

Chiral edge-state signatures of Berry monopole merging and splitting in a bilayer photonic crystal slab with synthetic momentum

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Berry monopoles are point degeneracies in parameter space where the Berry curvature emanates radially, analogous to magnetic monopoles. Here, we report the adiabatic merging and splitting of Berry monopoles in a bilayer photonic crystal slab combining one genuine and one synthetic momentum. Two monopoles approach, merge, and split within this hybrid parameter space. The process is described by an effective coupled-mode model and confirmed by full-wave simulations. We further propose a scheme to experimentally probe the Berry monopole merging and splitting by monitoring the evolution of chiral edge states localized at the interface between two topological phases with inverted band structures.

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I. INTRODUCTION

Dirac famously introduced the concept of a pointlike *magnetic monopole* as the magnetic counterpart to electric charge [1,2]. While such particles remain unobserved in high-energy physics [3], monopolelike quasiparticles have emerged in condensed-matter systems such as spin-ice pyrochlores [4–6] and chiral skyrmion lattices [7,8]. In 1984, Berry showed that adiabatic evolution of a quantum state gives rise to a geometric phase, equal to the flux of a fictitious magnetic field—the Berry curvature—originating from point degeneracies known as *Berry monopoles* [9–11]. Prominent examples are Weyl points in topological semimetals, acting as Berry monopoles in momentum space [12,13]; analogous Weyl points have also been realized in photonic [14] and phononic crystal systems [15]. As with electric charges,

Berry monopoles carrying the same topological charge are expected to repel. They cannot annihilate, but their merging produces a higher-charge object that is generically unstable. Synthetic quantum matter has emerged as a powerful framework for simulating exotic quantum systems in controllable settings [16–18], including higher-dimensional phenomena. The four-dimensional (4D) quantum Hall effect [19–24] has been explored through synthetic dimensions [18,25], topological pumping [22,23,26], and engineered connectivity [27,28]. While synthetic dimensions expand real space into higher dimensions, the emerging concept of *synthetic momenta* instead enlarges the momentum space by incorporating additional parameters into the crystal momentum. These include relative layer displacements [29–31], geometric modulations [32,33], and material-dependent properties [34]. Weyl points have been realized in such synthetic momentum spaces [31–33], making them a compelling platform for investigating Berry monopoles.

Here, we propose a photonic platform that enables the generation, merging, and splitting of same-charge Berry monopoles along distinct directions of the hybrid momentum space. The system is described by an effective model in a synthetic three-dimensional (3D) parameter space (k, q, m), where k is the genuine momentum, q is a synthetic momentum, and m is a tunable gap parameter; the parametric space is illustrated in Fig. 1. Berry monopoles appear at band degeneracy points located in the $m = 0$ plane, where they act as quantized sources of Berry curvature. When $m \neq 0$, the

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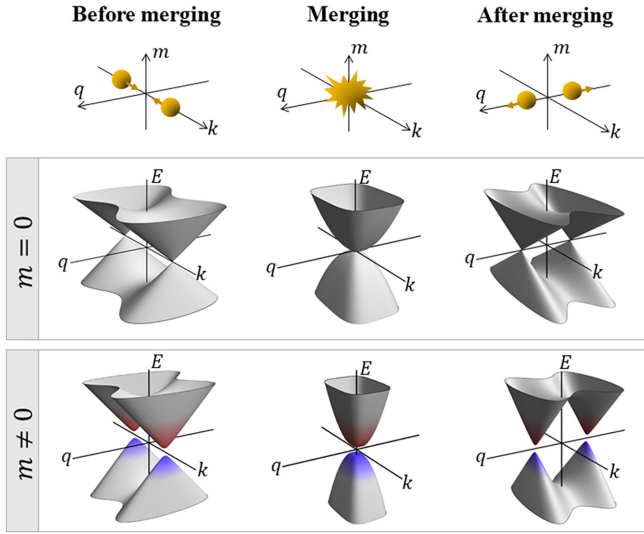


FIG. 1. Berry monopole merging and splitting in hybrid momentum space. Top: In the (k, q, m) space, two Berry monopoles with the same charge approach along the k axis, merge at $(0, 0, 0)$, and split along the q axis. Middle: At $m = 0$, the system is gapless with band degeneracies (linear and quadratic band-touching points) in the (k, q) plane, corresponding to Berry monopoles. Bottom: For $m \neq 0$, the bands are gapped and exhibit finite Berry curvature (red: positive; blue: negative).

system opens a gap and transitions into topological phases characterized by finite Chern numbers, a so-called Chern insulator. By continuously tuning the parameters, the monopoles move toward each other along the k axis, merge at the origin $(k, q) = (0, 0)$, and split along the q axis. We also demonstrate that chiral edge states emerge at a band-inverted interface, and that their evolution serves as a direct experimental probe of the Berry monopole merging and splitting. Finally, we discuss the experimental feasibility of this setup, laying the groundwork for realizing Berry monopole dynamics in photonic platforms.

