

Multiobjective tweezers in scattering media

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Radiation forces and torques enable the manipulation of objects with acoustic and electromagnetic waves. Yet, harnessing them in complex scattering media remains a formidable challenge, especially when multiple objects must be controlled under competing objectives. Here, we demonstrate that sound or light can be shaped to tailor momentum transfer to multiple objects simultaneously in a complex scattering medium. For a single object, our theory yields the maximal achievable force or torque; for multiple objects, it produces Pareto-optimal actuation and exact bounds on the simultaneous realization of incompatible objectives. This opens applications for wave tweezers, enabling selective and precise manipulation of objects within complex media, ranging from the handling of cells, organoids, or microrobots to targeted drug delivery in biological media.

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I. INTRODUCTION

When propagating through complex media, waves—acoustic or electromagnetic, among others—undergo repeated scattering events due to inhomogeneities, which challenges our ability to control and predict their propagation. Early work in acoustics showed that wavefronts can be time reversed to refocus through complex media, revealing that multiple scattering does not necessarily lead to information loss [1,2]. Subsequent optical experiments demonstrated that tailored wavefronts can focus light through opaque, strongly scattering media, marking the experimental inception of the field of wavefront shaping in optics [3,4], and further accelerated by the measurement of the optical transmission matrix [5]. A key concept in modern wavefront shaping is the use of the generalized Wigner-Smith (GWS) matrix, defined from the scattering matrix $\mathbf{S}$ as

$$\mathbf{Q}_\alpha = j\mathbf{S}^\dagger \partial_\alpha \mathbf{S}, \quad (1)$$

where α denotes a given parameter of the scattering region and $\dagger$ denotes the conjugate transpose (assuming the $+j\omega t$ time-harmonic convention). Originally introduced in quantum scattering [6–8], $\mathbf{Q}_\alpha$ has become an essential tool to construct incident states that extremize parametric responses [9,10], and has enabled a broad range of wave-control applications [11–21].

Beyond controlling the spatial distribution of wave amplitude or phase, many applications rely on the mechanical action that waves exert on objects. Contactless manipulation of microscopic objects is vital across biology, medicine, and materials science [22], but many envisioned uses—selective handling of cells and organoids, targeted drug delivery, elaborate microrobot actuation, automated biofabrication, and mechanobiology—remain constrained by the need for high precision in complex, inhomogeneous media. In acoustics, this precision has not been achieved in strongly scattering biological tissues, and existing *in vivo* demonstrations largely rely on focal targeting rather than programmable wave-momentum transfer [23–25]. High-fidelity operation typically remains tied to small, near-spherical, subwavelength objects, and concurrent manipulation in complex scattering environments is not yet broadly available [26–29]. In optics, holographic tweezers scale well in transparent media [30–32], and wavefront shaping has demonstrated trapping through scattering by iteratively reforming foci [33,34]. Yet, refocusing alone does not provide a general, exact, one-shot route to simultaneously tailor the momentum of multiple scattering objects. Prior work established that, for a single object undergoing a small translation or rotation, extremal GWS states can maximize the corresponding radiation force or torque in a complex environment [9,10,35,36]. What has been lacking, however, is a general theoretical framework that unifies these ideas for arbitrary acoustic or electromagnetic media and extends them to individual control of multiple objects of any shape, thus enabling a range of applications leveraging optimal tweezers.

Here, we develop such a framework based on variational relations linking parametric changes of the medium to modifications of the far-field scattering amplitudes. This naturally leads to GWS matrices that encode individual forces and torques on multiple objects at once. Our multiobjective formulation establishes that trade-offs between competing

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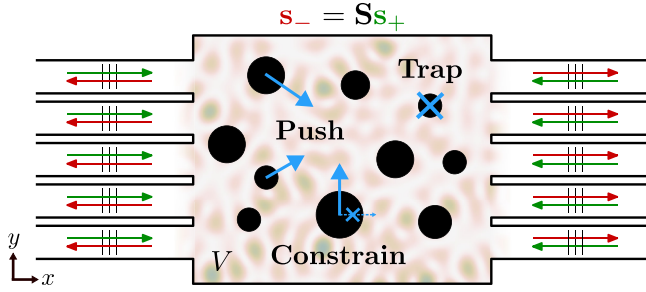


FIG. 1. Concept of multiobjective tweezers in scattering media. A multiport disordered cavity (scattering region V) is driven through accessible input channels, with incident $\mathbf{s}_+$ and outgoing $\mathbf{s}_-$ complex modal amplitudes connected through the $\mathbf{S}$ -matrix. By tailoring incident wavefronts, the interference inside the cavity is controlled so that radiation forces on embedded objects are simultaneously programmed: Multiple targets are optimally pushed; one object is trapped at a prescribed location; one object's motion is constrained to a single direction.

mechanical targets are inherently unavoidable, but also highly relevant in applications. The simultaneous optimization of such incompatible objectives is fundamentally constrained by a theoretical bound, which quantifies the achievable performance limits. Within this framework, the eigenstates of the GWS provide Pareto-optimal solutions, enabling the coordinated manipulation of multiple objects. In particular, we demonstrate how tailored input wavefronts can be used to achieve coordinated and selective control over the motion of multiple objects. Examples include simultaneously optimizing pushes on several objects along prescribed directions, trapping objects at chosen locations, and imposing directional constraints, such as maximizing the force along the y direction while suppressing motion along x , and keeping other objects still (Fig. 1).

