69), versus the phase θ_1 for $\mathbb{Z}_4$ parafermions (a) and (b), i.e., $m = 2$, and $\mathbb{Z}_6$ parafermions [(c) and (d)], i.e., $m = 3$, for different integer charge quantum numbers $n = 0$ [(a) and (c)] and $n = 1$ [(b) and (d)], and different fractional charge quantum numbers $q = 0, \dots, m - 1$. Depending on the parafermion integer and fractional charges n and q , respectively, the curve $\Delta E_n(\theta_1)$ has its extrema at different values of θ_1 and the height of the maxima differs. Therefore, by tuning θ_1 and measuring the experimentally accessible qubit frequency ΔE_n , the parafermion integer and fractional charges n and q , respectively, can be determined. The parameters for this figure are $t_0/\varepsilon = 1$, $t_1/\varepsilon = 0.1$, and $\eta_1/\varepsilon = 0.3$.

establish a topological parafermion qubit, it is necessary to allow only for superpositions of states with the *same* parity. In the first-quantization formalism, the effective Hamiltonian $\mathcal{H}_{\text{eff}}$ becomes

$$\mathcal{H}_{\text{eff}}^{(2Q)} = \sum_{\alpha, \beta=0}^3 J_{\alpha\beta}(q_l, q_r) \sigma^\alpha \tau^\beta, \quad (71)$$

where σ^α acts on the spin qubit of the quantum dot and τ^β acts on the odd-parity sector of the NWs, defined as $\tau^3|r\rangle = +|r\rangle$ and $\tau^3|l\rangle = -|l\rangle$. The anisotropic exchange couplings $J_{\alpha\beta}$ depend on the microscopic parameters of the Hamiltonian and are parametrized by the fractional charges q_l and q_r in the left and right NWs. Additionally, $J_{\alpha\beta}$ depends on the total $\mathbb{Z}_2$ parity of the system, $n_l + n_r$. The explicit form of $J_{\alpha\beta}$ is given in Appendix D.

For the following discussion, we restrict our system to the case $n_l + n_r = 1$, i.e., total even $\mathbb{Z}_2$ parity. Following Ref. [33], we assume that there is no magnetic field and that the nanowires are semi-infinite. The former assumption results in $\varepsilon_\uparrow = \varepsilon_\downarrow \equiv \varepsilon$, while the latter has two consequences. First, the PFs at two opposite NW ends do not hybridize, so that $\eta_{vk} = 0$ and $\delta_v(q) = 0$, as follows from Eqs. (34) and (51). Second, tunneling must not occur between the quantum dot and the far-away PFs (labeled as $\hat{\alpha}_{l,1}$ and $\hat{\alpha}_{r,2}$). Thus, using Eq. (37), the only nonzero hopping amplitudes t_{vk} are t_{lm} and t_{r0} and, therefore, using Eq. (53), we get

$$t_{l\pm} = \mp(\pm 1)^m (-1)^{q_l} i t_{lm}, \quad t_{r\pm} = \mp i t_{r0}. \quad (72)$$

For simplicity, let us assume $|t_{lm}| = |t_{r0}| = |t|$ and define their relative phase φ as $t_{r0} = e^{i\varphi} t_{lm}$. In this case, the only nonzero exchange-coupling constants $J_{\alpha\beta}$ in Eq. (71) are

$$J_{00} = \varepsilon + J_{10}, \quad J_{10} = 2|t|^2/\varepsilon, \quad (73)$$

$$J_{01} = J_{11} = (-1)^{q_l} \frac{1 - (-1)^m}{2} \cos \varphi J_{10}, \quad (74)$$

$$J_{02} = J_{12} = (-1)^{q_l+1} \frac{1 + (-1)^m}{2} \sin \varphi J_{10}. \quad (75)$$

Therefore, by properly tuning the relative phase φ between the hopping amplitudes t_{lm} and t_{r0} , one can bring the Hamiltonian $\mathcal{H}_{\text{eff}}^{(2Q)}$ into the form (up to single-qubit rotations in the case of m even)

$$\mathcal{H}_{\text{eff}}^{(2Q)} = \frac{2|t|^2}{\varepsilon} (1 + \sigma^1)(1 + \tau^1) + \varepsilon. \quad (76)$$

Up to the constant energy shift ε , this is equivalent to Eq. (7) in Ref. [33] and following the discussion outlined there, a hybrid CNOT gate can be constructed from this effective Hamiltonian.³ Furthermore, using the same arguments as in Ref. [33], a Hadamard gate and a phase gate can be implemented in this setup, thus resulting in a universal set of gates for the hybrid qubit system. Alternatively, a hybrid PF-spin-qubit network for universal quantum computation can be implemented, following Ref. [33].