The paper is organized as follows. Section II describes the bilayer photonic crystal slab by presenting an effective model derived from the coupled-wave theory. This section defines the synthetic momentum q that extends the hybrid momentum space and introduces the deformation parameter e that controls the Berry monopole motion. Section III describes the monopole motion in the parameter space by showing the photonic band structure and the Berry curvature maps in the gapped regime. Section IV illustrates the evolution of the chiral edge states, which can serve as experimental proof of the Berry monopole merging and splitting. Finally, Sec. V is dedicated to discussing the experimental feasibility of our setup.

II. BILAYER PHOTONIC CRYSTAL SLAB WITH SYNTHETIC MOMENTUM

Our system consists of a bilayer structure formed by two parallel two-dimensional (2D) photonic crystal slabs, separated by an interlayer distance d [Fig. 2(a)]. Each slab features a square lattice with identical lattice constant a , thickness h , and a unit cell comprising a square hole of edge length b .

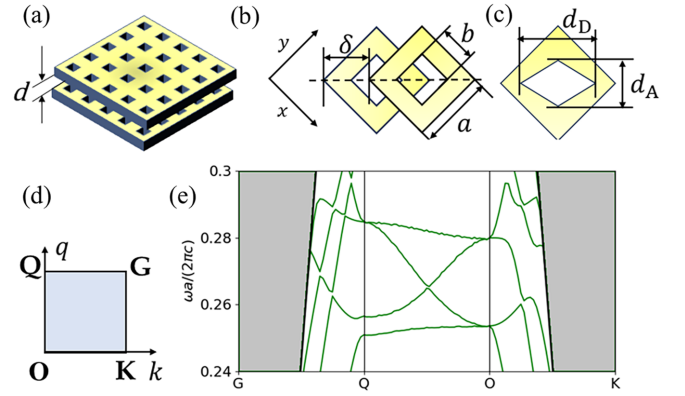


FIG. 2. Bilayer 2D photonic crystal slab with synthetic momentum. (a) The design of the bilayer 2D photonic crystal slab with square holes. (b) Top view of the unit cell. (c) Deforming the square hole into a rhombus of diagonals d_B and d_A . (d) The hybrid Brillouin zone. (e) The band structure in the hybrid Brillouin zone for the structure with $h/a = 0.35$ and $b/a = 0.38$ shows a quadratic band-touching point between the two lowest bands at the point O. The photonic crystal slab is made of silicon (refractive index $n_{\text{Si}} = 3.54$) and immersed in silica (refractive index $n_{\text{SiO}_2} = 1.46$.)

The two layers are laterally offset along their main diagonals [Fig. 2(b)], with a center-to-center displacement δ . The structure is periodic under translations $\delta \rightarrow \delta + \sqrt{2}a$, giving rise to a dimensionless synthetic momentum defined as

$$q = \frac{\delta}{\sqrt{2}a} - \frac{1}{2}, \quad (1)$$

with $q \in [-1/2, 1/2]$. This construction is analogous to the synthetic dimension used in one-dimensional multilayer grating systems [29,31]. By choosing $k_x = k_y = k/\sqrt{2} + 1/2$, the system can be reinterpreted as a hybrid $(1+1)$ D configuration, described by a genuine momentum k and a synthetic momentum q , rather than as a conventional 2D photonic crystal (Fig. 2). Interestingly, under the action of the time-reversal operator, the genuine momentum k changes sign ($k \rightarrow -k$) as expected when $t \rightarrow -t$, whereas the synthetic momentum q remains unchanged ($q \rightarrow q$) due to its geometric origin. It arises from the relative spatial shift between the two layers, rather than from a dynamical variable. As a result, when the system is viewed in the hybrid momentum space (k, q) , time-reversal symmetry is intrinsically broken. This symmetry breaking gives rise to nontrivial topological properties in the system.