II. RADIATION FORCES IN INHOMOGENEOUS MEDIA

We first set the stage by introducing time-averaged body force density expressions under harmonic excitation in lossless media. In acoustics [37],

$$\bar{\mathbf{f}}_{\text{ac}} = -\frac{1}{4}(|p_1|^2 \nabla \kappa_0 + |\mathbf{v}_1|^2 \nabla \rho_0), \quad (2)$$

where κ_0 , ρ_0 , p_1 , and $\mathbf{v}_1$ are the compressibility, density, first-order complex pressure, and acoustic velocity fields, respectively. Although initially derived for inhomogeneous fluids, it generally applies when solid scatterers can be modeled as equivalent fluids; i.e., transverse waves are neglected. A rule of thumb is to consider solid inclusions of characteristic size d , in which $\omega \ll \pi c_s/d$, where c_s is the shear speed of sound. Analogously, in electromagnetism [38],

$$\bar{\mathbf{f}}_{\text{em}} = -\frac{1}{4}(|\mathbf{E}|^2 \nabla \varepsilon + |\mathbf{H}|^2 \nabla \mu), \quad (3)$$

where ε , μ , $\mathbf{E}$, and $\mathbf{H}$ are the permittivity, permeability, complex electric, and magnetic vector fields, respectively. This is valid in linear, isotropic, nondispersive, and lossless inhomogeneous media with no free charges or currents. By exploiting the analogy between Eqs. (2) and (3), we formulate a general theory of radiation forces in scattering media from far-field quantities.

III. VARIATIONAL RELATIONS

Following circuit theory nomenclature [39,40], we introduce some scattering variables called immittance voltage u_m and current i_m , defined at a scattering port m as $u_m = \sqrt{2}(s_m^+ + s_m^-)$ and $i_m = \sqrt{2}(s_m^+ - s_m^-)$, where $s_m^\pm$ is the incident (+) and outgoing (−) complex amplitude of the wave, normalized such that $|s_m^\pm|^2$ is equal to the incident/outgoing power at the corresponding port. In the linear scattering regime, the $\mathbf{S}$ -matrix is defined as $\mathbf{s}_- = \mathbf{S}\mathbf{s}_+$, so the immittance vectors read

$$\mathbf{u} = \sqrt{2}(\mathbb{1} + \mathbf{S})\mathbf{s}_+, \quad \mathbf{i} = \sqrt{2}(\mathbb{1} - \mathbf{S})\mathbf{s}_+, \quad (4)$$

where $\mathbb{1}$ denotes the identity matrix. In practice, the N ports correspond to the asymptotic modes that couple the scattering region to its environment. In the open cavity of Fig. 1, they are the propagating waves in the monomode waveguides coupled to the cavity. The far-field information about the scattering medium of volume V is completely described by the $\mathbf{S}$ -matrix with respect to these ports (details in the Supplemental Material [41]).

We are interested in the far-field signature when a parameter α (e.g., the position of a target) in the system is slightly varied. In Appendix A, we show that the variations of the immittance vectors at the ports ($\delta \mathbf{i}$ and $\delta \mathbf{u}$) can be linked to the local changes in the scattering medium in both acoustics and electromagnetism, i.e.,

$$\frac{\mathbf{u}^\dagger \delta \mathbf{i} + \mathbf{i}^\dagger \delta \mathbf{u}}{j\omega} = \delta \alpha \cdot \int_V \left(\left| \frac{p_1}{\mathbf{E}} \right|^2 \nabla_{\delta \alpha} \left(\frac{\kappa_0}{\varepsilon} \right) + \left| \frac{\mathbf{v}_1}{\mathbf{H}} \right|^2 \nabla_{\delta \alpha} \left(\frac{\rho_0}{\mu} \right) \right) dV. \quad (5)$$

As we show below, these variations generate local radiation forces. We illustrate the argument with the density field ρ_0 ; the same construction applies to κ_0 , ε , and μ . Consider a system built from N_o arbitrarily shaped objects of constant density $\rho_{0,i}$ in a background $\rho_0^{(b)}(\mathbf{r})$, such that

$$\rho_0(\mathbf{r}) = \rho_0^{(b)}(\mathbf{r}) + \sum_{i=1}^{N_o} \chi_i(\mathbf{r})(\rho_{0,i} - \rho_0^{(b)}(\mathbf{r})). \quad (6)$$

