

Critical behavior of the multigel percolation model

Carlos G. L. Sousa ^{1,*} José S. Andrade Jr. ¹ Cesar I. N. Sampaio Filho ¹
Erika Eiser ² Alex Hansen ² Hans J. Herrmann ^{1,3} and André A. Moreira ¹

¹*Departamento de Física, Universidade Federal do Ceará, Caixa Postal 6030, Campus do Pici, 60455-760 Fortaleza, Ceará, Brazil*

²*PoreLab, Department of Physics, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway*

³*PMMH, ESPCI, Paris, 7 quai St. Bernard, 75005 Paris, France*



(Received 29 January 2026; accepted 2 June 2026; published 30 July 2026)

In a recently introduced bond-percolation model, Sousa *et al.* [*Phys. Rev. Res.* **7**, L022067 (2025)] showed that multiple spanning clusters can coexist in three dimensions, with their number remaining unbounded in the thermodynamical limit. Here, we investigate further this multigel percolation model by showing that the fractal dimension of the shortest path is the same as standard three-dimensional percolation, while the fractal dimension of the contact between percolating clusters at the critical point is consistent with that expected for the intersection of two fractal objects. When the initial density is treated as a control parameter, the behavior of both the largest cluster and the second moment follows the same scaling relations as in standard percolation. For systems with multiple species, the minimum density required for all clusters to percolate scales with system size in agreement with the fractal dimension. In the case of two species with different initial densities, we find a transition from a regime where only one species percolates to one where both percolate. These results strengthen the connection between multigel percolation and classical percolation theory and suggest possible routes for designing materials with multiple interpenetrating percolating structures, such as trigels and tetragels.

DOI: [10.1103/762d-x9gp](https://doi.org/10.1103/762d-x9gp)

I. INTRODUCTION

Percolation, introduced by Broadbent and Hammersley [1] as a paradigm for random connectivity, has since become one of the most widely applied statistical models in statistical physics [2,3]. It has been used successfully to describe a wide range of phenomena in nature, including the sol-gel transition [4,5], incipient flow through porous media [6–10], epidemics [11–14], even traffic flow [15–17], and network failure [18].

In a recent development [19,20], it was shown that a class of materials called “bigels,” which form through arrested demixing in binary colloidal mixtures, may develop two spanning clusters simultaneously, one for each species. Other studies have also reported experimental realizations of bipercolated structures [21,22] and tripercolated structures [23,24]. The emergence of multiple percolating clusters in these systems can happen only because aggregates of different species cannot merge. In fact, it has been proven [25,26] that for percolation with no restriction on the merging of clusters, the number of infinite clusters is zero, one, or infinity.

However, at the percolation threshold p_c , one must distinguish between infinite and spanning clusters [27–29]. In the case of spanning clusters, for $d > 1$, the probability of finding

more than one spanning cluster at the critical point p_c does not go to zero as $L \rightarrow \infty$ [27]. In fact, for $d = 2$, an exact result for the probability P_k of having k spanning clusters in two dimensions was derived using conformal invariance [30]. In another study [31], the go-with-the-winners strategy [32] was applied to a rectangular lattice with periodic boundaries in one direction to simulate five simultaneously spanning clusters [33] and calculate the probability of finding them.

To achieve more than one percolating cluster above the critical point, the polychromatic model was introduced [34], where nodes or bonds are differentiated by species, allowing n percolating clusters, provided that n times the critical density for standard percolation of a single species case remains below unity. This concept was expanded upon in models with species-dependent bond formation [35–37], correlated two-component percolation [38], site percolation with independent concentrations [39], AB percolation [40,41], colored percolation [42], two-color percolation [43], polychromatic random networks [44], and tricolor self-avoiding walks [45]. However, none of these models can generate more than $1/p_c$ percolating clusters.

In a recent paper [46], we introduced what is now referred to as the multigel percolation model (MPM), which allows for $k > 1/p_c$ spanning clusters to occur in three-dimensional (3D) lattices not only at the critical point but also above it (see Fig. 1). The purpose of the present work is to extend and deepen those results. We analyzed three types of lattices: the simple cubic lattice, the diamond lattice [47], and the simple cubic lattice with additional bonds between nearest and second-nearest neighbors [48]. We seek to identify the necessary conditions to obtain k spanning clusters and determine

*Contact author: germano@fisica.ufc.br

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

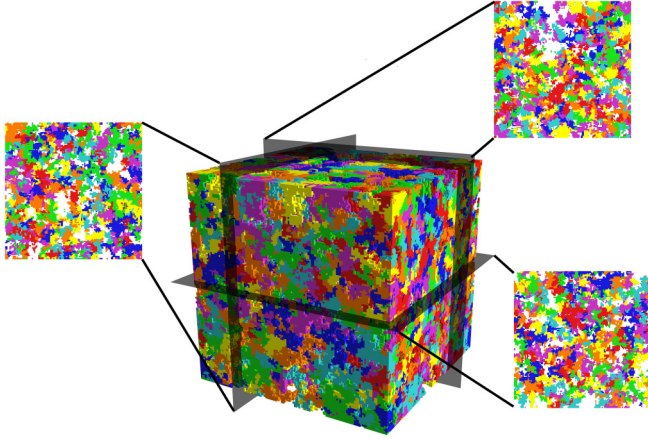


FIG. 1. Percolating clusters of seven species in a simulation of the multigel percolation model [46] on a simple cubic lattice with side length $L = 128$. Each color corresponds to a different species. The empty regions between the percolating clusters correspond to clusters that do not percolate by the end of the process. The black planes indicate the cuts of the configuration shown.

whether there is an upper limit to the number of percolating clusters. Additionally, we analyze the structure of these clusters. Our results suggest that it may be experimentally feasible to create systems exhibiting multiple percolating clusters.

The paper is organized as follows: In Sec. II, we reintroduce, for completeness, the MPM. In Sec. III, we discuss the critical properties of the model, including the second moment of the cluster-size distribution, the largest cluster, and the critical point. In Sec. IV, we analyze the topological properties, such as the fractal dimension, the shortest path, and the interfacial contact between the largest clusters at the critical point. In Sec. V, we study the minimum and maximum bounds of the initial density. In Sec. VI, we show how to determine the minimum side length required to find k percolating clusters. In Sec. VII, we examine the scenario in which two species have different initial densities. Finally, in Sec. VIII, we present the conclusions.

II. MODEL

We consider a simple cubic lattice of linear size L , in which all bonds are initially empty ($p = 0$). A fraction $k\rho \leq 1$ of the sites is permanently inhabited by k different species, chosen at random. In the initial configuration, each species inhabits an equal fraction ρ of the lattice sites, under the constraint that no site can be inhabited by more than one species. Throughout this work, the terms “empty” and “occupied” refer to bonds, while “inhabited” and “uninhabited” refer to sites. Following the usual bond-percolation procedure [49], bonds are randomly selected and occupied, ensuring that each bond is chosen only once. When an occupied bond connects a site inhabited by a given species to an uninhabited site, the latter, along with all uninhabited sites in its connected cluster, becomes inhabited by the same species. A cluster is defined as a group of connected sites that are either uninhabited or inhabited by a single species. This rule allows the spreading of a species into uninhabited regions through the aggregation

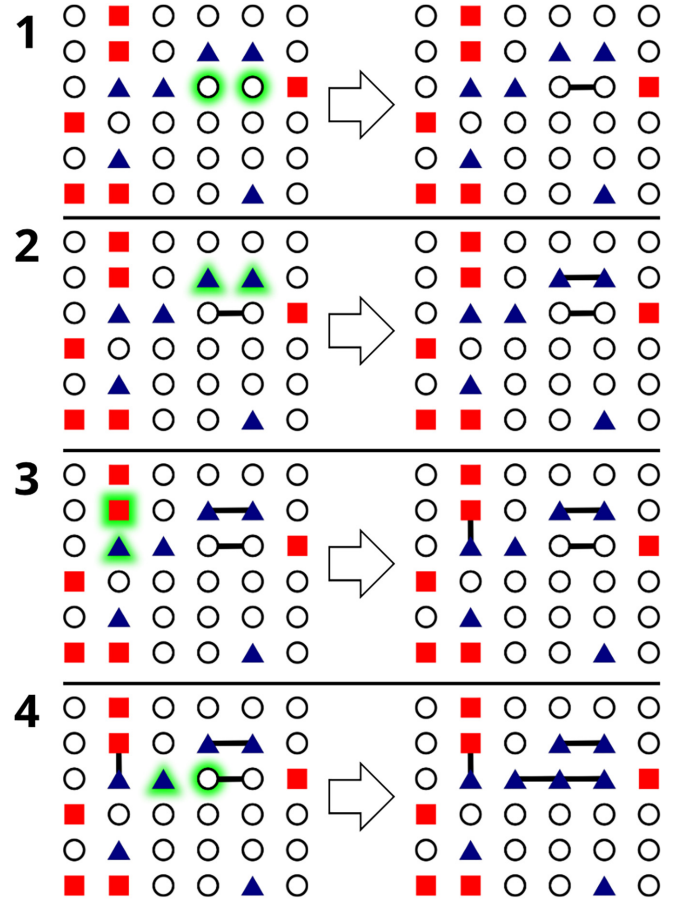


FIG. 2. Rules of the multigel percolation model [46]. Red squares and blue triangles represent nodes inhabited by two different species, while white circles denote uninhabited nodes. In each panel, the configuration on the left shows the system before updating, by occupying the bond connecting the two neighboring nodes highlighted in green, while the configuration on the right shows the system after this occupation. The four panels illustrate different cases: (1) When a bond connects two uninhabited nodes, they remain uninhabited; (2) when it connects two nodes inhabited by the same species, they belong to the same cluster; (3) when it connects two nodes inhabited by different species, both retain their identities and remain in separate clusters; and (4) when it connects an uninhabited node (or an uninhabited cluster) to a node inhabited by a given species, the entire uninhabited cluster adopts the species of the inhabited node.

of clusters, as illustrated in Fig. 2. A crucial rule of the model is that neighboring sites inhabited by different species retain their species identities even if they are connected by an occupied bond, and therefore belong to distinct clusters. The case $k = 1$ corresponds to standard bond percolation, whereas the limit $\rho k = 1$ represents the polychromatic model.

We implement periodic boundary conditions in the x and y directions and open boundaries in the z direction and define n_b as the number of bonds that have been occupied. Therefore, the probability of bond occupation is $p = n_b / (3L^3 - L^2)$, which is the variable controlling the percolation process. The properties of the MPM are investigated as a function of two parameters, namely, the number of different species k and the density ρ of nodes initially inhabited by each species. Except

for the cases shown in Sec. VII, only cases where the initial density ρ is the same for all species are considered.