Figure 2(e) shows the photonic band structure in the hybrid Brillouin zone for a representative bilayer structure, calculated using the plane wave expansion (PWE) method implemented in the MIT Photonic Bands (MPB) package [35]. The two lowest-energy bands exhibit a degeneracy at the high-symmetry point O ($k = q = 0$), marked by the blue arrow in the figure. This point corresponds to a quadratic band-touching point, with both bands displaying parabolic dispersion along the k and q directions. In the following, we focus on the physical phenomena that emerge near this degeneracy point.

Next, we deform each square hole into a rhombus whose diagonals are defined by

$$d_{D/A}^\ell = \sqrt{2}b_\ell \frac{1 \pm e}{1 \mp e}, \quad (2)$$

where e is the *diagonal deformation parameter*, constrained within $-1 < e < 1$ [Fig. 2(c)]; we allow the hole size to be different for the two layers: $\ell = \pm$, $+$ for the upper layer and $-$ for the lower one. This deformation preserves the hole area—and thus the filling factor—while allowing geometric anisotropy to be continuously tuned via e . Here, d_D^ℓ corresponds to the main diagonal along the diagonal $x = y$ direction, and d_A^ℓ to the transverse diagonal along the $x = -y$ direction. To further break the vertical symmetry, we slightly differentiate the hole sizes in the two layers, denoted by b_+ and b_- . Both layers, however, share the same deformation parameter e , which serves as a key control parameter for tuning the system's optical behavior. The lowest-energy transverse electric (TE) eigenmodes near the high-symmetry point O can be described by an effective Hamiltonian derived from coupled-wave theory [36–39]:

$$\mathcal{H}(k, q, m) = \begin{pmatrix} \Delta_+ & \Omega \\ \Omega^\dagger & \Delta_- \end{pmatrix}, \quad (3)$$

where Δ_ℓ is the Hamiltonian of individual layer ℓ ($\ell = +$ for the upper layer and $\ell = -$ for the lower layer [40]):

$$\Delta_\ell = \omega_\ell + \frac{v_\ell}{\sqrt{2}}k^2 + \begin{pmatrix} \eta + v_\ell k & W_\ell & W_\ell & U_\ell(1 + \alpha) \\ W_\ell & -\eta & U_\ell(1 - \alpha) & W_\ell \\ W_\ell & U_\ell(1 - \alpha) & -\eta & W_\ell \\ U_\ell(1 + \alpha) & W_\ell & W_\ell & \eta - v_\ell k \end{pmatrix}. \quad (4)$$

The asymmetry between the layers is captured by the *gap parameter* m , which modifies the coupling constants as $\omega_\pm = \omega(1 \pm r_\omega m)$, $v_\pm = v(1 \pm r_v m)$, $U_\pm = U(1 \pm m)$, $W_\pm = W(1 \pm r_W m)$. The parameters (k, q, m) define a three-dimensional space in which topological features emerge. When $m = 0$, the layers are identical; when $m \neq 0$, they differ in thickness and/or hole size. The parameters α and η originate from the deformation of the square hole into a rhombus shape and vary monotonically as functions of the deformation parameter e . Both vanish when the holes are square ($e = 0$).

The interlayer coupling is described by the block

$$\Omega = e^{-d/d_0} \times \text{diag}\{V_{00}e^{-i2\pi q}, V_{0,-1}, V_{-1,0}, V_{-1,-1}e^{i2\pi q}\},$$

in which the matrix elements are dependent on the genuine momentum k :

$$\begin{aligned} V_{00}(k) &= -\left(V + \beta k + \frac{\beta}{\sqrt{2}}k^2\right), \\ V_{0,-1}(k) &= V_{-1,0}(k) = V + \frac{\beta}{\sqrt{2}}k^2, \\ V_{-1,-1}(k) &= -\left(V - \beta k + \frac{\beta}{\sqrt{2}}k^2\right). \end{aligned} \quad (5)$$

The parameters of the effective model fitted with MPB calculations are $\omega = 0.2992$, $v = 0.3131$, $U = -0.01525$,

