

Inverse current in coupled transport: A quantum thermodynamic model

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The recent discovery of inverse current in coupled transport (ICC) in classical systems [Wang *et al.*, *Phys. Rev. Lett.* **124**, 110607 (2020)]—where an induced current flows opposite to two mutually parallel thermodynamic forces, yet remains consistent with the second law of thermodynamics—reveals a striking and counterintuitive transport phenomenon. Using an exactly solvable model of strongly coupled quantum dots, we develop a thermodynamic framework to describe the ICC phenomenon at the quantum level. By systematically connecting the microscopic and macroscopic formulations of the entropy production rate in terms of appropriate entropic biases and entropic fluxes, our analysis identifies the conditions under which a *genuine* ICC effect can arise in quantum thermal transport and highlights potential applications in autonomous quantum engines and refrigerators.

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I. INTRODUCTION

Force and flux are two central thermodynamic quantities, interlinked through a cause-and-effect relationship. A thermodynamic force ($\mathcal{F}$) engenders a response in the form of a flux (J) within the system, causing a departure from equilibrium. For a single driving force, the resulting flux always aligns with the direction of the applied force, establishing a new equilibrium, consistent with the positive entropy production rate $\dot{\Sigma} = J\mathcal{F} \geq 0$ [1,2]. The situation changes significantly in the presence of multiple force-flux pairs. For example, a thermal (energy) force $\mathcal{F}_E$ and a particle force $\mathcal{F}_N$ can drive both particle (J_N) and energy (J_E) currents. Consequently, the rate of entropy production near equilibrium is expressed as $\dot{\Sigma} = J_E\mathcal{F}_E + J_N\mathcal{F}_N$, where $J_E(J_N)$ is the thermodynamic flux conjugate to the thermodynamic force $\mathcal{F}_E(\mathcal{F}_N)$. This could lead to the possibility of *inverse current in coupled transport* (ICC) when both forces are *mutually parallel*, yet one of the induced currents flows *against* both forces. Although the sign of a current is a matter of convention, for ICC to occur, a current must flow simultaneously against both *mutually parallel* thermodynamic forces. Consequently, we set without loss any generality, both $\mathcal{F}_E > 0$ and $\mathcal{F}_N > 0$ in $\dot{\Sigma} = J_E\mathcal{F}_E + J_N\mathcal{F}_N$, so that a negative current signals ICC [3]. An immediate consequence is that simultaneous ICC in both fluxes is impossible, as it would violate the second law of thermodynamics. This is the result of Wang *et al.* [3], who recently demonstrated the effect of ICC in classical systems. Although the phenomenon appears highly counterintuitive, it is not forbidden, as long as the overall entropy production rate remains non-negative. Following the results of Wang *et al.* [3], multiple attempts

have been made to extend the concept of the ICC to quantum systems with coupled quantum dots (QDs) [4,5]. Although Refs. [4,5] numerically obtained the ICC effect, a rigorous analytical formulation in terms of well-defined thermodynamic forces and their corresponding fluxes is still missing for such coupled QD systems.

For this purpose, we propose an exactly solvable model of coupled QDs to demonstrate the ICC effect. Though our model is based on a three-terminal setup, we show that it can be effectively reduced to a *two-terminal configuration*, essential for *uniquely* identifying coupled transport driven by pairs of thermodynamic forces and their conjugate fluxes (see Fig. 3). However, the three-terminal structure remains indispensable because the QDs are capacitively coupled. Consequently, to generate both energy and particle currents within a minimal setup, one of the QDs must be simultaneously coupled to multiple thermal reservoirs. We further emphasize that interdot interactions play a crucial role in establishing ICC behavior under the near-equilibrium condition. Most importantly, we derive the conditions for genuine ICC in both energy and (spin-polarized) particle currents, fully consistent with the second law of thermodynamics.

The structure of the paper is as follows: Section II outlines our model based on coupled QDs and presents the system dynamics. In Sec. III, we analyze steady-state currents and examine entropy production, including both macroscopic and microscopic expressions for the entropic forces and fluxes. Section IV summarizes the differences between the ICC phenomenon and the normal cross effects observed in conventional QD thermoelectric setups. In Sec. V, we present our results and discuss genuine ICC in energy and spin-polarized particle currents. Finally, we conclude in Sec. VI.

II. MODEL AND SYSTEM DYNAMICS

Our model consists of two QDs that are strongly and capacitively coupled, labeled upper (QD_u) and bottom (QD_b), as illustrated in Fig. 1(a). The dots interact via a long-range Coulomb force [6–8], preventing particle exchange while

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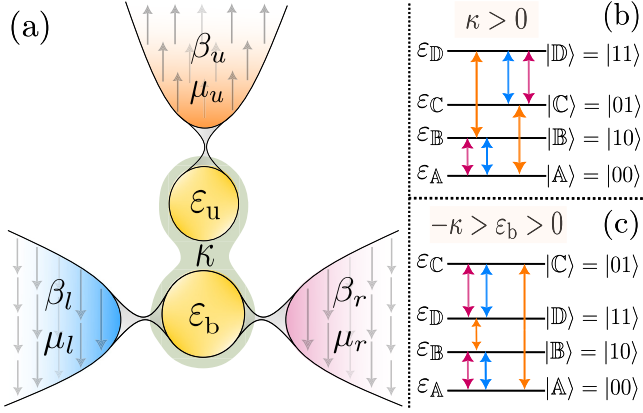


FIG. 1. (a) Schematic diagram of the three-terminal Coulomb-coupled QDs. The energy level diagram of the eigenstates of coupled QDs for (b) $\kappa > 0$ and (c) $\kappa < 0$; $|\kappa| > \varepsilon_b$, i.e., $-\kappa > \varepsilon_b > 0$. For specific values of bath parameters $\{\beta, \mu\}$, the above model reduces to a coupled transport phenomenon with a pair of thermodynamic forces (Fig. 3).

allowing energy exchange due to Coulomb interaction (κ_c). Since interdot particle hopping is restricted, generating particle current requires at least one QD to be simultaneously coupled to more than one reservoir [8–10]. We assume that QD_b is tunnel coupled to two fermionic leads, labeled as *l* (left) and *r* (right), respectively. The other dot, QD_u, is tunnel coupled to only a single reservoir, labeled as *u*. For spinless electrons, interdot interaction is always positive; however, we extend our analysis to spin-polarized [6,11] electrons, where we assume that QD_b is coupled to spin-down ($\downarrow$) fermionic reservoirs, and QD_u to spin-up ($\uparrow$) one [Fig. 1(a)]. The Hamiltonian of the coupled QD system is given by

$$H_S = \varepsilon_b \mathcal{N}_{b\downarrow} + \varepsilon_u \mathcal{N}_{u\uparrow} + \kappa_c \mathcal{N}_{b\downarrow} \mathcal{N}_{u\uparrow} + \kappa_s \sigma_{b\downarrow}^z \sigma_{u\uparrow}^z. \quad (1)$$

In Eq. (1), $\varepsilon_\alpha > 0$ ($\alpha = u, b$) denotes the single-particle energy level of the α th QD, which are taken to be positive without loss of generality. Since the electron density in the dots is low, the occupancy limits to either zero or one [6,7,12,13]. Thus, eigenstates of QD_{b(u)} are $|0\rangle$ and $|\downarrow\rangle$ ($|\uparrow\rangle$), with energy eigenvalues 0 and ε_b (ε_u), respectively. The number operators are $\mathcal{N}_{b\downarrow} = d_{b\downarrow}^\dagger d_{b\downarrow}$ and $\mathcal{N}_{u\uparrow} = d_{u\uparrow}^\dagger d_{u\uparrow}$, where $d_{\alpha\sigma}^\dagger$ ($d_{\alpha\sigma}$) represent the electron creation and annihilation operators with spin $\sigma = \{\uparrow, \downarrow\}$, obeying $\{d_{\alpha\sigma}, d_{\alpha'\sigma'}^\dagger\} = \delta_{\alpha\alpha'} \delta_{\sigma\sigma'}$. The spin-spin interaction term [14] in Eq. (1) is expressed as $\sigma_{b\downarrow}^z = 1 - 2\mathcal{N}_{b\downarrow}$ and $\sigma_{u\uparrow}^z = 2\mathcal{N}_{u\uparrow} - 1$, where the spin operator σ^z satisfies the relations $\sigma_{b\downarrow}^z |\downarrow\rangle = -1|\downarrow\rangle$ and $\sigma_{u\uparrow}^z |\uparrow\rangle = +1|\uparrow\rangle$.

With strong capacitive coupling [6,7,11], H_S is diagonal in the tensor product eigenbasis of the individual QD number operators. Thus, four eigenstates $\{|0\rangle, |\downarrow\rangle\} \otimes \{|0\rangle, |\uparrow\rangle\}$ are labeled by $|A\rangle = |00\rangle$, $|B\rangle = |\downarrow 0\rangle$, $|C\rangle = |0 \uparrow\rangle$, and $|D\rangle = |\downarrow \uparrow\rangle$ with corresponding energies (ε_i , $i = A, B, C, D$): $\varepsilon_A = 0$, $\varepsilon_B = \varepsilon_b$, $\varepsilon_C = \varepsilon_u$, and $\varepsilon_D = \varepsilon_b + \varepsilon_u + \kappa$ [Figs. 1(b) and 1(c)]. We assume, without loss of generality, that $\varepsilon_b < \varepsilon_u$. The total interaction, $\kappa = \kappa_c - \kappa_s$, can be either attractive ($\kappa < 0$) or repulsive ($\kappa > 0$), while κ_c and κ_s are both positive. If $\kappa < 0$ and $|\kappa| > \varepsilon_b$, the energy states $|C\rangle$ and $|D\rangle$ interchange their

positions [Fig. 1(c)]. Although an isolated electron produces a repulsive Coulomb potential, if it is embedded in a medium that flips the sign of its Coulomb potential, it could become attractive to other electrons [15–18]. Hamo *et al.* [15] recently demonstrated such attractions in a carbon-nanotube double QD coupled to a charge qubit. In their setup, the double QD and the qubit are fabricated on separate microchips and aligned perpendicularly. When the qubit is brought sufficiently close, it induces attraction between electrons in the double QD. The oscillating polarization field of the qubit “dresses” the electron potentials, favoring doubly occupied states over singly occupied ones [15–18], as shown in Fig. 1(c). Owing to its high degree of tunability, the above platform is well suited for realizing ICC in coupled QDs, as discussed in Sec. V.

Finally, electron (fermionic) reservoirs are characterized by temperature and chemical potential. The Hamiltonian for the λ th spin-polarized bath with spin $\sigma = \{\uparrow, \downarrow\}$ is defined as $H_B^\lambda \equiv H_B^{\lambda\sigma} = \sum_k (\epsilon_k^{\lambda\sigma} - \mu_{\lambda\sigma}) c_{\lambda\sigma k}^\dagger c_{\lambda\sigma k}$, where $\epsilon_k^{\lambda\sigma}$ is the energy of the noninteracting electrons, $\mu_{\lambda\sigma}$ is the chemical potential, and $c^\dagger$ (c) are creation (annihilation) operators, obeying anticommutation relation [19–24]. Finally, QDs are weakly coupled to the reservoirs, allowing sequential tunneling [6,7,13,25–27], where one QD interacts with one lead at a time. The tunnel-coupled Hamiltonians characterized by the coupling constants $t_k^{\alpha\sigma\lambda}$ are given by

$$H_T^{b\downarrow l(r)} = \hbar \sum_k [t_k^{b\downarrow l(r)} c_{l(r)\downarrow k}^\dagger d_{b\downarrow} + t_k^{b\downarrow l(r)*} d_{b\downarrow}^\dagger c_{l(r)\downarrow k}],$$

$$H_T^{u\uparrow u} = \hbar \sum_k [t_k^{u\uparrow r} c_{u\uparrow k}^\dagger d_{u\uparrow} + t_k^{u\uparrow u*} d_{u\uparrow}^\dagger c_{u\uparrow k}], \quad (2)$$

where $H_T^{b\downarrow l(r)}$ and $H_T^{u\uparrow u}$ describe the interaction between QD_b and QD_u with reservoirs *l* (*r*) via $\downarrow$ electrons and reservoir *u* via $\uparrow$ electrons, respectively. Coulomb blockade and the sequential tunneling approximation prohibit transitions $|B\rangle \leftrightarrow |C\rangle$ and $|A\rangle \leftrightarrow |D\rangle$, allowing four allowed transitions. Reservoirs *l*, *r* control $|A\rangle \leftrightarrow |B\rangle$, and $|C\rangle \leftrightarrow |D\rangle$, while *u* governs $|A\rangle \leftrightarrow |C\rangle$ and $|B\rangle \leftrightarrow |D\rangle$ as designated in Figs. 1(b) and 1(c). The transition energies are defined as $\omega_{ij} = \varepsilon_j - \varepsilon_i$, with $\omega_{AB} = \varepsilon_b$, $\omega_{AC} = \varepsilon_u$, $\omega_{CD} = \varepsilon_b + \kappa$, and $\omega_{BD} = \varepsilon_u + \kappa$ [Figs. 1(b) and 1(c)].

