

Universal quantum computation via superposed orders of single-qubit gates

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Superposed orders of quantum channels have already been proved—both theoretically and experimentally—to enable unparalleled opportunities in the quantum communication domain. Superposition of orders can be exploited within the quantum computing domain as well, by relaxing the (traditional) assumption underlying quantum computation about applying gates in a well-defined causal order. In this context, we address a fundamental question arising with quantum computing: Whether superposed orders of single-qubit gates can enable universal quantum computation. As shown in this paper, the answer to this key question is a definitive “yes.” Indeed, we prove that any two-qubit controlled quantum gate can be *deterministically* realized, including the so-called Barenco gate that alone enables universal quantum computation.

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I. INTRODUCTION

During the last 10 yr, there has been an increasing widespread interest in investigating the advantages arising from the superposition of traversing orders of quantum channels in the quantum communication domain, due to the outstanding possibilities arising from the nonclassical propagation of quantum carriers [1–7]. While classical carriers propagate through classical communication paths, quantum channels can be placed in a genuinely quantum setting with no counterpart in the classical world [8,9], giving rise to the concept of *quantum path* [10–12]. This nonclassical propagation has been experimentally validated [13,14], and it finds a possible physical realization in the device known as *quantum switch* [15,16], which enables a coherent superposition of alternative channel configurations. Specifically, by means of the quantum switch, the quantum carrier can travel through different communication channels in a quantum superposition of different causal orders. This, in turn, makes the order of the communication channels indefinite, and it returns disruptive advantages, for instance, for communications through noisy channels [17–19]. Furthermore, it has been recently demonstrated that the quantum switch can be exploited for generating different classes of genuine multipartite entangled states starting from separable inputs [20,21]. Ongoing

research is exploring further intriguing aspects of the unconventional resources enabled by the quantum switch [22–29]. Introducing the quantum switch to the engineering of quantum communications presents key advantages.

The possibilities enabled by the superposition of orders of quantum operations can be exploited within the quantum computing domain as well [30,31]. Specifically, it is possible to relax the (traditional) assumption underlying quantum computation, i.e., quantum gates applied in a well-defined causal order only [32]. By doing this, a more general computing framework arises, which has been applied to a number of problems, both theoretically [30,31,33–38] and experimentally [13,39].

In this context, we address a fundamental question arising with quantum computing, namely, whether superposed orders of quantum operations can constitute a paradigm for *universal quantum computing*.

As proved in the following, the answer to this key question is a definitive “yes.” Specifically, through the manuscript we show that any controlled quantum gate can be *deterministically* realized via superposition of the traversing orders of single-qubit gates. Notably, this includes the two-qubit gate known as *Barenco gate*, which *alone* enables universal quantum computation [40].

Contribution

Our contributions can be summarized as follows:

- (1) We provide a general framework for the realization of controlled quantum gates through superposed orders of single-qubit unitaries.
- (2) We prove that this framework enables the *deterministic* realization of any two-qubit controlled gate.
- (3) We specialize the framework for two-qubit controlled gates widely used in quantum computing, including CNOT, CZ,

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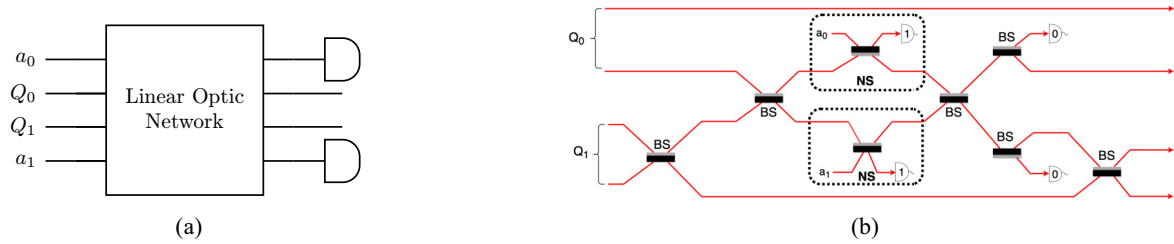


FIG. 1. Nondeterministic CNOT gate via linear optics. (a) Simplified diagram of a linear optic implementation of a CNOT logic gate, where Q_0 and Q_1 represent the control qubit and the target qubit and a_0 , a_1 are two ancillary photons, (b) KLM CNOT scheme with simplified NS gates. Q_0 and Q_1 denote the control qubit and the target qubit, respectively. This scheme assumes the logical qubits to be encoded through spatial modes (path encoding).

and the *universal* Barenco gate. Furthermore, we extend the implementation of controlled gates to d -dimensional systems.

We note that the relevance of our work can be linked to the photonic computing domain considering physical implementations of the quantum switch are mostly photonic. As a matter of fact, photonic quantum gates are either probabilistic or based on preshared multipartite-entangled states, as discussed in Sec. II A. Conversely, our framework enables the deterministic implementation of universal quantum computation by exploiting only single-qubit gates, without requiring preshared large entangled states. This approach provides a solid foundation for further theoretical investigations and experimental validations aimed at exploring superposed orders of quantum gates for photonic quantum computing.

II. BACKGROUND

In this section, we first summarize the main challenges arising with the implementation of controlled operations in optical setups. Then, we provide the reader with a concise guide through the supermap formalism required for modeling superposed orders of quantum gates.

It is important to discuss first the physical realizability of the quantum switch. It is known that if the underlying spacetime is classical, no experiment can realize the quantum switch deterministically [41]. This is consistent with Ref. [30] that shows the only possible circuit for a quantum switch is via a postselection that simulates a closed timelike curve. The known experimental realizations of quantum switch overcome this limitation by implementing the parties' local operations that are delocalized in time [42]. To do away with postselection, known experimental realizations of the quantum switch itself requires controlled unitary gates. These controlled unitary gates correspond to arbitrary single-qubit unitary gates that require “vacuum-extended” unitaries; i.e., in Fig. 2, $\mathcal{A}$ and $\mathcal{B}$ are part of $\tilde{\mathcal{A}} = \mathcal{A} \oplus I$ and $\tilde{\mathcal{B}} = \mathcal{B} \oplus I$ [47]. This requirement must be kept in mind in the following discussions of the implementations of two-qubit gates that use a quantum switch. Technically, because of the use of “vacuum-extended unitaries,” one may refer to experimental realizations of the quantum switch as simulations of the quantum switch. We will not dwell on that distinction in this current work. It is sufficient to say that the operations necessary for the current proposal—a quantum switch that takes two target qubits (Q_0 and Q_1)—can be realized experimentally following extensions of the implementations shown in Fig. 2.

Another implementation can be based on two target qubits and one control qubit on three different photons, although such realizations of the quantum switch have not been shown.

A. Optical Controlled Operations

It is widely recognized by both the industrial and academic communities that light represents the prominent candidate for quantum information carriers [48]. In more detail, the advantages arising from photons as quantum information carriers are manifold.

Photons hardly interact with the environment, and they do not require a complex cooling infrastructure or high vacuum chambers. Additionally, they support long-range transmission with low losses through optical fiber channels or waveguides. Photons exhibit multiple degrees of freedom that represent a resource for quantum information encoding. While many physical realizations use polarization, photons can also exist in a superposition of time bins, frequency bins, path, and transverse modes, which can also serve as qudits [49].

Unfortunately, regardless of the appealing features for quantum communications, photonic quantum technologies still represent a challenge from a quantum computing perspective. Indeed, such technologies suffer from the probabilistic nature of single-photon sources and photon-photon nonlinear interactions, which realize quantum logic gates.

However, *gate-based* photonic quantum computing can also be realized by exploiting linear optical elements [50]. Specifically, quantum logic operations can be obtained probabilistically through linear optical elements, ancillary photons, and postselection based on the output of single-photon detectors [51–56]. In this context, particularly challenging is the realization of the logic gate CNOT, whose scheme for optical implementation is represented in Fig. 1 and can be summarized as follows. The two input photons (control qubit Q_0 and target qubit Q_1 , respectively) and two additional ancilla photons (a_0 and a_1) are combined through a linear optic network of Beam Splitters (BS) as pictorially represented in Fig. 1(a). If both the ancilla qubits are detected at the output of the optical network—specifically, both the detectors signal a single-photon detection—then the target qubit has been successfully subjected to the CNOT logic operation. To better understand the nondeterministic nature of the optical implementation of the gate, we consider the well-known Knill-Laflamme-Milburn (KLM) CNOT scheme [50], represented in Fig. 1(b). The basic unit of the

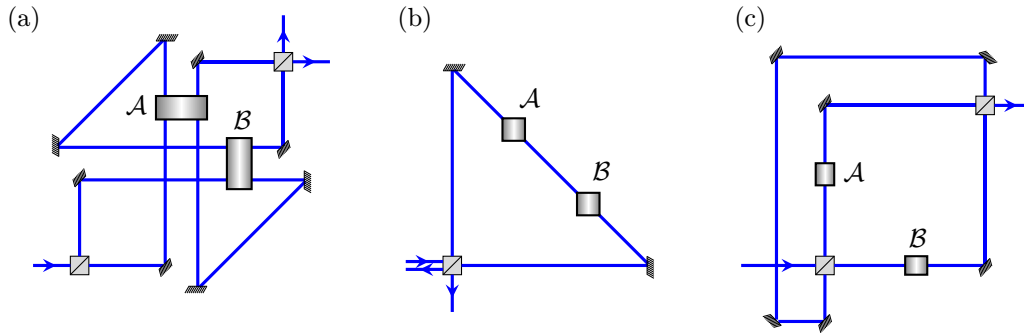


FIG. 2. Schematic diagram of some of the experimentally implemented architectures of the photonic quantum switch. (a) An implementation via a Mach-Zehnder geometry, where the target qubit is encoded in polarization of the photon, while the control qubit is mapped into its path degree of freedom using the first beam splitter and coherently recombining the paths $\mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{B} \rightarrow \mathcal{A}$ at the second beam splitter [13,14,16,43,44]. (b) An implementation via a Sagnac geometry, where the target qubit is encoded in polarization of the photon [as in panel (a)], whereas a single beam splitter introduces the path degree of freedom as control and completes superposition of causal orders of $\mathcal{A}$ and $\mathcal{B}$ [45]. (c) An implementation via a geometry, where the target qubit is encoded in the path degree of freedom of the photon, while the role of the control qubit is played by its polarization [15,46].

scheme is a nonlinear sign-shift gate (NS) which given the input state $|\mu\rangle = a|0\rangle + b|1\rangle + c|2\rangle$ returns the output state $|\mu'\rangle = a|0\rangle + b|1\rangle - c|2\rangle$, where $|0\rangle$, $|1\rangle$, and $|2\rangle$ denote the vacuum state, single-photon state, and two-photon state, respectively. Consider the qubits encoded through spatial modes (path), the NS gate is obtained through three BS and two number-resolving detectors. The NS gate successfully performs a heralded π phase rotation, when exactly one photon is detected on one detector and no photons are detected at the other. This event occurs with probability $1/4$. The KLM CNOT gate is constructed from two NS gates; hence, the CNOT success probability is $(1/4)^2 = 1/16$. We represent in Fig. 1(b) the KLM CNOT gate with two simplified NS gates, namely, the NS gates are implemented through one BS and one detector. The result is still a heralded sign shift, where the detection of one photon represents the herald event; however, the success probability is slightly decreased (from 0.25 to 0.23) [57].

For overcoming the nondeterministic nature of *gate-based* schemes, the so-called cluster-state-based photonic quantum computing has been developed [58]. The key idea is that, in the absence of deterministic two-photon operations, an initial cluster state can be built up offline using nondeterministic interactions. Successively, the computation progresses by manipulating the cluster state via deterministic single-qubit operations through optical elements [58].