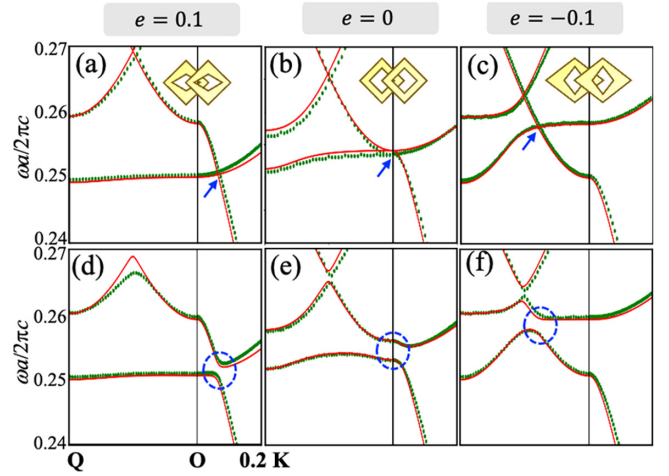


FIG. 3. Band structure comparison. Photonic band structures computed using the effective model (red lines) and the PWE method (green dots). Top row: Symmetric case with $b_+ = b_- = 0.38a$. Bottom row: Asymmetric case with $b_+ = 0.46a$ and $b_- = 0.30a$. Each column corresponds to a different deformation parameter: $e = 0.1$ (left), $e = 0$ (center), and $e = -0.1$ (right). Degeneracy points are marked by blue arrows; gap openings are highlighted by blue circles.

$W = 0.00176$; $r_\omega = 0.1353$, $r_v = 0.0892$, $r_W = 2.4215$, $m = 0.00045$; $d_0 = 0.35$, $d = 0.1$, $V = 0.039$, $\beta = -0.3$. $(\alpha, \eta) = (-0.062, 0.0032)$ for $e = 0.1$; $(\alpha, \eta) = (0.0, 0.0)$ for $e = 0.0$; $(\alpha, \eta) = (0.062, -0.0032)$ for $e = -0.1$. A full derivation of this effective model is given in the Supplemental Material [41].

III. BERRY MONOPOLE MERGING AND SPLITTING

Figures 3(a)–3(c) show the photonic band structures in the hybrid Brillouin zone (k, q) for the symmetric case where the two layers are identical ($b_+ = b_-$), corresponding to $m = 0$. In this configuration, the system remains gapless. For $e > 0$, two Dirac cones appear symmetrically along the k axis with respect to the origin O [Fig. 3(a)]. As e decreases, the cones move closer and merge when $e = 0$, forming a quadratic band-touching point at O [42] [Fig. 3(b)]. When $e < 0$, the cones reappear but now separate along the q axis [Fig. 3(c)]. Breaking inversion symmetry by making the hole sizes slightly different ($b_+ \neq b_-$, i.e., $m \neq 0$) opens a gap at both the Dirac cones and the quadratic band-touching point [Figs. 3(d)–3(f)]. In all cases, the effective model agrees perfectly with full numerical simulations using the MPB package.

Thanks to the intrinsic breaking of time-reversal symmetry—originating from the synthetic momentum q —the system becomes a Chern insulator as soon as the degeneracies are lifted. We compute the Berry curvature of band n ($1 \leq n \leq 8$) using [43]

$$\mathcal{F}_{kq}^{(n)}(k, q) = -2 \text{Im} \sum_{n' \neq n} \frac{\langle n | \partial_k H | n' \rangle \langle n' | \partial_q H | n \rangle}{(E_n - E_{n'})^2}. \quad (6)$$

Figure 4 presents the Berry curvature of the lowest band ($n = 1$) for $e > 0$, $e = 0$, and $e < 0$, under band inversion with $m = 0.1$ and $m = -0.1$. The Berry curvature changes sign across the transition, confirming the topological nature

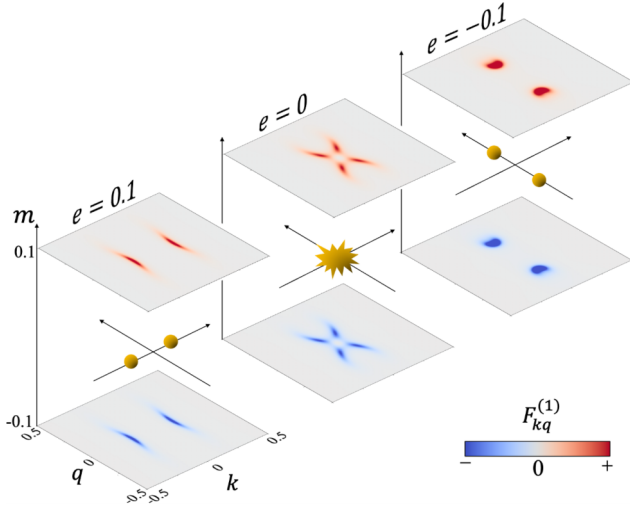


FIG. 4. The Berry curvature maps. The Berry curvature $\mathcal{F}_{kq}^{(1)}$ of band 1 for $m = \pm 0.1$ and $e = 0.1, 0, -0.1$. The results are calculated using the effective model with fitted parameters and MPB band-structure calculations.