The reduced state of the composite system is represented by the density matrix $\rho_s(t) = \text{Tr}_B[\rho_{\text{tot}}(t)]$, where $\rho_{\text{tot}}(t)$ is the total density matrix of the system and bath combined. Using strong-coupling formalism, the time evolution of $\rho_s(t)$ is governed by the Lindblad master equation (LME) under the Born, Markov, and Secular (BMS) approximations [8,28] (Appendix A):

$$\frac{d}{dt} \rho_s(t) = \sum_\lambda \mathcal{L}_\lambda[\rho_s(t)], \quad \lambda = l, r, u. \quad (3)$$

It is worth noting that the strong-coupling formalism pertains to the interaction between QDs, while we still assume the weak coupling between the system and its environment. This validates the use of the BMS approximation to derive Eq. (3) based on the eigenstates of H_S involving both QDs. Consequently, the dissipation of each QD depends not only on its bath coupling(s) but also on the interdot interactions, which

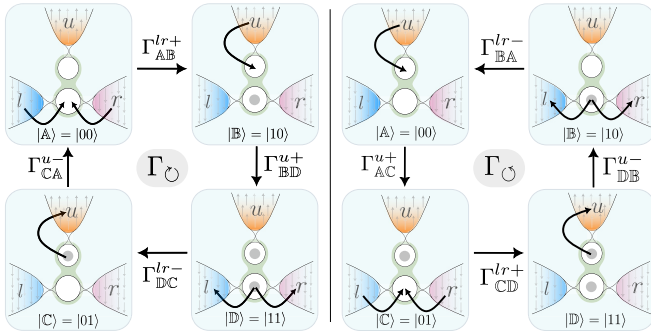


FIG. 2. Schematic representation of the clockwise (left) and anticlockwise (right) transition cycles, induced by energy flow from one bath to another, associated with the particle exchange between QDs and the coupled reservoirs. For the left (clockwise) cycle, the transition starts from the state $|A\rangle = |00\rangle$ (where both QDs are vacant) to the state $|B\rangle = |10\rangle$ (where the bottom QD is occupied), with a net transition rate Γ_{AB}^{lr+} (indicated above the arrow), induced by both the l and r reservoirs. Next, the system transitions from $|B\rangle = |10\rangle$ to $|D\rangle = |11\rangle$, where a spin-up electron enters the upper QD from reservoir u with a net transition rate Γ_{BD}^{u+} . The electron then leaves the bottom QD to either the l or r reservoir with rate Γ_{DC}^{lr-} , changing the state from $|D\rangle = |11\rangle$ to $|C\rangle = |01\rangle$. Finally, the cycle closes when the electron in the upper QD leaves with rate Γ_{CA}^{u-} , changing the state from $|C\rangle = |01\rangle$ back to $|A\rangle = |00\rangle$. A similar sequence occurs for the right transition cycle, but in the reverse direction. Finally, at steady state, all transition rates of the clockwise (anticlockwise) cycles are equal to each other and represented by $\Gamma_{\odot}(\Gamma_{\ominus})$ [cf. Eq. (16)].

are crucial for accurately modeling heat flow across the entire system [29]. The Lindblad superoperator $\mathcal{L}_\lambda[\rho_s(t)]$ in Eq. (3) is defined as

$$\begin{aligned} \mathcal{L}_\lambda[\rho_s(t)] = & \sum_{\{\omega_\alpha\} > 0} \left\{ \gamma_\lambda(\omega_\alpha) f_\lambda^+(\omega_\alpha) \left[d_{\alpha\sigma}^\dagger(\omega_\alpha) \rho_s d_{\alpha\sigma}(\omega_\alpha) \right. \right. \\ & - \frac{1}{2} \{ \rho_s, d_{\alpha\sigma}(\omega_\alpha) d_{\alpha\sigma}^\dagger(\omega_\alpha) \} \\ & + \gamma_\lambda(\omega_\alpha) f_\lambda^-(\omega_\alpha) \left[d_{\alpha\sigma}(\omega_\alpha) \rho_s d_{\alpha\sigma}^\dagger(\omega_\alpha) \right. \\ & \left. \left. - \frac{1}{2} \{ \rho_s, d_{\alpha\sigma}^\dagger(\omega_\alpha) d_{\alpha\sigma}(\omega_\alpha) \} \right] \right\}. \end{aligned} \quad (4)$$

The bare electron transfer rate $\gamma_\lambda(\omega_\alpha)$ between reservoir λ and QD $_\alpha$ is given by Fermi's golden rule $\gamma_\lambda(\omega_\alpha) \equiv \gamma_{\lambda\sigma}(\omega_\alpha) = 2\pi \sum_k |t_k^{\alpha\sigma\lambda}|^2 \delta(\omega_\alpha - \epsilon_k^{\lambda\sigma})$, where ω_α is the energy associated with transitions between QD $_\alpha$ and its coupled lead. For simplicity, we write $\gamma_{l(r)}(\omega_b) \equiv \gamma_{l(r)}$ and $\gamma_u(\omega_u) \equiv \gamma_u$, where $\omega_b = \{\omega_{AB}, \omega_{CD}\}$ and $\omega_u = \{\omega_{AC}, \omega_{BD}\}$, for excitation (particle is entering into the system), and similarly for deexcitation (particle leaving the system) (Fig. 2) [10]. To simplify calculations without relying on the specific ordering of system eigenenergies, we define the Fermi distribution function (FDF), $f_\lambda^\pm(\omega_{ij})$ governing transitions $|i\rangle \rightarrow |j\rangle$. The “+” sign represents excitation (particle entering the system), while “−” indicates deexcitation (particle leaving the system) [10], which is explicitly evident from Fig. 2. This notation is independent of eigenstate arrangements and satisfies

the relation [30]

$$f_\lambda^+(\omega_{ij}) = 1 - f_\lambda^-(\omega_{ji}) = \left[1 + \exp\left(\frac{\omega_{ij} - \mu_\lambda}{k_B T_\lambda}\right) \right]^{-1}, \quad (5)$$

where T_λ and μ_λ are the temperature and chemical potential of the λ th reservoir. From Fig. 2 and Eq. (4), the rate equations for occupation probabilities are

$$\begin{aligned} \dot{\rho}_A &= -\Gamma_{AB}^{lr+} - \Gamma_{AC}^{u+}, & \dot{\rho}_B &= \Gamma_{AB}^{lr+} - \Gamma_{BD}^{u+}, \\ \dot{\rho}_C &= -\Gamma_{CD}^{lr+} + \Gamma_{AC}^{u+}, & \dot{\rho}_D &= \Gamma_{CD}^{lr+} + \Gamma_{BD}^{u+}, \end{aligned} \quad (6)$$

where $\dot{\rho}_i = \sum_{\lambda=l,r,u} \langle i | \mathcal{L}_\lambda[\rho_s(t)] | i \rangle$ and $\Gamma_{ij}^{lr\pm} \equiv \Gamma_{ij}^{l\pm} + \Gamma_{ij}^{r\pm}$. The net transition rate $\Gamma_{ij}^{\lambda\pm}$ mediated by reservoir λ is expressed as

$$\Gamma_{ij}^{\lambda\pm} \equiv \Gamma_{i \rightarrow j}^{\lambda\pm} = k_{ij}^{\lambda\pm} \rho_i - k_{ji}^{\lambda\mp} \rho_j, \quad (7)$$

where

$$k_{ij}^{\lambda\pm} = \gamma_\lambda f_\lambda^\pm(\omega_{ij}), \quad k_{ji}^{\lambda\mp} = \gamma_\lambda f_\lambda^\mp(\omega_{ji}). \quad (8)$$

For fermionic reservoirs,

$$k_{ij}^{\lambda\pm} + k_{ji}^{\lambda\mp} = \gamma_\lambda. \quad (9)$$

Assuming all γ_λ are equal, we derive closed-form expressions for steady-state energy and particle currents in the following section. As a final remark, we emphasize that the quantum master equations (3) and (4) reduce to a rate equation [Eq. (6)] that appears formally “classical,” yet retains intrinsic quantum-mechanical content. This is evident from the expressions for the net transition rates $\Gamma_{ij}^{\lambda\pm}$, which depend on the Fermi-Dirac statistics of the associated electron reservoirs, the discrete energy levels of the coupled QDs, and the nature of the interaction between the spin-polarized electrons, which may be either attractive or repulsive. This has deeper implications, which are further illustrated at the end of Sec. III.

III. STEADY-STATE CURRENTS: ENTROPIC REPRESENTATION

To derive the expressions for steady-state currents under the grand canonical formalism, we consider the equilibrium initial density operator of the system $\rho_s(0)$ as [8]

$$\rho_s^{\text{eq}} = \rho_s(0) = \frac{e^{-\tilde{\beta}(H_s - \tilde{\mu}\mathcal{N})}}{\mathcal{Z}(\tilde{\beta}, \tilde{\mu})}, \quad (10)$$

where $\mathcal{Z}(\tilde{\beta}, \tilde{\mu}) = \text{Tr}[e^{-\tilde{\beta}(H_s - \tilde{\mu}\mathcal{N})}]$ is the grand canonical partition function and $\mathcal{N} = \mathcal{N}_{b\downarrow} + \mathcal{N}_{u\uparrow}$ being the total particle number operator of the two QDs. The $\tilde{\beta}$ and $\tilde{\mu}$ are the effective inverse temperature and chemical potential, $\tilde{\beta} = \sum_\lambda \beta_\lambda$ and $\tilde{\mu} = \sum_\lambda \mu_\lambda$, respectively. We assume that environmental interaction slightly perturbs the system from its initial equilibrium state $\rho_s(0) = \rho_s^{\text{eq}}$ to $\rho_s(t)$, such that $\delta\rho_s(t) = \rho_s(t) - \rho_s(0) \equiv \mathcal{O}(\xi)$, where ξ is a small expansion parameter. Following Ref. [8], equating the system von Neumann entropy (times k_B) with thermodynamic entropy near close to equilibrium [31,32]

$$S_s(t) = -k_B \text{Tr}_s[\rho_s(t) \ln \rho_s(t)], \quad (11)$$

and retaining only terms that are first order in ξ , the expression for $\Delta S_s = S_s(t) - S_s(0)$ can be derived as [8]

$$\begin{aligned}\Delta S_s(t) &\approx -k_B \text{Tr}_s[\delta \rho_s(t) \ln \rho_s^{\text{eq}}] \\ &= k_B \beta \text{Tr}_s[\delta \rho_s(t) H_s] - k_B \beta \mu \text{Tr}_s[\delta \rho_s(t) \mathcal{N}].\end{aligned}\quad (12)$$

Comparing Eq. (12) and the first law of thermodynamics in the presence of *chemical work done*, we identify change in energy as $\Delta E = \text{Tr}_s[\delta \rho_s(t) H_s]$ and change in particle number as $\Delta N = \text{Tr}_s[\delta \rho_s(t) \mathcal{N}]$. Noting that $\delta \rho_s(t) = \rho_s(t) - \rho_s(0)$, and $H_s, \mathcal{N}$ are time-independent, the net energy flux (J_E) and particle flux (J_N) are recognized as follows:

$$\begin{aligned}J_E(t) &= \text{Tr}_s[\dot{\rho}_s(t) H_s] = \sum_{\lambda=a,b,r} \text{Tr}_s[\mathcal{L}_\lambda[\rho_s(t)] H_s], \\ J_N(t) &= \text{Tr}_s[\dot{\rho}_s(t) \mathcal{N}] = \sum_{\lambda=a,b,r} \text{Tr}_s[\mathcal{L}_\lambda[\rho_s(t)] \mathcal{N}],\end{aligned}\quad (13)$$

where we use Eq. (3) for $\dot{\rho}_s(t)$. Identifying $J_{E(N)}$ as the sum of the contributions of energy (particle) flux associated with all three reservoirs: $J_E(t) = \sum_\lambda J_E^\lambda(t)$ and $J_N(t) = \sum_\lambda J_N^\lambda(t)$, where $J_{E(N)}^\lambda$ is positive if energy (particle) current flows from the reservoir λ to the system, we obtain

$$J_E^\lambda(t) = \text{Tr}_s[\mathcal{L}_\lambda[\rho(t)] H_s], \quad J_N^\lambda(t) = \text{Tr}_s[\mathcal{L}_\lambda[\rho(t)] \mathcal{N}]. \quad (14)$$