Here, $\chi_i(\mathbf{r})$ represents the indicator function of object i , taking a value of 1 inside the object and 0 outside. When object k undergoes a small displacement by a vector $\delta \alpha = \delta \mathbf{r}$, its indicator function shifts accordingly as $\chi_k(\mathbf{r}) \rightarrow \chi_k(\mathbf{r} - \delta \mathbf{r})$. The induced variation is supported only in a local region V_p containing the displaced interface, so the $\nabla_{\delta \alpha}$ terms in Eq. (5) vanish outside V_p and the integral may be restricted from V to V_p . Within V_p , we assume that $\rho_0^{(b)}$ is constant, and thus obtain $\nabla \rho_0 = -\nabla_{\delta \mathbf{r}} \rho_0$ with $\nabla_{\delta \mathbf{r}} = (\partial_{\delta x}, \partial_{\delta y}, \partial_{\delta z})$. As a result, the total time-averaged acoustic/electromagnetic radiation force $\bar{\mathbf{F}}$ on the volume V_p in the direction of a small shift $\delta \mathbf{r}$ is

$$\bar{\mathbf{F}} \cdot \delta \mathbf{r} = -\frac{j}{4\omega} (\mathbf{u}^\dagger \delta \mathbf{i} + \mathbf{i}^\dagger \delta \mathbf{u}). \quad (7)$$

We report below how this relation enables unprecedented control over local radiation forces in complex media.

IV. OPTIMAL MOTION OF ONE OBJECT

From Eq. (4),

$$\delta \mathbf{u} = \sqrt{2}[(\mathbb{1} + \mathbf{S})\delta \mathbf{s}_+ + \delta \mathbf{S} \mathbf{s}_+], \quad (8a)$$

$$\delta \mathbf{i} = \sqrt{2}[(\mathbb{1} - \mathbf{S})\delta \mathbf{s}_+ - \delta \mathbf{S} \mathbf{s}_+]. \quad (8b)$$

Since $\delta \mathbf{s}_+ = \mathbf{0}$ for a fixed chosen input state, the change in immittance vectors is only due to the parametric change of the $\mathbf{S}$ -matrix: $\mathbf{u}^\dagger \delta \mathbf{i} + \mathbf{i}^\dagger \delta \mathbf{u} = -4\mathbf{s}_+^\dagger \delta \mathbf{S} \mathbf{s}_+$. This suggests extending the GWS matrix defined in Eq. (1) to multidimensional parameters $\delta \mathbf{r}$ as

$$\mathbf{Q}_{\delta \mathbf{r}} = \mathbf{j} \mathbf{S}^\dagger (\hat{\mathbf{e}}_{\mathbf{r}} \cdot \nabla_{\delta \mathbf{r}}) \mathbf{S} = \frac{\delta x \mathbf{Q}_{\delta x} + \delta y \mathbf{Q}_{\delta y} + \delta z \mathbf{Q}_{\delta z}}{|\delta \mathbf{r}|}, \quad (9)$$

where $\hat{\mathbf{e}}_{\mathbf{r}} = \delta \mathbf{r}/|\delta \mathbf{r}|$. The expectation value over the input states is

$$\langle \mathbf{Q}_{\delta \mathbf{r}} \rangle_{\mathbf{s}_+} \equiv \mathbf{s}_+^\dagger \mathbf{Q}_{\delta \mathbf{r}} \mathbf{s}_+ = -\frac{\mathbf{j}}{4|\delta \mathbf{r}|} (\mathbf{u}^\dagger \delta \mathbf{i} + \mathbf{i}^\dagger \delta \mathbf{u}). \quad (10)$$

Using Eq. (7), we finally obtain

$$\frac{1}{\omega} \langle \mathbf{Q}_{\delta \mathbf{r}} \rangle_{\mathbf{s}_+} = \bar{\mathbf{F}} \cdot \hat{\mathbf{e}}_{\mathbf{r}}. \quad (11)$$

Equation (11) is the central equation of our article. It shows that each force component is encoded in how the scattering changes under an infinitesimal shift of the object. Since $\mathbf{Q}_{\delta \mathbf{r}}$ is Hermitian for a unitary $\mathbf{S}$ -matrix (lossless propagation), the projection of the radiation force can be written as a Rayleigh quotient. The Courant-Fischer min-max theorem then ensures that

$$\frac{\lambda_1}{\omega} |\mathbf{s}_+|^2 \leq \bar{\mathbf{F}} \cdot \hat{\mathbf{e}}_{\mathbf{r}} \leq \frac{\lambda_N}{\omega} |\mathbf{s}_+|^2, \quad (12)$$

where the eigenvalues of $\mathbf{Q}_{\delta \mathbf{r}}$ are ordered as $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$. Of particular interest are the extremal values of $\bar{\mathbf{F}} \cdot \hat{\mathbf{e}}_{\mathbf{r}}$, which are reached when the incident state coincides with the associated eigenvector, enabling the maximization or minimization of the applied force projection. More generally, if only L out of N ports are accessible, the largest force projection lies between λ_L/ω and λ_N/ω . In the following, all the expectation values $\langle \cdot \rangle$ are taken with respect to the normalized input state such that $|\mathbf{s}_+|^2 = 1W$.