We search for percolating clusters that span from the top to the bottom of the 3D lattice along the z axis. Growing clusters of different species act as obstacles to one another. In this way, each species races to reach the uninhabited nodes first while blocking the growth of competing clusters of other species.

In some cases, it is also possible to observe more than one percolating cluster of the same species. For example, for $\rho = 0.025$, $L = 512$, and $k = 4$, we observe, with a probability of approximately 0.47, at least two percolating clusters of the same species at the critical point. Most of these spanning clusters will merge into a single cluster with increasing p before the end of the occupation process. However, in some cases, it is even possible to observe two percolating clusters of the same species at $p = 1$. For instance, for $\rho = 3.89 \times 10^{-4}$, $L = 512$, and $k = 9$, this occurs in about 2% of the samples.

III. CRITICAL PROPERTIES OF THE MODEL

In this section, we study, on a simple cubic lattice, the second moment of the cluster-size distribution χ_i , the size of the largest cluster S_i , the total number of nodes N_i , and the percolation threshold $p_c^{(i)}$ for each species i , labeled in percolation order. We find that they exhibit the same critical exponents, γ and ν , as those of standard 3D percolation.

A. Second moment and largest cluster

The second moment of the cluster-size distribution is defined as [2]

$$\chi_i = \frac{\sum_l s_{li}^2 - S_i^2}{N_i}, \quad (1)$$

where s_{li} is the mass of cluster l inhabited by the i th species. Figure 3 illustrates the evolution of these quantities, and the inset shows that the curves for χ and S exhibit some dependence on the percolation order i near the critical point.

In the inset of Fig. 4, we show that, by finding the correct p_c and using the critical exponents $\gamma = 1.819 \pm 0.003$ [50] and $\nu = 0.875 \pm 0.001$ [51] of standard three-dimensional percolation, the curves of χ for different L collapse on top of each other when the data are rescaled. Furthermore, Fig. 4 shows, on a logarithmic scale, that χ follows the relation [52]

$$\chi(p) \propto |p - p_c|^{-\gamma}, \quad (2)$$

and that the curves display the expected critical exponent behavior.

B. Critical point

Figure 5 presents the average percolation thresholds $p_c^{(i)}(L)$ at which the i th species exhibits, for the first time, a cluster that reaches both the top and bottom faces of the system. We observe that the percolation thresholds $p_c^{(i)}(L)$ converge to the same point as the lattice size increases. The least-squares fit to the data is consistent with the following equation [52]:

$$p_c^{(i)}(L) = p_c(1 - a_i L^{-\frac{1}{\nu}}), \quad (3)$$

where a_i takes different values for different i , and extrapolating L to $L \rightarrow \infty$ gives $p_c = 0.321 \pm 0.001$, which

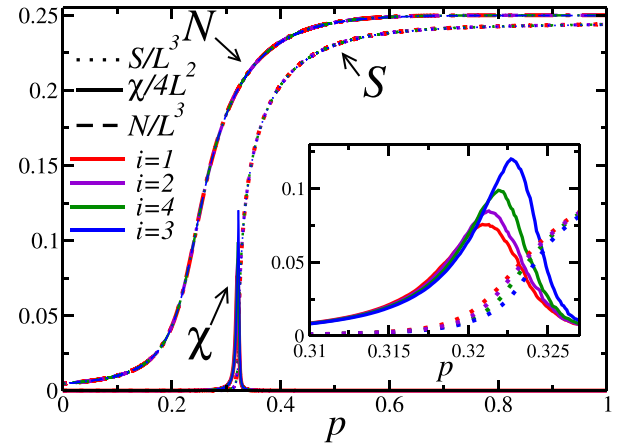


FIG. 3. Evolution of the properties of the cluster distribution during the bond occupation process of the MPM. Species are labeled in the order in which they percolate; that is, we label as 1 the species that percolates first, as 2 the second species to percolate, and so on. The curves are averages over 1000 samples. The colors of the curves indicate the order i in which the species percolated. The dotted lines correspond to the average size S of the largest cluster; the solid lines indicate the second moment of the cluster-size distribution χ , as defined in Eq. (1); and the dashed lines indicate the number of nodes N that have been occupied by each species. For convenience, each of the quantities was rescaled by the system size in a different way, as indicated in the legend. We used a simple cubic lattice with size $L = 512$, four species, and an initial density $\rho = 0.0047$. The inset shows that the curves for χ and S exhibit some dependence on the percolation order i .

is consistent with the scaling shown in Fig. 4. Note that p_c obtained is much larger than the threshold of standard bond percolation on the simple cubic lattice, which is $p_c = 0.248\,8126 \pm 0.000\,0005$ [51].

The critical threshold p_c depends on the initial density ρ and the number of species k . In Fig. 6, we show, for the first percolating cluster, p_c as a function of ρ for different numbers of species k . We observe that, for sufficiently large ρ , the average percolation threshold grows linearly with ρ as follows:

$$p_c(\rho) = C_0^k + C_k \rho, \quad (4)$$

where the constant C_0^k and the slope C_k are associated with k . Moreover, in the limit $\rho \rightarrow 0$, the linear relation no longer holds and p_c approaches the standard bond-percolation threshold. Furthermore, Fig. 7 shows that the slope C_k follows an exponential law,

$$C_k = C e^{\alpha k}, \quad (5)$$

where the least-squares fit to the data using this exponential law gives $C = 0.14 \pm 0.02$ and $\alpha = 0.82 \pm 0.03$.

The maximum percolation threshold, $p_c = 1$, corresponds to the maximum initial density $\rho_{\max}$, above which no percolating clusters can be obtained in the thermodynamic limit. For $k = 2$ and $k = 3$, p_c does not reach unity for any value of ρ , consistent with the behavior expected from the polychromatic model. Further details on this topic are discussed in Sec. V.

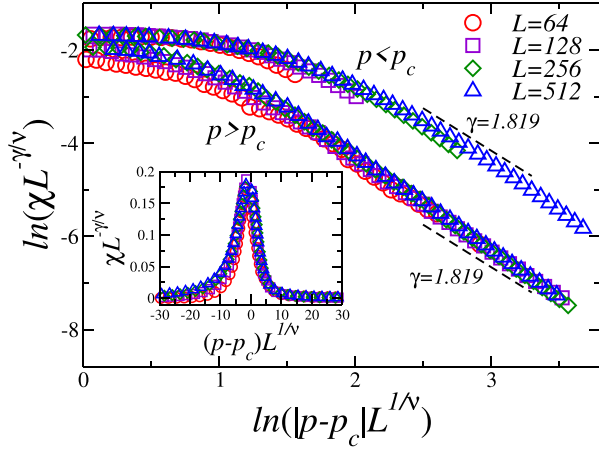


FIG. 4. Logarithmic plot of the finite-size scaling of χ for the first percolating cluster. We used the exponents $\gamma = 1.819 \pm 0.003$ [50] and $\nu = 0.875 \pm 0.001$ [51] of standard 3D percolation. The percolation threshold was $p_c = 0.321$. The slope of the dashed lines corresponds to the known value of γ . The curves asymptotically approach this slope as L increases. The averages in this figure were taken over 1000 samples on a simple cubic lattice for the sizes L shown in the legend, and we considered four species and $\rho = 0.0047$. The inset shows the same data on a linear scale.

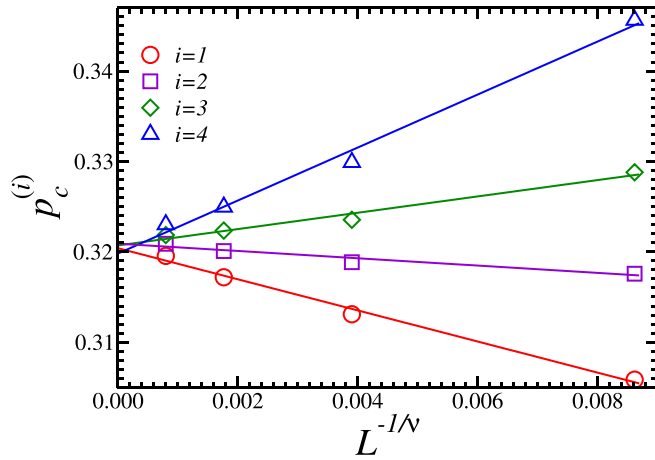


FIG. 5. Percolation threshold p_c as a function of $L^{-1/\nu}$ for four different species labeled in percolation order. For each realization, we record $p_c^{(i)}$ as the fraction p of occupied bonds at which the i th spanning cluster takes place. In cases where more than one cluster of the same species connects the boundaries, we record the first one to percolate. The thresholds $p_c^{(i)}$ are the averages over 1000 realizations for $L = 64$, and over 100 realizations for $L = 128, 256$, and 512 on simple cubic lattices. Here, we considered $\rho = 0.0047$, for which we always find four percolating species for all investigated system sizes. The curves follow the function given by Eq. (3), and the extrapolation of the straight lines suggests that, in the thermodynamic limit $L \rightarrow \infty$, all the clusters percolate at the same fraction $p_c = 0.321 \pm 0.001$. Here, we used the exponent $\nu = 0.875 \pm 0.001$ of standard 3D percolation [51]. Statistical errors are smaller than the symbol size.

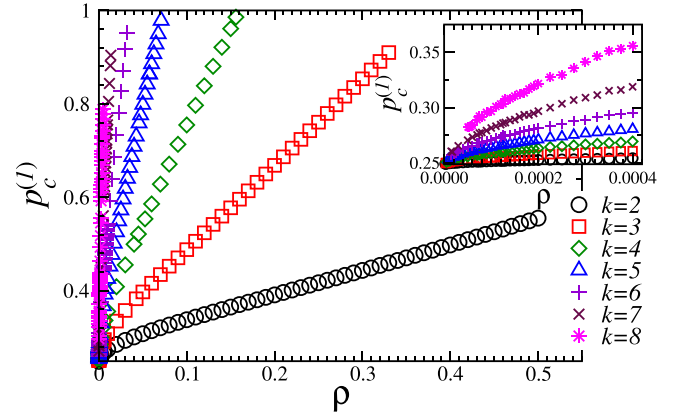


FIG. 6. Critical point of the first percolating cluster $p_c^{(1)}$ as a function of ρ . Each realization has k percolating clusters of different species, with symbols and colors indicating the corresponding k . We observe that, for sufficiently large ρ , the average percolation threshold grows linearly with ρ , as in Eq. (4), and exponentially with k , as in Eq. (5). The value of ρ at which $p_c^{(1)} = 1$ is the maximum initial density $\rho_{\max}$, above which no percolating cluster should be observed in the thermodynamic limit. Moreover, the inset shows that, in the limit $\rho \rightarrow 0$, the linear relation no longer holds and p_c approaches the standard bond-percolation threshold. The averages were computed over 100 samples on a simple cubic lattice of size $L = 512$. Statistical errors are smaller than the symbol size.