The heat current is obtained as $J_Q^\lambda(t) = J_E^\lambda(t) - \mu_\lambda J_N^\lambda(t)$, which at steady state simplifies to $J_Q^\lambda = J_E^\lambda - \mu_\lambda J_N^\lambda$. Here, $J_E^\lambda = \text{Tr}_s[\mathcal{L}_\lambda[\rho_{ss}] H_s]$ and $J_N^\lambda = \text{Tr}_s[\mathcal{L}_\lambda[\rho_{ss}] \mathcal{N}]$. Using Eq. (4) for the $\mathcal{L}_\lambda[\rho_{ss}]$, all three steady-state currents are given by [33,34]

$$\begin{aligned}J_E^\lambda &= \sum_{\{\omega_{ij}\}} \omega_{ij} \Gamma_{ij}^{\lambda+} = \sum_{\{\omega_{ji}\}} \omega_{ji} \Gamma_{ji}^{\lambda-}, \\ J_N^\lambda &= \sum_{\{\omega_{ij}\}} \Gamma_{ij}^{\lambda+} = \sum_{\{\omega_{ji}\}} \Gamma_{ji}^{\lambda-}, \\ J_Q^\lambda &= \sum_{\{\omega_{ij}\}} (\omega_{ij} - \mu_\lambda) \Gamma_{ij}^{\lambda+} = \sum_{\{\omega_{ji}\}} (\omega_{ji} - \mu_\lambda) \Gamma_{ji}^{\lambda-},\end{aligned}\quad (15)$$

where we use the relation $\Gamma_{ij}^{\lambda+} = -\Gamma_{ji}^{\lambda-}$. The explicit expressions for all three currents are given in Appendix B. At steady state ($\dot{\rho}_i = 0$), the transition rates satisfy

$$\begin{aligned}\Gamma_{AB}^{lr+} &= \Gamma_{BD}^{u+} = \Gamma_{DC}^{lr-} = \Gamma_{CA}^{u-} = \Gamma_{ABCD A} \equiv \Gamma_\circ, \\ \Gamma_{AC}^{u+} &= \Gamma_{CD}^{lr+} = \Gamma_{DB}^{u-} = \Gamma_{BA}^{lr-} = \Gamma_{ACDB A} \equiv \Gamma_\circ,\end{aligned}\quad (16)$$

implying $\Gamma_\circ = -\Gamma_\circ$ (Fig. 2). With the help of Eq. (15), one can verify

$$\begin{aligned}J_E^u &= \varepsilon_u \Gamma_\circ + (\varepsilon_u + \kappa) \Gamma_\circ = \kappa \Gamma_\circ, \\ J_N^u &= \Gamma_{AC}^{u+} + \Gamma_{BD}^{u+} = \Gamma_\circ + \Gamma_\circ = 0,\end{aligned}\quad (17)$$

as QD_u is coupled only to lead u , resulting in $J_N^u = 0$. This gives $J_Q^u = J_E^u = \kappa \Gamma_\circ$. However, QD_b is coupled to leads l and r , allowing steady-state (spin-polarized) particle flow between them. Conservation of total energy and particle currents at steady state, $\sum_\lambda J_E^\lambda = 0$ and $\sum_\lambda J_N^\lambda = 0$ gives

$$J_E^u = -J_E^{lr} = \kappa \Gamma_\circ, \quad J_N^l = -J_N^r. \quad (18)$$

The explicit form of $\Gamma_\circ$ is provided in Appendix C. Unlike energy, heat current is not conserved, as $J_Q^l + J_Q^r \equiv J_Q^{lr} \neq J_E^{lr} = -\kappa \Gamma_\circ$.

A. Entropy production rate: Entropic flux and bias

From Eq. (12), one can write [32]

$$\Delta S_s(t) = \Sigma(t) + \Phi(t), \quad (19)$$

where $\Sigma(t)$ is the *entropy production* and $\Phi(t)$ is the *entropy flux*. The explicit forms of $\Sigma(t)$ and the $\Phi(t)$ are derived following Ref. [8] (see Appendix D):

$$\begin{aligned}\Sigma(t) &= k_B \text{Tr}[\rho_{\text{tot}}(t) \ln \{\rho_{\text{tot}}(t)\}] \\ &\quad - k_B \text{Tr} \left[\rho_{\text{tot}}(t) \ln \left\{ \rho_s(t) \prod_\lambda \rho_\lambda^{\text{eq}} \right\} \right], \\ \Phi(t) &= k_B \sum_\lambda \text{Tr}_\lambda [\{\rho_\lambda(t) - \rho_\lambda^{\text{eq}}\} \ln \rho_\lambda^{\text{eq}}],\end{aligned}\quad (20)$$

where ρ_λ^{eq} and $\rho_\lambda(t)$ are the equilibrium and *near-equilibrium* density operators of reservoir λ , respectively. In the grand canonical ensemble, ρ_λ^{eq} is expressed as

$$\rho_\lambda^{\text{eq}} = \frac{e^{-\beta_\lambda H_\lambda^{\text{eq}}}}{\mathcal{Z}_\lambda(\beta_\lambda)}, \quad (21)$$

where $H_B^\lambda \equiv H_B^{\lambda\sigma} = \sum_k (\epsilon_k^{\lambda\sigma} - \mu_{\lambda\sigma}) c_{\lambda\sigma k}^\dagger c_{\lambda\sigma k}$ and $\mathcal{Z}_\lambda(\beta_\lambda) = \text{Tr}[e^{-\beta_\lambda H_B^\lambda}]$ is the grand canonical partition function. As a result, $\Phi(t)$ is identified as the reversible heat exchanged with the reservoir between some final and initial times:

$$\Phi(t) = -k_B \sum_\lambda \beta_\lambda [\langle H_B^\lambda \rangle_t - \langle H_B^\lambda \rangle_0] = k_B \sum_\lambda \beta_\lambda \Delta Q_\lambda(t), \quad (22)$$

where $\Delta Q_\lambda(t) = \langle H_B^\lambda \rangle_0 - \langle H_B^\lambda \rangle_t$ and $\langle H_B^\lambda \rangle_t = \text{Tr}_\lambda[\rho_\lambda(t) H_B^\lambda]$. Thus, we rewrite Eq. (19) as

$$\Delta S_s(t) = \Sigma(t) + k_B \sum_\lambda \beta_\lambda \Delta Q_\lambda(t), \quad (23)$$

where we replace the second term by Eq. (22). Therefore, the general definition of the entropy production rate can be deduced from the above equation as follows:

$$\begin{aligned}\dot{\Sigma}(t) &= \frac{d}{dt} \{\Delta S_s(t)\} - k_B \sum_\lambda \beta_\lambda \frac{d}{dt} \{\Delta Q_\lambda(t)\} \\ &= \frac{d}{dt} \{\Delta S_s(t)\} - k_B \sum_\lambda \beta_\lambda J_Q^\lambda(t).\end{aligned}\quad (24)$$

At steady state, the system's entropy remains constant, i.e., $\frac{d}{dt} \{\Delta S_s(t)\} = 0$, and $\dot{\Sigma}$ reaches its minimum [8,32]. So, Eq. (24) yields

$$\dot{\Sigma} = -k_B \sum_\lambda \beta_\lambda J_Q^\lambda. \quad (25)$$

The above equation expresses the entropy production rate in terms of the appropriate thermodynamic fluxes and the corresponding temperatures of the various leads. Substituting

$J_Q^\lambda = J_E^\lambda - \mu_\lambda J_N^\lambda$ into Eq. (25), the entropy production rate for our three-terminal system can be expressed as

$$\dot{\Sigma} = J_E^u \mathcal{F}_E^u + J_E^r \mathcal{F}_E^r + J_N^r \mathcal{F}_N^r, \quad (26)$$

where $J_N^u = 0$, since the upper QD is connected to only one lead [see also Eq. (17)]. In deriving Eq. (26), we have also utilized the conservation laws for the energy and particle currents: $J_E^u + J_E^l + J_E^r = 0$ and $J_N^l + J_N^r = 0$. The conjugate “forces” associated with the set of thermodynamic fluxes $\{J_E^u, J_E^r, J_N^r\}$ in Eq. (26) are given by

$$\begin{aligned} \mathcal{F}_E^u &= k_B(\beta_l - \beta_u), \\ \mathcal{F}_E^r &= k_B(\beta_l - \beta_r), \\ \mathcal{F}_N^r &= k_B(\beta_r \mu_r - \beta_l \mu_l). \end{aligned} \quad (27)$$

It is important to note that Eq. (26) is not a unique representation of the entropy production rate, because the three energy fluxes are subject to one constraint arising from energy conservation. Consequently, one may choose any two independent energy fluxes (e.g., $\{J_E^u, J_E^r\}$ or $\{J_E^u, J_E^l\}$ or $\{J_E^r, J_E^l\}$) together with one particle flux (either J_N^r or J_N^l), along with their respective conjugate forces, to write the entropy production rate. Therefore, there exist six equivalent representations of $\dot{\Sigma}$ that describe the same three-terminal system. The nonuniqueness in defining the entropic forces and corresponding fluxes does not signal any mathematical inconsistency; rather, it is an inherent feature of a general three-terminal transport. Although the form of energy biases depends on the choice of the force-flux representation, all such representations are thermodynamically equivalent and yield identical physical predictions across the different permissible formulations of entropy production rate. Therefore, without loss of generality, we adopt one particular form of $\dot{\Sigma}$ [cf. Eq. (26)] throughout the remainder of this paper. If a different form of $\dot{\Sigma}$ is chosen, the parameter regimes for the occurrence of ICC or any other thermodynamic effects in the corresponding entropy production rate can be readily determined from the relation between $\{\mathcal{F}_E^u, \mathcal{F}_E^r, \mathcal{F}_N^r\}$ in the present formulation [cf. Eq. (27)] and those in the alternative one. The quantities $\{\mathcal{F}_E^u, \mathcal{F}_E^r, \mathcal{F}_N^r\}$ defined in Eqs. (26) and (27) should not be

confused with the *real* thermal and particle forces acting on the system. They should instead be regarded as *entropic* energy and particle biases (i.e., energy and particle biases in the entropy representation) that are conjugate to the complete set of individual fluxes $\{J_E^u, J_E^r, J_N^r\}$. In the case of coupled transport involving a single pair of forces and fluxes, this correspondence becomes unique, and the entropic biases reduce to the real thermodynamic forces experienced by the system.

Before we analyze the coupled transport, let us first identify the microscopic descriptions of the above thermodynamic quantities. The ideal starting point could be the entropy production rate, which sets the connection between macroscopic and microscopic thermodynamic frameworks [35]. So, rewriting the von Neumann entropy defined in Eq. (11):

$$\mathcal{S}_s(t) = -k_B \sum_{\mathbf{i}} \rho_{\mathbf{i}}(t) \ln \rho_{\mathbf{i}}(t), \quad (28)$$

in terms of the microscopic populations of the different system eigenstates, $\{\rho_{\mathbf{i}}\}$ ($\mathbf{i} = \mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$), we evaluate the time evolution of the system’s entropy change as [35]

$$\begin{aligned} \frac{d}{dt} \Delta \mathcal{S}_s(t) &= k_B \left[\Gamma_{\mathbb{A}\mathbb{B}}^{l+} \ln \left(\frac{\rho_{\mathbb{A}}}{\rho_{\mathbb{B}}} \right) + \Gamma_{\mathbb{A}\mathbb{B}}^{r+} \ln \left(\frac{\rho_{\mathbb{A}}}{\rho_{\mathbb{B}}} \right) \right. \\ &\quad + \Gamma_{\mathbb{B}\mathbb{D}}^{u+} \ln \left(\frac{\rho_{\mathbb{B}}}{\rho_{\mathbb{D}}} \right) + \Gamma_{\mathbb{D}\mathbb{C}}^{l-} \ln \left(\frac{\rho_{\mathbb{D}}}{\rho_{\mathbb{C}}} \right) \\ &\quad \left. + \Gamma_{\mathbb{D}\mathbb{C}}^{r-} \ln \left(\frac{\rho_{\mathbb{D}}}{\rho_{\mathbb{C}}} \right) + \Gamma_{\mathbb{C}\mathbb{A}}^{u-} \ln \left(\frac{\rho_{\mathbb{C}}}{\rho_{\mathbb{A}}} \right) \right], \end{aligned} \quad (29)$$

where we use $\rho_{\mathbf{i}}(t)$ from Eq. (6) (see Appendix E). We emphasize that, since there are no off-diagonal (quantum coherence) elements in the reduced density matrix in Eqs. (3) and (4), the LME effectively reduces to a rate equation [cf. Eq. (6)] expressed in terms of occupation probabilities. As a result, the von Neumann entropy in Eq. (28) coincides with the Shannon entropy within its diagonalized description. Comparing the above equation with Eq. (19), we identify the microscopic version of the entropy production rate and entropy flux, as follows (Appendix E):