VIII. CONCLUSIONS

In this work, we have proposed a system that couples a singly occupied quantum dot (forming a spin qubit) to two

³In the presence of magnetic field, $\varepsilon_\uparrow \neq \varepsilon_\downarrow$, one can similarly realize a simple entangling gate. Indeed, let $\varepsilon_{\uparrow,\downarrow} = \varepsilon \pm \Delta$, and let us tune $\varepsilon = 0$. We again assume that the wires are semi-infinite so that $\eta_{vk} = 0$, $\delta_v = 0$, and the only nonzero tunneling amplitudes are t_{lm} and t_{r0} , which are related to each other as written after Eq. (72). Then, the effective two-qubit Hamiltonian becomes $\mathcal{H}_{\text{eff}}^{(2Q)} = J_{30}\sigma^3 + J_{3\beta}\sigma^3\tau^\beta$, where for odd (even) m one has $\beta = 1$ ($\beta = 2$) (see Appendix D). The coupling constants are given by $J_{30} = \Delta + (2|t|^2/\Delta)$, $J_{31} = (-1)^{q_l} \cos \varphi (2|t|^2/\Delta)$, and $J_{32} = -(-1)^{q_l} \sin \varphi (2|t|^2/\Delta)$. In both cases, the two-qubit Hamiltonian is equivalent to that in Eq. (76) up to a single-gate operation, and hence can be similarly used to construct a hybrid parafermion spin-qubit CNOT gate.

NWs hosting $\mathbb{Z}_{2m}$ PFs, where m is an arbitrary integer. Using the algebraic properties of $\mathbb{Z}_{2m}$ (Fock) PFs and extending the notion of particle-hole symmetry to PFs, we derive an effective two-qubit Hamiltonian, where the first qubit is encoded into the spin degree of freedom on the quantum dot, and the second qubit is constituted by the PFs on the two NWs. We demonstrate that, using this setup, a universal set of quantum gates can be realized. The special case $m = 1$ reproduces the results that have been derived for Majorana-spin qubits in Ref. [33]. For both even and odd values of m we have identified concrete physical systems where the proposed setup can be experimentally realized. Furthermore, we have demonstrated a method for reading out the integer and fractional charge of a PF by measuring the Larmor frequency (Zeeman splitting) of the quantum dot spin qubit [67,68].

Given that $\mathbb{Z}_{2m}$ parafermions can also be utilized to encode logical $2m$ -state qudits, it would be interesting to extend our results to a hybrid qudit-qubit system. In particular, it raises the question of whether coupling with the quantum dot could enable a universal set of gates for quantum computation with even-dimensional qudits (multilevel generalizations of two-level qubits). Similarly, it would be interesting to extend the present setup to the case of odd $\mathbb{Z}_{2m+1}$ parafermions which would allow for CNOT quantum gates by topologically protected braiding of the PFs alone (without the need of projective measurements) [27]. However, these problems lie beyond the scope of this work and we leave it for future study.

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APPENDIX A: MATRIX REPRESENTATIONS

For the sake of completeness, in this appendix we present an explicit matrix representation for the operators acting on the mixed Hilbert space of spin- $\frac{1}{2}$ fermions and $\mathbb{Z}_d$ parafermions.