IV. TOPOLOGICAL PROPERTIES

In this section, we show that the fractal dimension d_f of the largest cluster at the critical percolation threshold p_c , as well as the fractal dimension of its shortest path, coincide with the established values for 3D percolation. Furthermore, we show that the number of interfacial contacts between the coexisting percolating clusters follows a power-law scaling.

A. Fractal dimension

Next, we study the fractal dimension of the percolating clusters segregated by percolation order i . For each species,

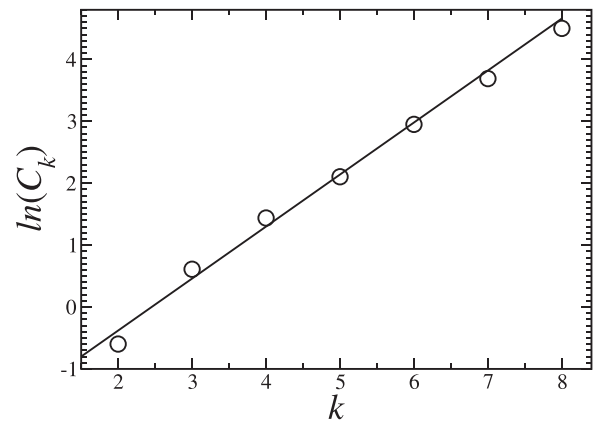


FIG. 7. Semilogarithmic plot of the slope C_k as a function of k for the first percolating cluster. The quantity C_k grows exponentially with k , according to Eq. (5). Statistical errors are smaller than the symbol size.

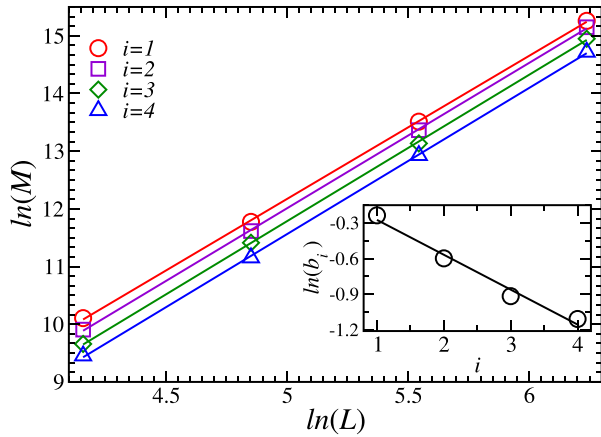


FIG. 8. Logarithmic plot of the mass M of the first (red circles), second (purple squares), third (green diamonds), and fourth (blue triangles) percolating clusters at p_c as a function of L . We consider four species, labeled in percolation order, with $\rho = 0.0047$ for each species, averaged over 1000 realizations for all sizes L . The fractal dimensions were obtained by fitting the mass of the largest clusters at $p_c = 0.321$ to the power law $M_i = b_i L^{d_f}$ using a least-squares method, yielding the exponents $d_f^{(1)} = 2.50 \pm 0.03$ for the first, $d_f^{(2)} = 2.52 \pm 0.02$ for the second, $d_f^{(3)} = 2.54 \pm 0.02$ for the third, and $d_f^{(4)} = 2.53 \pm 0.02$ for the fourth percolating cluster. Within error bars, our results indicate that the fractal dimensions are the same regardless of percolation order and are consistent with the fractal dimension of standard 3D percolation, $d_f = 2.5227$ [56,57]. The inset shows, in a semilogarithmic plot, that the prefactors b_i decrease with percolation order i , as described by Eq. (7). Statistical errors are smaller than the symbol size.

we consider the size of the largest cluster at the extrapolated critical threshold p_c [53,54]. We define M_i as the mass of the cluster of the species that percolates at the i th threshold $p_c^{(i)}$. The results presented in Fig. 8 show that M_i scales with system size as [55]

$$M_i = b_i L^{d_f}, \quad (6)$$

where, within error bars, the exponent d_f is the same for all species and consistent with the classical 3D percolation, $d_f = 2.5227$ [56,57]. The prefactors b_i , shown in the inset of Fig. 8, decrease with i according to the exponential law

$$b_i = A e^{-Bi}, \quad (7)$$

where the least-squares fit to the data yields $A = 1.02 \pm 0.08$ and $B = 0.29 \pm 0.03$, indicating that the masses of the largest clusters at p_c decrease exponentially with the percolation order.

B. Interfacial contact between the largest clusters at the critical point

Figure 9(a) shows the number of occupied bonds between the largest clusters, n_{pc} , at p_c . For each contact between two species i and j , n_{pc} follows a power law, $n_{pc}^{ij} \propto L^{d_f^{\cap}}$, where, within error bars, $d_f^{\cap}$ corresponds to

$$d_f^{\cap} = 2d_f - d, \quad (8)$$

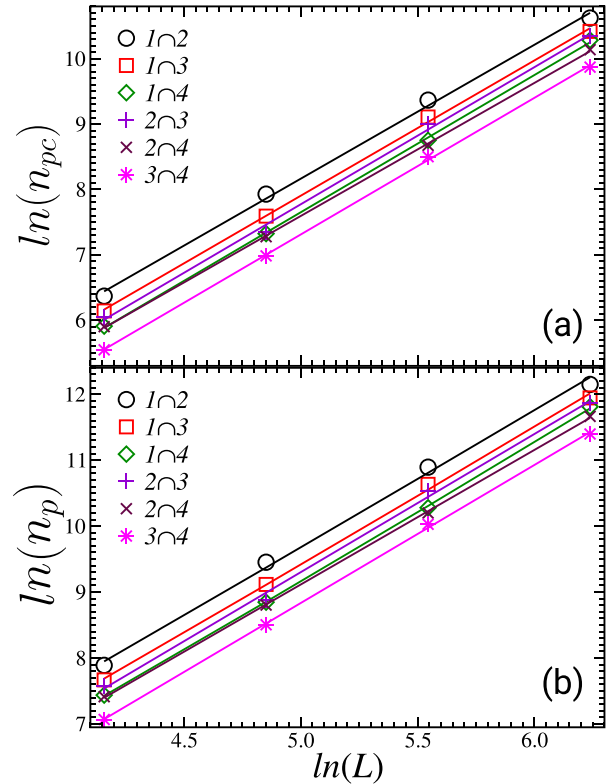


FIG. 9. Logarithmic plot of (a) the number of occupied bonds between the largest clusters, n_{pc} , at p_c and (b) n_p , the number of bonds between the largest clusters at p_c , regardless of whether the bonds had been occupied until p_c or not, both as functions of system size L . We consider four species, labeled in percolation order, with $\rho = 0.025$, averaged over 1000 realizations on simple cubic lattices. The different symbols represent the values of n_{pc} and n_p between the two largest clusters i and j . The exponents were obtained by fitting n_{pc} and n_p at $p_c = 0.434$ to the power law $n \propto L^{d_f^{\cap}}$. For n_{pc} , the exponents are $d_f^{1\cap 2} = 2.05 \pm 0.04$, $d_f^{1\cap 3} = 2.07 \pm 0.04$, $d_f^{1\cap 4} = 2.09 \pm 0.05$, $d_f^{2\cap 3} = 2.09 \pm 0.06$, $d_f^{2\cap 4} = 2.04 \pm 0.02$, and $d_f^{3\cap 4} = 2.09 \pm 0.06$. Similarly, for n_p , the exponents are $d_f^{1\cap 2} = 2.06 \pm 0.04$, $d_f^{1\cap 3} = 2.07 \pm 0.04$, $d_f^{1\cap 4} = 2.09 \pm 0.05$, $d_f^{2\cap 3} = 2.09 \pm 0.06$, $d_f^{2\cap 4} = 2.04 \pm 0.02$, and $d_f^{3\cap 4} = 2.09 \pm 0.06$. Within error bars, the exponents are consistent with the fractal dimension of the intersection of two fractal objects, as given by Eq. (8). Statistical errors are smaller than the symbol size.

as expected for the intersection between two fractal objects [58].

The same behavior and corresponding fractal dimension are observed in Fig. 9(b), where n_p denotes the number of bonds between the largest clusters at p_c , regardless of whether the bonds had been occupied until p_c or not.

C. Shortest path

Next, we study the fractal dimension of the shortest path. For each species, we consider the largest cluster at the extrapolated critical threshold p_c . We define l_i as the size of the shortest path between the top and bottom of the lattice along the z axis for the cluster that percolates at the i th threshold $p_c^{(i)}$. Note that, at p_c , not all species have percolated yet, so we selected only those clusters that had percolated. We

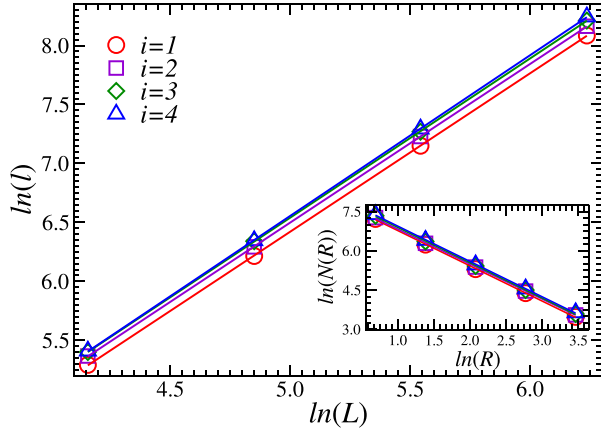


FIG. 10. Logarithmic plot of the shortest path l as a function of L . l is calculated as the number of sites in the shortest path from the top to the bottom of the cluster that has just percolated at p_c . We considered four species, labeled in percolation order, with $\rho = 0.025$ for each species and averaged over 10 000 realizations for all system sizes L on simple cubic lattices. The fractal dimensions are obtained through a least-squares fit of l at $p_c = 0.434$ for each species to a power law, $l_i \propto L^{d_{\min}^{(i)}}$, yielding the exponents $d_{\min}^{(1)} = 1.35 \pm 0.03$ for the first, $d_{\min}^{(2)} = 1.35 \pm 0.03$ for the second, $d_{\min}^{(3)} = 1.36 \pm 0.02$ for the third, and $d_{\min}^{(4)} = 1.36 \pm 0.02$ for the fourth percolating cluster. The inset shows, on a logarithmic scale, the results of the yardstick method for the shortest paths of the percolating clusters on a cubic lattice with $L = 512$. The vertical axis shows the number of covering spheres, $N(R)$, and the horizontal axis the sphere radius, R . The least-squares fit of $N(R)$ at p_c to a power law, $N(R) \propto R^{-d_{\min}}$, gives the exponents $d_{\min}^{(1)} = 1.39 \pm 0.02$, $d_{\min}^{(2)} = 1.39 \pm 0.02$, $d_{\min}^{(3)} = 1.39 \pm 0.02$, and $d_{\min}^{(4)} = 1.39 \pm 0.02$ for the first to fourth percolating clusters, respectively. Our results indicate that the fractal dimensions of the shortest paths are equal within error bars, regardless of percolation order, and are consistent with the fractal dimension of standard 3D percolation, $d_{\min} = 1.3756 \pm 0.0006$ [61]. Statistical errors are smaller than the symbol size.