Our framework merges both the appealing features of *gate-based* and *cluster-state-based* photonic computing. Specifically, our framework overcomes the limitations of the former schemes by enabling deterministic computing. And it also overcomes the limitations exhibited by the latter, since it does not require any preshared multipartite-entangled state, whose generation requires offline nondeterministic interactions.

B. Superposed Orders of Quantum Operations via Quantum Switch

While the quantum circuit model is one of the most widely used paradigms of quantum computations, a lot of effort has been put into extending it to computations of higher order. Such objects as quantum combs, which can be seen as

quantum circuits with open slots for arbitrary quantum gates, have allowed one to solve problems not achievable with the quantum circuit model [59–61]. Nevertheless, quantum combs, which put the gates into a certain order on a circuit board, are not the most general computational model of higher order that can be achieved within quantum mechanics. Indeed, it allows for higher-order operations putting quantum gates into configurations—such as the superpositions of causal orders discussed in the following—that cannot be reduced to their well-ordered compositions [9,30,31,61,62].

Superposition of causal orders of quantum operations can be realized via the quantum switch, i.e., a quantum device already implemented in numerous table-top optical and nuclear magnetic resonance (NMR) experiments [13–16,43–46,63–67] as schematized in Fig. 2. In what follows, we provide the reader with the mathematical description of the quantum switch.

Given input and output systems $\mathbf{I}$ and $\mathbf{O}$, any quantum operation transforming the former to the latter can be represented by a completely positive trace-preserving map $\mathcal{A} : \mathcal{L}(\mathcal{H}_{\mathbf{I}}) \rightarrow \mathcal{L}(\mathcal{H}_{\mathbf{O}})$, where $\mathcal{H}_{\mathbf{I/O}}$ denotes the Hilbert space of system $\mathbf{I}$ or $\mathbf{O}$, respectively, while $\mathcal{L}(\mathcal{H}_{\mathbf{I/O}})$ is the set of density operators over $\mathcal{H}_{\mathbf{I/O}}$.

The quantum switch is an example of a supermap that sends any N quantum operations $\mathcal{A}_1[\cdot], \dots, \mathcal{A}_N[\cdot]$ to a quantum operation $\mathcal{S}(\mathcal{A}_1, \dots, \mathcal{A}_N)[\cdot]$. More into details, given an ancillary quantum system $\mathbf{C}$, the quantum switch is constructed as a supermap that uses a state ω of $\mathbf{C}$ to coherently control the order in which $\mathcal{A}_1[\cdot], \dots, \mathcal{A}_N[\cdot]$ acts on the input system in the state ρ :

$$\mathcal{S}(\mathcal{A}_1, \dots, \mathcal{A}_N)[\rho \otimes \omega] = \sum_{i_1 \dots i_N} K_{i_1 \dots i_N}(\rho \otimes \omega) K_{i_1 \dots i_N}^\dagger, \quad (1)$$

with

$$K_{i_1 \dots i_N} = \sum_k \mathcal{P}_k(A_{i_1}^{(1)} \dots A_{i_N}^{(N)}) \otimes |k\rangle\langle k| \quad (2)$$

denoting the Kraus operators of the output quantum operation of the quantum switch. In Eq. (2), $\{A_{i_j}^{(j)}\}_i$ denotes the set of

Kraus operators of $\mathcal{A}_j[\cdot]$, $\mathcal{P}_k$ denotes a k th permutation, and $|k\rangle$ is the k th basis state of system $\mathbf{C}$.

By restricting the number of controlled operations to two operations $\mathcal{A}[\cdot]$ and $\mathcal{B}[\cdot]$, the supermap in Eq. (1) exhibits Kraus operators given by $K_{ij} = A_i B_j \otimes |0\rangle\langle 0| + B_j A_i \otimes |1\rangle\langle 1|$, with $\{A_i\}_i$ and $\{B_j\}_j$ being Kraus operators of $\mathcal{A}[\cdot]$ and $\mathcal{B}[\cdot]$. This operation implemented by the quantum switch can be represented explicitly in a simple form as [5,29,68]

$$\begin{aligned} \mathcal{S}(\mathcal{A}, \mathcal{B})[\rho \otimes \omega] = & \frac{1}{4} \sum_{ij} (\{A_i, B_j\} \rho \{A_i, B_j\}^\dagger \otimes \omega \\ & + \{A_i, B_j\} \rho [A_i, B_j]^\dagger \otimes \omega Z \\ & + [A_i, B_j] \rho \{A_i, B_j\}^\dagger \otimes Z \omega \\ & + [A_i, B_j] \rho [A_i, B_j]^\dagger \otimes Z \omega Z), \end{aligned} \quad (3)$$

where $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ denote a commutator and an anticommutator [69], respectively, and $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ is the Pauli Z operator.

III. CONTROLLED GATES VIA SUPERPOSED ORDERS

In this section, we first exploit the quantum switch to realize superposed orders of arbitrary unitary gates. Then, we exploit these preliminary results to realize controlled gates by resorting to single-qubit gates only.

A. Superposed orders of arbitrary gates

Given two arbitrary unitary gates A and B , the execution of each of them on the input system $\mathbf{I}$ in state ρ can be seen as the action of quantum operations $\mathcal{A}[\rho] = A\rho A^\dagger$ and $\mathcal{B}[\rho] = B\rho B^\dagger$. Therefore, we can realize a superposition of causal orders between the two gates A and B by simplifying Eq. (3) as

$$\begin{aligned} \mathcal{S}(\mathcal{A}, \mathcal{B})[\rho \otimes \omega] = & \frac{1}{4} (\{A, B\} \rho \{A, B\}^\dagger \otimes \omega \\ & + \{A, B\} \rho [A, B]^\dagger \otimes \omega Z \\ & + [A, B] \rho \{A, B\}^\dagger \otimes Z \omega \\ & + [A, B] \rho [A, B]^\dagger \otimes Z \omega Z). \end{aligned} \quad (4)$$

Equation (4) can be equivalently interpreted as the execution of a unitary gate $S(A, B)$ —acting on the overall system composed by the input system $\mathbf{I}$ and the ancilla $\mathbf{C}$ —given by

$$S(A, B) = \frac{1}{2} [\{A, B\} \otimes I_C + [A, B] \otimes Z_C], \quad (5)$$

where I_C and Z_C denote identity and Z -Pauli operators, respectively, that act on $\mathbf{C}$. For taking full advantage of the indefinite causal order among the unitaries, we set system $\mathbf{C}$ in the pure state $\omega = |+\rangle\langle +|$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, so that the input system evolves into an even superposition of the two causal orders among the unitaries.

By assuming for the sake of simplicity that the input system $\mathbf{I}$ is in a pure [70] initial state $\rho = |\psi\rangle\langle\psi|$, the action of the unitary gate $S(A, B)$ is given by

$$\begin{aligned} S(A, B)(|\psi\rangle \otimes |+\rangle) = & \frac{1}{2} ([\{A, B\} |\psi\rangle] \otimes |+\rangle \\ & + ([A, B] |\psi\rangle) \otimes |-\rangle). \end{aligned} \quad (6)$$

It is straightforward to see that tracing out the ancillary qubit $\mathbf{C}$ results in a probabilistic gate that chooses between well-ordered sequences AB or BA randomly with probability $1/2$. On the other hand, a measurement of the state of $\mathbf{C}$ in the basis spanned by

$$|\mu(\theta)\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle - i \sin\left(\frac{\theta}{2}\right)|1\rangle, \quad (7)$$

$$|\mu^\perp(\theta)\rangle = -i \sin\left(\frac{\theta}{2}\right)|0\rangle + \cos\left(\frac{\theta}{2}\right)|1\rangle, \quad (8)$$

leaves the system in the following states with probability $1/2$

$$|\psi_+(\theta)\rangle = \left[\cos\left(\frac{\theta}{2}\right)AB + i \sin\left(\frac{\theta}{2}\right)BA \right] |\psi\rangle, \quad (9)$$

$$|\psi_-(\theta)\rangle = \left[i \sin\left(\frac{\theta}{2}\right)AB + \cos\left(\frac{\theta}{2}\right)BA \right] |\psi\rangle. \quad (10)$$

Accordingly, once the ancillary qubit is measured, with probability $1/2$, one of two different gates $S_+^{A,B}(\theta)$ and $S_-^{A,B}(\theta)$ is realized:

$$S_+^{A,B}(\theta) = \cos\left(\frac{\theta}{2}\right)AB + i \sin\left(\frac{\theta}{2}\right)BA, \quad (11)$$

$$S_-^{A,B}(\theta) = i \sin\left(\frac{\theta}{2}\right)AB + \cos\left(\frac{\theta}{2}\right)BA. \quad (12)$$

B. Superposed orders of single-qubit gates

A prerequisite for the realization of any universal set of quantum gates is the ability of implementing a multiqubit gate that cannot be reduced to a single tensor product of single-qubit gates only. With the following lemma and corollary, we prove that the overall multiqubit gate—implemented by combining single-qubit gates via the quantum switch—satisfies the aforementioned property for nontrivial choices of the single-qubit gates and the ancillary measurement bases.

Lemma 1. Combining N single-qubit gates $A = \bigotimes_{i=1}^N A_i$ and N single-qubit gates $B = \bigotimes_{i=1}^N B_i$ via quantum switch implements one of the following two N -qubit unitaries:

$$S_+^{A,B}(\theta) = \cos\left(\frac{\theta}{2}\right) \bigotimes_{i=1}^N A_i B_i + i \sin\left(\frac{\theta}{2}\right) \bigotimes_{i=1}^N B_i A_i, \quad (13)$$

$$S_-^{A,B}(\theta) = i \sin\left(\frac{\theta}{2}\right) \bigotimes_{i=1}^N A_i B_i + \cos\left(\frac{\theta}{2}\right) \bigotimes_{i=1}^N B_i A_i, \quad (14)$$

with the actual implemented gate depending on whether the ancillary qubit is measured as Eq. (7) or (8).

Proof. See Appendix A. ■

We are now ready to provide the main result of this section with the following corollary.

Corollary 1. $S_+^{A,B}(\theta)$ and $S_-^{A,B}(\theta)$ in Eqs. (13) and (14) cannot be reduced to a single tensor product of single-qubit gates only, unless either: (1) $\theta = \pi k$ with $k \in \mathbb{Z}$ or (2) $[A_i, B_i] = 0$ for any $N - 1$ gates A_i and B_i .

Proof. The proof follows directly from Lemma 1. ■

C. Realization of controlled gates

The realization of controlled gates is key in the quantum domain. Indeed, not only is entanglement usually generated

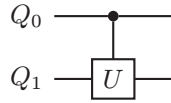


FIG. 3. The CU (controlled U) logic gate.

within the quantum circuit model via controlled gates (typically, using a controlled-not CNOT), but also—even more relevant from our perspective—there exists a class of two-qubit controlled gates any one of which is universal for quantum computation [40].

For this, in the following we restrict our attention on two-qubit controlled gates. Accordingly, we denote the controlled gate for an arbitrary single-qubit unitary gate U as CU (controlled U), which is formally defined as

$$CU = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U. \quad (15)$$

Gate CU acts on two input qubits, with the qubit acting as control denoted as Q_0 and the qubit acting as target denoted with Q_1 , as depicted in Fig. 3.

In the following, we aim at proving that any arbitrary CU function can be realized with the gates in Eqs. (13) and (14), namely, by combining single-qubit gates via the quantum switch. Accordingly, given the two input qubits in an initial state ρ , the quantum switch combines two-qubit gates $A = A_0 \otimes A_1$ and $B = B_0 \otimes B_1$ in superposed orders. As represented in Fig. 4, it is worthwhile to note that A_0, B_0 denote the single-qubit unitaries acting on the first qubit Q_0 of the input state ρ , whereas A_1, B_1 denote the single-qubit unitaries acting on the second qubit Q_1 of the input state ρ .