$$\begin{aligned} \dot{\Sigma}(t) &= k_B \left[(k_{\mathbb{A}\mathbb{B}}^{l+} \rho_{\mathbb{A}} - k_{\mathbb{B}\mathbb{A}}^{l-} \rho_{\mathbb{B}}) \ln \left(\frac{k_{\mathbb{A}\mathbb{B}}^{l+} \rho_{\mathbb{A}}}{k_{\mathbb{B}\mathbb{A}}^{l-} \rho_{\mathbb{B}}} \right) + (k_{\mathbb{A}\mathbb{B}}^{r+} \rho_{\mathbb{A}} - k_{\mathbb{B}\mathbb{A}}^{r-} \rho_{\mathbb{B}}) \ln \left(\frac{k_{\mathbb{A}\mathbb{B}}^{r+} \rho_{\mathbb{A}}}{k_{\mathbb{B}\mathbb{A}}^{r-} \rho_{\mathbb{B}}} \right) \right. \\ &\quad + (k_{\mathbb{B}\mathbb{D}}^{u+} \rho_{\mathbb{B}} - k_{\mathbb{D}\mathbb{B}}^{u-} \rho_{\mathbb{D}}) \ln \left(\frac{k_{\mathbb{B}\mathbb{D}}^{u+} \rho_{\mathbb{B}}}{k_{\mathbb{D}\mathbb{B}}^{u-} \rho_{\mathbb{D}}} \right) \\ &\quad \left. + (k_{\mathbb{D}\mathbb{C}}^{l-} \rho_{\mathbb{D}} - k_{\mathbb{C}\mathbb{D}}^{l+} \rho_{\mathbb{C}}) \ln \left(\frac{k_{\mathbb{D}\mathbb{C}}^{l-} \rho_{\mathbb{D}}}{k_{\mathbb{C}\mathbb{D}}^{l+} \rho_{\mathbb{C}}} \right) + (k_{\mathbb{D}\mathbb{C}}^{r-} \rho_{\mathbb{D}} - k_{\mathbb{C}\mathbb{D}}^{r+} \rho_{\mathbb{C}}) \ln \left(\frac{k_{\mathbb{D}\mathbb{C}}^{r-} \rho_{\mathbb{D}}}{k_{\mathbb{C}\mathbb{D}}^{r+} \rho_{\mathbb{C}}} \right) \right. \\ &\quad \left. + (k_{\mathbb{C}\mathbb{A}}^{u-} \rho_{\mathbb{C}} - k_{\mathbb{A}\mathbb{C}}^{u+} \rho_{\mathbb{A}}) \ln \left(\frac{k_{\mathbb{C}\mathbb{A}}^{u-} \rho_{\mathbb{C}}}{k_{\mathbb{A}\mathbb{C}}^{u+} \rho_{\mathbb{A}}} \right) \right], \\ \dot{\Phi}(t) &= -k_B \left[\Gamma_{\mathbb{A}\mathbb{B}}^{l+} \ln \left(\frac{k_{\mathbb{A}\mathbb{B}}^{l+}}{k_{\mathbb{B}\mathbb{A}}^{l-}} \right) + \Gamma_{\mathbb{A}\mathbb{B}}^{r+} \ln \left(\frac{k_{\mathbb{A}\mathbb{B}}^{r+}}{k_{\mathbb{B}\mathbb{A}}^{r-}} \right) + \Gamma_{\mathbb{B}\mathbb{D}}^{u+} \ln \left(\frac{k_{\mathbb{B}\mathbb{D}}^{u+}}{k_{\mathbb{D}\mathbb{B}}^{u-}} \right) \right. \\ &\quad \left. + \Gamma_{\mathbb{D}\mathbb{C}}^{l-} \ln \left(\frac{k_{\mathbb{D}\mathbb{C}}^{l-}}{k_{\mathbb{C}\mathbb{D}}^{l+}} \right) + \Gamma_{\mathbb{D}\mathbb{C}}^{r-} \ln \left(\frac{k_{\mathbb{D}\mathbb{C}}^{r-}}{k_{\mathbb{C}\mathbb{D}}^{r+}} \right) + \Gamma_{\mathbb{C}\mathbb{A}}^{u-} \ln \left(\frac{k_{\mathbb{C}\mathbb{A}}^{u-}}{k_{\mathbb{A}\mathbb{C}}^{u+}} \right) \right]. \end{aligned} \quad (30)$$

In Eq. (30), each term of the Schnakenberg entropy production rate $\dot{\Sigma}$ [35] has the form $(a - b) \ln(\frac{a}{b})$, ensuring non-negativity of the total entropy production rate. At steady state, this leads to (Appendix E)

$$\begin{aligned} \dot{\Sigma}(t) &= -\dot{\Phi}(t) = \kappa \Gamma_{\odot} \left[\left(\frac{k_B}{\kappa} \right) \ln \left(\frac{k_{\mathbb{A}\mathbb{B}}^{l+} k_{\mathbb{B}\mathbb{D}}^{u+} k_{\mathbb{D}\mathbb{C}}^{l-} k_{\mathbb{C}\mathbb{A}}^{u-}}{k_{\mathbb{B}\mathbb{A}}^{l-} k_{\mathbb{D}\mathbb{B}}^{u-} k_{\mathbb{C}\mathbb{D}}^{l+} k_{\mathbb{A}\mathbb{C}}^{u+}} \right) \right] + (\varepsilon_b \Gamma_{\mathbb{A}\mathbb{B}}^{r+} - (\varepsilon_b + \kappa) \Gamma_{\mathbb{D}\mathbb{C}}^{r-}) \left[\left(\frac{k_B}{\kappa} \right) \ln \left(\frac{k_{\mathbb{B}\mathbb{A}}^{r-} k_{\mathbb{A}\mathbb{B}}^{l+} k_{\mathbb{C}\mathbb{D}}^{r+} k_{\mathbb{D}\mathbb{C}}^{l-}}{k_{\mathbb{D}\mathbb{C}}^{r-} k_{\mathbb{C}\mathbb{D}}^{l+} k_{\mathbb{A}\mathbb{B}}^{r+} k_{\mathbb{B}\mathbb{A}}^{l-}} \right) \right] \\ &\quad + (\Gamma_{\mathbb{A}\mathbb{B}}^{r+} - \Gamma_{\mathbb{D}\mathbb{C}}^{r-}) \left[k_B (1 + \theta) \ln \left(\frac{k_{\mathbb{A}\mathbb{B}}^{r+} k_{\mathbb{B}\mathbb{A}}^{l-}}{k_{\mathbb{B}\mathbb{A}}^{r-} k_{\mathbb{A}\mathbb{B}}^{l+}} \right) + k_B \theta \ln \left(\frac{k_{\mathbb{D}\mathbb{C}}^{r-} k_{\mathbb{C}\mathbb{D}}^{l+}}{k_{\mathbb{C}\mathbb{D}}^{r+} k_{\mathbb{D}\mathbb{C}}^{l-}} \right) \right] \equiv J_E^u \mathcal{F}_E^u + J_E^r \mathcal{F}_E^r + J_N^r \mathcal{F}_N^r, \end{aligned} \quad (31)$$

where $\theta = (\frac{\varepsilon_b}{\kappa})$ is the scaled system parameter. The sign of θ plays a crucial role in determining the ICC behavior in energy and (spin-polarized) particle currents, as discussed in Sec. V. From the above equation, all associated fluxes and their conjugate forces (entropic bias) are identified as follows:

$$\begin{aligned} J_E^u &= \kappa \Gamma_{\odot}, \quad J_N^r = (\Gamma_{AB}^{r+} + \Gamma_{CD}^{r+}), \\ J_E^r &= \varepsilon_b \Gamma_{AB}^{r+} + (\varepsilon_b + \kappa) \Gamma_{CD}^{r+} = \varepsilon_b J_N^r + \kappa \Gamma_{CD}^{r+}, \end{aligned} \quad (32)$$

and

$$\begin{aligned} \mathcal{F}_E^u &= \left(\frac{k_B}{\kappa} \right) \ln \left(\frac{k_{AB}^{l+} k_{BD}^{u+} k_{DC}^{l-} k_{CA}^{u-}}{k_{BA}^{l-} k_{DB}^{u-} k_{CD}^{l+} k_{AC}^{u+}} \right) \\ &= k_B (\beta_l - \beta_u), \end{aligned} \quad (33a)$$

$$\mathcal{F}_E^r = \left(\frac{k_B}{\kappa} \right) \ln \left(\frac{k_{BA}^{r-} k_{AB}^{l+} k_{CD}^{r+} k_{DC}^{l-}}{k_{DC}^{r-} k_{CD}^{l+} k_{AB}^{r+} k_{BA}^{l-}} \right) \quad (33b)$$

$$= \left(\frac{k_B}{\kappa} \right) \ln \left(\frac{\mathcal{N}}{\mathcal{M}} \right) = k_B (\beta_l - \beta_r), \quad (33c)$$

$$\begin{aligned} \mathcal{F}_N^r &= k_B (1 + \theta) \ln \left(\frac{k_{AB}^{r+} k_{BA}^{l-}}{k_{BA}^{l-} k_{AB}^{r+}} \right) - k_B \theta \ln \left(\frac{k_{CD}^{r+} k_{DC}^{l-}}{k_{DC}^{r-} k_{CD}^{l+}} \right) \\ &= k_B (1 + \theta) \ln \mathcal{M} - k_B \theta \ln \mathcal{N} \end{aligned} \quad (33d)$$

$$= k_B \ln(\mathcal{M}) - \varepsilon_b \mathcal{F}_E^r = k_B (\beta_r \mu_r - \beta_l \mu_l), \quad (33e)$$

where we have defined two important factors as follows:

$$\mathcal{M} = \left(\frac{k_{AB}^{r+} k_{BA}^{l-}}{k_{BA}^{l-} k_{AB}^{r+}} \right), \quad \mathcal{N} = \left(\frac{k_{CD}^{r+} k_{DC}^{l-}}{k_{DC}^{r-} k_{CD}^{l+}} \right). \quad (34)$$

Now, Eq. (32) is identical to Eq. (27). The equivalence between macroscopic and microscopic expressions of the above entropic forces [cf. Eq. (33)] can be verified using Eq. (8) and the corresponding FDFs. Equations (32) and (33) are the first major results of our analysis that follow the linearity of the underlying quantum master equation [cf. Eq. (6)]. Here, it is useful to highlight a point that may at first appear paradoxical. From Eq. (33), it might seem that although the microscopic expressions of the entropic biases depend on the sign of κ , their macroscopic expressions are independent of it. This arises because $k_{ij}^{\lambda\pm}$ [and consequently $\mathcal{M}$ and $\mathcal{N}$ in Eq. (34)] depend on the sign of κ via ω_{ij} , which appears in FDF [cf. Eq. (8)]. Therefore, microscopically, depending on the sign of κ , the signs of $k_{ij}^{\lambda\pm}$ within the logarithm (or equivalently $\mathcal{M}$ and $\mathcal{N}$) adjust such that the same macroscopic entropic bias emerges, irrespective of the nature of κ . This freedom in the microscopic realization of the entropic biases allows us to establish the counterintuitive transport phenomenon of ICC in *nanoscale quantum setting* with capacitively coupled QDs, as considered in the following section. In this precise microscopic sense, ICC in quantum systems is fundamentally different from the classical ICC model put forward by Wang *et al.* [3], where it originates from self-organization within the system in response to the applied force. However, from a thermodynamic perspective, the ICC phenomenon in both classical and quantum systems exhibits the same richness: One induced current must flow against two mutually parallel thermodynamic forces without violating the laws of thermodynamics.

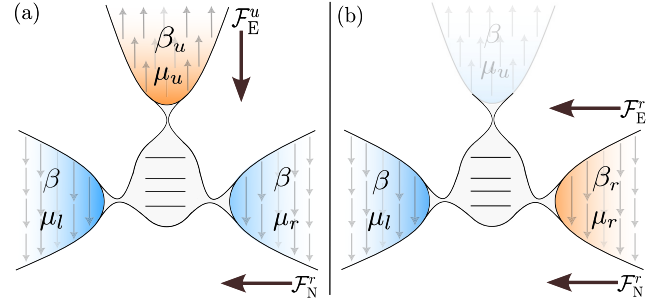


FIG. 3. (a) Coupled transport phenomena in QD thermoelectric system. A three-terminal energy harvester based on capacitively coupled QDs in a 2DEG setup, experimentally realized by Thierschmann *et al.* [27], where a thermal bias between the upper and lower terminals drives a current against a chemical potential bias. (b) QD ICC effect as a kind of coupled transport phenomenon, where both thermodynamic forces are mutually parallel to each other, in contrast to the QD thermoelectric setup shown in Fig. 3(a).