evaluated $d_{\min}$ using two independent methods. In the first, we analyzed how the shortest path scales with system size, where the least-squares fit of l_i to a power law on L follows [52]

$$l_i \propto L^{d_{\min}}. \quad (9)$$

In the second method, we employed the yardstick approach [59,60] applied to a cubic lattice with $L = 512$. In this approach, we count how many spheres, $N(R)$, of radius R are required to cover the entire shortest path. Specifically, we begin the process by selecting the initial node as the center of the first sphere, then place the next sphere at a distance of $2R$ along the path, and continue this procedure until the path is fully covered. By repeating this for different values of R , we observe the following power-law behavior:

$$N(R) \propto R^{-d_{\min}}. \quad (10)$$

The results presented in Fig. 10 show that, within error bars, both methods yield the same exponent $d_{\min}$ for all species, consistent with the shortest path in standard 3D percolation, $d_{\min} = 1.3756 \pm 0.0006$ [61].

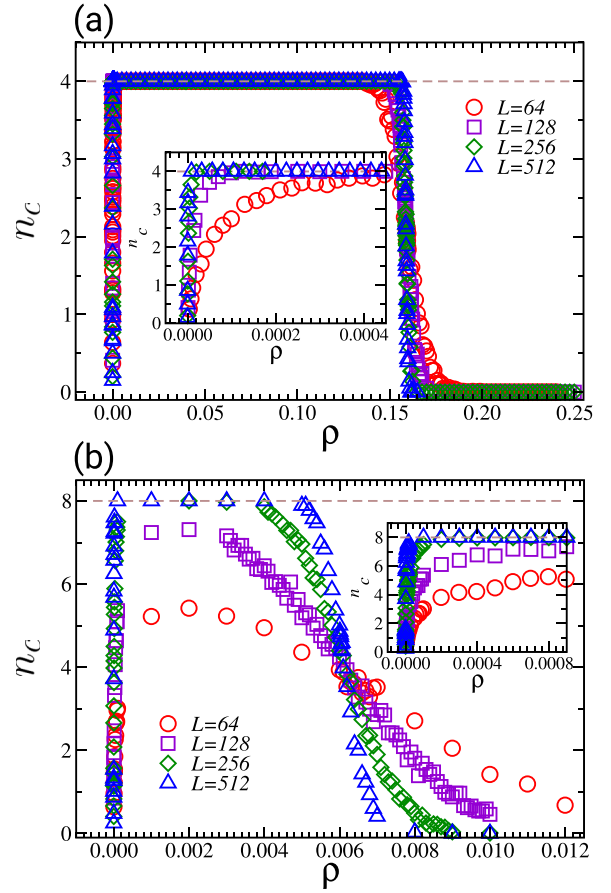


FIG. 11. Average number of percolating clusters of different species, n_c , as a function of ρ , for (a) four and (b) eight species. The legend indicates the side length L of the simple cubic lattice. Each data point represents an average over 100 realizations. A horizontal dashed line marks the upper limit of n_c . For all investigated system sizes, ρ must lie within a specific range, $\rho_{\min} < \rho < \rho_{\max}$, to reach this limit. For eight species, only lattices with $L \geq 256$ achieve it. The insets show that, near $\rho_{\min}$, n_c depends on the system size L .

V. BOUNDS OF THE INITIAL DENSITY

In this section, we investigate the conditions required for all species to percolate simultaneously. Figure 11 shows that this depends on the system size L , the number of species k , and, most importantly, on the initial density ρ . For fixed L and k , percolation of every species occurs only when ρ lies within a specific range, $\rho_{\min} < \rho < \rho_{\max}$.

At the maximum initial density, denoted by $\rho_{\max}$, a transition occurs that is analogous to the percolation transition at p_c and $\rho_{\max}$. Conversely, the minimum initial density, $\rho_{\min}(L)$, is size dependent and vanishes as $L \rightarrow \infty$, implying that, in the thermodynamic limit, any nonzero initial density is sufficient to ensure percolation for all species.

A. Maximum initial density $\rho_{\max}$

Since the threshold for site percolation on a simple cubic lattice is $p_c = 0.311\,607\,68 \pm 0.000\,000\,15 < 1/3$ [56], simultaneous percolating clusters occur for $k = 2$ and $k = 3$ even when all sites are initially inhabited ($\rho k = 1$), as in the

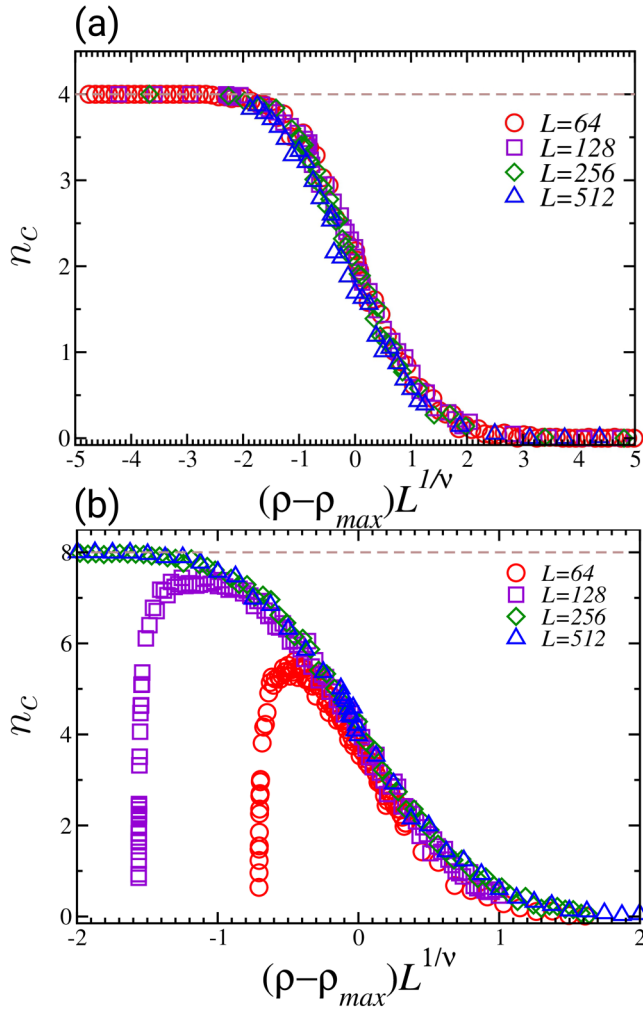


FIG. 12. Finite-size scaling of the average number of percolating clusters, n_c , as a function of ρ for (a) four and (b) eight species in the region near $\rho_{\max}$. The legend indicates the side length L of each simple cubic lattice. A scaling exponent $\nu = 0.875 \pm 0.001$ [51] is used, with $\rho_{\max} = 0.159 \pm 0.001$ for panel (a) and $\rho_{\max} = 0.0061 \pm 0.0001$ for panel (b).

polychromatic model [34]. For $k > 3$, however, there exists a $\rho_{\max} < 1/k$, above which clusters hinder each other's growth because too few uninhabited nodes remain for aggregation. Figure 12 shows that, at this density $\rho_{\max}$, a transition takes place from a regime where all species percolate to one where none of them do. As the system size L increases, this transition in ρ becomes sharper, following a finite-size scaling of $L^{-1/\nu}$, with $\nu = 0.875 \pm 0.001$ being the exponent of standard 3D percolation [51], suggesting that it is a classical percolation transition.

In the thermodynamic limit, the probability $W(\rho)$ of having the maximal number of percolating clusters becomes a step function, $\Theta(\rho)$, defined as

$$\Theta(\rho) = \begin{cases} 1, & \text{if } \rho < \rho_{\max}, \\ 0, & \text{if } \rho > \rho_{\max}. \end{cases} \quad (11)$$

Furthermore, for each species, we examine the largest cluster size S and the second moment χ as functions of ρ at $p = 1$,

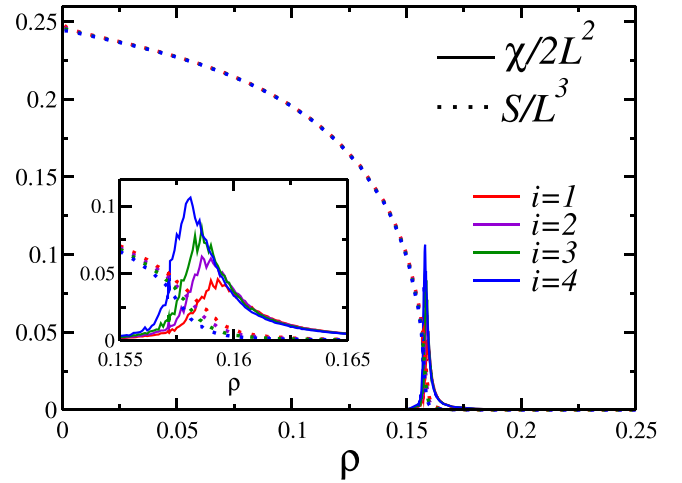


FIG. 13. Evolution of the properties of the cluster distribution at $p = 1$ as a function of ρ . Unlike in the other figures, the species are labeled according to the size of their largest cluster; that is, species 1 corresponds to the largest cluster, species 2 to the second largest, and so on. The curves represent averages over 1000 samples, and their colors indicate the size order i . The dotted lines correspond to the average size S of the largest cluster, while the solid lines indicate the second moment of the cluster-size distribution, χ , as defined in Eq. (1). For clarity, each quantity has been rescaled by the system size in a different way, as indicated in the legend. The data correspond to a system of size $L = 512$ on a simple cubic lattice with four species. The inset shows that the curves for χ and S exhibit some dependence on the size order i .