First, let us discuss the local Hilbert space for $\mathbb{Z}_d$ parafermions. Choosing the basis spanned by the states $\{|k\rangle\}_{k=0}^{p-1}$, for the generalized Pauli matrices we choose the following d -dimensional representation:

$$X = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \omega & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \omega^{p-1} \end{pmatrix}, \quad (\text{A1})$$

where $\omega = e^{2\pi i/d}$. One can easily check that the generalized Pauli matrices satisfy $XZ = \omega ZX$. Then, for a single Fock parafermion one has

$$\mathcal{F} = \frac{d-1}{d}X - \frac{1}{d}X \sum_{r=1}^{d-1} Z^r = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \quad (\text{A2})$$

It is straightforward to check that the representation (A2) satisfies the relations (7) and (8). Note that for $d = 2$, one obtains $X = \sigma^x$, $Z = \sigma^z$, and $\mathcal{F} = f$, where

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (\text{A3})$$

In the case of a mixed Hilbert space containing both fermionic and parafermionic degrees of freedom, the only nontrivial question is how to ensure the commutation relations between the fermions and parafermions, as given by Eq. (43).

To illustrate the construction of matrix representation of operators acting on a mixed fermion-parafermion Hilbert space, we focus on the case of two $\mathbb{Z}_{2m}$ Fock parafermion modes and two spin- $\frac{1}{2}$ fermions. This corresponds to a setup similar to that in Fig. 1 if one allows for double occupancy of the quantum dot. Introducing the basis states $|\mathcal{N}_l\rangle \otimes |\mathcal{N}_r\rangle \otimes |n_\uparrow\rangle \otimes |n_\downarrow\rangle$, we write the following representation for the operators acting on the mixed Hilbert space:

$$\mathcal{F}_l = \mathcal{F} \otimes \mathbb{1}_{2m} \otimes \mathbb{1}_2 \otimes \mathbb{1}_2, \quad (\text{A4})$$

$$\mathcal{F}_r = Z \otimes \mathcal{F} \otimes \mathbb{1}_2 \otimes \mathbb{1}_2, \quad (\text{A5})$$

$$d_\uparrow = Z \otimes Z \otimes f \otimes \mathbb{1}_2, \quad (\text{A6})$$

$$d_\downarrow = Z \otimes Z \otimes \sigma^z \otimes f, \quad (\text{A7})$$

where $\mathcal{F}_l$ ($\mathcal{F}_r$) acts on the left (right) nanowire and d_σ with $\sigma = \uparrow, \downarrow$ act on the quantum dot. In Eqs. (A4)–(A7) the matrices Z , $\mathcal{F}$, σ_z , and f are given by Eqs. (A1)–(A3) and $\mathbb{1}_n$ is the n -dimensional identity matrix. It is easy to check that the matrices in Eqs. (A4)–(A7) satisfy the correct commutation relations for fermions and parafermions. For the mixed commutation relations one finds

$$d_\sigma \mathcal{F}_v = e^{-i\pi/m} \mathcal{F}_v d_\sigma, \quad (\text{A8})$$

where $\sigma = \uparrow, \downarrow$ and $v = l, r$. Thus, Eq. (A8) coincides with the required form of the mixed commutation relations (43), with the statistical angle $\chi = -\pi/m$. Generalization to a larger number of fermions and parafermions is straightforward.

APPENDIX B: PARTICLE-HOLE SYMMETRY FOR THE TUNNELING HAMILTONIAN

In this appendix we discuss the particle-hole symmetry for the tunneling Hamiltonian

$$\hat{V} = \sum_{v=l,r} \sum_{\sigma=\uparrow,\downarrow} \hat{V}_{v\sigma}, \quad (\text{B1})$$

with $\hat{V}_{v\sigma}$ given in Eq. (38) in the main text. For the tunneling Hamiltonian $\hat{V}$ to be particle-hole symmetric, one should have

$$\hat{C}\hat{V}_{v\sigma}\hat{C}^{-1} = -\hat{V}_{v\sigma}, \quad (\text{B2})$$

where $\hat{C}$ is the particle-hole transformation operator. One can always write it as

$$\hat{C} = \hat{U}_f \hat{U}_{pf} \hat{K}, \quad (\text{B3})$$

where $\hat{U}_f$ and $\hat{U}_{pf}$ are unitary operators acting on both the fermionic and parafermionic degrees of freedom, correspondingly, and $\hat{K}$ is the complex conjugation. Our goal is to find the explicit form of $\hat{C}$.