V. OPTIMAL MOTION OF MULTIPLE OBJECTS

When N_d objects are displaced, the variational integrals split into nonoverlapping contributions over the disjoint volumes $V_{p,i}$. The total quantity

$$\frac{1}{\omega} \langle \mathbf{Q}_{\delta \mathbf{r}_{\text{tot}}} \rangle = \frac{1}{|\delta \mathbf{r}_{\text{tot}}|} \sum_{i=1}^{N_d} |\delta \mathbf{r}_i| \bar{\mathbf{F}}_i \cdot \hat{\mathbf{e}}_{\mathbf{r}_i} \quad (13)$$

therefore measures the infinitesimal (virtual) work of the radiation force on the system, normalized by the stacked displacement vector $\delta \mathbf{r}_{\text{tot}} = (\delta \mathbf{r}_1, \dots, \delta \mathbf{r}_{N_d})$. $\bar{\mathbf{F}}_i \cdot \hat{\mathbf{e}}_{\mathbf{r}_i}$ is the time-averaged force on the i th object in the direction of its push. Without loss of generality, we can focus on the generic case of two objects ($N_d = 2$). Because $\mathbf{Q}_{\delta \mathbf{r}_1}$ and $\mathbf{Q}_{\delta \mathbf{r}_2}$ typically do not commute, they cannot, in general, be diagonalized simultaneously; hence, no single incident state maximizes both projections $\bar{\mathbf{F}}_1 \cdot \hat{\mathbf{e}}_{\mathbf{r}_1}$ and $\bar{\mathbf{F}}_2 \cdot \hat{\mathbf{e}}_{\mathbf{r}_2}$ [42]. Yet,

the best eigenvector of $\mathbf{Q}_{\delta \mathbf{r}_{\text{tot}}} = (|\delta \mathbf{r}_1| \mathbf{Q}_{\delta \mathbf{r}_1} + |\delta \mathbf{r}_2| \mathbf{Q}_{\delta \mathbf{r}_2})/|\delta \mathbf{r}_{\text{tot}}|$ maximizes the sum of the projections. Consequently, the multiobject formulation can be directly interpreted through the lens of multiobjective optimization [43]. Each $\bar{\mathbf{F}}_i \cdot \hat{\mathbf{e}}_{\mathbf{r}_i}$ defines an objective to be maximized and $\langle \mathbf{Q}_{\delta \mathbf{r}_{\text{tot}}} \rangle$ corresponds to a so-called linear scalarization of this problem (i.e., forming a weighted sum to be maximized), with the coefficients $|\delta \mathbf{r}_i|$ playing the role of optimization weights. An extension of the theory to optimal torque on objects is proposed in Appendix B. For two objectives—such as displacing/rotating two objects along/about one direction each—the feasible set of force components corresponds to the joint numerical range of two Hermitian matrices, which is always convex [44]. This is also true for three objectives when the number of accessible ports is $N > 2$ [45]. In these cases, it is known that linear scalarization generates an entire Pareto front [43]—that is, the set of solutions where no objective can be improved without degrading another. We show here that the GWS explores the full set of such compromises. For four or more objectives, convexity can fail, and the GWS recovers only the exposed subset of the Pareto front, leaving nonconvex regions inaccessible.

As an example, a biobjective Pareto front for the simultaneous translation of two objects in a 20-port disordered two-dimensional acoustic cavity is simulated in Fig. 2(a) (red solid line). For comparison, we also show the corresponding force projections for 10^7 random inputs represented by the distribution of dots, the color of which indicates the number of occurrences. Strikingly, the Pareto front obtained from the GWS procedure greatly outperforms all the random inputs, thus demonstrating that naive Monte Carlo exploration of the input space cannot realistically be implemented to tailor radiation forces in complex media. In the Supplemental Material [41], we conjecture that the force projections—whose respective probability distributions are B -splines [46–49]—can never compete with the GWS procedure for a sufficiently large number of ports. Moreover, the Pareto front is compared to the result of a heuristic procedure also based on the GWS matrix, where a linear combination of eigenvectors of $\mathbf{Q}_{\delta \mathbf{r}_1}$ and $\mathbf{Q}_{\delta \mathbf{r}_2}$ is performed (green dashed line). In terms of maximum force projections, the procedure consistently outperforms this benchmark technique [36]. This demonstrates that our approach can improve existing manipulation schemes. The result of the Pareto optimization is shown in Fig. 2(b), where blue arrows correspond to the objectives and black arrows correspond to the net forces exerted on the objects, computed via the Brillouin stress tensor. We see that the optimized wavefield yields net forces in the intended directions, which confirms that our framework translates directly into the mechanical action predicted by the Brillouin stress tensor—validating that it is possible to shape the local momentum of waves for complex tweezing scenarios from far-field actuation only. Further examples can be found in the Supplemental Material [41].