The following preliminary definitions are needed.

Definition 1. A single-qubit gate $R_{\mathbf{n}}(\theta)$ that performs a rotation on angle θ around the $\mathbf{n}$ axis defined by the Bloch vector $\mathbf{n} = (n_x, n_y, n_z)$ is given by

$$R_{\mathbf{n}}(\theta) = \cos\left(\frac{\theta}{2}\right)I - i \sin\left(\frac{\theta}{2}\right)(\mathbf{n} \cdot \boldsymbol{\sigma}), \quad (16)$$

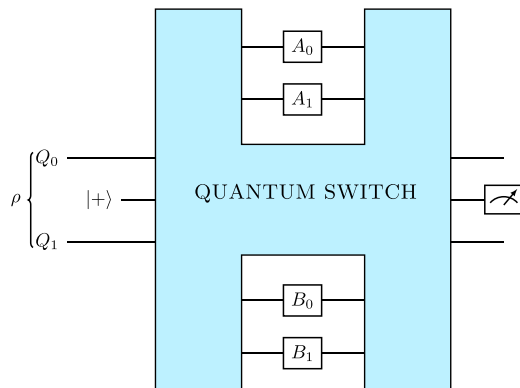


FIG. 4. Representation of two-qubit controlled logic gate via quantum switch, with the switch represented as an H-shape blue box as in Refs. [4,26,42,71]. The quantum switch combines two-qubit gates $A = A_0 \otimes A_1$ and $B = B_0 \otimes B_1$ in superposed orders, with A_0, B_0 denoting the single-qubit unitaries acting on the first qubit Q_0 and A_1, B_1 denoting the single-qubit unitaries acting on the second qubit Q_1 .

with I and $\boldsymbol{\sigma} = (X, Y, Z)$ denoting the identity matrix and a vector of Pauli matrices, respectively.

Accordingly, gates $R_X(\theta)$, $R_Y(\theta)$, and $R_Z(\theta)$ denote the rotation gate given in Eq. (16) with respect to Bloch vectors $\mathbf{n} = (1, 0, 0)$, $\mathbf{n} = (0, 1, 0)$, and $\mathbf{n} = (0, 0, 1)$, respectively.

Definition 2. A two-qubit rotation gate $R_{\tilde{\mathbf{n}}\mathbf{n}}(\theta)$ with respect to angle θ and Bloch vectors $\tilde{\mathbf{n}}$ and $\mathbf{n}$ is given by

$$R_{\tilde{\mathbf{n}}\mathbf{n}}(\theta) = \cos\left(\frac{\theta}{2}\right)I \otimes I - i \sin\left(\frac{\theta}{2}\right)(\tilde{\mathbf{n}} \cdot \boldsymbol{\sigma}) \otimes (\mathbf{n} \cdot \boldsymbol{\sigma}). \quad (17)$$

Definition 3. Two-qubit gates A and B are locally equivalent [72] if they can be mapped to one another by a tensor product of single-qubit unitary gates $\{V_i\}_{i=1,2}$ and $\{\tilde{V}_i\}_{i=1,2}$:

$$A = (V_1 \otimes V_2)B(\tilde{V}_1 \otimes \tilde{V}_2). \quad (18)$$

Lemma 2. The controlled two-qubit gate CU given by

$$CU = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U, \quad (19)$$

which is locally equivalent to sequences of single-qubit gates put into superposed orders via quantum switch.

Proof. See Appendix B. ■

Lemma 2 shows that any two-qubit controlled gate can be implemented using *single-qubit gates only* by combining them in superposed orders. By exploiting this result, the following theorem provides a recipe for the implementation of any arbitrary CU gate.

Theorem 1. The arbitrary two-qubit controlled gate $CU(\alpha, \theta, \mathbf{n})$

$$CU(\alpha, \theta, \mathbf{n}) = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U(\alpha, \theta, \mathbf{n}), \quad (20)$$

with $U(\alpha, \theta, \mathbf{n})$ defined as

$$U(\alpha, \theta, \mathbf{n}) = \exp[i(\alpha I + \theta(\mathbf{n} \cdot \boldsymbol{\sigma}))], \quad (21)$$

can be deterministically realized through:

(1) superposed orders of single-qubit gates $A(\mathbf{n}) = A_0 \otimes A_1(\mathbf{n})$ and $B(\mathbf{n}) = B_0 \otimes B_1(\mathbf{n})$ combined via a quantum switch,

(2) preceded by a preprocessing phase $P(\mathbf{n}) = P_0 \otimes P_1(\mathbf{n})$, implemented by single-qubit gates $\{P_0, P_1(\mathbf{n})\}$,

(3) followed by a postprocessing phase $F_{\pm}(\alpha, \theta, \mathbf{n})$, implemented by single-qubit gates set accordingly to the ancillary-qubit measurement results in the basis spanned by Eq. (7),

as follows:

$$CU(\alpha, \theta, \mathbf{n}) = F_{\pm}(\alpha, \theta, \mathbf{n})S_{\pm}^{A(\mathbf{n}), B(\mathbf{n})}(\theta)P(\mathbf{n}). \quad (22)$$

where

$$F_{\pm}(\alpha, \theta, \mathbf{n}) = e^{i\frac{\alpha}{2}}\left(R_Z\left(\alpha \pm \frac{\pi}{2}\right) \otimes R_{\mathbf{n}}\left(-\theta \pm \frac{\pi}{2}\right)\right), \quad (23)$$

$$A(\mathbf{n}) = A_0 \otimes A_1(\mathbf{n}) = X \otimes (\mathbf{n}^{\perp} \cdot \boldsymbol{\sigma}), \quad (24)$$

$$B(\mathbf{n}) = B_0 \otimes B_1(\mathbf{n}) = R_Z\left(\frac{\pi}{2}\right) \otimes R_{\mathbf{n}}\left(\frac{\pi}{2}\right), \quad (25)$$

$$P(\mathbf{n}) = P_0 \otimes P_1(\mathbf{n}) = X \otimes (\mathbf{n}^{\perp} \cdot \boldsymbol{\sigma}), \quad (26)$$

with $\mathbf{n}^{\perp}$ denoting the Bloch vector perpendicular to $\mathbf{n}$.

Proof. The proof follows by exploiting the result of Lemma 2, by plugging Eqs. (23)–(26) into the decomposition

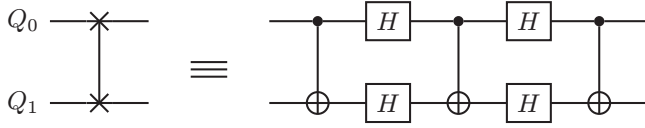


FIG. 5. Example of equivalent quantum circuit for a SWAP gate exploiting the CNOT logic gate implementation.

[Eq. (22)], and by comparing it with the decomposition

$$\text{CU}(\alpha, \theta, \mathbf{n}) = e^{i\frac{\alpha}{2}} (R_Z(\alpha) \otimes R_{\mathbf{n}}(-\theta)) R_{Z\mathbf{n}}(\theta) \quad (27)$$

of the CU gate provided in Appendix B. ■

Remark 1. It is worthwhile to note that the deterministic realization of an arbitrary controlled gate CU via superposed orders of single-qubit gates imposes different constraints on the gates acting on the control qubit with respect to the gates acting on the target qubit. Specifically, the single-qubit gates acting on the target—i.e., A_1 and B_1 —depend on the actual definition of gate U. Conversely, the single-qubit gates acting on the control qubit—i.e., A_0 and B_0 —do not depend on the gate U.

Note that the preprocessing and postprocessing gates specified in Theorem 1 are single-qubit, local unitaries that map the quantum-switch construction to the exact controlled operation CU. Omitting these local operations yields output states that differ from the target only by local unitaries; i.e., they are local-unitary-equivalent (LU-equivalent) to the output obtained when the pre- and postprocessing operations are included. This shows that the essential “two-parties” interaction of the implemented controlled operation is entirely captured by the superposed-order construction itself.

Moreover, while the above discussion focuses on controlled two-qubit unitaries, the framework naturally extends to arbitrary two-qubit gates through compilation [76]. In particular, a compiler can transpile any target unitary into a circuit composed solely of single-qubit and controlled-unitary operations, each of which can be realized deterministically using the proposed superposed-order primitives, as a consequence of the universality. Consequently, noncontrolled gates such as the SWAP can also be implemented within our scheme. Specifically, since a SWAP gate is equivalently realized by a quantum circuit utilizing three CNOT operations [77], as shown in Fig. 5, the SWAP can be realized through three invocations of the superposed-order primitive combined with local single-qubit gates.

Theorem 1 is the main instrument, exploited in what follows for the realization of several important examples of two-qubit controlled gates belonging to—or constituting alone as for the Barenco gate—universal sets.

IV. UNIVERSAL QUANTUM COMPUTATION

In this section, we provide several examples of popular quantum gates that are used to construct universal sets, and we detail how they can be synthesized from single-qubit gates in superposed orders.

A. Controlled NOT

We start by considering the gate CNOT (controlled NOT), a paramount example of a gate widely used to construct more complex gates and lying at the core of fundamental quantum protocols such as entanglement generation, quantum teleportation, and telegate [76,78–80].

Formally, CNOT is a controlled-X gate which acts on two qubits as

$$\text{CNOT} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X. \quad (28)$$

Crucially, CNOT appears in several universal sets as a unique multiqubit gate together with certain single-qubit gates. Indeed, any other unitary gate can be expressed as a sequence of CNOT gates and some single-qubit gates [69]. Hence, an efficient implementation of the CNOT gate is of paramount importance when it comes to universal quantum computation. To this aim, the following proposition demonstrates that the CNOT gate can be realized using only simple, widely used single-qubit gates by properly placing some of them into a superposition of orders.

Proposition 1. The CNOT gate can be deterministically realized by combining in superposition of orders the following single-qubit gates:

$$A = X \otimes Z, \quad (29)$$

$$B = R_Z\left(\frac{\pi}{2}\right) \otimes R_X\left(\frac{\pi}{2}\right). \quad (30)$$

Proof. The proof follows from Theorem 1, by recognizing that the CNOT gate is given by the decomposition in Eq. (22) for $\alpha = -\frac{\pi}{2}$, $\theta = \frac{\pi}{2}$, and $\mathbf{n} = (1, 0, 0)$ and choosing $\mathbf{n}^\perp = (0, 0, 1)$. ■

Proposition 1 provides us with the realization of the CNOT gate via superposed orders of single-qubit gates. Specifically, as shown in Fig. 6, we first preprocess the input state—namely, control and target qubits Q_0 and Q_1 —with a sequence of single-qubit gates $P = X \otimes Z$ in accordance with Eq. (26). Then, the resulting state goes into the quantum switch, which processes it according to simple, widely used single-qubit gates as in Eqs. (29) and (30). A measurement of the ancillary qubit C in the basis spanned by states [Eqs. (7) and (8)]—which we denote in this case as $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ and $|-i\rangle = \frac{1}{\sqrt{2}}(-i|0\rangle + |1\rangle)$ —realizes the gates $S_{\pm}^{A,B}(\frac{\pi}{2})$ with probability 1/2 in accordance with Eqs. (13) and (14). Finally, depending on the outcome of the measurement, we perform a postprocessing by applying another sequence of single-qubit gates, namely, (1) $F_-(-\pi/2, \pi/2, \mathbf{n}) = -e^{-i\frac{\pi}{4}}(Z \otimes X)$ whenever the ancillary qubit is found in the state $|-i\rangle$ or (2) the identity $F_+(-\pi/2, \pi/2, \mathbf{n}) = e^{-i\frac{\pi}{4}}(I \otimes I)$ otherwise. Accordingly, the overall deterministic implementation of the CNOT via superposed orders can be expressed as

$$\text{CNOT} = \begin{cases} e^{-i\frac{\pi}{4}} S_+^{A,B}(\frac{\pi}{2})(X \otimes Z), & \text{if ancilla in } |+i\rangle, \\ -e^{-i\frac{\pi}{4}} (Z \otimes X) S_-^{A,B}(\frac{\pi}{2})(X \otimes Z), & \text{otherwise,} \end{cases} \quad (31)$$

with, again, A, B given in Eqs. (29) and (30).