First, let us rewrite the tunneling Hamiltonian in a more convenient form. Using the expression for $\hat{\alpha}_{v,1}^{m-k} \hat{\alpha}_{v,2}^k$ written after Eq. (40), we obtain

$$\begin{aligned} \hat{V}_{v\sigma} = & \sum_{k=0}^{2m-1} (-1)^k t_{v\sigma}^* \hat{K}_v^{(m)} ((\hat{\mathcal{F}}_v^\dagger)^m + \hat{\mathcal{F}}_v^m) e^{i\frac{k\pi}{m} \hat{N}_v} \hat{d}_\sigma \\ & - \sum_{k=0}^{2m-1} t_{v\sigma} \hat{K}_v^{(m)} ((\hat{\mathcal{F}}_v^\dagger)^m + \hat{\mathcal{F}}_v^m) e^{-i\frac{k\pi}{m} \hat{N}_v} \hat{d}_\sigma^\dagger, \end{aligned} \quad (\text{B4})$$

where we used Eq. (42), took into account that the Klein factor $\hat{K}_v^{(m)}$ is Hermitian [see Eq. (55)], and used the relation

$$e^{-i\frac{k\pi}{m} \hat{N}_v} \hat{\mathcal{F}}_v^m = (-1)^k \hat{\mathcal{F}}_v^m e^{-i\frac{k\pi}{m} \hat{N}_v}, \quad (\text{B5})$$

which follows directly from the commutation relations (11).

We then take into account that regardless of the specific form of $\hat{U}_{pf}$ the particle-hole transformation must yield

$$\hat{C} \hat{N}_v \hat{C}^{-1} = m - 2m(\hat{\mathcal{F}}_v^\dagger)^m \hat{\mathcal{F}}_v^m + \hat{N}_v, \quad (\text{B6})$$

$$\hat{C} (\hat{\mathcal{F}}_v^\dagger)^m \hat{\mathcal{F}}_v^m \hat{C}^{-1} = 1 - \hat{\mathcal{F}}_v^m (\hat{\mathcal{F}}_v^\dagger)^m, \quad (\text{B7})$$

so that one necessarily has

$$\hat{C} e^{-i\frac{k\pi}{m} \hat{N}_v} \hat{C}^{-1} = (-1)^k e^{i\frac{k\pi}{m} \hat{N}_v}, \quad (\text{B8})$$

$$\hat{C} \hat{K}_v^{(m)} \hat{C}^{-1} = (-1)^{(m-1)\delta_{v,r}} \hat{K}_v^{(m)}. \quad (\text{B9})$$

On the other hand, the action of $\hat{C}$ on the ladder operators is defined up to a sign

$$\hat{C} \hat{d}_\sigma \hat{C}^{-1} = (-1)^{a_\sigma} \hat{d}_\sigma^\dagger, \quad (\text{B10})$$

$$\hat{C} \hat{\mathcal{F}}_v^m \hat{C}^{-1} = (-1)^{b_v^{(m)}} (\hat{\mathcal{F}}_v^\dagger)^m, \quad (\text{B11})$$

where a_σ and $b_v^{(m)}$ are integers. From Eqs. (B2) and (B4) we then obtain the following condition for the particle-hole symmetry of the tunneling Hamiltonian $\hat{V}$:

$$a_\sigma + b_v^{(m)} + (m-1)\delta_{v,r} = 0 \pmod{2}. \quad (\text{B12})$$

It is easy to check that the unitary operators

$$\hat{U}_f = (\hat{d}_\uparrow + \hat{d}_\uparrow^\dagger)(\hat{d}_\downarrow + \hat{d}_\downarrow^\dagger), \quad (\text{B13})$$

$$\hat{U}_{pf} = (\hat{\mathcal{F}}_l^m + \hat{\mathcal{F}}_l^{\dagger m}) e^{i(m-1)\pi \hat{\mathcal{F}}_l^m \hat{\mathcal{F}}_l^{\dagger m}} (\hat{\mathcal{F}}_r^m + \hat{\mathcal{F}}_r^{\dagger m}) \quad (\text{B14})$$