VI. UNCERTAINTY RELATION

In biobjective optimization with GWS matrices, one seeks to simultaneously maximize the expectation values of the matrices $\mathbf{Q}_\alpha$ and $\mathbf{Q}_\beta$, corresponding for instance to distinct forces

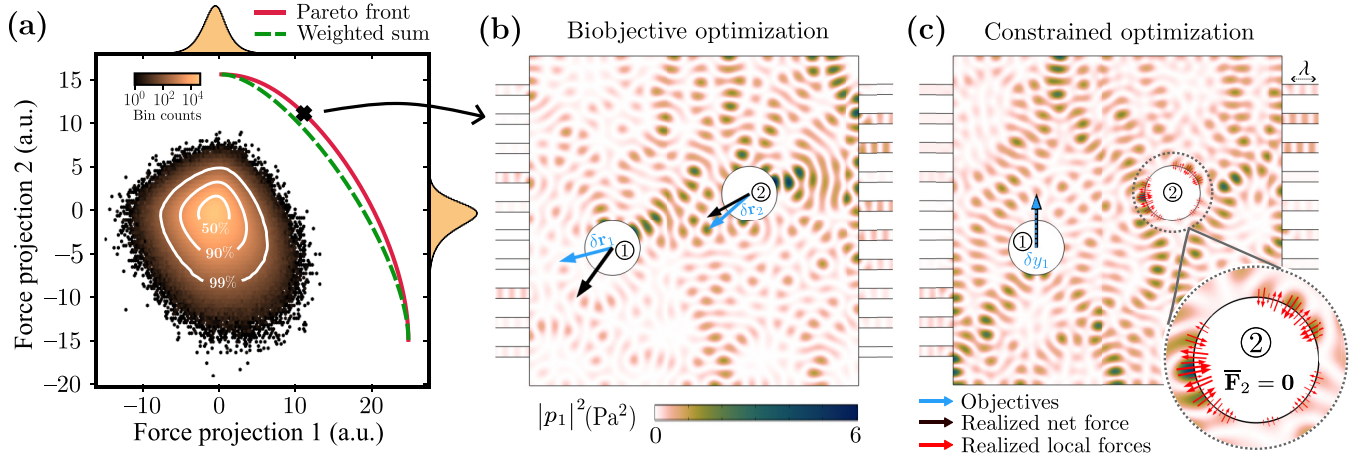


FIG. 2. Collective and selective manipulation of objects in scattering media. (a) Biobjective optimization: probability distribution of the force projections on two objects for 10^7 random input states. The marginal distributions form B -splines. The Pareto front (red solid line) is compared to a heuristic weighted sum of eigenstates (green dashed line). (b) Full wave simulation of the squared acoustic pressure field corresponding to the Pareto-optimal solution for which the force projections are equal, i.e., $\langle \mathbf{Q}_{\delta r_1} \rangle = \langle \mathbf{Q}_{\delta r_2} \rangle$. The blue arrows are the objectives and the black arrows are the net forces exerted on the objects. (c) Constrained optimization: The object ① is displaced along the vertical direction only (net force in black perfectly overlapping with the objective in blue), while the object ② experiences local forces (small red arrows with their width and length logarithmically scaling with the local force magnitude), which globally compensate to obtain zero net force. The wavelength λ is shown on the right.

or torques on one or multiple objects. The central question is how incompatible the objectives can be, and what bound captures the resulting trade-off. To answer this, we define the deficits from utopia values (highest eigenvalues) for a given input state $\mathbf{s}_+$ as

$$\epsilon_\alpha = \lambda_N(\mathbf{Q}_\alpha) - \langle \mathbf{Q}_\alpha \rangle, \quad \epsilon_\beta = \lambda_N(\mathbf{Q}_\beta) - \langle \mathbf{Q}_\beta \rangle. \quad (14)$$

A direct application of the Schrödinger-Robertson uncertainty relation [50], combined with the Bhatia-Davis inequality [51], yields the following bound:

$$\epsilon_\alpha \epsilon_\beta \geq \frac{\frac{1}{4} |\langle [\mathbf{Q}_\alpha, \mathbf{Q}_\beta] \rangle|^2 + \text{Cov}(\mathbf{Q}_\alpha, \mathbf{Q}_\beta)^2}{(\langle \mathbf{Q}_\alpha \rangle - \lambda_1(\mathbf{Q}_\alpha))(\langle \mathbf{Q}_\beta \rangle - \lambda_1(\mathbf{Q}_\beta))}. \quad (15)$$