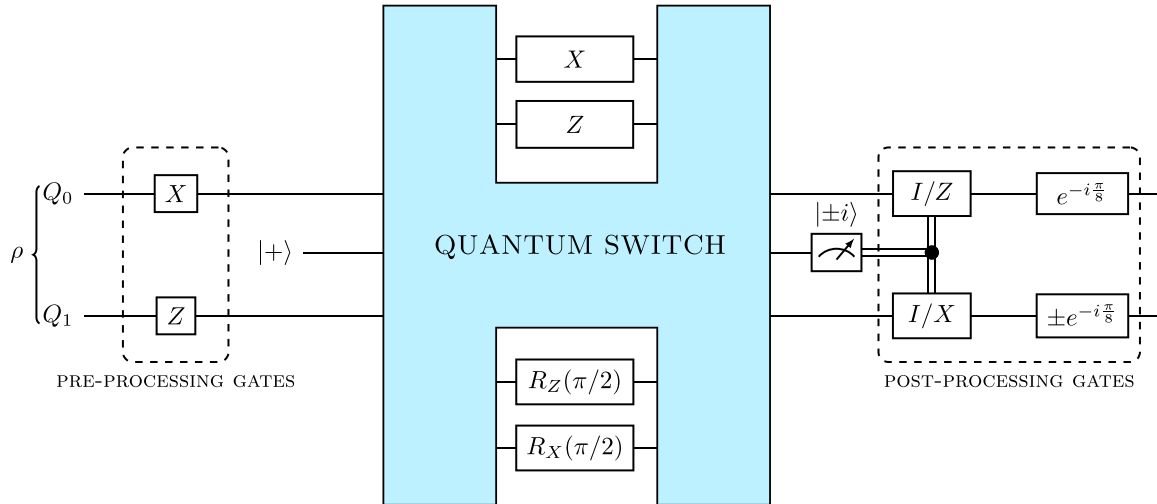


FIG. 6. Abstract representation of the CNOT gate implemented via superposed orders of single-qubit gates, with Q_0 denoting the control qubit and Q_1 denoting the target qubit, respectively. Specifically, the order between the gates is controlled by an ancillary qubit in state $|+\rangle$ that—after a preprocessing phase, implemented by single-qubit gates $P = X \otimes Z$ —implements an even superposition of causal orders between single-qubit gates $A = X \otimes Z$ and $B = R_Z(\frac{\pi}{2}) \otimes R_X(\frac{\pi}{2})$. Once the ancillary qubit is measured in the basis spanned by $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ and $|-i\rangle = \frac{1}{\sqrt{2}}(-i|0\rangle + |1\rangle)$, qubits Q_0 and Q_1 are postprocessed by $F_- = -e^{-i\frac{\pi}{4}}(Z \otimes X)$ or $F_+ = e^{-i\frac{\pi}{4}}(I \otimes I)$, depending on whether the ancilla is measured as $|-i\rangle$ or $|+i\rangle$.

B. Controlled Z

While CNOT gate can be directly associated with a classical reversible XOR gate, a logic gate without classical counterpart known as CZ (controlled Z) gate is also frequently used due to its diagonal form and can constitute a universal set together with the corresponding single-qubit gates [69]. Formally, CZ acts on two qubits as

$$\text{CZ} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z, \quad (32)$$

and is locally equivalent to the CNOT gate via Hadamard gates, as shown in Fig. 7:

$$\text{CZ} = (I \otimes H)\text{CNOT}(I \otimes H). \quad (33)$$

The following proposition demonstrates that the CZ gate, similarly to the CNOT gate, can be synthesized directly using only single-qubit gates by putting some of them in superposed orders.

Proposition 2. The CZ gate can be deterministically realized by combining in superposition of orders the following single-qubit gates:

$$A = X \otimes X, \quad (34)$$

$$B = R_Z\left(\frac{\pi}{2}\right) \otimes R_Z\left(\frac{\pi}{2}\right). \quad (35)$$

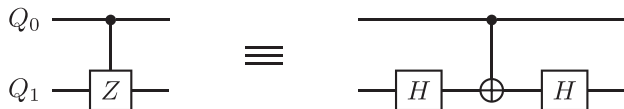


FIG. 7. Example of equivalent quantum circuit for a CZ logic gate exploiting the CNOT logic gate implementation.

Proof. The proof follows from Theorem 1, by recognizing that the CZ gate is given by the decomposition in Eq. (22) for $\alpha = -\frac{\pi}{2}$, $\theta = \frac{\pi}{2}$, and $\mathbf{n} = (0, 0, 1)$, and choosing $\mathbf{n}^\perp = (1, 0, 0)$. ■

As represented in Fig. 8, Proposition 2 proves that the CZ gate is obtained via superposed orders of single-qubit gates as follows. We first preprocess the input state with a sequence of single-qubit gates $P = X \otimes X$ in accordance with Eq. (26), then the resulting state goes into the quantum switch, which processes it according to the gates in Eqs. (34) and (35). A measurement of the ancillary qubit $\mathbf{C}$ in basis spanned by the states [Eqs. (7) and (8)], which we denote in this case as $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ and $|-i\rangle = \frac{1}{\sqrt{2}}(-i|0\rangle + |1\rangle)$, realizes the gates $S_{\pm}^{A,B}(\frac{\pi}{2})$ with probability 1/2 in accordance with Eqs. (13) and (14). Finally, depending on the outcome of the measurement, we perform a postprocessing by applying another sequence of single-qubit gates, namely, $F_-(-\pi/2, \pi/2, \mathbf{n}) = -e^{-i\frac{\pi}{4}}(Z \otimes Z)$, if the ancillary qubit is found in the state $|-i\rangle$ or $F_+(-\pi/2, \pi/2, \mathbf{n}) = e^{-i\frac{\pi}{4}}(I \otimes I)$ otherwise:

$$\text{CZ} = \begin{cases} e^{-i\frac{\pi}{4}} S_+^{A,B}(\frac{\pi}{2})(X \otimes X), & \text{if ancilla in } |+i\rangle, \\ -e^{-i\frac{\pi}{4}} (Z \otimes Z) S_-^{A,B}(\frac{\pi}{2})(X \otimes X), & \text{otherwise.} \end{cases} \quad (36)$$

The discussed implementation of the qubit gates CNOT and CZ naturally extends to higher-dimensional systems ($d > 2$). The corresponding d -dimensional counterparts are [81–83]

$$\text{SUM}_d = \sum_{k=0}^{d-1} |k\rangle\langle k| \otimes X_d^m, \quad (37)$$

$$\text{PHASE}_d = \sum_{k=0}^{d-1} |k\rangle\langle k| \otimes Z_d^m, \quad (38)$$

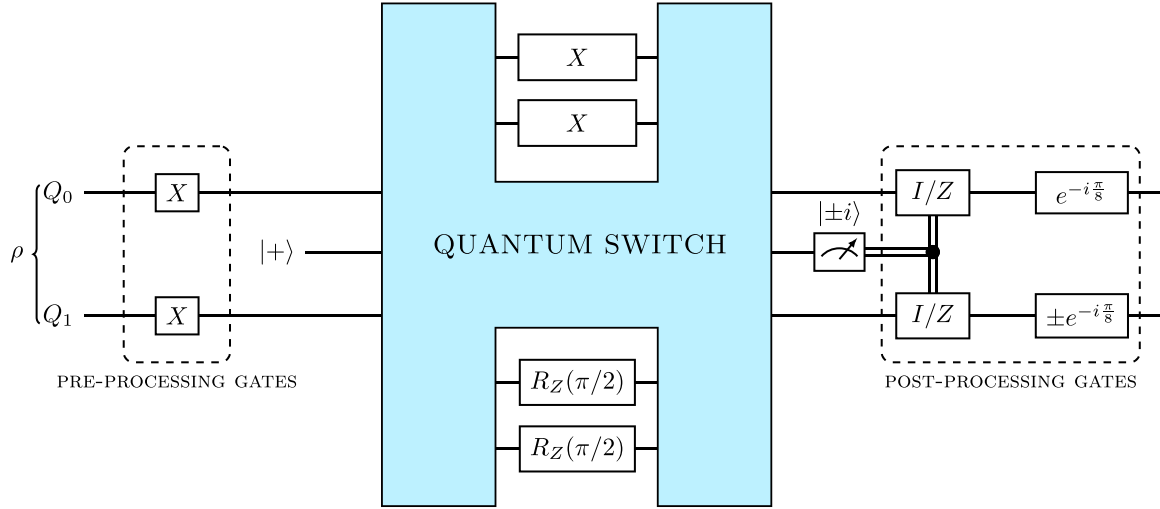


FIG. 8. Abstract representation of the CZ gate realized via superposed orders of single-qubit gates. As in Fig. 6, the ancillary qubit is set in $|+\rangle$ to implement an even superposition of causal orders, and the preprocessing phase is implemented by single-qubit gates $P = X \otimes X$. The single-qubit gates in superposed orders via quantum switch are $A = X \otimes X$ and $B = R_Z(\frac{\pi}{2}) \otimes R_Z(\frac{\pi}{2})$, respectively. Once the ancillary qubit is measured in the basis spanned by $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ and $|-i\rangle = \frac{1}{\sqrt{2}}(-i|0\rangle + |1\rangle)$, qubits Q_0 and Q_1 are postprocessed by $F_- = -e^{-i\frac{\pi}{4}}(Z \otimes Z)$ or $F_+ = e^{-i\frac{\pi}{4}}(I \otimes I)$, depending on whether the ancilla is measured as $|-i\rangle$ or $|+i\rangle$.

where

$$X_d = \sum_{k=0}^{d-1} |k\rangle\langle k+1 \pmod{d}|, \quad (39)$$

$$Z_d = \sum_{k=0}^{d-1} \omega^k |k\rangle\langle k|, \quad (40)$$

and $\omega = e^{i\frac{2\pi}{d}}$. For $d = 2$, these reduce, respectively, to the qubit CNOT and CZ gates. While a comprehensive discussion on higher-dimensional quantum computation lies beyond the main scope of the present work, we provide a possible implementation of Eqs. (37) and (38) via the superposed-order framework in Appendix C. As future work, we will investigate suitable strategies to reduce the number of required quantum switches for realizing controlled unitaries in higher dimensions.

C. Barenco gate

Although CNOT and CZ gates are widely used in quantum computing, they still require additional single-qubit gates to realize an arbitrary unitary gate, i.e., to realize universal quantum computation. Differently, there exists a gate named *Barenco gate*—denoted in the following as $\text{BAR}(\alpha_B, \phi_B, \theta_B)$ —which alone is sufficient for universal quantum computation [40]. In other words, Barenco gate forms by itself a universal set of quantum gates. Formally, the Barenco gate is a controlled rotation gate, which acts on two qubits as

$$\text{BAR}(\alpha_B, \phi_B, \theta_B) = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes e^{i\alpha_B} R_{\mathbf{n}(\phi_B)}(2\theta_B), \quad (41)$$

and is parameterized by the angles $\alpha_B, \phi_B, \theta_B \in [0, 2\pi]$, with $\mathbf{n}(\phi_B) = (\cos(\phi_B), \sin(\phi_B), 0)$. Though its universality, practical implementation of the Barenco gate is known to be highly

challenging [84,85]. The following proposition demonstrates that the $\text{BAR}(\alpha_B, \phi_B, \theta_B)$ gate can be realized for any choice of $\alpha_B, \phi_B, \theta_B$ by using only single-qubit gates in a superposition of orders.