act on the ladder operators as

$$\hat{C} \hat{d}_\sigma \hat{C}^{-1} = -\hat{d}_\sigma^\dagger, \quad (\text{B15})$$

$$\hat{C} \hat{\mathcal{F}}_v^m \hat{C}^{-1} = -(-1)^{(m-1)\delta_{v,r}} \hat{\mathcal{F}}_v^{\dagger m}, \quad (\text{B16})$$

so that the particle-hole symmetry condition (B12) is satisfied with

$$a_\sigma = 1, \quad b_v^{(m)} = (m-1)\delta_{v,r} + 1. \quad (\text{B17})$$

Note that the particle-hole transformation (B3) satisfies $\hat{C}^2 = 1$ since one has $\hat{U}_f^2 = \hat{U}_{pf}^2 = -1$ and $\hat{K}^2 = 1$.

APPENDIX C: EFFECTIVE HAMILTONIAN

For completeness, in this appendix we outline the derivation of the effective Hamiltonian for our setup using the Schrieffer-Wolff (SW) transformation. The discussion is equivalent to that in Ref. [33].

1. Schrieffer-Wolff transformation

Given a Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{H}_0$ defines the diagonal part and $\hat{V}$ is an off-diagonal perturbation, the Schrieffer-Wolff (SW) transformation is [48,49]

$$\hat{H}_{\text{eff}} = e^{\hat{S}} \hat{H} e^{-\hat{S}} = \hat{H} + [\hat{S}, \hat{H}] + \frac{1}{2} [\hat{S}, [\hat{S}, \hat{H}]] + \dots, \quad (\text{C1})$$

where the generator of the transformation $\hat{S}$ is an anti-Hermitian operator and we performed the expansion of the adjoint action. The goal is to choose $\hat{S}$ such that terms linear in $\hat{V}$ are eliminated from the effective Hamiltonian $\hat{H}_{\text{eff}}$. This means that the operator $\hat{S}$ must satisfy the relation

$$[\hat{S}, \hat{H}_0] = -\hat{V}. \quad (\text{C2})$$

Then, to the second order in $\hat{V}$ the effective Hamiltonian becomes

$$\hat{H}_{\text{eff}} = \hat{H}_0 + \frac{1}{2} [\hat{S}, \hat{V}]. \quad (\text{C3})$$

The main advantage of the SW transformation is that one may continue the procedure in a controlled way [49].

In our case, from Eqs. (36) and (50) for the diagonal part of the Hamiltonian we have

$$\hat{H}_0 = \sum_{\sigma=\uparrow,\downarrow} \varepsilon_\sigma \hat{N}_\sigma + U \hat{N}_\uparrow \hat{N}_\downarrow + \sum_{v=l,r} \delta_v(q_v)(2\hat{n}_v - 1). \quad (\text{C4})$$

The off-diagonal perturbation $\hat{V}$ that couples the dot and the edge states is given by Eqs. (37) and (56).

It is easy to check that the SW generator can be chosen as

$$\hat{S} = \sum_{v=l,r} \hat{S}_v, \quad \hat{S}_v = \hat{A}_v + \hat{B}_v - \text{H.c.}, \quad (\text{C5})$$

where the operators $\hat{A}_v$ and $\hat{B}_v$ are given by

$$\begin{aligned} \hat{A}_v = & -it_{v-}^*(q_v) \sum_{\sigma} \left(\frac{1}{\varepsilon_\sigma - 2\delta_v(q_v)} \right. \\ & \left. - \frac{U \hat{N}_{\bar{\sigma}}}{[\varepsilon_\sigma - 2\delta_v(q_v)][\varepsilon_\sigma + U - 2\delta_v(q_v)]} \right) \hat{f}_v^\dagger \hat{d}_\sigma, \end{aligned} \quad (\text{C6})$$

$$\begin{aligned} \hat{B}_v = & it_{v+}^*(q_v) \sum_{\sigma} \left(\frac{1}{\varepsilon_\sigma + 2\delta_v(q_v)} \right. \\ & \left. - \frac{U \hat{N}_{\bar{\sigma}}}{[\varepsilon_\sigma + 2\delta_v(q_v)][\varepsilon_\sigma + U + 2\delta_v(q_v)]} \right) \hat{f}_v \hat{d}_\sigma. \end{aligned} \quad (\text{C7})$$