Here, the symmetrized covariance is

$$\text{Cov}(\mathbf{Q}_\alpha, \mathbf{Q}_\beta) = \frac{1}{2} \langle \{ \mathbf{Q}_\alpha - \langle \mathbf{Q}_\alpha \rangle, \mathbf{Q}_\beta - \langle \mathbf{Q}_\beta \rangle \} \rangle, \quad (16)$$

and $[\mathbf{A}, \mathbf{B}]$ and $\{\mathbf{A}, \mathbf{B}\}$ denote the commutator and anticommutator, respectively. The right-hand side of Eq. (15) cannot vanish unless the expectation value of the commutator vanishes and the two GWS matrices are uncorrelated in the chosen input state. In particular, the case $\epsilon_\alpha = \epsilon_\beta = 0$ requires the two matrices to share the same eigenstate associated with their respective maximal eigenvalues, a highly nongeneric condition. Equation (15) therefore provides a rigorous state-dependent witness of incompatibility: Driving one objective closer to its optimum generically restricts how closely the other can approach its own. The Pareto front then identifies the optimal achievable compromises within this constrained landscape, while the corresponding GWS solutions generate strongly directed forces that are exceptionally unlikely to arise from random inputs.

VII. CONSTRAINED OPTIMIZATION

So far, our theory allows us to maximize multiple objectives simultaneously. For example, in Fig. 2(b), the wavefront is the best one to move both objects along the blue arrows. However, the optimal forces are not exactly in the desired directions, because the orthogonal force components are left unconstrained. A natural question we ask now is: Can we generalize the framework to handle additional constraints, such as no motion of certain objects in particular directions? Formally, consider the problem of maximizing the expectation value of a given $\mathbf{Q}_\alpha$ while enforcing N_c constraints on the expectation value of other objectives $\mathbf{Q}_{\beta_i}$. This is known as a complex-valued homogeneous quadratically constrained quadratic program, and is generally an NP-hard problem for $N_c > 2$ [52,53]. However, established algorithms can easily find local optima, allowing us to include constraints within our theory. As an example, we show in Fig. 2(c) the manipulation of object ① in the y direction only (constraining the x motion to zero), while object ② is kept completely still, i.e., $\langle \mathbf{Q}_{\delta y_1} \rangle$ is maximized, while $\langle \mathbf{Q}_{\delta x_1} \rangle$, $\langle \mathbf{Q}_{\delta x_2} \rangle$, and $\langle \mathbf{Q}_{\delta y_2} \rangle$ are forced to be close to zero. The Gurobi Optimization solver is used for this task [54]. The resulting wave field indeed manages to compensate local forces acting on an object, as indicated by the red arrows in the inset of Fig. 2(c), and cancels the prescribed force projections (more details in the Supplemental Material [41]). This represents another multiobjective tweezing task of high practical relevance.

VIII. DISCUSSION

We have shown that local radiation forces on targets embedded in complex scattering media can be optimized through the GWS matrix: Its extremal eigenstates maximize force or torque projections on a single object, while in the multiobject

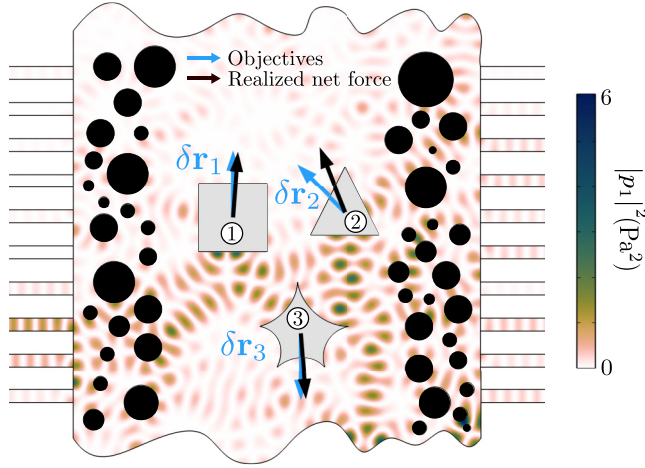


FIG. 3. Example of triobjective optimization through a turbid medium (fixed black circular scatterers). A single incident wavefront is optimized to simultaneously impose independent radiation force objectives on three targets of arbitrary shape (in gray: square, triangle, and star). The background colormap shows the resulting squared pressure field; arrows indicate the prescribed force directions (blue) and the corresponding realized net forces on each object (black).

case they realize Pareto-optimal trade-offs among competing objectives. Together with a realistic demonstration of constrained optimization of objects' motion along prescribed directions, our article establishes a unified framework across acoustics and electromagnetism, and provides a principled route to coordinated manipulation in strongly scattering environments. In addition, the uncertainty relation of Eq. (15) quantifies the fundamental trade-offs that arise when simultaneously optimizing force or torque components and shows that Pareto optimization in wave momentum shaping mirrors the mathematical structure associated with incompatible quantum observables [50].