of the gap. Moreover, the Berry curvature *hot spots*—regions of concentrated curvature—follow the same trajectory as the Dirac cones in the gapless regime: For $e > 0$, they lie on the k axis; as e decreases, they merge at the origin when $e = 0$, forming a four-lobed pattern; for $e < 0$, they split and move along the q axis.

The theory of Berry phase [9] provides a unified interpretation of these phenomena through the concept of Berry monopoles. These monopoles are located in the plane $m = 0$ of the parameter space (k, q, m) , where the two bands are degenerate and act as sources of the Berry field $\mathcal{F} = (\mathcal{F}_{qm}^{(1)}, \mathcal{F}_{mk}^{(1)}, \mathcal{F}_{kq}^{(1)})$. The sign of $\mathcal{F}_{kq}^{(1)}$ changes across $m = 0$: It is positive for $m > 0$ and negative for $m < 0$, indicating that the Berry field radiates outward from these monopoles. The monopole strength, analogous to magnetic charge, is obtained by integrating the Berry flux over a closed surface enclosing the monopole [9,10] $g = \frac{1}{4\pi} \oint \mathcal{F}(k, q, m) \cdot d\mathbf{S}$. This yields $g = +1/2$ for each Dirac cone and $g = +1$ for the quadratic band-touching point. As a result, the Chern number of the lowest band is $+1$ for $m > 0$ and -1 for $m < 0$, regardless of the deformation parameter e . Overall, two Berry monopoles approach each other along the genuine momentum (k) axis, merge at the origin (O), and split along the synthetic momentum (q) axis.

Unlike the familiar scenario in which Berry monopoles with opposite charges merge and annihilate [44], corresponding to the opening of a gap, in our model, same-charge Berry monopoles split apart after having merged into a single charge-2 monopole. In the gapless case ($m = 0$), no gap opens after the transition, and the dispersion remains semimetallic, with two Dirac cones separating along a trajectory set by the Berry monopoles. The merging and splitting of two Berry monopoles of the same charge can be considered as a particular case of “two-node process” in Ref. [45]. The analogous splitting of a charge-2 double-Weyl point into two charge-1 Weyl points was experimentally observed in a 3D chiral woodpile photonic crystal via Fourier-transform infrared

(FTIR) spectroscopy [46]. Crucially, the Berry monopoles in Refs. [45,46] live in a genuine momentum space, where the merging of topological charges is defined with respect to ordinary crystal momenta. In contrast, the Berry monopoles in our work live in a hybrid momentum space (k, q) , in which the construction breaks the time-reversal symmetry intrinsically, enabling nontrivial Chern phases on either side of the transition.

IV. CHIRAL EDGE STATE TRANSFORMATION

We propose a scheme to probe Berry monopole merging and splitting by tracking the evolution of chiral edge states at the interface between two topological photonic phases with inverted band structures. This can be realized by constructing a heterojunction with a zigzag interface [Fig. 5(a)] between two bilayer photonic crystal slabs that differ only by the sign of the gap parameter m . Although the bulk band structures of both slabs are identical, their Chern numbers are opposite ($+1$ versus -1), resulting in the emergence of two topologically protected edge states at the interface.

Figure 5(b) shows the band dispersion of the heterojunction, calculated using the effective model for three representative cases: $e > 0$, $e = 0$, and $e < 0$. Both bulk bands and chiral edge states are clearly visible. To assess the experimental feasibility of this scheme, we perform finite difference time domain (FDTD) simulations using the MEEP package [47] [see Fig. 5(a) for the setup]. The resulting transmission spectra, shown in Fig. 5(c), agree well with the effective model and confirm the presence of robust chiral edge states. These results demonstrate that the edge states persist under structural deformation and, more importantly, serve as a direct probe of the process of merging and splitting of the Berry monopole. As the deformation parameter e varies, the edge-state evolution tracks the trajectory of Berry monopoles in synthetic momentum space, as discussed below.