following the same procedure as in Sec. III. Figure 13 shows the curves labeled according to the order of the size of the largest cluster, revealing behavior similar to that shown in Fig. 3. The results shown in Fig. 14 show that a finite-size

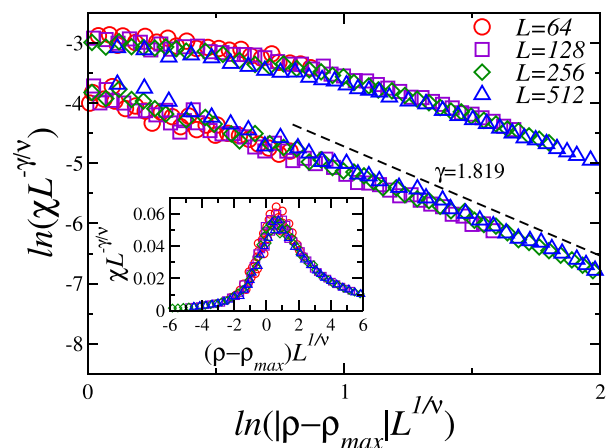


FIG. 14. Logarithmic plot of the finite-size scaling of χ for the largest cluster. The critical exponents $\gamma = 1.819 \pm 0.003$ [50], represented by the dashed line, and $\nu = 0.875 \pm 0.001$ [51] correspond to standard 3D percolation. Averages were taken over 1000 samples on simple cubic lattices with $L = 512$ and four species. The axes indicate how the curves were rescaled, and the inset shows the same data on a linear scale.

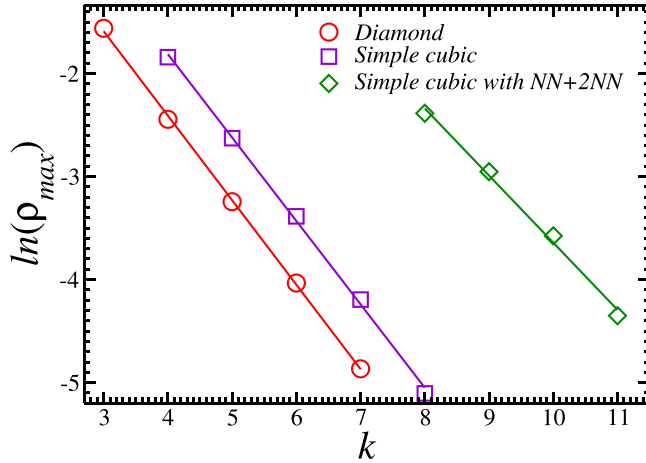


FIG. 15. Semilogarithmic plot of $\rho_{\max}$ as a function of k . Symbols correspond to results for different 3D lattices. The data were obtained as follows: For each k , $\rho_{\max}$ was determined as the value at which finite-size scaling, shown in Fig. 12, can be performed. The results show that $\rho_{\max}$ exhibits an exponential dependence on k , as described by Eq. (12), with distinct parameters for each lattice type, as listed in Table I. Statistical errors are smaller than the symbol size.

scaling of χ yields a convincing collapse when using the critical exponents γ and ν of standard 3D percolation.

Finally, Fig. 15 shows that $\rho_{\max}$ decreases exponentially with k as

$$\rho_{\max} = A_{\max} e^{-\alpha_{\max} k}, \quad (12)$$

and a least-squares fit to the data using this exponential law for a simple cubic lattice, a diamond lattice [47], and a simple cubic lattice with bonds between nearest and second-nearest neighbors [48] yields the parameters listed in Table I.

Although obtained independently, the parameters α and $\alpha_{\max}$ are numerically equal within the margin of error. Using Eqs. (4), (5), and (12), and given that $p_c(\rho_{\max}) = 1$, we obtain

$$e^{(\alpha - \alpha_{\max})k} = \frac{1 - C_0^k}{CA_{\max}},$$

which implies that, if $\alpha = \alpha_{\max}$, then C_0^k is independent of k . However, we do not claim this conclusively, as the error bars of C_0^k are large, on the order of 25%.

B. Minimum density

Let us consider the case $k = 1$. When ρ is below a certain threshold ρ_* , uninhabited clusters can grow significantly without connecting to any inhabited node and may eventually percolate before ever being inhabited. We define ρ_* as the minimal initial density at which the percolating cluster is inhabited at p_c . This quantity decreases with system size L , as

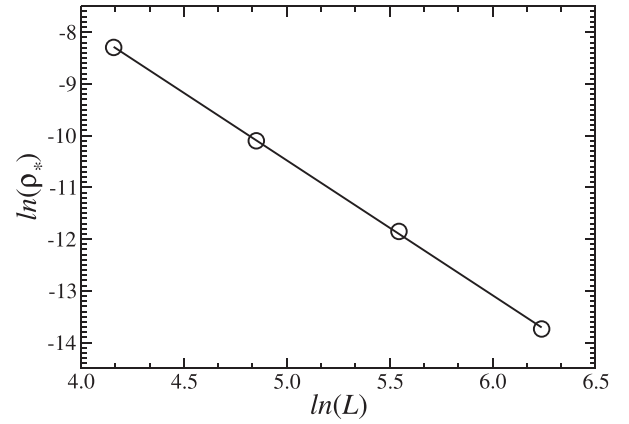


FIG. 16. Logarithmic plot of ρ_* as a function of L . A least-squares fit of the data to a power law, $\rho_*(L) \propto L^{-d_f}$, yields the exponent $d_f = 2.56 \pm 0.04$. The averages were taken over 1000 realizations for $L = 64$, 100 realizations for $L = 128$ and $L = 256$, and 10 realizations for $L = 512$, on simple cubic lattices. Statistical errors are smaller than the symbol size.

shown in Fig. 16, and can be interpreted as the probability that an inhabited node belongs to the percolating cluster. Consequently, this probability should be proportional to the fraction of nodes in the percolating cluster, leading to the relation $\rho_* \propto L^{-d_f}$. A least-squares fit of the data to this power law yields $d_f = 2.56 \pm 0.04$, consistent with the fractal dimension of standard 3D percolation.

On the other hand, for $k > 1$, when $\rho < \rho_{\min}$, at least one species never percolates, and the inset of Fig. 11 shows that $\rho_{\min}$, required to obtain the maximum number of k percolating clusters, depends on L . Moreover, Fig. 17 reveals that, in the region near $\rho_{\min}$, the values of n_c for different L collapse onto a single curve when rescaled using the exponent $d_f = 2.5227$ for densities $\rho < \rho_{\min}$.

VI. MINIMUM SYSTEM SIZE FOR k SPANNING CLUSTERS

The minimum density $\rho_{\min}$, corresponding to the side length $L_{\min}$ required for k species to percolate, decreases exponentially with k according to

$$\rho_{\min} = A_{\min} e^{-\alpha_{\min} k}. \quad (13)$$

A least-squares fit of the data to this exponential form for a simple cubic lattice, a diamond lattice [47], and a simple cubic lattice with bonds between nearest and second-nearest neighbors [48] yields the parameters listed in Table II. The results are shown in Fig. 18.

We determined the minimum side length $L_{\min}$ required to obtain k percolating clusters of different species. The results,

TABLE I. Parameters of Eq. (12) for different 3D lattices.

Lattice	Diamond	Simple cubic	Cubic NN + 2NN
$A_{\max}$	2.39 ± 0.04	4.16 ± 0.14	17.7 ± 0.9
$\alpha_{\max}$	0.82 ± 0.01	0.80 ± 0.02	0.65 ± 0.04

TABLE II. Parameters of Eq. (13) for different 3D lattices.

Lattice	Diamond	Simple cubic	Cubic NN + 2NN
$A_{\min}$	0.46 ± 0.08	1.4 ± 0.1	2.37 ± 0.16
$\alpha_{\min}$	0.81 ± 0.02	0.87 ± 0.02	0.56 ± 0.02

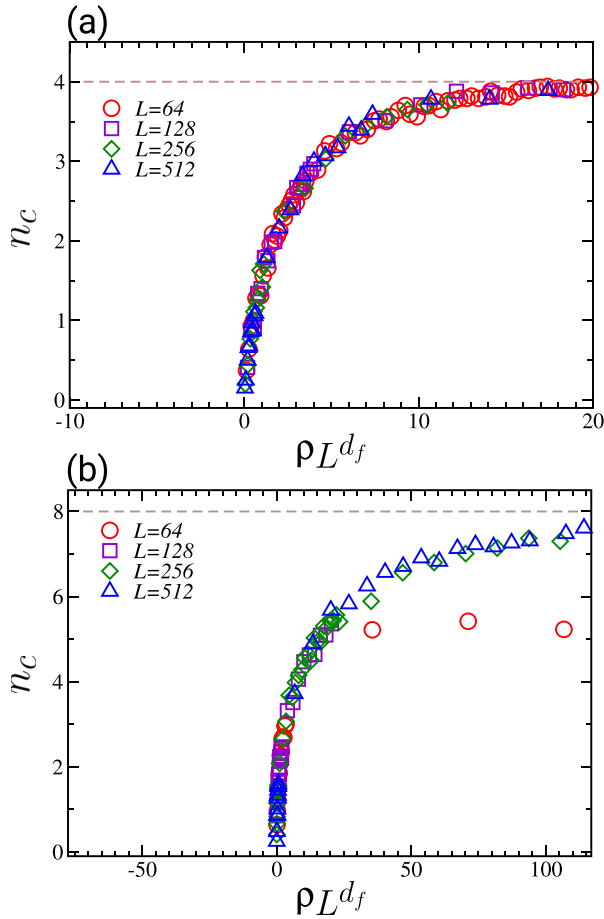


FIG. 17. Average number of percolating clusters for (a) four and (b) eight species, plotted against the rescaled initial density ρ around the region of $\rho_{\min}$. The legend indicates the side length of each simple cubic lattice, and the horizontal axis specifies the rescaling using the fractal dimension for standard 3D percolation, $d_f = 2.5227$ [56,57].