Beyond fundamental interest, our results may contribute to the development of *in vivo* precision therapy and drug delivery, where navigation through deep tissues requires controlling radiation forces in turbid media [23–25, 55–59]. From a broader perspective, our framework applies across wave platforms and frequency ranges, from acoustofluidic manipulation to electromagnetic actuation. As a final illustration of the generality of the method, Fig. 3 shows how a single incident wavefront can simultaneously enforce multiple independent radiation force objectives on multiple targets of arbitrary geometry, despite the presence of a turbid scattering medium. We also note interesting connections between our findings and a recent article [60], which investigates the simultaneous control of transmission, reflection, and absorption. Our article strongly suggests that the notion of Pareto optimality for multiobjective wavefront shaping introduced here can serve as a common framework for such joint-control scenarios beyond forces and torques.

An important next step is the experimental validation of our multiobjective predictions, including direct tests of Pareto fronts and constrained actuation in realistic multiscatterer environments. A second direction is the theoretical extension to lossy media, for which the unitary foundation established here

provides the natural and rigorous starting point. Promisingly, experiments already indicate that protocols based on GWS matrices remain highly effective in practice even in the presence of absorption [36]. Together, these developments open a clear path toward the ultimate goal: collective and selective manipulation of multiple objects in lossy, strongly scattering media.

ACKNOWLEDGMENT

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DATA AVAILABILITY

The data that support the findings of this article are openly available [61].

APPENDIX A: DERIVATION DETAILS FOR EQ. (5)

We show here how Eq. (5) is derived. It extends an important variational integral relation [62, 63] connecting parametric changes occurring in the scattering region to changes of the immittance voltages and currents in the asymptotic regions. Defining the complex Poynting vector as

$$\mathbf{\Pi} = \begin{cases} \frac{1}{2} p_1 \mathbf{v}_1^* & (\text{in acoustics}), \\ \frac{1}{2} \mathbf{E} \times \mathbf{H}^* & (\text{in electromagnetism}), \end{cases} \quad (\text{A1a})$$

it can be shown that in both acoustics and electromagnetism [64]

$$-2 \int_A (\mathbf{\Pi} + \mathbf{\Pi}^*) \cdot d\mathbf{A} = \mathbf{u}^\dagger \mathbf{i} + \mathbf{i}^\dagger \mathbf{u} \quad (\text{A2})$$

holds, where $\mathbf{A}$ is the boundary of the scattering volume V , oriented away from the scattering region. Equation (A2) holds whenever no reactive wave enters the volume V ; i.e., the scattering region is only excited via propagating waves from the outside. Experimentally, this often requires placing sources in the far field, so that all evanescent waves are sufficiently decayed. For a given physical state a , Eq. (A2) reads in acoustics

$$P(a) \equiv - \int_A (p_{1a} \mathbf{v}_{1a}^* + p_{1a}^* \mathbf{v}_{1a}) \cdot d\mathbf{A} = \mathbf{u}_a^\dagger \mathbf{i}_a + \mathbf{i}_a^\dagger \mathbf{u}_a. \quad (\text{A3})$$

By extension, it is possible to compose any other physical state $a \pm b$ or $a \pm jb$. Using the linearity of p_1 , $\mathbf{v}_1$, $\mathbf{u}$, and $\mathbf{i}$, one can verify the so-called polarization identity [65]:

$$\frac{P(a+b) + jP(a+jb) - [P(a-b) + jP(a-jb)]}{4} = - \int_A (p_{1a} \mathbf{v}_{1b}^* + p_{1b}^* \mathbf{v}_{1a}) \cdot d\mathbf{A} = \mathbf{u}_b^\dagger \mathbf{i}_a + \mathbf{i}_b^\dagger \mathbf{u}_a. \quad (\text{A4})$$

Taking the complex conjugate of Eq. (A4) for the states $a \rightarrow (p_1, \mathbf{v}_1, \mathbf{u}, \mathbf{i})$ and $b \rightarrow (\delta p_1, \delta \mathbf{v}_1, \delta \mathbf{u}, \delta \mathbf{i})$, we obtain

$$- \int_A (p_1^* \delta \mathbf{v}_1 + \delta p_1 \mathbf{v}_1^*) \cdot d\mathbf{A} = \mathbf{u}^\dagger \delta \mathbf{i} + \mathbf{i}^\dagger \delta \mathbf{u}. \quad (\text{A5})$$

Now, we use δ to denote small variations of fields or parameters due to perturbations such that $\delta f = \nabla_{\delta \alpha} f \cdot \delta \alpha$. In

acoustics,

$$\nabla \cdot \mathbf{v}_1 = -j\omega\kappa_0 p_1 \text{ and } \nabla p_1 = -j\omega\rho_0 \mathbf{v}_1. \quad (\text{A6})$$