Before collision ($e > 0$): the two edge states are “entangled”—both originate from the same local valley ($q < 0$) in the lower bulk band and terminate at the same point ($q > 0$) in the upper band. In the bulk, this configuration corresponds to two Berry monopoles separated along the genuine momentum (k) axis. However, across the interface, translational symmetry is broken, so k is no longer a good quantum number. As a result, the separation of the monopoles along k becomes ill-defined, and the two associated edge states appear intertwined in synthetic momentum space. At collision ($e = 0$): the two edge states begin to “disentangle”. One originates from a local valley ($q < 0$), while the other starts at $q = 0$ in the lower bulk band. Nevertheless, they still enter the upper bulk band at the same point. This configuration corresponds to the collision of the Berry monopoles at the origin of the (k, q) plane. After collision ($e < 0$): the disentanglement is complete. The two edge states now originate from distinct valleys of the lower bulk band and terminate at different q values in the upper band. This maps directly to the postcollision configuration, where the Berry monopoles have split and are now separated along the synthetic momentum axis q . This proposed scenario provides a powerful and experimentally accessible route to visualize Berry monopole

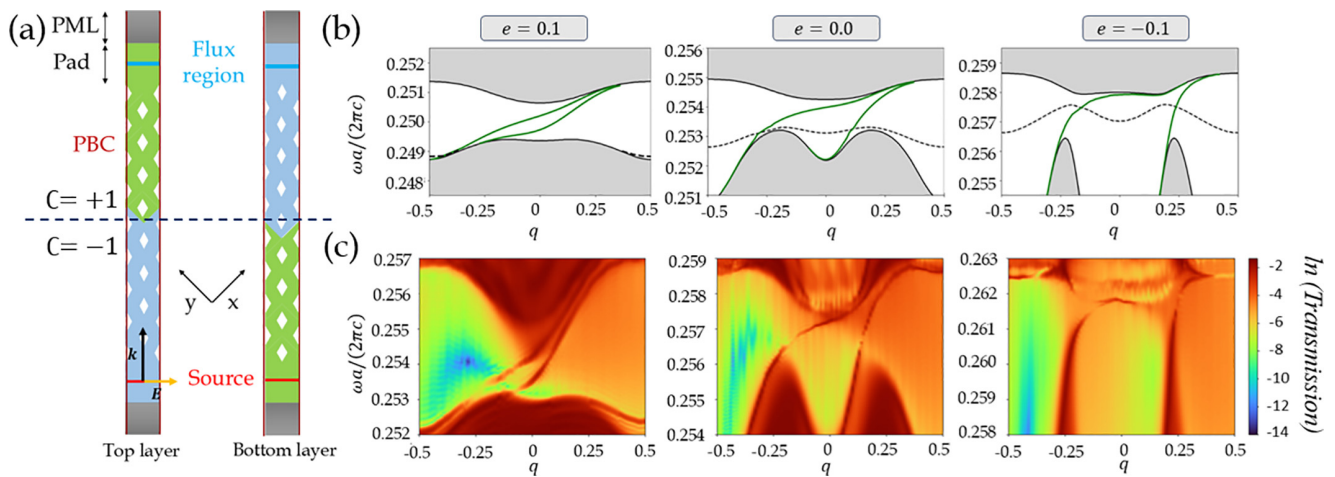


FIG. 5. Edge states spectra. (a) Schematic of the FDTD simulation setup. The heterojunction is formed by joining a bilayer, each layer having 12 periods, with thickness $h = 0.35a$ (Chern number $C = -1$) to its inverted counterpart (with the two layers swapped, $C = +1$), separated by an interface (dashed line). The hole size is $b = 0.30a$ for the blue regions and $b = 0.46a$ for the green regions. Periodic boundary conditions are applied along the interface, while perfectly matched layers are used perpendicular to it. A linearly polarized TE source (red) is placed at one end of the structure, and the transmitted flux is measured at the opposite end using a detector region (blue). (b) Band structure of the heterojunction interface computed using the effective model, showing chiral edge states (green lines) and bulk bands (gray). A third edge state (dashed black line), which remains within the lower bulk band and does not connect the two bands, is predicted by the model. This state is topologically trivial and does not appear in the FDTD transmission spectra. Parameters used are the same as in Fig. 3. (c) FDTD transmission spectra, plotted on a logarithmic scale. Both the effective model and the FDTD simulations show the presence of two robust chiral edge states, in agreement with the difference $\Delta C = +2$ in the Chern number between the two sides of the interface.

merging and splitting, using edge-state evolution as a topological probe in the extended (k, q, m) parameter space.