IV. INVESTIGATING COUPLED TRANSPORT

Now, there are two effective approaches to establish our three-terminal setup for a coupled transport with one energy and one particle force. The third possibility, in which both energy forces are present, could lead to anomalous energy flow; this has been studied in recent work [36] and is therefore not discussed in this paper.

A. $\beta_l = \beta_r = \beta$: QD thermoelectric setup

Once we set $\beta_l = \beta_r = \beta$, we can immediately identify one energy force, $\mathcal{F}_E^u$, and one particle force, $\mathcal{F}_N^r$, operating exactly as shown in Fig. 3(a). This is already evident from Eq. (33). Under the above choice of parameters, one of the entropic forces, $\mathcal{F}_E^r$, becomes zero, and the remaining two entropic biases unambiguously reduce to the energy force $\mathcal{F}_E^u = k_B (\beta - \beta_u)$ and the particle force $\mathcal{F}_N^r = k_B \beta (\mu_r - \mu_l)$, respectively, as depicted in Fig. 3(a). This configuration is equivalent to the Sánchez-Büttiker model [10], which is widely used to study thermoelectric effects in coupled QD systems [10–12,27,37,38], as well as various other quantum-transport and thermal-rectification phenomena [7,30,39–43].

In this limit, Eq. (26) simplifies to

$$\dot{\Sigma} = J_E^u \mathcal{F}_E^u + J_N^r \mathcal{F}_N^r. \quad (35)$$

It is important to emphasize that although Eq. (26) is not, in general, a unique representation of the entropy production rate, once one of the entropic biases becomes zero (here, $\mathcal{F}_E^r = 0$), Eq. (26) describes coupled transport for physical systems with well-defined thermodynamic forces.

However, the configuration shown in Fig. 3(a) cannot exhibit ICC because an induced current cannot simultaneously flow against both thermodynamic forces, $\mathcal{F}_E^u$ and $\mathcal{F}_N^r$, as they are not *mutually parallel*. Nevertheless, the setup in Fig. 3(a) can still operate as a thermoelectric engine or as a refrigerator. For example, when a particle current J_N^r , induced by the energy force $\mathcal{F}_E^u$, flows against the particle force $\mathcal{F}_N^r$, the device functions as an engine. In contrast, when an energy current J_E^u , induced by $\mathcal{F}_N^r$, flows against its conjugate force

$\mathcal{F}_E^u$, the device acts as a refrigerator. This behavior relies on the well-known principle of thermodynamic cross effects [6,10,11,27], where an applied force drives a current that can oppose its conjugate force without violating the positivity of the total entropy production rate. An analogous model, shown in Fig. 3(a), was theoretically proposed by Sánchez and Büttiker [10] and experimentally realized by Thierschmann *et al.* [27] as a three-terminal energy harvester based on capacitively coupled QDs in a two-dimensional electron gas (2DEG) setup. In the experiment, a thermal bias of approximately 100 mK is applied between the upper and lower terminals. As a result, a current on the order of pA is measured in the lower terminals against a chemical potential bias of approximately 10 μ V. An extension of this model in the presence of spin-bias voltages has also been proposed by the present authors [6].

In contrast to the present case—where an induced current can oppose only one thermodynamic force at a time (for example, in a thermoelectric refrigerator, J_E^u opposes $\mathcal{F}_E^u$ while J_N^r still aligns with $\mathcal{F}_N^r$, otherwise they will violate the second law)—the configuration in Fig. 3(b) features mutually parallel thermodynamic forces. In that scenario, a current can flow against both forces while still satisfying $\dot{\Sigma} \geq 0$. This is the hallmark of the ICC effect proposed by Wang *et al.* [3] in classical settings.

B. $\beta_l = \beta_u = \beta$: QD ICC effect

If we set $\beta_l = \beta_u = \beta$, this is equivalent to imposing that the other entropic energy bias $\mathcal{F}_E^u$ is identically zero. Under this choice, Eq. (25) simplifies to

$$\dot{\Sigma} = J_E^r \mathcal{F}_E^r + J_N^r \mathcal{F}_N^r \geq 0, \quad (36)$$

where the two real thermodynamic forces are given by $\mathcal{F}_E^r = k_B(\beta - \beta_r)$ and $\mathcal{F}_N^r = k_B(\beta_r \mu_r - \beta \mu_l)$, respectively [Fig. 3(b)]. Without loss of any generality, if they are mutually parallel with $\mathcal{F}_E^r > 0$ and $\mathcal{F}_N^r > 0$, either J_E^r or J_N^r can flow opposite to both forces while the second flux compensates for the resulting negative contribution, ensuring that the total entropy production still satisfies $\dot{\Sigma} \geq 0$. As a result, Fig. 3(b) represents the ideal configuration for achieving ICC as a distinct thermodynamic phenomenon: A regime where one current can flow against two mutually parallel thermodynamic forces, without violating the laws of thermodynamics.

A special case of ICC is *pseudo-ICC*, which occurs when one of the forces is set to zero. For example, if $\mathcal{F}_E^r = 0$, then the induced current J_E^r may flow along or against $\mathcal{F}_N^r$ without affecting the nonnegativity of the entropy production rate, i.e., $\dot{\Sigma} = J_N^r \mathcal{F}_N^r \geq 0$ [cf. Eq. (36)]. In such a situation, if J_E^r opposes $\mathcal{F}_N^r$, it signals ICC, and we refer to it as “pseudo-ICC”—a precursor to genuine ICC, which is in complete accordance with second law of thermodynamics (see also Sec. V A). Following Wang *et al.* [3], here, we must distinguish ICC from normal cross effects. For example, ICC in the energy (particle) current is realized by $J_{E(N)}^r < 0$ when both $\mathcal{F}_E^r > 0$ and $\mathcal{F}_N^r > 0$ (see Fig. 4). In contrast, for a conventional cross effect, $\mathcal{F}_E^r$ and $\mathcal{F}_N^r$ have opposite signs, i.e., the two forces oppose each other. Then, the cross effect in the energy current (characterized by $J_E^r < 0$) is induced by the particle force (characterized by $\mathcal{F}_N^r < 0$), flow against the energy force $\mathcal{F}_E^r > 0$ (see Fig. 4). The same is true for the

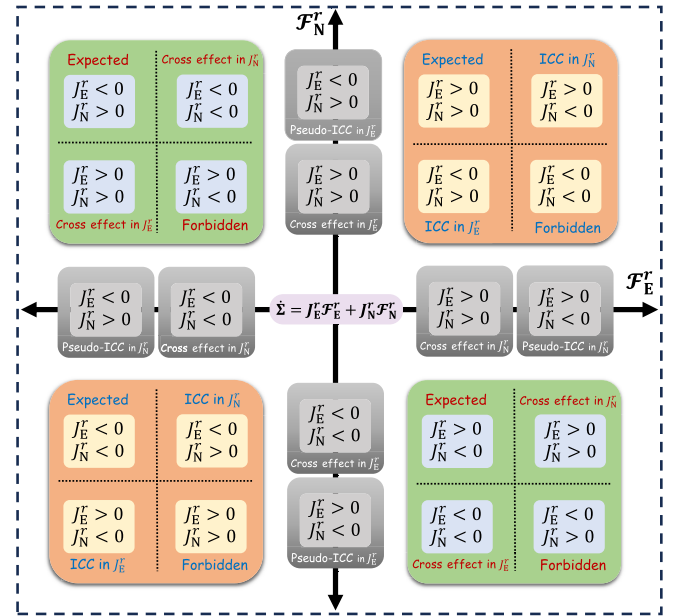


FIG. 4. Pictorial representation of various thermodynamic phenomena governed by the entropy production rate $\dot{\Sigma} = J_E^r \mathcal{F}_E^r + J_N^r \mathcal{F}_N^r$ [cf. Eq. (36)], i.e., normal cross effect (in green shaded region), ICC (orange shaded region), pseudo-ICC, and Seebeck and Peltier effects (gray shaded blocks along X and Y axis) that can be obtained.

particle current with $J_N^r < 0$, when $\mathcal{F}_E^r < 0$ but $\mathcal{F}_N^r > 0$ (see Fig. 4).

In summary, while Fig. 3(a) illustrates the conventional thermoelectric effect, Fig. 3(b) captures the essence of ICC, characterized by mutually parallel thermodynamic forces [cf. Eq. (36)]. Moreover, any alternative formulation of the entropy production rate involving two mutually parallel force-flux pairs is isomorphic to Fig. 3(b) and Eq. (36). Thus, we may drop (for convenience of the later sections, we still keep) the superscripts in Eq. (36), while still capturing all possible behaviors in the $(\mathcal{F}_E, \mathcal{F}_N)$ plane, as summarized schematically in Fig. 4. As shown there, ICC occurs in the first and third quadrants, whereas cross effects arise in the second and fourth quadrants.

In the following section, we analyze this reduced two-force-flux setup as a prototype system to realize ICC. For all numerical plots, we set the tunneling rates equal, $\gamma_l = \gamma_r = \gamma_u \equiv \gamma$, and employ appropriately scaled dimensionless forces and fluxes. All fluxes are expressed in units of $\hbar\gamma^2$, while the energy and particle forces are measured in units of $k_B/(\hbar\gamma)$ and k_B , respectively. Since the ICC appears only in the first and third quadrants (Fig. 4), in the following we may, without loss of generality, choose the scaled forces to be non-negative and restrict our discussion to the first quadrant.

V. EXAMINING ICC: RESULTS AND DISCUSSION

As discussed in the previous section, considering $\beta_l = \beta_u = \beta$, we set the entropic bias $\mathcal{F}_E^u$ to zero. Now, from Eq. (32), it is clear that to determine the currents $J_N^r = (\Gamma_{AB}^{r+} + \Gamma_{CD}^{r+})$ and $J_E^r = \varepsilon_b J_N^r + \kappa \Gamma_{CD}^{r+}$, we need to understand the signs of the rates Γ_{AB}^{r+} and Γ_{CD}^{r+} . To this end, we define two

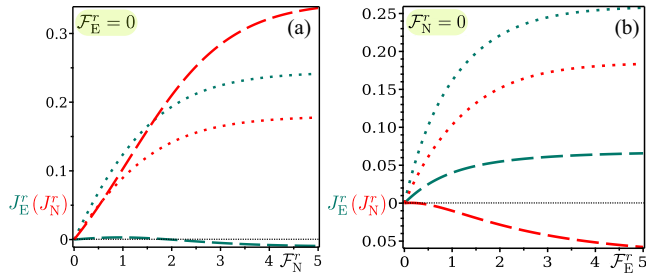


FIG. 5. Pseudo-ICC in J_E^r (bluegreen line) and J_N^r (red line) against the thermodynamic forces (a) $\mathcal{F}_N^r$ while $\mathcal{F}_E^r = 0$ and (b) $\mathcal{F}_E^r$ while $\mathcal{F}_N^r = 0$, respectively. For all cases, the (dashed) dotted lines correspond to $\kappa = +1.5\hbar\gamma$ ($\kappa = -1.5\hbar\gamma$), validating the condition $|\kappa| > \varepsilon_b$. Other system and bath parameters: $\varepsilon_b = 1.0\hbar\gamma$, $\varepsilon_u = 2.5\hbar\gamma$, $\beta_r = 1/\hbar\gamma$, and $\mu_r = 1.0\hbar\gamma$.

variables:

$$\mathcal{X} \stackrel{\text{def}}{=} \Gamma_{AB}^{r+} - \Gamma_{AB}^{l+}, \quad \mathcal{Y} \stackrel{\text{def}}{=} \Gamma_{CD}^{r+} - \Gamma_{CD}^{l+}. \quad (37)$$

In terms of which, the rates are given by

$$\Gamma_{AB}^{r+} = \frac{1}{2}(\mathcal{X} + \Gamma_\odot), \quad \Gamma_{CD}^{r+} = \frac{1}{2}(\mathcal{Y} - \Gamma_\odot), \quad (38)$$

and the currents are given by

$$J_N^r = \frac{1}{2}(\mathcal{X} + \mathcal{Y}), \quad J_E^r = \frac{\varepsilon_b}{2}(\mathcal{X} + \mathcal{Y}) + \frac{\kappa}{2}(\mathcal{Y} - \Gamma_\odot). \quad (39)$$

Thus, by simply knowing the signs of $\mathcal{X}$, $\mathcal{Y}$, and $\Gamma_\odot$, we can determine the sign of the particle current J_N^r and the energy current J_E^r . Now, the variables $\mathcal{X}, \mathcal{Y}$ are directly related to $\mathcal{M}, \mathcal{N}$ of Eq. (34). It is straightforward to show that $\mathcal{M} \geq 1$ and $\mathcal{N} \geq 1$ corresponds to $\mathcal{X} \geq 0$ and $\mathcal{Y} \geq 0$, respectively, whereas $\mathcal{M} = \mathcal{N} = 1$ implies $\mathcal{X} = \mathcal{Y} = 0$. The determination of the sign of $\Gamma_\odot$ is little more involved and we refer to Appendix F for details. This is our general recipe for determining the signs of both energy and particle currents. As a simple consistency check of the above procedure, we have shown in Appendix F, when both forces are zero, both currents vanish, as expected.