At first variational order,

$$\nabla \cdot (\delta \mathbf{v}_1) = -j\omega\delta\alpha \cdot \nabla_{\delta\alpha}(\kappa_0 p_1), \quad (\text{A7a})$$

$$\nabla(\delta p_1) = -j\omega\delta\alpha \cdot \nabla_{\delta\alpha}(\rho_0 \mathbf{v}_1). \quad (\text{A7b})$$

To relate this to the immittance vectors, Eqs. (A7a) and (A7b) may be combined as follows:

$$-\nabla \cdot (p_1^* \delta \mathbf{v}_1 + \delta p_1 \mathbf{v}_1^*) = j\omega\delta\alpha \cdot (|p_1|^2 \nabla_{\delta\alpha} \kappa_0 + |\mathbf{v}_1|^2 \nabla_{\delta\alpha} \rho_0). \quad (\text{A8})$$

It is then possible to compute the volume integral of Eq. (A8) and use Eq. (A5) to obtain

$$\mathbf{u}^\dagger \delta \mathbf{i} + \mathbf{i}^\dagger \delta \mathbf{u} = j\omega\delta\alpha \cdot \int_V (|p_1|^2 \nabla_{\delta\alpha} \kappa_0 + |\mathbf{v}_1|^2 \nabla_{\delta\alpha} \rho_0) dV. \quad (\text{A9})$$

In electromagnetism, Maxwell-Faraday and Maxwell-Ampère laws in frequency domain give

$$\nabla \times \delta \mathbf{E} = -j\omega\delta\alpha \cdot \nabla_{\delta\alpha}(\mu \mathbf{H}), \quad (\text{A10a})$$

$$\nabla \times \delta \mathbf{H} = j\omega\delta\alpha \cdot \nabla_{\delta\alpha}(\varepsilon \mathbf{E}), \quad (\text{A10b})$$

and

$$\begin{aligned} -\nabla \cdot (\mathbf{E}^* \times \delta \mathbf{H} + \delta \mathbf{E} \times \mathbf{H}^*) \\ = j\omega\delta\alpha \cdot (|\mathbf{E}|^2 \nabla_{\delta\alpha} \varepsilon + |\mathbf{H}|^2 \nabla_{\delta\alpha} \mu), \end{aligned} \quad (\text{A11})$$

from which we obtain

$$\mathbf{u}^\dagger \delta \mathbf{i} + \mathbf{i}^\dagger \delta \mathbf{u} = j\omega\delta\alpha \cdot \int_V (|\mathbf{E}|^2 \nabla_{\delta\alpha} \varepsilon + |\mathbf{H}|^2 \nabla_{\delta\alpha} \mu) dV. \quad (\text{A12})$$

This is completely analogous to the acoustic case and yields the same end result.

APPENDIX B: OPTIMAL TORQUE

Rotations enter the variational identities [Eqs. (A9) and (A12)] through the generator of infinitesimal rigid rotations: A small angle $\delta\theta$ about a pivot point $\mathbf{r}_0$ moves each material point by $\delta\mathbf{r} = \delta\theta \times (\mathbf{r} - \mathbf{r}_0)$. This gives the parametric derivative $\nabla_{\delta\theta} = -(\mathbf{r} - \mathbf{r}_0) \times \nabla$. Defining the time-averaged torque $\bar{\tau}(\mathbf{r}_0)$ as

$$\bar{\tau}(\mathbf{r}_0) = \int_V (\mathbf{r} - \mathbf{r}_0) \times \bar{\mathbf{f}} dV, \quad (\text{B1})$$

one obtains

$$\bar{\tau}(\mathbf{r}_0) \cdot \delta\theta = -\frac{j}{4\omega} (\mathbf{u}^\dagger \delta \mathbf{i} + \mathbf{i}^\dagger \delta \mathbf{u}). \quad (\text{B2})$$

Then, in analogy with the linear force case, the GWS reads

$$\frac{1}{\omega} \langle \mathbf{Q}_{\delta\theta} \rangle = \bar{\tau}(\mathbf{r}_0) \cdot \hat{\mathbf{n}} \quad (\text{B3})$$

and

$$\frac{1}{\omega} \langle \mathbf{Q}_{\delta\theta}^{\mathbf{r}_{0,i}} \rangle = \frac{1}{|\delta\theta_{\text{tot}}|} \sum_{i=1}^{N_d} |\delta\theta_i| \bar{\tau}_i(\mathbf{r}_{0,i}) \cdot \hat{\mathbf{n}}_i, \quad (\text{B4})$$

where $\hat{\mathbf{n}} = \delta\theta/|\delta\theta|$. Interestingly, for axisymmetric objects around θ (such as the disks in Fig. 2), $\mathbf{Q}_{\delta\theta} = \mathbf{0}$, and no input state can transfer angular momentum to the objects. This counterintuitive result, holding for any lossless scattering, was already discovered in both acoustics and electromagnetism via other methods; see, for example, Refs. [66–73].

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