Proposition 3. The $\text{BAR}(\alpha_B, \phi_B, \theta_B)$ gate can be deterministically realized by combining in superposition of orders the following single-qubit gates:

$$A = X \otimes Z, \quad (42)$$

$$B = R_Z\left(\frac{\pi}{2}\right) \otimes R_{\mathbf{n}(\phi_B)}\left(\frac{\pi}{2}\right), \quad (43)$$

where $\mathbf{n}(\phi_B) = (\cos(\phi_B), \sin(\phi_B), 0)$.

Proof. The proof follows from Theorem 1, by recognizing that the $\text{BAR}(\alpha_B, \phi_B, \theta_B)$ gate is given by the decomposition in Eq. (22) for $\alpha = \alpha_B, \theta = -\theta_B$, and $\mathbf{n} = (\cos(\phi_B), \sin(\phi_B), 0) \equiv \mathbf{n}(\phi_B)$, and choosing $\mathbf{n}^\perp = (0, 0, 1)$. ■

As shown in Fig. 9, Proposition 3 proves that the $\text{BAR}(\alpha_B, \phi_B, \theta_B)$ gate is obtained via superposed orders of single-qubit gates as follows. We first preprocess the input state with a sequence of single-qubit gates $P = X \otimes Z$ in accordance with Eq. (26). Then, the resulting state goes into the quantum switch, which processes it according to the gates [Eqs. (42) and (43)]. A measurement of the ancillary qubit $\mathbf{C}$ in a basis spanned by the states [Eqs. (7) and (8)], with $\theta = -\theta_B$, realizes the gates $S_{\pm}^{A,B}(-\theta_B)$ with probability 1/2 in accordance with Eqs. (13) and (14). Finally, depending on the outcome of the measurement, we perform a postprocessing by applying another sequence of single-qubit gates, namely, (1) $F_+(\alpha_B, -\theta_B, \mathbf{n}(\phi_B)) = e^{i\frac{\alpha_B}{2}} R_Z(\alpha_B + \frac{\pi}{2}) \otimes R_{\mathbf{n}(\phi_B)}(\theta_B + \frac{\pi}{2})$, whenever the ancillary qubit is found in the state $|\mu(-\theta_B)\rangle$ or (2) $F_-(\alpha_B, -\theta_B, \mathbf{n}(\phi_B)) = e^{i\frac{\alpha_B}{2}} (R_Z(\alpha_B - \frac{\pi}{2}) \otimes R_{\mathbf{n}(\phi_B)}(\theta_B - \frac{\pi}{2}))$ otherwise. Accordingly, the overall deterministic implementation of the $\text{BAR}(\alpha_B, \phi_B, \theta_B)$ gate via

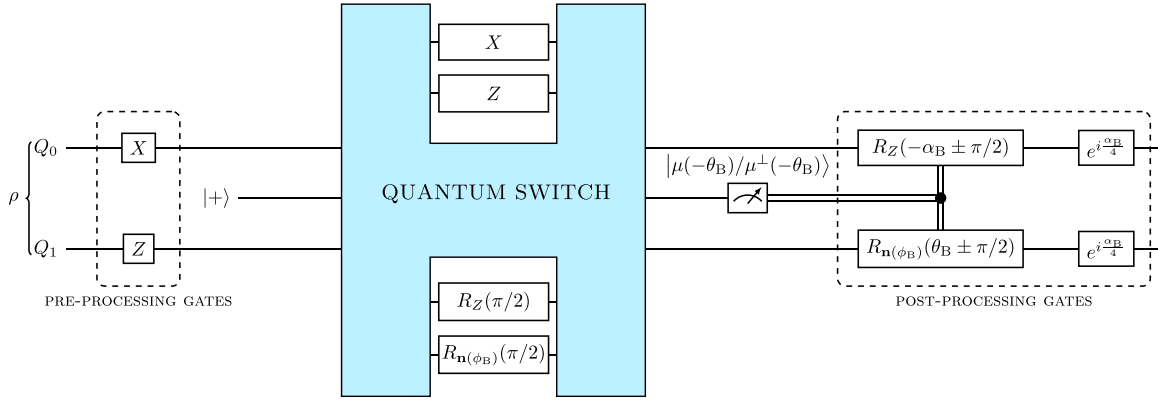


FIG. 9. Abstract representation of the $\text{BAR}(\alpha_B, \phi_B, \theta_B)$ gate realized via superposed orders of single-qubit gates. As in Fig. 6, the ancillary qubit is set in $\omega = |+\rangle$ to implement an even superposition of causal orders, and the preprocessing phase is implemented by single-qubit gates $P = X \otimes Z$. The single-qubit gates in superposed orders via quantum switch are $A = X \otimes Z$ and $B = R_Z(\frac{\pi}{2}) \otimes R_{\mathbf{n}(\phi_B)}(\frac{\pi}{2})$, with $\mathbf{n}(\phi_B) = (\cos(\phi_B), \sin(\phi_B), 0)$. Once the ancillary qubit is measured in the basis spanned by the states [Eqs. (7) and (8)] with $\theta = -\theta_B$, qubits Q_0 and Q_1 are postprocessed by either $F_+ = e^{i\frac{\alpha_B}{2}} (R_Z(\alpha_B + \frac{\pi}{2}) \otimes R_{\mathbf{n}(\phi_B)}(\theta_B + \frac{\pi}{2}))$ or $F_- = e^{i\frac{\alpha_B}{2}} (R_Z(\alpha_B - \frac{\pi}{2}) \otimes R_{\mathbf{n}(\phi_B)}(\theta_B - \frac{\pi}{2}))$, depending on the ancilla qubit measurement result.

superposed orders can be expressed as

$$\text{BAR}(\alpha_B, \phi_B, \theta_B) = e^{i\frac{\alpha_B}{2}} \left(R_Z\left(\alpha_B \pm \frac{\pi}{2}\right) \otimes R_{\mathbf{n}(\phi_B)}\left(\theta_B \pm \frac{\pi}{2}\right) \right) \times \cdot S_{\pm}^{A,B}(-\theta_B)(X \otimes Z). \quad (44)$$

V. CONCLUSIONS

In this paper, we prove that quantum gates in superposed orders via the quantum switch give birth to a paradigm for universal quantum computation. Specifically, the quantum switch enables a framework able to implement any two-qubit controlled gate in a deterministic manner, by exploiting only single-qubit gates in superposition of causal orders. The result proposed here paves the way for unleashing the advantages provided by the engineering of the unconventional quantum propagation phenomena toward a computing model based on higher-order quantum operations. And we do hope that this manuscript can fuel further research—both theoretical and experimental—about the powerful setup enabled by superposed orders of quantum gates for photonic quantum computing.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: PROOF OF LEMMA 1

For $A = \bigotimes_{i=1}^N A_i$ and $B = \bigotimes_{i=1}^N B_i$, the quantum switch implements the unitary gate given in Eq. (5), i.e.,

$$S(A, B) = \frac{1}{2} \left[\left\{ \bigotimes_{i=1}^N A_i, \bigotimes_{i=1}^N B_i \right\} \otimes I_C + \left[\bigotimes_{i=1}^N A_i, \bigotimes_{i=1}^N B_i \right] \otimes Z_C \right]. \quad (A1)$$

Hence, when the input N qubits are in a pure initial state $\rho = |\psi\rangle\langle\psi|$ and the ancillary system C is in the pure state $\omega = |+\rangle\langle+|$, the output of the unitary in Eq. (A1) is equal to

$$S(A, B)(|\psi\rangle \otimes |+\rangle) = \frac{1}{2} \left[\left\{ \bigotimes_{i=1}^N A_i, \bigotimes_{i=1}^N B_i \right\} |\psi\rangle \otimes |+\rangle + \left[\bigotimes_{i=1}^N A_i, \bigotimes_{i=1}^N B_i \right] |\psi\rangle \otimes |-\rangle \right]. \quad (A2)$$

Measurement of the ancillary qubit in the basis spanned by the states [Eqs. (7) and (8)] leaves the input N qubits in one of the two following states:

$$|\psi_+(\theta)\rangle = \left[\cos\left(\frac{\theta}{2}\right) \bigotimes_{i=1}^N A_i B_i + i \sin\left(\frac{\theta}{2}\right) \bigotimes_{i=1}^N B_i A_i \right] |\psi\rangle, \quad (A3)$$

$$|\psi_-(\theta)\rangle = \left[i \sin\left(\frac{\theta}{2}\right) \bigotimes_{i=1}^N A_i B_i + \cos\left(\frac{\theta}{2}\right) \bigotimes_{i=1}^N B_i A_i \right] |\psi\rangle, \quad (A4)$$

with probability $1/2$. It is straightforward to observe that this is equivalent to the implementation of one of the gates in Eqs. (13) and (14), each with probability $1/2$, and hence the proof follows.

APPENDIX B: PROOF OF LEMMA 2

An arbitrary unitary gate U can be represented by a Hermitian operator as $U = e^{iH}$. Accordingly, applying this representation to the CU gate, it results that CU can be expressed as

$$CU = e^{\frac{i}{2}(I-Z) \otimes H}. \quad (B1)$$

For a single-qubit gate U , the corresponding Hermitian operator H can be expanded into the Pauli basis as

$$H = \alpha I + \theta(\mathbf{n} \cdot \boldsymbol{\sigma}), \quad (B2)$$

where $\alpha, \theta \in \mathbb{R}$, $\mathbf{n}$ denotes a real-valued unit vector known as the Bloch vector [69], and $\boldsymbol{\sigma} = (X, Y, Z)$ is a vector of Pauli matrices. Plugging Eq. (B2) into Eq. (B1), after some algebraic manipulations, the CU gate can be decomposed into a sequence of gates as

$$CU = e^{i\frac{\alpha}{2}} R_Z(\alpha) \otimes R_{\mathbf{n}}(-\theta) R_{Z\mathbf{n}}(\theta), \quad (B3)$$

where $R_{\mathbf{n}}(\theta) = \cos(\frac{\theta}{2})I - i\sin(\frac{\theta}{2})(\mathbf{n} \cdot \boldsymbol{\sigma})$ is the rotation operator with respect to the Bloch vector $\mathbf{n}$ defined in Eq. (16), $R_Z(\theta)$ is the rotation operator with respect to $\mathbf{n} = (0, 0, 1)$, and $R_{Z\mathbf{n}}(\theta)$ is the two-qubit rotation gate given in Eq. (17) when the Bloch vector $\tilde{\mathbf{n}}$ is set as $\tilde{\mathbf{n}} = (0, 0, 1)$. Clearly, from Eq. (27), it is easy to recognize that the CU gate is locally equivalent to the two-qubit rotation gate $R_{Z\mathbf{n}}(\theta)$ given in Eq. (17).