A. Pseudo-ICC

Now, we analyze pseudo-ICC in the energy and particle currents setting $\mathcal{F}_E^r = 0$ and $\mathcal{F}_N^r = 0$, respectively, since pseudo-ICC serves as a precursor to genuine ICC.

1. Setting $\mathcal{F}_E^r = 0$: Pseudo-ICC in J_E^r

When $\mathcal{F}_N^r$ is the only nonzero force, the spin-polarized particle current J_N^r is always positive; while the energy current J_E^r remains positive for $\kappa > 0$, but can change sign when $\varepsilon_b + \kappa < 0$ (Appendix F). A positive J_E^r corresponds to a cross effect, whereas a negative J_E^r signals pseudo-ICC, since the current flows against the nonconjugate particle force $\mathcal{F}_N^r$. Thus, genuine ICC can arise in the energy current within this model [Fig. 5(a)].

2. Setting $\mathcal{F}_N^r = 0$: Pseudo-ICC in J_N^r

When $\mathcal{F}_E^r$ is the only driving force, the energy current J_E^r remains positive [Fig. 5(b)]. For $\kappa > 0$, the particle cur-

rent J_N^r is also positive, indicating no ICC. However, when $\varepsilon_b + \kappa < 0$, J_N^r may become positive or negative (Appendix F), enabling pseudo-ICC in the spin-polarized particle current [Fig. 5(b)].

It follows from the above discussion that the occurrence of negative cross effects provides a strong indication for the emergence of a genuine ICC. As a result, an unconventional Seebeck or unconventional Peltier effect where an induced current flows against the existing thermodynamic force can be regarded as manifestations of pseudo-ICC (see Ref. [44] for further details). Nevertheless, this scenario does not correspond to a true or genuine ICC, since the conjugate force attributed to the current is absent. In this sense, the pseudo-ICC regime serves as a precursor for the realization of genuine ICC behavior. Achieving genuine ICC is therefore considerably more challenging than observing pseudo-ICC, making it a particularly counterintuitive and nontrivial manifestation of coupled transport phenomena.

B. Genuine ICC: When both forces are nonzero

From the previous two cases, pseudo-ICC in both energy and particle currents occurs only when $\varepsilon_b + \kappa < 0$, i.e., with negative κ . Therefore, the search for genuine ICC is restricted to the regime $-\kappa > \varepsilon_b > 0$ (equivalently $-1 < \theta < 0$). Also, without loss of generality—we focus only on the first quadrant of Fig. 4, where both forces ($\mathcal{F}_E^r, \mathcal{F}_N^r$) are positive and mutually parallel. Because $\kappa < 0$, Eq. (33c) implies $\mathcal{M} > \mathcal{N}$. The positivity of the particle force $\mathcal{F}_N^r$ further requires $\mathcal{M} > 1$, which in turn yields $\mathcal{X} > 0$. Imposing positivity of both forces and using Eq. (33d), we obtain

$$(\varepsilon_b + \kappa) \ln \mathcal{M} < \varepsilon_b \ln \mathcal{N}, \quad (40)$$

which allows $\mathcal{N}$ to be either greater or less than unity, i.e., $\mathcal{N} \geq 1$, corresponding to $\mathcal{Y} \geq 0$. We therefore arrive at two possible subcases: (1) $\mathcal{M} > 1$, $\mathcal{N} > 1$ and (2) $\mathcal{M} > 1$, $\mathcal{N} < 1$, which ultimately determine the conditions for achieving genuine ICC in J_E^r and J_N^r , respectively.

1. ICC in energy current J_E^r

Since both $\mathcal{M}, \mathcal{N} > 1$ (equivalently $\mathcal{X}, \mathcal{Y} > 0$), it follows directly that $J_N^r = \frac{1}{2}(\mathcal{X} + \mathcal{Y}) > 0$. For $\kappa < 0$ and $\mathcal{F}_E^r > 0$, we have $\Gamma_\odot > 0$, which implies the sign of J_E^r is not guaranteed, and the energy current may take either sign, $J_E^r \geq 0$. Genuine ICC appears precisely when $J_E^r < 0$ [Fig. 6(a)], indicating energy flow against both positive forces $\mathcal{F}_N^r$ and $\mathcal{F}_E^r$. Thus, our present model can function as an autonomous thermoelectric refrigerator [5], driving energy flux against thermal gradient with a co-efficient of performance (COP) given by

$$\zeta = \frac{-J_E^r \beta_r}{J_N^r \mathcal{F}_N^r} = \zeta_R \left(1 - \frac{\dot{\Sigma}}{J_N^r \mathcal{F}_N^r} \right) \leq \zeta_R, \quad (41)$$

where ζ_R is the COP of an ideal refrigerator, defined as $\zeta_R = T_{\text{cold}}/(T_{\text{hot}} - T_{\text{cold}})$.

2. ICC in particle current J_N^r

In the second case, defined by $\mathcal{M} > 1$ and $\mathcal{N} < 1$ (i.e., $\mathcal{X} > 0$ and $\mathcal{Y} < 0$), the particle current $J_N^r = \frac{1}{2}(\mathcal{X} + \mathcal{Y})$ can be either positive or negative, $J_N^r \geq 0$. Genuine ICC

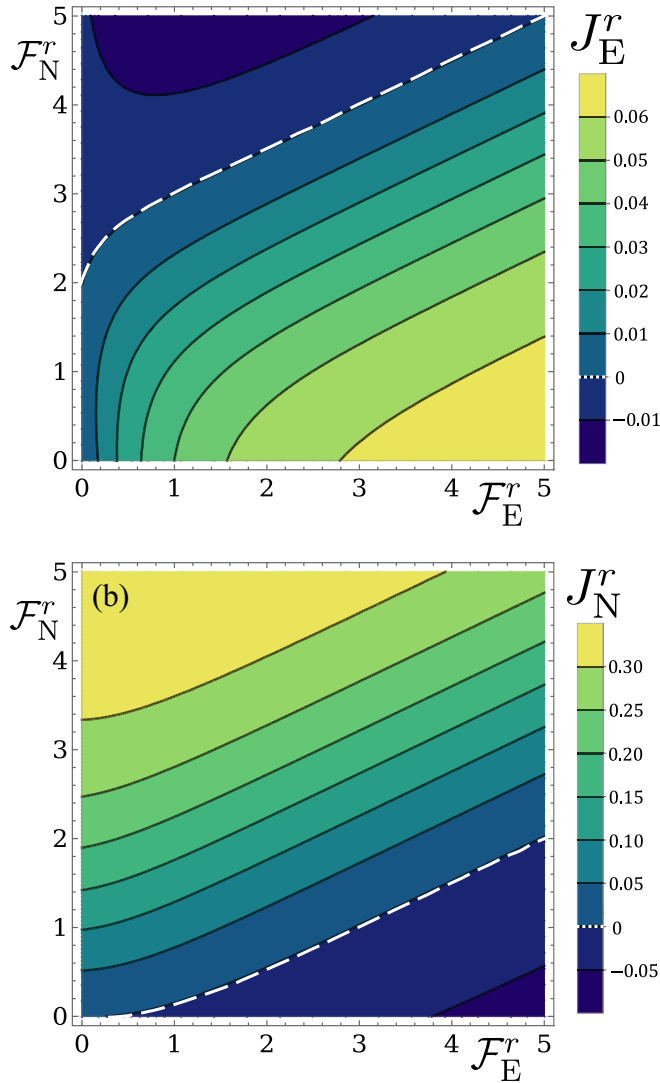


FIG. 6. Variation of (a) the spin-polarized particle current J_E^r and (b) the energy current J_N^r with both the thermodynamic forces for $\kappa = -1.5\hbar\gamma$. Genuine ICC for J_E^r occurs in the area above the white dashed line in panel (a), while it occurs for J_N^r below the white dashed line in panel (b). Other system and bath parameters: $\varepsilon_b = 1.0\hbar\gamma$, $\varepsilon_u = 2.5\hbar\gamma$, $\beta_r = 1/\hbar\gamma$, and $\mu_r = 1.0\hbar\gamma$.

occurs when $J_N^r < 0$ [Fig. 6(b)], corresponding to particle flow against both positive forces $\mathcal{F}_N^r$ and $\mathcal{F}_E^r$.

Thus, our reduced three-terminal model can be implemented as an autonomous (spin-thermoelectric) heat engine [45], driving spin-polarized particle flux against the particle force, under the influence of the nonconjugate energy force $\mathcal{F}_E^r$. The output power is defined as $|(J_N^r \mathcal{F}_N^r / \beta)|$, and the device efficiency is given by

$$\eta = \frac{-J_N^r \mathcal{F}_N^r}{\beta J_E^r} = \eta_C - \frac{\dot{\Sigma}}{\beta J_E^r} \leq \eta_C, \quad (42)$$

where $\eta_C = (T_{\text{hot}} - T_{\text{cold}})/T_{\text{hot}}$.

To summarize, a genuine ICC is achieved in both energy- and spin-polarized particle currents when the two *mutually parallel* forces $\mathcal{F}_E^r$ and $\mathcal{F}_N^r$ are positive. This allows the model

to function as both a spin-thermoelectric heat engine and a refrigerator. The key takeaway from the present model is that the condition $-\kappa > \varepsilon_b > 0$ is essential for observing ICC in energy and particle currents, representing the swapping of energy levels between $|0 \uparrow\rangle(|C\rangle)$ and $|\downarrow \uparrow\rangle(|D\rangle)$. For $\kappa > 0$, the eigenstates are arranged in such a way [Fig. 1(b)] that the particle (de)excitation implies energy (de)excitation. When $-\kappa > \varepsilon_b > 0$, however, the reordering of the eigenstates [Fig. 1(c)] creates an asymmetry between the particle and energy exchange, namely, the particle excitation between $|0 \uparrow\rangle \rightarrow |\downarrow \uparrow\rangle$ implies the energetic deexcitation and vice versa. It is worth mentioning that Wang *et al.* [3] also identified analogous symmetry breaking as a guiding factor for the emergence of ICC in classical Hamiltonian systems. Thus, it is intuitively evident that some form of symmetry breaking is necessary to achieve inverse current in coupled quantum transport. In the present model with double QDs, an attractive interdot ($\kappa < 0$) interaction [15,16] serves as the necessary criterion, while $-\kappa > \varepsilon_b > 0$ acts as the sufficient condition for ICC by breaking the energy-particle exchange symmetry. Furthermore, the present results validate the second law of thermodynamics by ensuring that the two ICC regions do not overlap (Fig. 6). Specifically, ICC in J_N^r occurs when both $\mathcal{M}, \mathcal{N}$ are positive, whereas for J_E^r , $\mathcal{N}$ must be negative, in accordance with the sufficient criteria for genuine ICC. Thus, our analysis highlights ICC as a counterintuitive thermodynamic phenomenon distinct from normal cross effect, and particularly significant for coupled transport with two *mutually parallel* thermodynamic forces.

VI. CONCLUSION

To conclude, identifying the correct entropic bias and its conjugate fluxes in systems with multiple reservoirs is crucial for comprehending complex thermodynamic phenomena. Reducing the three-terminal model to a two-force transport system clarifies the distinction between ICC and normal cross effects, since the three-terminal case, with its *nonphysical* entropic bias, obscures the unambiguous identification of ICC. Above all, the present work develops a comprehensive quantum thermodynamic framework for the ICC, where one induced current flows counter to two mutually parallel thermodynamic forces in the system. Using a simple model of Coulomb-coupled QDs, we investigate the intriguing inverse current behavior in both energy and spin-polarized particle currents under near-equilibrium conditions. Our study examines the entropy production rate at both macroscopic and microscopic levels, leveraging the grand-canonical formalism of the Lindblad master equation and Schnakenberg's entropy formulation. The linear structure of the quantum master equation allows for exact analytical expressions for entropic bias and entropic fluxes, incorporating macroscopic reservoir parameters and microscopic system details. This approach uniquely identifies all possible entropic force-flux pairs for both general and coupled transport systems with *mutually parallel* forces, enabling a systematic exploration of genuine ICC phenomena in energy and spin-polarized particle currents.