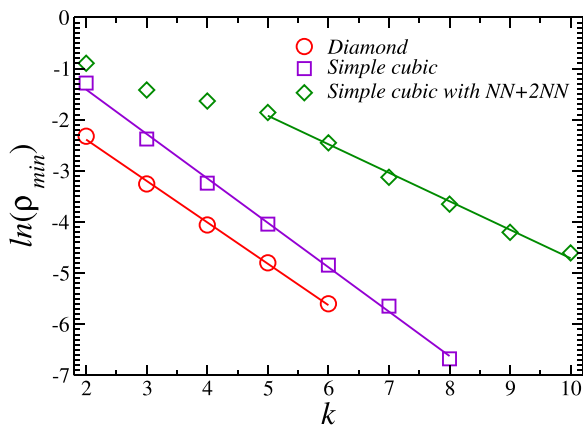


FIG. 18. Semilogarithmic plot of the minimum density $\rho_{\min}$, corresponding to the minimum side length $L_{\min}$ required for k species to percolate, as a function of the number of species k . Symbols correspond to results for different 3D lattices. In determining $L_{\min}$, we also obtain $\rho_{\min}$, as described in Sec. VI. All curves follow an exponential law, as given by Eq. (13), with parameters listed in Table II that depend on the lattice structure. Statistical errors are smaller than the symbol size.

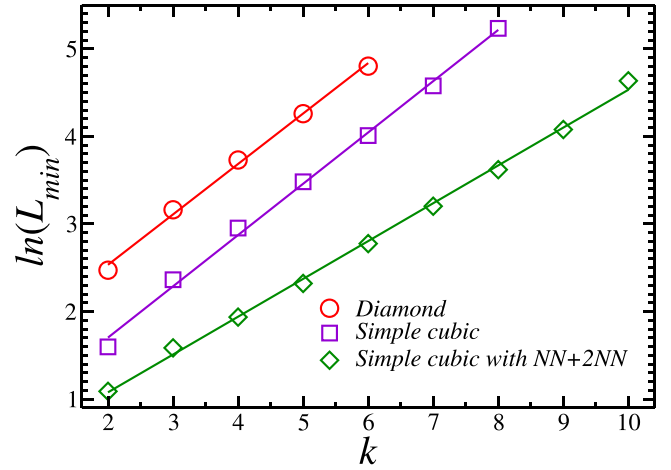


FIG. 19. Semilogarithmic plot of the minimum lattice side length $L_{\min}$ required to obtain k percolating clusters of different species for three 3D lattices, represented by different symbols. The data were obtained as follows: For each number of species k , we determined the minimum side length $L_{\min}$ for which all species percolate across all realizations. The minimum size $L_{\min}$ follows the exponential dependence on k given by Eq. (14). Statistical errors are smaller than the symbol size.

shown in Fig. 19, indicate that $L_{\min}$ increases exponentially with k according to

$$L_{\min} = A_L e^{\alpha_L k}. \quad (14)$$

A least-squares fit of the data to Eq. (14) yields the parameters listed in Table III for the three different 3D cubic lattices.

We conclude that the number of percolating clusters is unbounded in principle. However, the characteristic system size increases exponentially with the number of clusters. Moreover, Eqs. (14) and (13) yield a direct relation between $\rho_{\min}$ and $L_{\min}$, given by

$$\rho_{\min} = A_{\min} \left(\frac{L_{\min}}{A_L} \right)^{-\frac{\alpha_{\min}}{\alpha_L}}. \quad (15)$$

VII. TWO SPECIES WITH DIFFERENT INITIAL DENSITIES

Up to this point, we have analyzed only cases in which all species share the same initial density, ρ . In this section, however, we focus on a system composed of two species, considering the case where the initial densities satisfy $\rho_{\text{large}} > \rho_{\text{small}}$.

A. Evolution of the cluster properties

Figure 20 shows the evolution of S and χ , as described in Sec. III, for two species with different initial densities,

TABLE III. Parameters of Eq. (14) for different 3D lattices.

Lattice	Diamond	Simple cubic	Cubic NN + 2NN
A_L	3.98 ± 0.08	1.70 ± 0.01	1.24 ± 0.05
α_L	0.58 ± 0.02	0.59 ± 0.01	0.43 ± 0.01

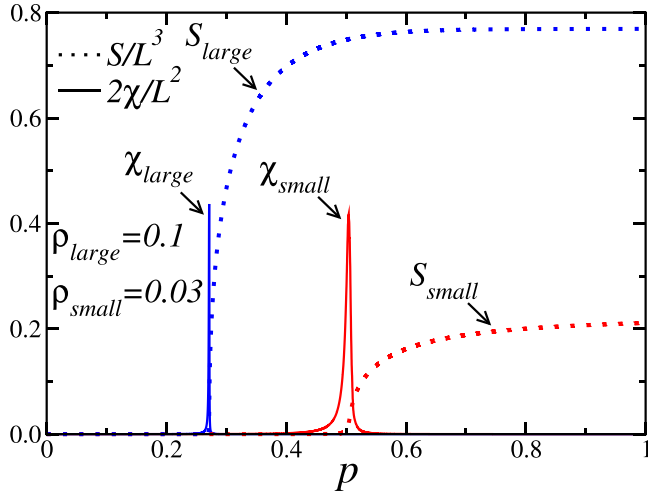


FIG. 20. Evolution of the properties of the cluster distribution during the bond occupation process of the MPM for two different initial densities, $\rho_{\text{large}} = 0.1$ and $\rho_{\text{small}} = 0.03$. This figure presents results similar to those shown in Fig. 3, with different colors representing the results for different species. As observed, the critical points for ρ_{small} and ρ_{large} differ, with the species having the higher initial density percolating first and maintaining a larger mass until the end of the process. Averaging was done over 1000 samples of a simple cubic lattice of size $L = 512$. For convenience, each of the quantities was rescaled by the system size, as indicated in the legend.

$\rho_{\text{large}} = 0.1$ and $\rho_{\text{small}} = 0.03$. We observe that the species with the higher initial density, ρ_{large} , exhibits a lower percolation threshold, indicated by the maximum of χ . Moreover, its largest cluster is greater from the point at which it first becomes nonzero.

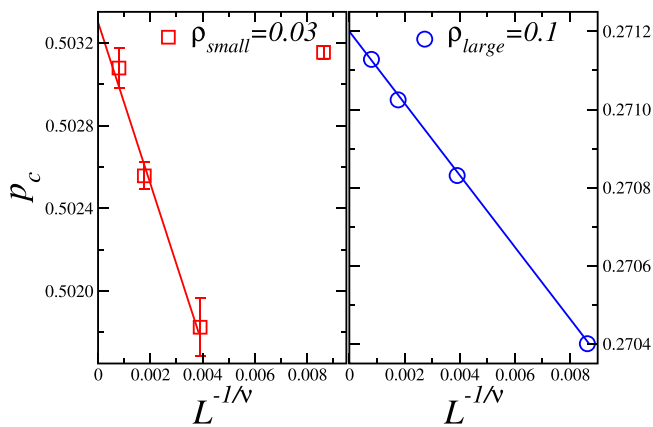


FIG. 21. Percolation threshold, p_c , as a function of $L^{-1/\nu}$ for $\rho_{\text{small}} = 0.03$ (left) and $\rho_{\text{large}} = 0.1$ (right). All realizations have the two percolating clusters, and the curves follow Eq. (3), yielding in the thermodynamic limit $L \rightarrow \infty$, $p_c^{\text{small}} = 0.5033 \pm 0.0001$ and $p_c^{\text{large}} = 0.2712 \pm 0.0001$. We averaged over 10 000 realizations for $L = 64$ and over 1000 for the other lattice sizes. We used the critical exponent $\nu = 0.875 \pm 0.001$ of standard 3D percolation [51].

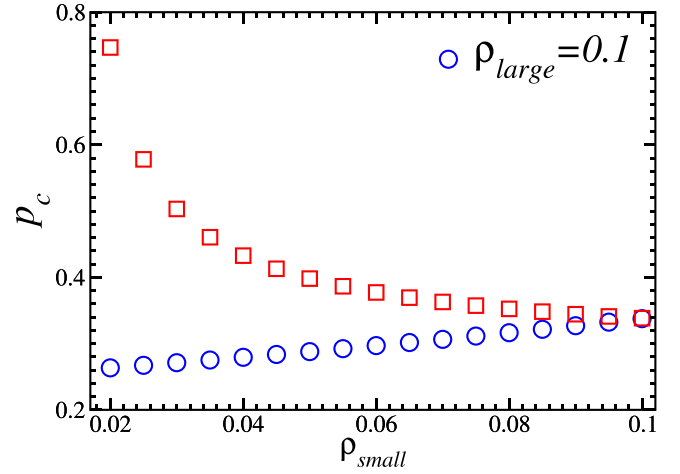


FIG. 22. Percolation thresholds for two different initial densities, ρ_{small} (squares) and $\rho_{\text{large}} = 0.1$ (circles). Each percolation threshold is averaged over 1000 realizations. The results were obtained on a cubic lattice of size $L = 512$. We observe that the thresholds become closer as ρ_{small} increases and coincide when the initial densities are equal. Statistical errors are smaller than the symbol size.

B. Percolation threshold

Figure 21 presents the average percolation thresholds, p_c^{small} and p_c^{large} , for $\rho_{\text{small}} = 0.03$ and $\rho_{\text{large}} = 0.1$, at which each species first exhibits a cluster that spans from the bottom to the top faces of the system. In contrast to the results of Sec. III B, in the case of two species with different densities, the percolation thresholds do not converge to the same value as L increases. However, the least-squares fit to the data is consistent with Eq. (3), and extrapolating to $L \rightarrow \infty$ yields $p_c^{\text{small}} = 0.5033 \pm 0.0001$ and $p_c^{\text{large}} = 0.2712 \pm 0.0001$.

The percolation thresholds of the two species depend on their respective initial densities. Figure 22 shows the results for a fixed $\rho_{\text{large}} = 0.1$ and varying ρ_{small} . As ρ_{small} increases toward ρ_{large} , the two thresholds approach the same value.

C. Fractal dimension

Next, we study the fractal dimension of the percolating clusters. As in Sec. IV A, we consider, for each species, the size of the largest cluster at the extrapolated critical points, $p_c^{\text{small}} = 0.5033$ and $p_c^{\text{large}} = 0.2712$ [53,54]. We define M as the mass of the largest cluster for the species with initial densities $\rho_{\text{small}} = 0.03$ and $\rho_{\text{large}} = 0.1$. The results presented in Fig. 23 show that, at the respective critical points p_c , the least-squares fit of the cluster mass M follows Eq. (6), where, within error bars, the fractal dimension d_f is consistent with 3D percolation, $d_f = 2.5227$ [56,57].