From this, let us demonstrate that the two-qubit rotation $R_{Z\mathbf{n}}(\theta)$ is locally equivalent to:

(1) single-qubit gates $A(\mathbf{n}) = A_0 \otimes A_1(\mathbf{n})$ and $B = B_0 \otimes B_1(\mathbf{n})$ combined in superposition of orders via a quantum switch,

(2) preceded by a sequence $P(\mathbf{n}) = P_0 \otimes P_1(\mathbf{n})$ of single-qubit *preprocessing* gates,

(3) followed by a sequence $F(\mathbf{n}) = F_0 \otimes F_1(\mathbf{n})$ of single-qubit *postprocessing* gates

Accordingly, by exploiting Lemma 1, it results that the overall gate implemented by the above-described superposition of orders is either

$$S_+^{A,B}(\theta)P = \cos\left(\frac{\theta}{2}\right)A_0B_0P_0 \otimes A_1B_1P_1 + i\sin\left(\frac{\theta}{2}\right)B_0A_0P_0 \otimes B_1A_1P_1, \quad (B4)$$

$$S_-^{A,B}(\theta)P = i\sin\left(\frac{\theta}{2}\right)A_0B_0P_0 \otimes A_1B_1P_1 + \cos\left(\frac{\theta}{2}\right)B_0A_0P_0 \otimes B_1A_1P_1, \quad (B5)$$

depending on the measurement of the ancillary qubit $\mathbf{C}$ in the basis spanned by vectors (7) and (8), where the parameter θ matches the corresponding parameter in decomposition (B2).

By setting the single-qubit gates as

$$P(\mathbf{n}) = P_0 \otimes P_1(\mathbf{n}) = X \otimes (\mathbf{n}^\perp \cdot \boldsymbol{\sigma}), \quad (B6)$$

$$A(\mathbf{n}) = A_0 \otimes A_1(\mathbf{n}) = X \otimes (\mathbf{n}^\perp \cdot \boldsymbol{\sigma}), \quad (B7)$$

$$B(\mathbf{n}) = B_0 \otimes B_1(\mathbf{n}) = R_Z\left(\frac{\pi}{2}\right) \otimes R_{\mathbf{n}}\left(\frac{\pi}{2}\right), \quad (B8)$$

where $\mathbf{n}^\perp$ denotes the Bloch vector perpendicular to $\mathbf{n}$ (i.e., $\mathbf{n}^\perp \cdot \mathbf{n} = 0$), the operations arising from the causal order AB read as

$$A_0B_0P_0 = XR_Z\left(\frac{\pi}{2}\right)X = R_Z\left(-\frac{\pi}{2}\right), \quad (B9)$$

$$A_1(\mathbf{n})B_1(\mathbf{n})P_1(\mathbf{n}) = (\mathbf{n}^\perp \cdot \boldsymbol{\sigma})R_{\mathbf{n}}\left(\frac{\pi}{2}\right)(\mathbf{n}^\perp \cdot \boldsymbol{\sigma}) = R_{\mathbf{n}}\left(-\frac{\pi}{2}\right), \quad (B10)$$

while the ones arising from the causal order BA are

$$B_0A_0P_0 = R_Z\left(\frac{\pi}{2}\right)XX = R_Z\left(\frac{\pi}{2}\right), \quad (B11)$$

$$B_1(\mathbf{n})A_1(\mathbf{n})P_1(\mathbf{n}) = R_{\mathbf{n}}\left(\frac{\pi}{2}\right)(\mathbf{n}^\perp \cdot \boldsymbol{\sigma})(\mathbf{n}^\perp \cdot \boldsymbol{\sigma}) = R_{\mathbf{n}}\left(\frac{\pi}{2}\right). \quad (B12)$$

By substituting the above equations in Eqs. (B4) and (B5), it results

$$S_+^{A,B}(\theta)P(\mathbf{n}) = \cos\left(\frac{\theta}{2}\right)R_Z\left(-\frac{\pi}{2}\right) \otimes R_{\mathbf{n}}\left(-\frac{\pi}{2}\right) + i\sin\left(\frac{\theta}{2}\right)R_Z\left(\frac{\pi}{2}\right) \otimes R_{\mathbf{n}}\left(\frac{\pi}{2}\right), \quad (B13)$$

$$S_-^{A,B}(\theta)P(\mathbf{n}) = i\sin\left(\frac{\theta}{2}\right)R_Z\left(-\frac{\pi}{2}\right) \otimes R_{\mathbf{n}}\left(-\frac{\pi}{2}\right) + \cos\left(\frac{\theta}{2}\right)R_Z\left(\frac{\pi}{2}\right) \otimes R_{\mathbf{n}}\left(\frac{\pi}{2}\right). \quad (B14)$$

Finally, by setting the single-qubit postprocessing gates $F(\mathbf{n}) = F_0 \otimes F_1(\mathbf{n})$ equal either to

$$F_{\pm}(\mathbf{n}) = F_{\pm,0} \otimes F_{\pm,1}(\mathbf{n}) = R_Z\left(\pm\frac{\pi}{2}\right) \otimes R_{\mathbf{n}}\left(\pm\frac{\pi}{2}\right), \quad (B15)$$

depending on whether the ancillary qubit measurement result is $|\mu(\theta)\rangle$ or $|\mu^\perp(\theta)\rangle$, the following two-qubit rotation is implemented:

$$F_{\pm}(\mathbf{n})S_{\pm}^{A,B}(\theta)P(\mathbf{n}) = R_{Z\mathbf{n}}(\theta). \quad (B16)$$

In Eq. (B16), we exploited the equality $R_Z(\pm\pi) \otimes R_{\mathbf{n}}(\pm\pi) = -Z \otimes (\mathbf{n} \cdot \boldsymbol{\sigma})$. As both $P(\mathbf{n})$ and $F_{\pm}(\mathbf{n})$ are tensor products of single-qubit case, it results that $S_{\pm}^{A,B}(\theta)$ —hence—is locally (single-qubit) equivalent to the two-qubit rotation gate $R_{Z\mathbf{n}}(\theta)$ via Eq. (B16), which in turn is locally equivalent to the CU gate via Eq. (27). Hence, the proof follows.

APPENDIX C: REALIZATION OF SUM_d AND PHASE_d GATES

The gates SUM_d and PHASE_d are locally equivalent [82]:

$$\text{SUM}_d = (I \otimes \mathcal{F}_d^\dagger) \text{PHASE}_d (I \otimes \mathcal{F}_d), \quad (\text{C1})$$

where the discrete Fourier transform is defined by

$$\mathcal{F}_d = \frac{1}{\sqrt{d}} \sum_{k,k'=0}^{d-1} \omega^{kk'} |k\rangle \langle k'|, \quad \omega = e^{2\pi i/d}. \quad (\text{C2})$$

In particular, for $d = 2$, Eq. (C1) reduces to the local equivalence (33) between CNOT and CZ. We thus focus on the implementation of PHASE_d, which, according to Eq. (38), acts as

$$\text{PHASE}_d(|m\rangle \otimes |n\rangle) = \omega^{mn} |m\rangle \otimes |n\rangle. \quad (\text{C3})$$

Let the label of each computational basis state be expressed in binary form as

$$m = \sum_{b=0}^{s-1} m_b 2^b, \quad n = \sum_{c=0}^{s-1} n_c 2^c, \quad (\text{C4})$$

where $s = \lceil \log_2 d \rceil$ is the number of binary digits required to encode d distinct states, and $m_b, n_c \in \{0, 1\}$ denote the b 'th and c 'th binary digits of m and n , respectively. Equivalently, each label can be represented as the binary string $(m_{s-1} \cdots m_1 m_0) \equiv m$, where m_b specifies the value (0 or 1) of the b 'th binary component. Using the binary expansions in Eq. (C4), the phase factor appearing in PHASE_d can then be expressed as

$$\begin{aligned} \omega^{mn} &= e^{i \frac{2\pi}{d} \sum_{b,c=0}^{s-1} 2^{b+c} m_b n_c} \\ &= \prod_{b,c=0}^{s-1} e^{i \phi_{b,c} m_b n_c}, \end{aligned} \quad (\text{C5})$$

where $\phi_{b,c} = \frac{2^{b+c+1}\pi}{d}$. Therefore, each bit pair (b, c) contributes an independent phase whenever $m_b = n_c = 1$. To promote this expression to operator form, we define projectors onto the subspaces where the b 'th bit equals 1:

$$\Pi_b = \sum_{\{m: m_b=1\}} |m\rangle \langle m|. \quad (\text{C6})$$

For a fixed b , they act as identity on the basis states with $m_b = 1$ and annihilate those with $m_b = 0$. In turn, $\Pi_b \otimes \Pi_c$ projects onto all states $|m\rangle \otimes |n\rangle$ for which $m_b = n_c = 1$, contributing to a phase $e^{i\phi_{b,c}}$ as in Eq. (C5). Therefore, PHASE_d can be decomposed as

$$\text{PHASE}_d = \prod_{\substack{b,c=0 \\ 2^{b+c} < d}}^{s-1} e^{i\phi_{b,c} \Pi_b \otimes \Pi_c}, \quad (\text{C7})$$

where the condition $2^{b+c} < d$ reflects that, for dimensions $d = 2^s$ (i.e., when the qudit can be viewed as a register of s qubits), the corresponding phases $\phi_{b,c} = 2^{b+c-s+1}\pi$ lead to $e^{i\phi_{b,c} \Pi_b \otimes \Pi_c} = I$ whenever $b + c \geq s$. Each exponential in Eq. (C7) acts nontrivially only on those two-qudit basis states whose binary digits (m_b, n_c) are both equal to one. Using

$\Pi_b = \frac{1}{2}(I - \tilde{Z}_b)$, with

$$\tilde{Z}_b = \sum_{m=0}^{d-1} (-1)^{m_b} |m\rangle \langle m|, \quad (\text{C8})$$

which acts as a Pauli-Z operator on each pair of states that differ only by the b 'th binary digit, every term in Eq. (C7) can be rewritten as

$$e^{i\phi_{b,c} \Pi_b \otimes \Pi_c} = e^{i \frac{\phi_{b,c}}{4}} (e^{-i \frac{\phi_{b,c}}{4} \tilde{Z}_b} \otimes e^{-i \frac{\phi_{b,c}}{4} \tilde{Z}_c}) e^{i \frac{\phi_{b,c}}{4} \tilde{Z}_b \otimes \tilde{Z}_c}. \quad (\text{C9})$$

Each exponential factor $e^{i \frac{\phi_{b,c}}{4} \tilde{Z}_b \otimes \tilde{Z}_c}$ can be implemented using a single quantum switch. We choose the single-qudit gates as

$$P^{(b,c)} = P_0^{(b,c)} \otimes P_1^{(b,c)} = \tilde{X}_b \otimes \tilde{X}_c, \quad (\text{C10})$$

$$A^{(b,c)} = A_0^{(b,c)} \otimes A_1^{(b,c)} = \tilde{X}_b \otimes \tilde{X}_c, \quad (\text{C11})$$

$$B^{(b,c)} = B_0^{(b,c)} \otimes B_1^{(b,c)} = e^{i \frac{\pi}{4} \tilde{Z}_b} \otimes e^{i \frac{\pi}{4} \tilde{Z}_c}, \quad (\text{C12})$$

where the operator

$$\tilde{X}_b = \sum_{\substack{m=0 \\ m \oplus 2^b < d}} (|m\rangle \langle m \oplus 2^b| + |m \oplus 2^b\rangle \langle m|) + \sum_{m: m \oplus 2^b \geq d} |m\rangle \langle m| \quad (\text{C13})$$

acts as a generalized bit flip on the b 'th binary digit of m , and the $\oplus$ denotes bitwise addition modulo two acting on the binary representation of m ,

$$m \oplus 2^b \equiv (m_{s-1} \cdots m_{b+1} \bar{m}_b m_{b-1} \cdots m_0), \quad (\text{C14})$$

where $\bar{m}_b = 1 - m_b$ indicates that the b 'th binary digit is flipped. Hence, $\tilde{X}_b$ acts as a Pauli-X operator on each pair of states that differ only by the b 'th binary digit, so that

$$\tilde{X}_b \tilde{Z}_b = -\tilde{Z}_b \tilde{X}_b, \quad \tilde{X}_b^2 = I, \quad (\text{C15})$$

so that

$$A^{(b,c)} B^{(b,c)} P^{(b,c)} = e^{-i \frac{\pi}{4} \tilde{Z}_b} \otimes e^{-i \frac{\pi}{4} \tilde{Z}_c}, \quad (\text{C16})$$

$$B^{(b,c)} A^{(b,c)} P^{(b,c)} = e^{i \frac{\pi}{4} \tilde{Z}_b} \otimes e^{i \frac{\pi}{4} \tilde{Z}_c}. \quad (\text{C17})$$