This could open potential applications for unconventional autonomous nano-thermoelectric engines and refrigerators based on the ICC effect in coupled QD systems. Notably,

our analysis indicates that an autonomous QD refrigerator or engine requires attractive interactions between QDs as a necessary criterion [15–18] and highlights the possibility for more versatile spin-thermoelectric devices compared to traditional designs, with the novelty rooted in the counterintuitive thermodynamic behavior of inverse current. A similar concept for an autonomous circular heat engine, inspired by the ICC effect in classical systems [3], has already been recently proposed by Benenti *et al.* [45]. We also believe our findings will play a pivotal role in advancing ICC-assisted unconventional autonomous quantum thermal devices in the near future.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: DERIVATION OF THE LINDBLAD MASTER EQUATION

We are going to derive the master equation in the interaction picture for studying the dynamical evolution of the system due to interaction with the reservoir through the exchange of both particles and energy. Let us start with the tunneling Hamiltonian, which essentially signifies the interaction,

$$H_T = H_T^{b\downarrow l} + H_T^{b\downarrow r} + H_T^{u\uparrow u}, \quad (A1)$$

where $H_T^{b\downarrow l(r)}$ signifies the interaction of QD_b with the reservoir $l(r)$ through the exchange of spin-down electrons and similarly $H_T^{u\uparrow u}$ signifies the interaction of QD_u with the reservoir u through the exchange of spin-up electrons, which are given in Eq. (2). To derive the master equation, we start with the von Neumann equation for the total density matrix ρ_{tot} in the interaction picture

$$\frac{d}{dt}\rho_{\text{tot}} = -\frac{i}{\hbar}[H_T(t), \rho_{\text{tot}}(t)]. \quad (A2)$$

Integrating the above equation and taking a trace over the bath degrees of freedom, one obtains

$$\frac{\partial}{\partial t}\rho_s(t) = \frac{1}{(i\hbar)^2} \int_0^t dt' \text{Tr}_{l,r,u}[H_T(t), [H_T(t-t'), \rho_{\text{tot}}(t')]], \quad (A3)$$

where $\text{Tr}_{l,r,u}$ refers to the trace over each bath degrees of freedom and $\text{Tr}_{l,r,u}\{\rho_{\text{tot}}(t)\} = \rho_s(t)$ denotes the reduced density operator for the system. We also assume that $\text{Tr}_{l,r,u}[H_T(t), \rho_{\text{tot}}(0)] = 0$. Under the Born-Markov approximation, the above equation can be rewritten as [7,28,46]

$$\dot{\rho}_s(t) = \frac{1}{(i\hbar)^2} \sum_{\lambda,\lambda'} \int_0^\infty dt' \text{Tr}_{l,r,u}[H_T^\lambda(t), [H_T^{\lambda'}(t-t'), \rho_s(t) \otimes \rho_l \otimes \rho_r \otimes \rho_u]], \quad (A4)$$

where we use the following properties of the bath operators $\text{Tr}_\lambda\{c_\lambda(t)\rho_\lambda\} = 0 = \text{Tr}_\lambda\{c_\lambda^\dagger(t)\rho_\lambda\}$ and $\text{Tr}_{l,r,u}\{[H_T^\lambda(t), [H_T^{\lambda'}(t-t'), \rho_s(t) \otimes \rho_l \otimes \rho_r \otimes \rho_u]]\} = 0; \lambda \neq \lambda'; \lambda, \lambda' = l, r, u$. Now, in the above equation, we use the interaction picture system and bath operators

$$\begin{aligned} d_\alpha(t) &= e^{iH_S t/\hbar} d_\alpha e^{-iH_S t/\hbar} = \sum_{\omega_{ij} > 0} e^{-i\omega_{ij} t/\hbar} d_\alpha, \quad \alpha = b, u, \\ c_\lambda(t) &= e^{iH_B t/\hbar} c_\lambda e^{-iH_B t/\hbar} = \sum_k e^{-i(\epsilon_k^\lambda - \mu_\lambda)t/\hbar} c_\lambda, \end{aligned} \quad (A5)$$

and their hermitian adjoints, where ω_{ij} is defined as the transition energy for the transition between the system eigenstates $|\hat{i}\rangle$ and $|\hat{j}\rangle$. Eliminating the high-frequency oscillating terms by the standard procedure of secular approximation, one can finally derive the master equation in the following form:

$$\dot{\rho}_s(t) = \sum_\lambda \mathcal{L}_\lambda[\rho_s(t)], \quad (A6)$$

where the Lindblad operators $\mathcal{L}_\lambda[\rho_s(t)]$ are given by

$$\begin{aligned} \mathcal{L}_\lambda[\rho_s(t)] &= \sum_{\{\omega_\alpha\} > 0} \mathcal{G}_\lambda(\omega_\alpha) \left[d_\alpha^\dagger(\omega_\alpha) \rho_s d_\alpha(\omega_\alpha) - \frac{1}{2} \{ \rho_s, d_\alpha(\omega_\alpha) d_\alpha^\dagger(\omega_\alpha) \} \right] \\ &+ \mathcal{G}_\lambda(-\omega_\alpha) \left[d_\alpha(\omega_\alpha) \rho_s d_\alpha^\dagger(\omega_\alpha) - \frac{1}{2} \{ \rho_s, d_\alpha^\dagger(\omega_\alpha) d_\alpha(\omega_\alpha) \} \right]. \end{aligned} \quad (A7)$$

In the above equation, we define the temperature-dependent bath spectral functions as

$$\mathcal{G}_\lambda(\omega_\alpha) = \gamma_\lambda(\omega_\alpha)f_\lambda^+(\omega_\alpha), \quad \mathcal{G}_\lambda(-\omega_\alpha) = \gamma_\lambda(\omega_\alpha)f_\lambda^-(\omega_\alpha), \quad (\text{A8})$$

where $\gamma_\lambda(\omega_\alpha)$ is the bare electron transfer rate between the reservoir λ and coupled QD $_\alpha$. The explicit forms in terms of the system-reservoir coupling constants can be calculated using Fermi's golden rule, as $\gamma_\lambda(\omega_\alpha) \equiv \gamma_\lambda(\omega_\alpha) = 2\pi \sum_k |t_k^{\alpha\lambda}|^2 \delta(\omega_\alpha - \epsilon_k^\lambda)$, where ω_α represents the required amount of energy associated with the transition between QD $_\alpha$ and its coupled lead. The function $f_\lambda^\pm(\omega_{ij})$ represents the (FDF), which are obtained by tracing over the bath density operator; for example, $f_\lambda^+(\omega_{ij}) = \text{Tr}_\lambda(c_\lambda^\dagger c_\lambda \rho_\lambda)$ and $f_\lambda^-(\omega_{ij}) = \text{Tr}_\lambda(c_\lambda c_\lambda^\dagger \rho_\lambda)$, where the bath operators $c_\lambda^\dagger$ and c_λ obey anticommutation relation and the reservoir governs the transition between eigenstate $|i\rangle$ to $|j\rangle$, which costs ω_{ij} amount of energy. The explicit expressions of the FDFs for various transitions are mentioned in the main text.

Now, to derive the forms of the Lindbladians, we need to express the creation and the annihilation operators in terms of the system eigenstates in the following forms:

$$\begin{aligned} \sum_{\{w_b\}} d_b^\dagger(w_b) &= d_b^\dagger(\omega_{AB}) + d_b^\dagger(\omega_{CD}) = |\mathbb{B}\rangle\langle\mathbb{A}| + |\mathbb{D}\rangle\langle\mathbb{C}|, & \sum_{\{w_b\}} d_b(w_b) &= d_b(\omega_{BA}) + d_b(\omega_{DC}) = |\mathbb{A}\rangle\langle\mathbb{B}| + |\mathbb{C}\rangle\langle\mathbb{D}|, \\ \sum_{\{w_u\}} d_u^\dagger(w_u) &= d_u^\dagger(\omega_{AC}) + d_u^\dagger(\omega_{BD}) = |\mathbb{C}\rangle\langle\mathbb{A}| + |\mathbb{D}\rangle\langle\mathbb{B}|, & \sum_{\{w_u\}} d_u(w_u) &= d_u(\omega_{CA}) + d_u(\omega_{DB}) = |\mathbb{A}\rangle\langle\mathbb{C}| + |\mathbb{B}\rangle\langle\mathbb{D}|. \end{aligned} \quad (\text{A9})$$

So, the time evolution of the occupation probabilities, which are essentially the diagonal elements of the reduced density matrix $\rho_{ii} = \langle i | \rho_s(t) | i \rangle$, can be obtained using the LME [Eq. (A7)] as follows:

$$\dot{\rho}_{ii} \equiv \frac{d\rho_{ii}}{dt} = \langle i | \dot{\rho}_s(t) | i \rangle = \sum_\lambda \langle i | \mathcal{L}_\lambda[\rho_s(t)] | i \rangle. \quad (\text{A10})$$

APPENDIX B: EXPLICIT EXPRESSIONS OF VARIOUS CURRENTS

From Eq. (15), one can evaluate the explicit expressions of both energy and particle currents for each reservoir in the following way:

$$\begin{aligned} J_E^l &= \omega_{AB}\Gamma_{AB}^{l+} + \omega_{CD}\Gamma_{CD}^{l+} = \varepsilon_b\Gamma_{AB}^{l+} + (\varepsilon_b + \kappa)\Gamma_{CD}^{l+}, & J_E^r &= \omega_{AB}\Gamma_{AB}^{r+} + \omega_{CD}\Gamma_{CD}^{r+} = \varepsilon_b\Gamma_{AB}^{r+} + (\varepsilon_b + \kappa)\Gamma_{CD}^{r+}, \\ J_E^u &= \omega_{AC}\Gamma_{AC}^{u+} + \omega_{BD}\Gamma_{BD}^{u+} = \varepsilon_u\Gamma_{AC}^{u+} + (\varepsilon_u + \kappa)\Gamma_{BD}^{u+}, & J_N^l &= \Gamma_{AB}^{l+} + \Gamma_{CD}^{l+}, & J_N^r &= \Gamma_{AB}^{r+} + \Gamma_{CD}^{r+}, & J_N^u &= \Gamma_{AC}^{u+} + \Gamma_{BD}^{u+}. \end{aligned} \quad (\text{B1})$$

Inserting the above relations, the expression of the heat current associated with each reservoir can be evaluated as

$$J_Q^l = (\varepsilon_b - \mu_l)\Gamma_{AB}^{l+} + (\varepsilon_b + \kappa - \mu_l)\Gamma_{CD}^{l+}, \quad J_Q^r = (\varepsilon_b - \mu_r)\Gamma_{AB}^{r+} + (\varepsilon_b + \kappa - \mu_r)\Gamma_{CD}^{r+}, \quad J_Q^u = (\varepsilon_u - \mu_u)\Gamma_{AC}^{u+} + (\varepsilon_u + \kappa - \mu_u)\Gamma_{BD}^{u+}. \quad (\text{B2})$$

APPENDIX C: EXPRESSION OF THE STEADY-STATE TRANSITION RATE

To determine the steady-state transition rate, we rewrite Eqs. (6)–(8) as

$$\mathcal{M} \begin{bmatrix} \rho_A \\ \rho_B \\ \rho_C \\ \rho_D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad (\text{C1})$$

subject to the condition $\rho_A + \rho_B + \rho_C + \rho_D = 1$, and

$$\mathcal{M} = \begin{bmatrix} -f_u^{1+} - f_a^{1+} - f_r^{1+} & f_l^{1-} + f_r^{1-} & f_u^{1+} & 0 \\ f_l^{1+} + f_r^{1+} & -f_l^{1-} - f_r^{1-} - f_u^{2+} & 0 & f_u^{2-} \\ f_u^{1+} & 0 & -f_l^{2+} - f_r^{2+} - f_u^{1-} & f_l^{2-} + f_r^{2-} \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad (\text{C2})$$

where, for the sake of simplicity of our analysis, we assume that $\gamma_l \simeq \gamma_r \simeq \gamma_u \equiv \gamma$. Solving Eq. (C1) with the above matrix $\mathcal{M}$, one can obtain the steady-state population $\{\rho_i\}$ in terms of which we can evaluate the explicit expression

$$\Gamma_\odot = -\Gamma_\ominus = \gamma \left[\frac{f_{lr}^{1+} [f_{lr}^{2+} (f_u^{2+} - f_u^{1+}) + 2f_u^{2+} (f_u^{1+} - 1)] - 2f_u^{1+} f_{lr}^{2+} (f_u^{2+} - 1)}{3f_{lr}^{1+} (f_u^{1+} - f_u^{2+}) - 6 + 3f_{lr}^{2+} (f_u^{2+} - f_u^{1+})} \right], \quad (\text{C3})$$

where we define $f_{lr}^{1+(2+)} = f_l^{1+(2+)} + f_r^{1+(2+)}.$

APPENDIX D: NON-NEGATIVITY OF ENTROPY PRODUCTION RATE

One can evaluate the expression of the entropy production (Σ) from the entropy change of the system [$\Delta S_s = S_s(t) - S_s(0)$], which is defined as

$$\Delta S_s(t) = -k_B \text{Tr}_s[\rho_s(t) \ln \rho_s(t)] + k_B \text{Tr}_s[\rho_s(0) \ln \rho_s(0)] = -k_B \text{Tr}[\rho_{\text{tot}}(t) \ln \rho_s(t)] + k_B \text{Tr}[\rho_{\text{tot}}(0) \ln \rho_s(0)]. \quad (\text{D1})$$