The mass M of the largest cluster at p_c for both species depends on their initial densities. Figure 24 shows the results for systems with $\rho_{\text{large}} = 0.1$ fixed and varying ρ_{small} . As ρ_{small} approaches ρ_{large} , the values of M for both species converge to the same value.

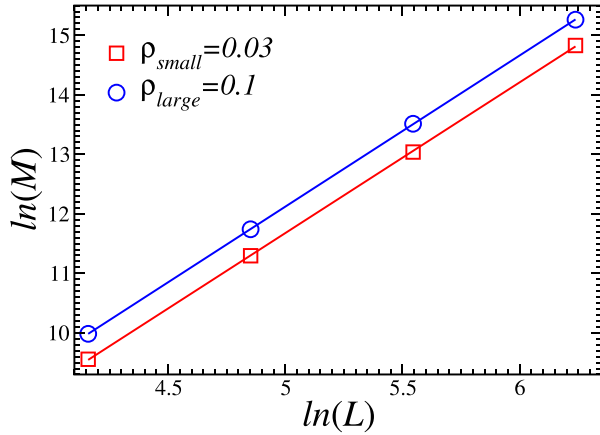


FIG. 23. Fractal dimension for two species with $\rho_{\text{small}} = 0.03$ and $\rho_{\text{large}} = 0.1$. For these values, all realizations have two percolating clusters of different species. The fractal dimensions are obtained through a least-squares fit of the size of the largest cluster of each species at p_c to a power law, $M_i = b_i L^{d_f}$, yielding the exponents $d_f^{\text{small}} = 2.53 \pm 0.02$ and $d_f^{\text{large}} = 2.54 \pm 0.02$, respectively. Within error bars, our results show that the fractal dimensions are the same regardless of the initial densities, and the observed exponents are consistent with the fractal dimension of standard 3D percolation. Results were averaged over 1000 realizations for all sizes L on simple cubic lattices.

D. Influence of initial density on the number of percolating clusters

To study how the number of percolating clusters of different species, n_c , changes when two species have different initial densities, we analyzed n_c for a fixed $\rho_{\text{large}} = 0.1$ while varying ρ_{small} . We find a transition in which, as ρ_{small} increases, the system evolves from a state with only one percolating species (ρ_{large}) to one with two percolating species. Moreover,

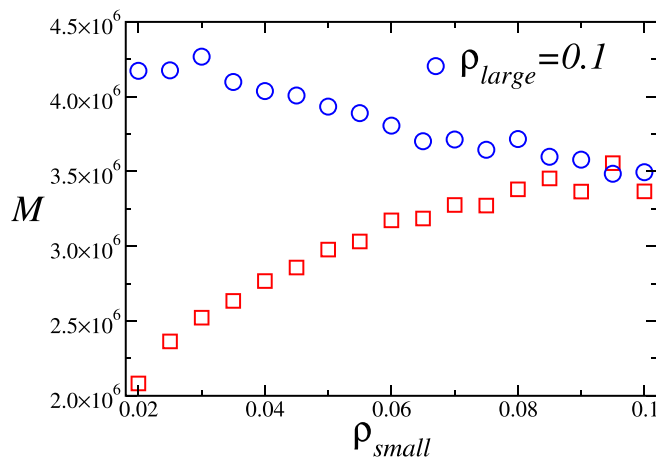


FIG. 24. Mass of the largest cluster at spanning for two different initial densities, ρ_{small} (squares) and $\rho_{\text{large}} = 0.1$ (circles). The averages were taken over 2000 realizations, measured when a cluster first spans the lattice from bottom to top for a simple cubic system of size $L = 512$. We observe that the masses of the largest clusters become closer as ρ_{small} increases and coincide when the initial densities are equal. Statistical errors are smaller than the symbol size.

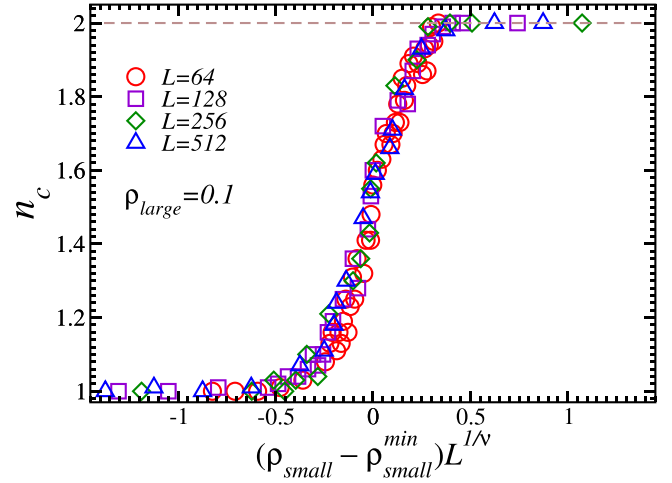


FIG. 25. Finite-size scaling of the average number of percolating clusters of different species, n_c , for fixed $\rho_{\text{large}} = 0.1$ and varying ρ_{small} . Each point represents an average over 100 realizations. For a given ρ_{large} , there exists a minimum density, $\rho_{\text{small}}^{\text{min}}$, above which two percolating clusters of different species appear. The horizontal axis displays the rescaled variable, using the scaling exponent $\nu = 0.875 \pm 0.001$ [51] and $\rho_{\text{small}}^{\text{min}} = 0.071 \pm 0.001$. The symbols on the curves indicate the lattice sizes L of the simple cubic system, as shown in the legend.

as the system size L increases, this transition becomes more pronounced, following a finite-size scaling in $L^{-1/\nu}$, with $\nu = 0.875 \pm 0.001$ being the critical exponent for standard 3D percolation [51]. This behavior suggests that a classical percolation transition occurs at $\rho_{\text{small}}^{\text{min}}$. Figure 25 shows the corresponding finite-size scaling with $\rho_{\text{small}}^{\text{min}} = 0.071 \pm 0.001$.

Figure 26 presents the phase diagram of n_c as a function of the two different initial densities, ρ_1 and ρ_2 . The values of ρ_1 and ρ_2 within the red region correspond to the presence of two percolating clusters ($n_c = 2$), while in the blue regions only

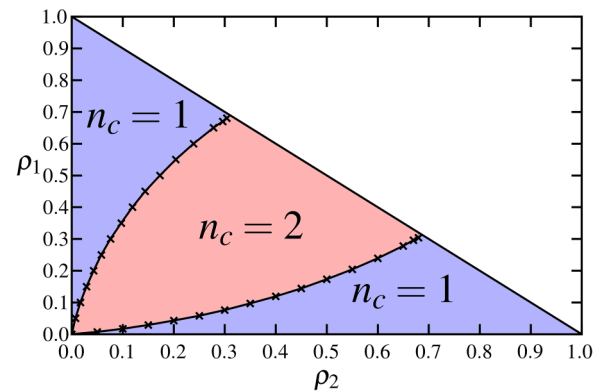


FIG. 26. Phase diagram of the number of percolating clusters, n_c , for two species with different initial densities, ρ_1 and ρ_2 . In the red region, two percolating clusters coexist, while in the blue regions only one cluster percolates. The percolating species in the blue region is the one with the higher initial density. The points at the boundary between the blue and red regions were obtained from the finite-size scaling shown in Fig. 25. The white region is unphysical, as it corresponds to $\rho_1 + \rho_2 > 1$.

the species with the higher density percolates ($n_c = 1$). The white region is unphysical, as it corresponds to $\rho_1 + \rho_2 > 1$.

VIII. CONCLUSIONS

In this work, we have extended the results of the multigel percolation model presented in Ref. [46], showing that the fractal dimension of the shortest path coincides with that of standard 3D percolation and that the fractal dimension of the contact between percolating clusters at the critical point is consistent with that expected for the intersection of two fractal objects. By treating ρ as a control parameter, we observed that the behavior of both the largest cluster and the second moment follows the same scaling relations as in classical 3D percolation. For a given k , the minimum density $\rho_{\min}$ required for all clusters to percolate scales as L^{d_f} , with d_f consistent with standard 3D percolation. Furthermore, for the case $k = 2$, when ρ differs between species, fixing one of the densities, ρ_{large} , there exists a minimum density, $\rho_{\text{small}}^{\min}$, above which a transition occurs from a regime where only one species percolates to another where both percolate. The findings of

our study strongly suggest that it should be possible to experimentally realize systems exhibiting multiple percolating clusters, such as trigels, tetragels, and beyond.

ACKNOWLEDGMENTS

We thank the Brazilian agencies CNPq, CAPES, FUNCAP, and the National Institute of Science and Technology for Complex Systems (INCT-SC) for the financial support. This work was also partly supported by the Research Council of Norway (Norges forskningsråd), Grant No. 262644 PoreLab. Further support, also from the Research Council of Norway (Norges forskningsråd), Grant No. 309139 INTPART Non-Newtonian fluids in porous media. A.H. furthermore acknowledges funding from the European Research Council (ERC) Grant No. 101141323 AGIPORE.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