We now proceed with constructing the effective Hamiltonian (C3):

$$\hat{H}_{\text{eff}} = \hat{H}_0 + \frac{1}{2}[\hat{S}, \hat{V}] = \hat{H}_0 + \sum_{\mu, \nu=l, r} \frac{1}{2}[\hat{S}_\mu, \hat{V}_\nu]. \quad (\text{C8})$$

2. Regime of a singly occupied quantum dot

Since we are interested in the regime of a singly occupied quantum dot, the effective Hamiltonian $\hat{H}_{\text{eff}}$ must be projected onto the corresponding sector of the Hilbert space. Thus, taking the limit $U \rightarrow +\infty$ and projecting the Hamiltonian (C8) onto the subspace without the doubly occupied quantum dot, we obtain

$$\hat{\mathcal{H}}_{\text{eff}} = \hat{H}_D + \hat{H}_{\text{PF}} + \hat{\mathcal{H}}_T, \quad (\text{C9})$$

where $\hat{H}_D$, $\hat{H}_{\text{PF}}$, and $\hat{\mathcal{H}}_T$ are the terms corresponding to the quantum dot, nanowires, and their coupling, respectively. The terms $\hat{H}_D$ and $\hat{H}_{\text{PF}}$ are given by Eqs. (36) and (50), respectively. Our primary interest is in the effective coupling term $\hat{\mathcal{H}}_T$ since it allows us to realize two-qubit gates. It is convenient to write it as

$$\hat{\mathcal{H}}_T = \hat{\mathcal{H}}_T^{(s)} + \hat{\mathcal{H}}_T^{(e)} + \hat{\mathcal{H}}_T^{(o)}, \quad (\text{C10})$$

[see discussion after Eq. (60) in the main text for the physical meaning of different terms in Eq. (C10).

Using the results of Ref. [33], in the regime of a singly occupied quantum dot, the effective Hamiltonian (C10) can be written as

$$\hat{\mathcal{H}}_T^{(s)} = \frac{1}{2} \sum_{\nu, \sigma} \left(\frac{|t_{\nu+}(q_\nu)|^2}{\varepsilon_\sigma + 2\delta_\nu(q_\nu)} \hat{f}_\nu^\dagger \hat{f}_\nu + \frac{|t_{\nu-}(q_\nu)|^2}{\varepsilon_\sigma - 2\delta_\nu(q_\nu)} \hat{f}_\nu \hat{f}_\nu^\dagger \right) (\hat{N}_\sigma + \hat{d}_\sigma^\dagger \hat{d}_\sigma + \text{H.c.}), \quad (\text{C11})$$

$$\hat{\mathcal{H}}_T^{(o)} = \frac{1}{2} \sum_{\nu, \sigma} \left[t_{\bar{\nu}-}^*(q_{\bar{\nu}}) t_{\nu-}(q_\nu) \left(\frac{\hat{N}_\sigma + \hat{d}_\sigma^\dagger \hat{d}_\sigma}{\varepsilon_\sigma - 2\delta_\nu(q_\nu)} + \frac{\hat{N}_\sigma + \hat{d}_\sigma^\dagger \hat{d}_\sigma}{\varepsilon_\sigma - 2\delta_{\bar{\nu}}(q_{\bar{\nu}})} \right) \hat{f}_\nu \hat{f}_\nu^\dagger + t_{\bar{\nu}+}^*(q_{\bar{\nu}}) t_{\nu+}(q_\nu) \left(\frac{\hat{N}_\sigma + \hat{d}_\sigma^\dagger \hat{d}_\sigma}{\varepsilon_\sigma + 2\delta_\nu(q_\nu)} + \frac{\hat{N}_\sigma + \hat{d}_\sigma^\dagger \hat{d}_\sigma}{\varepsilon_\sigma + 2\delta_{\bar{\nu}}(q_{\bar{\nu}})} \right) \hat{f}_\nu^\dagger \hat{f}_{\bar{\nu}} \right], \quad (\text{C12})$$