The effective model gives an additional M-shaped edge state shown as dashed line in Fig. 5(b). As e decreases from 0.1 down to -0.1 , this edge state shifts upward in energy and moves inside the gap from the lower band to the upper band. However, it does not connect the two bands, so at the current stage, we conclude that it is topologically trivial. Notably, the FDTD transmission spectra do not reproduce this additional edge state. It is possible that this edge state is strongly localized at the interface and does not couple to the other end of the heterojunction to be detectable in the transmission spectra (see the Supplemental Material [41] for more detailed discussion).

V. EXPERIMENTAL FEASIBILITY

Here, we address the possible experimental demonstration, in a photonic platform, of the process of merging and splitting of Berry monopoles we have introduced so far. A first demonstration does not require continuous actuation of the interlayer displacement: One may fabricate a series of bilayer samples with different prescribed relative shifts δ or assemble two separately fabricated slabs with controlled lateral offsets, and then reconstruct the monopole merging and splitting sequence from the measured spectra. This static approach is already sufficient to compare the three representative regimes before, at, and after the collision. The bilayer architecture itself is well within current experimental capabilities. Related bilayer photonic structures have already been realized in silicon moiré photonic crystals, on-chip optical bilayer photonic crystals, bilayer semiconductor metasurfaces, and WS₂ bilayer metasurfaces [48–51]. In addition, shift-dependent optical responses have recently been measured in bilayer metasurfaces,

further supporting the practical controllability of the interlayer displacement in coupled photonic platforms [52]. For a reconfigurable implementation, the bilayer can be integrated with micro-electromechanical systems (MEMS) actuation. Recent MEMS-integrated moiré photonic devices have demonstrated *in situ* tunability together with nanometer-scale positioning of coupled optical layers [53]. In our system, a residual displacement uncertainty $\Delta\delta$ translates into an uncertainty in synthetic momentum, $\Delta q = \frac{\Delta\delta}{\sqrt{2}a}$. For a lattice constant a of a few hundred nanometers, even a displacement uncertainty of a few to a few tens of nanometers corresponds to a relatively small uncertainty in q . Moreover, the experiment can be performed quasistatically by stepping the actuator to successive setpoints and recording the optical response only after mechanical settling. In this sense, the proposed measurement does not rely on coherent fast-time adiabatic motion, but rather on reproducible static or quasistatic sampling of q . Finally, the robustness observed in our FDTD calculations in the presence of structural perturbations further supports the proposal's feasibility. Altogether, these considerations indicate that Berry monopole merging and splitting should be observable either in a multisample static implementation or in a single MEMS-reconfigurable device.

VI. CONCLUSION AND OUTLOOK

In conclusion, we theoretically and numerically study the phenomenon of Berry monopole merging and splitting in a topological photonic system with synthetic momenta. We propose a practical scheme to probe this phenomenon via the formation and evolution of two chiral edge states at the interface between bilayers with opposite Chern numbers. These edge states provide a direct experimental signature

of monopole merging and splitting in momentum space. Our system is experimentally feasible, thanks to standard nanofabrication techniques and MEMS technology. Our findings provide a viable topological photonic platform to study Berry monopole physics with both theoretical significance and experimental feasibility. This 2D photonic crystal setup can be exploited to explore the physics in 4D. The two relative displacements along the x and y directions allow us to define two synthetic momenta (q_x, q_y). This permits defining a $(2+2)$ D hybrid momentum space with two genuine momenta (k_x, k_y) and two synthetic momenta. The physics in 4D is potentially rich for both Hermitian and non-Hermitian regimes, promising unusual properties that do not exist in two or three dimensions. We aim to explore the 4D analogs of the quantum Hall effect, Weyl semimetals, and tensor monopoles [54–56] within synthetic momentum space. Moreover, synthetic momenta would permit going beyond the familiar toroidal momentum space and exploring the properties of 4D nonorientable momentum spaces [33,57]. Finally, one can study polaritonic Chern insulator [58] in high dimensions by adding nonlinearity to this photonic platform with synthetic momenta.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [66,67].

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