Applying Lemma 1, the outputs of the switch are

$$\begin{aligned} S_+^{A^{(b,c)}, B^{(b,c)}}(\theta) P_{bc} &= \cos\left(\frac{\theta}{2}\right) e^{-i \frac{\pi}{4} \tilde{Z}_b} \otimes e^{-i \frac{\pi}{4} \tilde{Z}_c} \\ &\quad + i \sin\left(\frac{\theta}{2}\right) e^{i \frac{\pi}{4} \tilde{Z}_b} \otimes e^{i \frac{\pi}{4} \tilde{Z}_c}, \end{aligned} \quad (\text{C18})$$

$$\begin{aligned} S_-^{A^{(b,c)}, B^{(b,c)}}(\theta) P_{bc} &= i \sin\left(\frac{\theta}{2}\right) e^{-i \frac{\pi}{4} \tilde{Z}_b} \otimes e^{-i \frac{\pi}{4} \tilde{Z}_c} \\ &\quad + \cos\left(\frac{\theta}{2}\right) e^{i \frac{\pi}{4} \tilde{Z}_b} \otimes e^{i \frac{\pi}{4} \tilde{Z}_c}. \end{aligned} \quad (\text{C19})$$

Setting the postprocessing gates $F_\pm^{(b,c)} = e^{\pm i \frac{\pi}{4} \tilde{Z}_b} \otimes e^{\pm i \frac{\pi}{4} \tilde{Z}_c}$, we obtain

$$\begin{aligned} F_\pm^{(b,c)} S_\pm^{A^{(b,c)}, B^{(b,c)}}(\theta) P^{(b,c)} &= \cos\left(\frac{\theta}{2}\right) I \otimes I \\ &\quad + i \sin\left(\frac{\theta}{2}\right) \tilde{Z}_b \otimes \tilde{Z}_c \\ &= e^{i \frac{\theta}{2} \tilde{Z}_b \otimes \tilde{Z}_c}. \end{aligned} \quad (\text{C20})$$

Comparing this with Eq. (C9), we conclude that each factor in the decomposition can be implemented by measuring the ancillary system **C** in the basis spanned by Eqs. (7) and (8) with $\theta = \frac{\phi_{b,c}}{2}$, so that

$$e^{i\phi_{b,c}\Pi_b\otimes\Pi_c} = e^{i\frac{\phi_{b,c}}{4}}(e^{-i\frac{\phi_{b,c}\mp\pi}{4}}\tilde{Z}_b \otimes e^{-i\frac{\phi_{b,c}\mp\pi}{4}}\tilde{Z}_c) \times S_{\pm}^{A^{(b,c)}, B^{(b,c)}}\left(\frac{\phi_{b,c}}{2}\right)P^{(b,c)}. \quad (C21)$$

Therefore, the overall deterministic implementation of the gates SUM_d and PHASE_d via superposed orders can be expressed as

$$\text{SUM}_d = \prod_{\substack{b,c=0 \\ 2^{b+c} < d}}^{s-1} e^{i\frac{\phi_{b,c}}{4}}(e^{-i\frac{\phi_{b,c}\mp\pi}{4}}\tilde{Z}_b \otimes \mathcal{F}_d^\dagger e^{-i\frac{\phi_{b,c}\mp\pi}{4}}\tilde{Z}_c) \times S_{\pm}^{A^{(b,c)}, B^{(b,c)}}\left(\frac{\phi_{b,c}}{2}\right)(\tilde{X}_b \otimes \tilde{X}_c \mathcal{F}_d), \quad (C22)$$

$$\text{PHASE}_d = \prod_{\substack{b,c=0 \\ 2^{b+c} < d}}^{s-1} e^{i\frac{\phi_{b,c}}{4}}(e^{-i\frac{\phi_{b,c}\mp\pi}{4}}\tilde{Z}_b \otimes e^{-i\frac{\phi_{b,c}\mp\pi}{4}}\tilde{Z}_c) \times S_{\pm}^{A^{(b,c)}, B^{(b,c)}}\left(\frac{\phi_{b,c}}{2}\right)(\tilde{X}_b \otimes \tilde{X}_c), \quad (C23)$$

where each $S_{\pm}^{A^{(b,c)}, B^{(b,c)}}(\frac{\phi_{b,c}}{2})$ is implemented by one use of the quantum switch, so that the total number of required uses is

$$N = \begin{cases} \frac{s(s+1)}{2}, & \text{if } d = 2^s, \\ s^2, & \text{otherwise.} \end{cases} \quad (C24)$$

For $d = 2$, one has $s = 1$, so that $b, c = 0$ and $\phi_{b,c} = \pi$. In this case, the operators $\tilde{Z}_{b/c}$ and $\tilde{X}_{b/c}$ reduce to the standard Pauli operators Z and X , while the discrete Fourier transform becomes $\mathcal{F}_2 = H$. Consequently, Eqs. (C22) and (C23) recover the qubit implementations (31) and (36), respectively.

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- [1] M. Caleffi and A. S. Cacciapuotì, Quantum internet architecture: Unlocking quantum-native routing via quantum addressing, *IEEE Trans. Commun.* **74** 3577 (2026).
 - [2] G. Chiribella and H. Kristjánsson, Quantum Shannon theory with superpositions of trajectories, *Proc. R. Soc. A* **475**, 20180903 (2019).
 - [3] A. A. Abbott, J. Wechs, D. Horsman, M. Mhalla, and C. Branciard, Communication through coherent control of quantum channels, *Quantum* **4**, 333 (2020).
 - [4] H. Kristjánsson, G. Chiribella, S. Salek, D. Ebler, and M. Wilson, Resource theories of communication, *New J. Phys.* **22**, 073014 (2020).
 - [5] G. Chiribella, M. Banik, S. S. Bhattacharya, T. Guha, M. Alimuddin, A. Roy, S. Saha, S. Agrawal, and G. Kar, Indefinite causal order enables perfect quantum communication with zero capacity channels, *New J. Phys.* **23**, 033039 (2021).
 - [6] A. S. Cacciapuotì, C. Pellitteri, J. Illiano, L. d'Avozza, F. Mazza, S. Chen, and M. Caleffi, Quantum data centers: Why entanglement changes everything, *Philos. Trans. R. Soc. A* (2025).
 - [7] C. Pellitteri, M. Caleffi, and A. S. Cacciapuotì, Quantum paths: A quantum walk approach, in *2025 IEEE International Conference on Quantum Computing and Engineering (QCE)*, Albuquerque, NM, USA (IEEE, 2025), pp. 1248–1251.
 - [8] G. Chiribella, Perfect discrimination of no-signalling channels via quantum superposition of causal structures, *Phys. Rev. A* **86**, 040301(R) (2012).
 - [9] O. Oreshkov, F. Costa, and Č. Brukner, Quantum correlations with no causal order, *Nat. Commun.* **3**, 1092 (2012).
 - [10] J. Illiano, M. Caleffi, A. Manzalini, and A. S. Cacciapuotì, Quantum internet protocol stack: A comprehensive survey, *Comput. Netw.* **213**, 109092 (2022).
 - [11] A. S. Cacciapuotì, J. Illiano, S. Koudia, K. Simonov, and M. Caleffi, The quantum internet: Enhancing classical internet services one qubit at a time, *IEEE Netw.* **36**, 6 (2022).
 - [12] A. S. Cacciapuotì, J. Illiano, and M. Caleffi, Quantum internet addressing, *IEEE Netw.* **38**, 104 (2023).
 - [13] L. M. Procopio, A. Moqanaki, M. Araújo, F. Costa, I. Alonso Calafell, E. G. Dowd, D. R. Hamel, L. A. Rozema, Č. Brukner, and P. Walther, Experimental superposition of orders of quantum gates, *Nat. Commun.* **6**, 7913 (2015).
 - [14] G. Rubino, L. A. Rozema, A. Feix, M. Araújo, J. M. Zeuner, L. M. Procopio, Č. Brukner, and P. Walther, Experimental verification of an indefinite causal order, *Sci. Adv.* **3**, e1602589 (2017).
 - [15] K. Goswami, C. Giarmatzis, M. Kewming, F. Costa, C. Branciard, J. Romero, and A. G. White, Indefinite causal order in a quantum switch, *Phys. Rev. Lett.* **121**, 090503 (2018).
 - [16] Y. Guo, X.-M. Hu, Z.-B. Hou, H. Cao, J.-M. Cui, B.-H. Liu, Y.-F. Huang, C.-F. Li, G.-C. Guo, and G. Chiribella, Experimental transmission of quantum information using a superposition of causal orders, *Phys. Rev. Lett.* **124**, 030502 (2020).
 - [17] M. Caleffi and A. S. Cacciapuotì, Quantum switch for the quantum internet: Noiseless communications through noisy channels, *IEEE J. Sel. Areas Commun.* **38**, 575 (2020).
 - [18] D. Chandra, M. Caleffi, and A. S. Cacciapuotì, The entanglement-assisted communication capacity over quantum trajectories, *IEEE Trans. Wireless Commun.* **21**, 3632 (2021).
 - [19] M. Caleffi, K. Simonov, and A. S. Cacciapuotì, Beyond Shannon limits: Quantum communications through quantum paths, *IEEE J. Sel. Areas Commun.* **41**, 2707 (2023).
 - [20] S. Koudia, A. S. Cacciapuotì, and M. Caleffi, Deterministic generation of multipartite entanglement via causal activation in the quantum internet, *IEEE Access* **11**, 73863 (2023).
 - [21] Y. Chen and Y. Hasegawa, Fundamental trade-off relation in probabilistic entanglement generation, *arXiv:2112.03233*.
 - [22] D. Felce and V. Vedral, Quantum refrigeration with indefinite causal order, *Phys. Rev. Lett.* **125**, 070603 (2020).
 - [23] T. Guha, M. Alimuddin, and P. Parashar, Thermodynamic advancement in the causally inseparable occurrence of thermal maps, *Phys. Rev. A* **102**, 032215 (2020).
 - [24] G. Chiribella, M. Wilson, and H. F. Chau, Quantum and classical data transmission through completely depolarizing channels in a superposition of cyclic orders, *Phys. Rev. Lett.* **127**, 190502 (2021).