-
- [1] S. R. Broadbent and J. M. Hammersley, Percolation processes: I. Crystals and mazes, in *Mathematical Proceedings of the Cambridge Philosophical Society* (Cambridge University Press, Cambridge, UK, 1957), Vol. 53, pp. 629–641.
 - [2] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1991).
 - [3] M. Sahimi, *Applications of Percolation Theory* (CRC Press, London, 1994).
 - [4] C. J. Brinker and G. W. Scherer, *Sol-Gel Science: The Physics and Chemistry of Sol-Gel Processing* (Academic Press, Cambridge, 2013).
 - [5] J. E. Martin and J. P. Wilcoxon, Critical dynamics of the sol-gel transition, *Phys. Rev. Lett.* **61**, 373 (1988).
 - [6] H. E. Stanley and A. Coniglio, Flow in porous media: The “backbone” fractal at the percolation threshold, *Phys. Rev. B* **29**, 522 (1984).
 - [7] J. S. Andrade, S. V. Buldyrev, N. Dokholyan, S. Havlin, P. R. King, Y. Lee, G. Paul, and H. E. Stanley, Flow between two sites on a percolation cluster, *Phys. Rev. E* **62**, 8270 (2000).
 - [8] P. R. King, J. S. Andrade, S. V. Buldyrev, N. Dokholyan, Y. Lee, S. Havlin, and H. Stanley, Predicting oil recovery using percolation, *Physica A* **266**, 107 (1999).
 - [9] J. S. Andrade, D. A. Street, T. Shinohara, Y. Shibusa, and Y. Arai, Percolation disorder in viscous and nonviscous flow through porous media, *Phys. Rev. E* **51**, 5725 (1995).
 - [10] J. S. Andrade Jr., D. A. Street, Y. Shibusa, S. Havlin, and H. E. Stanley, Diffusion and reaction in percolating pore networks, *Phys. Rev. E* **55**, 772 (1997).
 - [11] R. Pastor-Satorras and A. Vespignani, Epidemic spreading in scale-free networks, *Phys. Rev. Lett.* **86**, 3200 (2001).
 - [12] R. Cohen, S. Havlin, and D. Ben Avraham, Efficient immunization strategies for computer networks and populations, *Phys. Rev. Lett.* **91**, 247901 (2003).
 - [13] P. Grassberger, On the critical behavior of the general epidemic process and dynamical percolation, *Math. Biosci.* **63**, 157 (1983).
 - [14] C. Moore and M. E. J. Newman, Epidemics and percolation in small-world networks, *Phys. Rev. E* **61**, 5678 (2000).
 - [15] F. Wang, D. Li, X. Xu, R. Wu, and S. Havlin, Percolation properties in a traffic model, *Europhys. Lett.* **112**, 38001 (2015).
 - [16] H. Hamedmoghadam, N. Zheng, D. Li, and H. L. Vu, Percolation-based dynamic perimeter control for mitigating congestion propagation in urban road networks, *Transp. Res. Part C* **145**, 103922 (2022).
 - [17] D. Li, B. Fu, Y. Wang, G. Lu, Y. Berezin, H. E. Stanley, and S. Havlin, Percolation transition in dynamical traffic network with evolving critical bottlenecks, *Proc. Natl. Acad. Sci. USA* **112**, 669 (2015).
 - [18] A. A. Moreira, J. S. Andrade, H. J. Herrmann, and J. O. Indekeu, How to make a fragile network robust and vice versa, *Phys. Rev. Lett.* **102**, 018701 (2009).
 - [19] L. Di Michele, D. Fiocco, F. Varrato, S. Sastry, E. Eiser, and G. Foffi, Aggregation dynamics, structure, and mechanical properties of bigels, *Soft Matter* **10**, 3633 (2014).
 - [20] F. Varrato, L. Di Michele, M. Belushkin, N. Dorsaz, S. H. Nathan, E. Eiser, and G. Foffi, Arrested demixing opens route to bigels, *Proc. Natl. Acad. Sci. USA* **109**, 19155 (2012).
 - [21] B. Verhaeghe, F. Louchet, Y. Bréchet, and J. Massoud, Damage and rupture mechanisms in a bipercolated material: The example of austenoferritic duplex steels, *Mater. Sci. Eng. A* **234-236**, 279 (1997).
 - [22] N. Zhang, W. Li, C. Chen, and J. Zuo, Molecular dynamics simulation of aggregation in dimethyl sulfoxide–water binary mixture, *Comput. Theor. Chem.* **1017**, 126 (2013).
 - [23] Y. Shao, Z.-X. Yang, B.-W. Deng, B. Yin, and M.-B. Yang, Tuning PVDF/PS/HDPE polymer blends to tri-continuous morphology by grafted copolymers as the compatibilizers, *Polymer* **140**, 188 (2018).
 - [24] Z. Gan, C. Zhang, Z.-R. Zhang, Z.-J. Chen, and L. Liu, Crystallization-dependent transition of corrosion resistance of an Fe-based bulk metallic glass under hydrostatic pressures, *Corros. Sci.* **179**, 109098 (2021).

- [25] C. Newman and L. Schulman, Infinite clusters in percolation models, *J. Stat. Phys.* **26**, 613 (1981).
- [26] C. Newman and L. Schulman, Number and density of percolating clusters, *J. Phys. A: Math. Gen.* **14**, 1735 (1981).
- [27] M. Aizenman, On the number of incipient spanning clusters, *Nucl. Phys. B* **485**, 551 (1997).
- [28] C. I. N. Sampaio Filho, J. S. Andrade, H. J. Herrmann, and A. A. Moreira, Elastic backbone defines a new transition in the percolation model, *Phys. Rev. Lett.* **120**, 175701 (2018).
- [29] W. R. Sena, J. S. Andrade, H. J. Herrmann, and A. A. Moreira, Decomposing the percolation backbone reveals novel scaling laws of the current distribution, *Front. Phys.* **11**, 1335339 (2023).
- [30] J. Cardy, The number of incipient spanning clusters in two-dimensional percolation, *J. Phys. A: Math. Gen.* **31**, L105 (1998).
- [31] P. Grassberger and W. Nadler, ‘Go with the winners’ simulations, in *Computational Statistical Physics: From Billiards to Monte Carlo*, edited by K. H. Hoffmann and M. Schreiber (Springer, Berlin, 2002), pp. 169–190.
- [32] D. Aldous and U. Vazirani, “Go with the winners” algorithms, in *Proceedings 35th Annual Symposium on Foundations of Computer Science* (IEEE Computer Society Press, Los Alamitos, CA, USA, 1994), pp. 492–501.
- [33] L. N. Shchur and T. Rostunov, On the Aizenman exponent in critical percolation, *JETP Lett.* **76**, 475 (2002).
- [34] R. Zallen, Polychromatic percolation: Coexistence of percolating species in highly connected lattices, *Phys. Rev. B* **16**, 1426 (1977).
- [35] M. Giona and A. Adrover, Multicomponent percolation: Probabilistic properties and application to nonisothermal reactions in granular materials, *Phys. Rev. E* **49**, 5287 (1994).
- [36] A. S. Ioselevich, Multicomponent percolation criterion and its application to hopping in disordered conductors, *Phys. Rev. Lett.* **74**, 1411 (1995).
- [37] H. M. Harreis and W. Bauer, Phase transitions in a two-component site-bond percolation model, *Phys. Rev. B* **62**, 8719 (2000).
- [38] I. V. Petrov, I. I. Stoynev, and F. V. Babalievskii, Correlated two-component percolation, *J. Phys. A: Math. Gen.* **24**, 4421 (1991).
- [39] J. Lin, A generalized model for site percolation with two independent concentrations, Technical Report No. IC-87/63 (International Centre for Theoretical Physics, Trieste, Italy, 1987).
- [40] J. C. Wierman, AB percolation: A brief survey, *Banach Cent. Publ.* **25**, 241 (1989).
- [41] H. Kesten and Z. Su, Some remarks on AB-percolation in high dimensions, *J. Math. Phys.* **41**, 1298 (2000).
- [42] S. Kundu and S. S. Manna, Colored percolation, *Phys. Rev. E* **95**, 052124 (2017).
- [43] V. Avetisov, A. Gorsky, S. Nechaev, and O. Valba, Spontaneous symmetry breaking and phase coexistence in two-color networks, *Phys. Rev. E* **93**, 012302 (2016).
- [44] V. Avetisov, A. Gorsky, S. Nechaev, and O. Valba, Finite plateau in spectral gap of polychromatic constrained random networks, *Phys. Rev. E* **96**, 062309 (2017).
- [45] R. M. Bradley, P. N. Strenski, and J.-M. Debierre, A growing self-avoiding walk in three dimensions and its relation to percolation, *Phys. Rev. A* **45**, 8513 (1992).
- [46] C. G. L. Sousa, J. S. Andrade, A. A. Moreira, C. I. N. Sampaio Filho, E. Eiser, A. Hansen, and H. J. Herrmann, Unlimited number of spanning clusters in three-dimensional percolation above the critical point, *Phys. Rev. Res.* **7**, L022067 (2025).
- [47] C. Kittel and P. McEuen, *Introduction to Solid State Physics* (John Wiley & Sons, Hoboken, NJ, USA, 2018).
- [48] Z. Xun and R. M. Ziff, Bond percolation on simple cubic lattices with extended neighborhoods, *Phys. Rev. E* **102**, 012102 (2020).
- [49] M. E. J. Newman and R. M. Ziff, Efficient Monte Carlo algorithm and high-precision results for percolation, *Phys. Rev. Lett.* **85**, 4104 (2000).
- [50] S. Fayfar, A. Bretaña, and W. Montfrooij, Protected percolation: A new universality class pertaining to heavily-doped quantum critical systems, *J. Phys. Commun.* **5**, 015008 (2021).
- [51] C. D. Lorenz and R. M. Ziff, Precise determination of the bond percolation thresholds and finite-size scaling corrections for the sc, fcc, and bcc lattices, *Phys. Rev. E* **57**, 230 (1998).
- [52] L. Böttcher and H. J. Herrmann, *Computational Statistical Physics* (Cambridge University Press, Cambridge, United Kingdom, 2021).
- [53] P. Sen, Nature of the largest cluster size distribution at the percolation threshold, *J. Phys. A: Math. Gen.* **34**, 8477 (2001).
- [54] N. Jan, D. Stauffer, and A. Aharony, An infinite number of effectively infinite clusters in critical percolation, *J. Stat. Phys.* **92**, 325 (1998).
- [55] A. Kapitulnik, A. Aharony, G. Deutscher, and D. Stauffer, Self similarity and correlations in percolation, *J. Phys. A: Math. Gen.* **16**, L269 (1983).
- [56] X. Xu, J. Wang, J.-P. Lv, and Y. Deng, Simultaneous analysis of three-dimensional percolation models, *Front. Phys.* **9**, 113 (2014).
- [57] Y. Deng and H. W. J. Blöte, Monte Carlo study of the site-percolation model in two and three dimensions, *Phys. Rev. E* **72**, 016126 (2005).
- [58] S. Miyazima and H. E. Stanley, Intersection of two fractal objects: Useful method of estimating the fractal dimension, *Phys. Rev. B* **35**, 8898 (1987).
- [59] B. Mandelbrot, *The Fractal Geometry of Nature* (W.H. Freeman, New York, NY, 1982), p. 460.
- [60] L. M. Abril, E. A. Oliveira, A. A. Moreira, J. S. Andrade, and H. J. Herrmann, Coastlines violate the Schramm–Loewner evolution, *Physica A* **653**, 130066 (2024).
- [61] Z. Zhou, J. Yang, Y. Deng, and R. M. Ziff, Shortest-path fractal dimension for percolation in two and three dimensions, *Phys. Rev. E* **86**, 061101 (2012).