$$\begin{aligned} \hat{\mathcal{H}}_T^{(e)} = & -\frac{1}{2} \sum_{\nu, \sigma} \left[t_{\bar{\nu}+}^*(q_{\bar{\nu}}) t_{\nu-}(q_\nu) \left(\frac{\hat{N}_\sigma + \hat{d}_\sigma^\dagger \hat{d}_\sigma}{\varepsilon_\sigma - 2\delta_\nu(q_\nu)} + \frac{\hat{N}_\sigma + \hat{d}_\sigma^\dagger \hat{d}_\sigma}{\varepsilon_\sigma + 2\delta_{\bar{\nu}}(q_{\bar{\nu}})} \right) \hat{f}_\nu \hat{f}_{\bar{\nu}} \right. \\ & \left. + t_{\bar{\nu}-}^*(q_{\bar{\nu}}) t_{\nu+}(q_\nu) \left(\frac{\hat{N}_\sigma + \hat{d}_\sigma^\dagger \hat{d}_\sigma}{\varepsilon_\sigma + 2\delta_\nu(q_\nu)} + \frac{\hat{N}_\sigma + \hat{d}_\sigma^\dagger \hat{d}_\sigma}{\varepsilon_\sigma - 2\delta_{\bar{\nu}}(q_{\bar{\nu}})} \right) \hat{f}_\nu^\dagger \hat{f}_{\bar{\nu}}^\dagger \right]. \end{aligned} \quad (\text{C13})$$

Note that our notation differs from the one in Ref. [33] by the factor of $\frac{1}{2}$ in the definition of $\hat{\mathcal{H}}_T^{(\alpha)}$.

APPENDIX D: FULL EXCHANGE HAMILTONIAN

We then project the effective Hamiltonian (C10) on the subspace with the fixed total fermion parity, write its matrix representation, and write it as a two-qubit Hamiltonian. This gives us Eq. (71) in the main text, where the exchange constants $J_{\alpha\beta}$ are given below.

For $\mathbb{Z}_{2m}$ parafermions in the sector with $n_l + n_r = 0 \pmod 2$ (even $\mathbb{Z}_2$ parity in the NWs and odd total $\mathbb{Z}_2$ parity), we obtain

$$\begin{aligned} J_{10} &= \frac{1}{4} \sum_{\sigma \nu s} \frac{|t_{\nu s}|^2}{\varepsilon_\sigma + 2s\delta_\nu}, \quad J_{20} = 0, \\ J_{30} &= \frac{\varepsilon_\uparrow - \varepsilon_\downarrow}{2} + \frac{1}{4} \sum_{\sigma \nu s} \text{sign}(\sigma) \frac{|t_{\nu s}|^2}{\varepsilon_\sigma + 2s\delta_\nu}, \end{aligned} \quad (\text{D1})$$

$$\begin{aligned} J_{01} &= J_{11} = -\frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(\nu s) \frac{\text{Re}\{t_{\nu s}^* t_{\bar{\nu} \bar{s}}\}}{\varepsilon_\sigma + 2s\delta_\nu}, \\ J_{21} &= -\frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(\sigma \nu s) \frac{\text{Im}\{t_{\nu s}^* t_{\bar{\nu} \bar{s}}\}}{\varepsilon_\sigma + 2s\delta_\nu}, \\ J_{31} &= -\frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(\sigma \nu s) \frac{\text{Re}\{t_{\nu s}^* t_{\bar{\nu} \bar{s}}\}}{\varepsilon_\sigma + 2s\delta_\nu}, \end{aligned} \quad (\text{D2})$$

$$\begin{aligned} J_{02} &= J_{12} = -\frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(\nu) \frac{\text{Im}\{t_{\nu s}^* t_{\bar{\nu} \bar{s}}\}}{\varepsilon_\sigma + 2s\delta_\nu}, \\ J_{22} &= \frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(\sigma \nu) \frac{\text{Re}\{t_{\nu s}^* t_{\bar{\nu} \bar{s}}\}}{\varepsilon_\sigma + 2s\delta_\nu}, \\ J_{32} &= -\frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(\sigma \nu) \frac{\text{Im}\{t_{\nu s}^* t_{\bar{\nu} \bar{s}}\}}{\varepsilon_\sigma + 2s\delta_\nu}, \end{aligned} \quad (\text{D3})$$