- [25] K. Simonov, G. Francica, G. Guarnieri, and M. Paternostro, Work extraction from coherently activated maps via quantum switch, *Phys. Rev. A* **105**, 032217 (2022).
- [26] S. Milz, J. Bavaresco, and G. Chiribella, Resource theory of causal connection, *Quantum* **6**, 788 (2022).
- [27] A. G. Maity and S. Bhattacharya, Activating information backflow with the assistance of quantum switch, *J. Phys. A: Math. Theor.* **57**, 215302 (2024).
- [28] S. Mukherjee, B. Mallick, S. Yanamandra, S. Bhattacharya, and A. G. Maity, Interplay between the Hilbert-Space dimension of a control system and the memory induced by a quantum switch, *Phys. Rev. A* **110**, 042624 (2024).
- [29] K. Simonov, S. Roy, T. Guha, Z. Zimborás, and G. Chiribella, Activation of thermal states by coherently controlled thermalization processes, *New J. Phys.* **27**, 074502 (2025).
- [30] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, Beyond quantum computers, *arXiv:0912.0195v1*.
- [31] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, Quantum computations without definite causal structure, *Phys. Rev. A* **88**, 022318 (2013).
- [32] J. Wechs, H. Dourdent, A. A. Abbott, and C. Branciard, Quantum circuits with classical versus quantum control of causal order, *PRX Quantum* **2**, 030335 (2021).
- [33] T. Colnaghi, G. M. D'Ariano, S. Facchini, and P. Perinotti, Quantum computation with programmable connections between gates, *Phys. Lett. A* **376**, 2940 (2012).
- [34] M. Araújo, F. Costa, and Č. Brukner, Computational advantage from quantum-controlled ordering of gates, *Phys. Rev. Lett.* **113**, 250402 (2014).
- [35] M. M. Taddei, R. V. Nery, and L. Aolita, Quantum superpositions of causal orders as an operational resource, *Phys. Rev. Res.* **1**, 033174 (2019).
- [36] M. J. Renner and Č. Brukner, Computational advantage from a quantum superposition of qubit gate orders, *Phys. Rev. Lett.* **128**, 230503 (2022).
- [37] J. Escandón-Monardes, A. Delgado, and S. P. Walborn, Practical computational advantage from the quantum switch on a generalized family of promise problems, *Quantum* **7**, 945 (2023).
- [38] W.-Q. Liu, Z. Meng, B.-W. Song, J. Li, Q.-Y. Wu, X.-X. Chen, J.-Y. Hong, A.-N. Zhang, and Z.-Q. Yin, Experimentally demonstrating indefinite causal order algorithms to solve the generalized Deutsch's problem, *Adv. Quantum Technol.* **7**, 2400181 (2024).
- [39] M. M. Taddei, J. Cariñe, D. Martínez, T. García, N. Guerrero, A. A. Abbott, M. Araújo, C. Branciard, E. S. Gómez, S. P. Walborn, L. Aolita, and G. Lima, Computational advantage from quantum superposition of multiple temporal orders of photonic gates, *PRX Quantum* **2**, 010320 (2021).
- [40] A. Barenco, A universal two-bit gate for quantum computation, *Proc. R. Soc. Lond. A* **449**, 679 (1995).
- [41] V. Vilasini and R. Renner, Fundamental limits for realizing quantum processes in spacetime, *Phys. Rev. Lett.* **133**, 080201 (2024).
- [42] O. Oreshkov, Time-delocalized quantum subsystems and operations: On the existence of processes with indefinite causal structure in quantum mechanics, *Quantum* **3**, 206 (2019).
- [43] L. M. Procopio, F. Delgado, M. Enríquez, N. Belabas, and J. A. Levenson, Sending classical information via three noisy channels in superposition of causal orders, *Phys. Rev. A* **101**, 012346 (2020).
- [44] G. Rubino, L. A. Rozema, F. Massa, M. Araújo, M. Zych, Č. Brukner, and P. Walther, Experimental entanglement of temporal orders, *Quantum* **6**, 621 (2022).
- [45] T. Strömberg, P. Schiavsky, R. W. Peterson, M. T. Quintino, and P. Walther, Demonstration of a quantum switch in a Sagnac configuration, *Phys. Rev. Lett.* **131**, 060803 (2023).
- [46] K. Goswami, Y. Cao, G. A. Paz-Silva, J. Romero, and A. G. White, Increasing communication capacity via superposition of order, *Phys. Rev. Res.* **2**, 033292 (2020).
- [47] A. Vanrietvelde and G. Chiribella, Universal control of quantum processes using sector-preserving channels, *Quantum Inf. Comput.* **21**, 1320 (2021).
- [48] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Experimental realization of any discrete unitary operator, *Phys. Rev. Lett.* **73**, 58 (1994).
- [49] J. Wang, F. Sciarrino, A. Laing, and M. G. Thompson, Integrated photonic quantum technologies, *Nat. Photon.* **14**, 273 (2020).
- [50] E. Knill, R. Laflamme, and G. J. Milburn, A scheme for efficient quantum computation with linear optics, *Nature (London)* **409**, 46 (2001).
- [51] X.-H. Bao, T.-Y. Chen, Q. Zhang, J. Yang, H. Zhang, T. Yang, and J.-W. Pan, Optical nondestructive controlled-not gate without using entangled photons, *Phys. Rev. Lett.* **98**, 170502 (2007).
- [52] J. Zeuner, A. N. Sharma, M. Tillmann, R. Heilmann, M. Gräfe, A. Moqanaki, A. Szameit, and P. Walther, Integrated-optics heralded controlled-not gate for polarization-encoded qubits, *npj Quantum Inf.* **4**, 13 (2018).
- [53] W.-Q. Liu, H.-R. Wei, and L.-C. Kwek, Low-cost Fredkin gate with auxiliary space, *Phys. Rev. Appl.* **14**, 054057 (2020).
- [54] J.-P. Li, X. Gu, J. Qin, D. Wu, X. You, H. Wang, C. Schneider, S. Höfling, Y.-H. Huo, C.-Y. Lu, N.-L. Liu, L. Li, and J.-W. Pan, Heralded nondestructive quantum entangling gate with single-photon sources, *Phys. Rev. Lett.* **126**, 140501 (2021).
- [55] W.-Q. Liu and H.-R. Wei, Linear optical universal quantum gates with higher success probabilities, *Adv. Quantum Technol.* **6**, 2300009 (2023).
- [56] F. Pegoraro, P. Held, J. Lammers, B. Brecht, and C. Silberhorn, Demonstration of a photonic time-multiplexed C-NOT gate, *arXiv:2412.02478*.
- [57] R. Okamoto, J. L. O'Brien, H. F. Hofmann, and S. Takeuchi, Realization of a Knill-Laflamme-Milburn controlled-not photonic quantum circuit combining effective optical nonlinearities, *Proc. Natl. Acad. Sci. USA* **108**, 10067 (2011).
- [58] S. Slussarenko and G. J. Pryde, Photonic quantum information processing: A concise review, *Appl. Phys. Rev.* **6**, 041303 (2019).
- [59] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Quantum circuit architecture, *Phys. Rev. Lett.* **101**, 060401 (2008).
- [60] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Theoretical framework for quantum networks, *Phys. Rev. A* **80**, 022339 (2009).

- [61] A. Bisio and P. Perinotti, Theoretical framework for higher-order quantum theory, *Proc. R. Soc. A* **475**, 20180706 (2019).
- [62] L. Apadula, A. Bisio, and P. Perinotti, No-signalling constraints quantum computation with indefinite causal structure, *Quantum* **8**, 1241 (2024).
- [63] K. Goswami and J. Romero, Experiments on quantum causality, *AVS Quantum Sci.* **2**, 037101 (2020).
- [64] H. Cao, N.-N. Wang, Z. Jia, C. Zhang, Y. Guo, B.-H. Liu, Y.-F. Huang, C.-F. Li, and G.-C. Guo, Quantum simulation of indefinite causal order induced quantum refrigeration, *Phys. Rev. Res.* **4**, L032029 (2022).
- [65] X. Nie, X. Zhu, K. Huang, K. Tang, X. Long, Z. Lin, Y. Tian, C. Qiu, C. Xi, X. Yang, J. Li, Y. Dong, T. Xin, and D. Lu, Experimental realization of a quantum refrigerator driven by indefinite causal orders, *Phys. Rev. Lett.* **129**, 100603 (2022).
- [66] M. Antesberger, M. T. Quintino, P. Walther, and L. A. Rozema, Higher-order process matrix tomography of a passively-stable quantum switch, *PRX Quantum* **5**, 010325 (2024).
- [67] L. A. Rozema, T. Strömberg, H. Cao, Y. Guo, B.-H. Liu, and P. Walther, Experimental aspects of indefinite causal order in quantum mechanics, *Nat. Rev. Phys.* **6**, 483 (2024).
- [68] N. Gao, D. Li, A. Mishra, J. Yan, K. Simonov, and G. Chiribella, Measuring incompatibility and clustering quantum observables with a quantum switch, *Phys. Rev. Lett.* **130**, 170201 (2023).
- [69] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th ed. (Cambridge University Press, Cambridge, 2011).
- [70] The results obtained in what follows can be straightforwardly extended to arbitrary (mixed) state ρ by considering the action of $S(A, B)$ on it as $S(A, B)\rho S^\dagger(A, B)$.
- [71] T. van der Lugt, J. Barrett, and G. Chiribella, Device-independent certification of indefinite causal order in the quantum switch, *Nat. Commun.* **14**, 5811 (2023).
- [72] We note that the notion of equivalence given in Definition 3, also referred to in literature as *LU equivalence* [73–75], restricts the allowed unitary operators to tensor product of single-qubit unitary gates only. The rationale for this constraint lies in the aim of enabling universal quantum computing via superposed orders of single-qubits gates.
- [73] J. Zhang, J. Vala, S. Sastry, and K. B. Whaley, Geometric theory of nonlocal two-qubit operations, *Phys. Rev. A* **67**, 042313 (2003).
- [74] M. Musz, M. Kuś, and K. Życzkowski, Unitary quantum gates, perfect entanglers, and unistochastic maps, *Phys. Rev. A* **87**, 022111 (2013).
- [75] Y. Shen, L. Chen, and L. Yu, Classification of Schmidt-rank-two multipartite unitary gates by singular number, *J. Phys. A: Math. Theor.* **55**, 465302 (2022).
- [76] M. Caleffi, M. Amoretti, D. Ferrari, J. Illiano, A. Manzalini, and A. S. Cacciapuoti, Distributed quantum computing: A survey, *Comput. Netw.* **254**, 110672 (2024).
- [77] A. Hashim, R. Rines, V. Omole, R. K. Naik, J. M. Kreikebaum, D. I. Santiago, F. T. Chong, I. Siddiqi, and P. Gokhale, Optimized swap networks with equivalent circuit averaging for qaoa, *Phys. Rev. Res.* **4**, 033028 (2022).
- [78] C. G. Almudéver, A quantum computer architecture perspective: Executing quantum applications on real quantum processors, in *Proceedings of SPIE 11918, Photonics for Quantum* (2020), p. 1191809.
- [79] C. G. Almudever and E. Alarcon, Structured optimized architecting of full-stack quantum systems in the NISQ era, in *2021 Design, Automation & Test in Europe Conference & Exhibition (DATE)* (Grenoble, France, 2021), pp. 762–767.
- [80] C. G. Almudever, R. Wille, F. Sebastiano, N. Haider and E. Alarcon, From Designing Quantum Processors to Large-Scale Quantum Computing Systems, in *2024 Design, Automation & Test in Europe Conference & Exhibition (DATE)* (Valencia, Spain, 2024), pp. 1–10.
- [81] D. Gottesman, Fault-tolerant quantum computation with higher-dimensional systems, *Chaos Solitons Frac.* **10**, 1749 (1999).
- [82] J. Daboul, X. Wang, and B. C. Sanders, Quantum gates on hybrid qudits, *J. Phys. A: Math. Gen.* **36**, 2525 (2003).
- [83] Y. Wang, Z. Hu, B. C. Sanders, and S. Kais, Qudits and high-dimensional quantum computing, *Front. Phys.* **8**, 589504 (2020).
- [84] C. P. Williams, *Explorations in Quantum Computing* (Springer, London, 2011), p. 93.
- [85] X.-F. Shi, Universal Barenco quantum gates via a tunable non-collinear interaction, *Phys. Rev. A* **97**, 032310 (2018).