$$\begin{aligned} J_{03} &= J_{13} + \delta_l + \delta_r = \delta_l + \delta_r + \frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(s) \frac{|t_{\nu s}|^2}{\varepsilon_\sigma + 2s\delta_\nu}, \\ J_{23} &= 0, \quad J_{33} = \frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(\sigma s) \frac{|t_{\nu s}|^2}{\varepsilon_\sigma + 2s\delta_\nu}, \end{aligned} \quad (\text{D4})$$

where $\sigma \in \{\uparrow, \downarrow\}$, $\nu \in \{l, r\}$, and $s \in \{+, -\}$.

Similarly, for $\mathbb{Z}_{2m}$ parafermions in the sector with $n \equiv n_l + n_r = 1 \pmod 2$ (odd $\mathbb{Z}_2$ parity in the NWs and even total $\mathbb{Z}_2$ parity), one finds

$$\begin{aligned} J_{10} &= \frac{1}{4} \sum_{\sigma \nu s} \frac{|t_{\nu s}|^2}{\varepsilon_\sigma + 2s\delta_\nu}, \quad J_{20} = 0, \\ J_{30} &= \frac{\varepsilon_\uparrow - \varepsilon_\downarrow}{2} + \frac{1}{4} \sum_{\sigma \nu s} \text{sign}(\sigma) \frac{|t_{\nu s}|^2}{\varepsilon_\sigma + 2s\delta_\nu}, \end{aligned} \quad (\text{D5})$$

$$\begin{aligned}
J_{01} &= J_{11} = \frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(s) \frac{\text{Re}\{t_{\nu s}^* t_{\bar{\nu} s}\}}{\varepsilon_\sigma + 2s\delta_\nu}, \\
J_{21} &= \frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(\sigma s) \frac{\text{Im}\{t_{\nu s}^* t_{\bar{\nu} s}\}}{\varepsilon_\sigma + 2s\delta_\nu}, \\
J_{31} &= \frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(\sigma s) \frac{\text{Re}\{t_{\nu s}^* t_{\bar{\nu} s}\}}{\varepsilon_\sigma + 2s\delta_\nu}, \\
J_{02} &= J_{12} = \frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(\nu) \frac{\text{Im}\{t_{\nu s}^* t_{\bar{\nu} s}\}}{\varepsilon_\sigma + 2s\delta_\nu}, \\
J_{22} &= -\frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(\sigma \nu) \frac{\text{Re}\{t_{\nu s}^* t_{\bar{\nu} s}\}}{\varepsilon_\sigma + 2s\delta_\nu},
\end{aligned} \tag{D6}$$

$$J_{32} = \frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(\sigma \nu) \frac{\text{Im}\{t_{\nu s}^* t_{\bar{\nu} s}\}}{\varepsilon_\sigma + 2s\delta_\nu}, \tag{D7}$$

$$\begin{aligned}
J_{03} &= \delta_r - \delta_l + J_{13} = \delta_r - \delta_l + \frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(s\nu) \frac{|t_{\nu s}|^2}{\varepsilon_\sigma + 2s\delta_\nu}, \\
J_{23} &= 0, \quad J_{33} = \frac{1}{4} \sum_{\sigma \nu s} \text{sgn}(\sigma \nu s) \frac{|t_{\nu s}|^2}{\varepsilon_\sigma + 2s\delta_\nu}.
\end{aligned} \tag{D8}$$

J_{00} is an irrelevant constant term, which can therefore be neglected. Note that for the special case $\delta_l = \delta_r = \delta$ in Eqs. (D5)–(D8), we recover the corresponding expressions for $J_{\alpha\beta}$ in Ref. [33], up to terms arising due to $\hat{H}_D + \hat{H}_{\text{PF}}$, which were not explicitly considered in Ref. [33].

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