We assume that the initial equilibrium state, $\rho_{\text{tot}}(0)$, does not display any entanglement or correlation between the system and the environment. Therefore,

$$\rho_{\text{tot}}(0) = \rho_s(0) \prod_{\lambda} \rho_{\lambda}^{\text{eq}}. \quad (\text{D2})$$

Inserting Eq. (D2) into Eq. (D1), we can continue as follows:

$$\begin{aligned} \Delta S_s(t) &= -k_B \text{Tr}[\rho_{\text{tot}}(t) \ln \rho_s(t)] + k_B \text{Tr}[\rho_{\text{tot}}(0) \ln \rho_{\text{tot}}(0)] - k_B \sum_{\lambda} \text{Tr}[\rho_{\text{tot}}(0) \ln \rho_{\lambda}^{\text{eq}}] \\ &= -k_B \text{Tr} \left[\rho_{\text{tot}}(t) \ln \left\{ \rho_s(t) \prod_{\lambda} \rho_{\lambda}^{\text{eq}} \right\} \right] + k_B \text{Tr}[\rho_{\text{tot}}(0) \ln \rho_{\text{tot}}(0)] + k_B \sum_{\lambda} \text{Tr}_{\lambda} [\{\rho_{\lambda}(t) - \rho_{\lambda}^{\text{eq}}\} \ln \rho_{\lambda}^{\text{eq}}]. \end{aligned} \quad (\text{D3})$$

Again, $\rho_{\text{tot}}(t)$ and $\rho_{\text{tot}}(0)$ are related through unitary evolution $\rho_{\text{tot}}(t) = \mathcal{U} \rho_{\text{tot}}(0) \mathcal{U}^{\dagger}$, which implies $\text{Tr}[\rho_{\text{tot}}(t) \ln \rho_{\text{tot}}(t)] = \text{Tr}[\rho_{\text{tot}}(0) \ln \rho_{\text{tot}}(0)]$. Applying this relation in the above equation, the final expression of the entropy change of the system can be calculated as

$$\Delta S_s(t) = -k_B \text{Tr} \left[\rho_{\text{tot}}(t) \ln \left\{ \rho_s(t) \prod_{\lambda} \rho_{\lambda}^{\text{eq}} \right\} \right] + k_B \text{Tr}[\rho_{\text{tot}}(t) \ln \rho_{\text{tot}}(t)] + k_B \sum_{\lambda} \text{Tr}_{\lambda} [\{\rho_{\lambda}(t) - \rho_{\lambda}^{\text{eq}}\} \ln \rho_{\lambda}^{\text{eq}}]. \quad (\text{D4})$$

The last term of the above equation can be identified as the *entropy flow* (Φ), representing the reversible contribution to the system entropy change due to heat exchange with the reservoirs. A comparison of the above equation with Eq. (19) defines the *entropy production*, representing the irreversible contribution to the entropy change of the system and the *entropy flow* as

$$\Sigma(t) = k_B \text{Tr}[\rho_{\text{tot}}(t) \ln \{\rho_{\text{tot}}(t)\}] - k_B \text{Tr} \left[\rho_{\text{tot}}(t) \ln \left\{ \rho_s(t) \prod_{\lambda} \rho_{\lambda}^{\text{eq}} \right\} \right], \quad \Phi(t) = k_B \sum_{\lambda} \text{Tr}_{\lambda} [\{\rho_{\lambda}(t) - \rho_{\lambda}^{\text{eq}}\} \ln \rho_{\lambda}^{\text{eq}}]. \quad (\text{D5})$$

The above expression of the *entropy production* can be expressed in terms of the relative entropy:

$$\Sigma(t) \equiv \mathcal{D} \left[\rho_{\text{tot}}(t) \parallel \left\{ \rho_s(t) \prod_{\lambda} \rho_{\lambda}^{\text{eq}} \right\} \right], \quad (\text{D6})$$

where $\mathcal{D}[\rho \parallel \rho']$ is the quantum relative entropy between two density matrices ρ and ρ' , defined via

$$\mathcal{D}[\rho \parallel \rho'] := \text{Tr}[\rho \ln \rho] - \text{Tr}[\rho \ln \rho']. \quad (\text{D7})$$

The non-negativity of relative entropy is affirmed, attaining a value of zero solely in the case of complete matrix identity. While the non-negativity of entropy production does not imply the same for its rate, in the limit of large reservoirs, $\Sigma(t)$ is expected to converge to a convex, monotonically increasing function of time [31]. In the same limit, if the system dynamics are described by a Markovian quantum Lindblad master equation, implying entropy production as a convex functional of the system density matrix [28,31], the entropy production rate $\dot{\Sigma}(t)$ would eventually be positive, only reaching zero for the equilibrium state.

APPENDIX E: DERIVATION OF THE MICROSCOPIC DEFINITION OF THE ENTROPY PRODUCTION RATE

Considering the von Neumann entropy defined as $S_s(t) = -k_B \sum_{\mathfrak{i}} \rho_{\mathfrak{i}}(t) \ln \rho_{\mathfrak{i}}(t)$, where $\rho_{\mathfrak{i}}$ signifies populations of the system eigenstates ($\mathfrak{i} = \mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$), the change in the system entropy is given by

$$\Delta S_s(t) = S_s(t) - S_s(0) = -k_B \sum_{\mathfrak{i}} \rho_{\mathfrak{i}}(t) \ln \rho_{\mathfrak{i}}(t) + k_B \sum_{\mathfrak{i}} \rho_{\mathfrak{i}}(0) \ln \rho_{\mathfrak{i}}(0). \quad (\text{E1})$$

So, the time evolution of the entropy change can be evaluated as

$$\frac{d}{dt} \Delta S_s(t) = -k_B \sum_{\mathfrak{i}} \dot{\rho}_{\mathfrak{i}}(t) \ln \rho_{\mathfrak{i}}(t) + k_B \sum_{\mathfrak{i}} \dot{\rho}_{\mathfrak{i}}(0) \ln \rho_{\mathfrak{i}}(0) = -k_B \sum_{\mathfrak{i}} \dot{\rho}_{\mathfrak{i}}(t) \ln \rho_{\mathfrak{i}}(t) \equiv -k_B \sum_{\mathfrak{i}} \dot{\rho}_{\mathfrak{i}} \ln \rho_{\mathfrak{i}}. \quad (\text{E2})$$

Now, using Eq. (6) for ρ_i , we recover Eq. (29):

$$\frac{d}{dt} \Delta S_s(t) = k_B \left[\Gamma_{AB}^{l+} \ln \left(\frac{\rho_A}{\rho_B} \right) + \Gamma_{AB}^{r+} \ln \left(\frac{\rho_A}{\rho_B} \right) + \Gamma_{BD}^{u+} \ln \left(\frac{\rho_B}{\rho_D} \right) + \Gamma_{DC}^{l-} \ln \left(\frac{\rho_D}{\rho_C} \right) + \Gamma_{DC}^{r-} \ln \left(\frac{\rho_D}{\rho_C} \right) + \Gamma_{CA}^{u-} \ln \left(\frac{\rho_C}{\rho_A} \right) \right]. \quad (E3)$$

By comparing the above equation with Eq. (19), we obtain Eq. (30) for the general expressions of $\dot{\Sigma}(t)$ and $\dot{\Phi}(t)$:

$$\begin{aligned} \dot{\Sigma}(t) &= k_B \left[\Gamma_{AB}^{l+} \ln \left(\frac{k_{AB}^{l+} \rho_A}{k_{BA}^{l-} \rho_B} \right) + \Gamma_{AB}^{r+} \ln \left(\frac{k_{AB}^{r+} \rho_A}{k_{BA}^{r-} \rho_B} \right) + \Gamma_{BD}^{u+} \ln \left(\frac{k_{BD}^{u+} \rho_B}{k_{DB}^{u-} \rho_D} \right) + \Gamma_{DC}^{l-} \ln \left(\frac{k_{DC}^{l-} \rho_D}{k_{CD}^{l+} \rho_C} \right) \right. \\ &\quad \left. + \Gamma_{DC}^{r-} \ln \left(\frac{k_{DC}^{r-} \rho_D}{k_{CD}^{r+} \rho_C} \right) + \Gamma_{CA}^{u-} \ln \left(\frac{k_{CA}^{u-} \rho_C}{k_{AC}^{u+} \rho_A} \right) \right] \\ &= k_B \left[(k_{AB}^{l+} \rho_A - k_{BA}^{l-} \rho_B) \ln \left(\frac{k_{AB}^{l+} \rho_A}{k_{BA}^{l-} \rho_B} \right) + (k_{AB}^{r+} \rho_A - k_{BA}^{r-} \rho_B) \ln \left(\frac{k_{AB}^{r+} \rho_A}{k_{BA}^{r-} \rho_B} \right) + (k_{BD}^{u+} \rho_B - k_{DB}^{u-} \rho_D) \ln \left(\frac{k_{BD}^{u+} \rho_B}{k_{DB}^{u-} \rho_D} \right) \right. \\ &\quad \left. + (k_{DC}^{l-} \rho_D - k_{CD}^{l+} \rho_C) \ln \left(\frac{k_{DC}^{l-} \rho_D}{k_{CD}^{l+} \rho_C} \right) + (k_{DC}^{r-} \rho_D - k_{CD}^{r+} \rho_C) \ln \left(\frac{k_{DC}^{r-} \rho_D}{k_{CD}^{r+} \rho_C} \right) + (k_{CA}^{u-} \rho_C - k_{AC}^{u+} \rho_A) \ln \left(\frac{k_{CA}^{u-} \rho_C}{k_{AC}^{u+} \rho_A} \right) \right], \\ \dot{\Phi}(t) &= -k_B \left[\Gamma_{AB}^{l+} \ln \left(\frac{k_{AB}^{l+}}{k_{BA}^{l-}} \right) + \Gamma_{AB}^{r+} \ln \left(\frac{k_{AB}^{r+}}{k_{BA}^{r-}} \right) + \Gamma_{BD}^{u+} \ln \left(\frac{k_{BD}^{u+}}{k_{DB}^{u-}} \right) + \Gamma_{DC}^{l-} \ln \left(\frac{k_{DC}^{l-}}{k_{CD}^{l+}} \right) + \Gamma_{DC}^{r-} \ln \left(\frac{k_{DC}^{r-}}{k_{CD}^{r+}} \right) + \Gamma_{CA}^{u-} \ln \left(\frac{k_{CA}^{u-}}{k_{AC}^{u+}} \right) \right]. \end{aligned} \quad (E4)$$

There is no net entropy change in the system at the steady state, which reduces Eq. (31) for the form of the entropy production rate:

$$\begin{aligned} \dot{\Sigma}(t) = -\dot{\Phi}(t) &= k_B \left[\Gamma_{AB}^{l+} \ln \left(\frac{k_{AB}^{l+}}{k_{BA}^{l-}} \right) + \Gamma_{AB}^{r+} \ln \left(\frac{k_{AB}^{r+}}{k_{BA}^{r-}} \right) + \Gamma_{BD}^{u+} \ln \left(\frac{k_{BD}^{u+}}{k_{DB}^{u-}} \right) + \Gamma_{DC}^{l-} \ln \left(\frac{k_{DC}^{l-}}{k_{CD}^{l+}} \right) \right. \\ &\quad \left. + \Gamma_{DC}^{r-} \ln \left(\frac{k_{DC}^{r-}}{k_{CD}^{r+}} \right) + \Gamma_{CA}^{u-} \ln \left(\frac{k_{CA}^{u-}}{k_{AC}^{u+}} \right) \right] \\ &= k_B \left[\Gamma_{AB}^{l+} \ln \left(\frac{k_{AB}^{l+}}{k_{BA}^{l-}} \right) + \Gamma_{BD}^{u+} \ln \left(\frac{k_{BD}^{u+}}{k_{DB}^{u-}} \right) + \Gamma_{DC}^{l-} \ln \left(\frac{k_{DC}^{l-}}{k_{CD}^{l+}} \right) + \Gamma_{CA}^{u-} \ln \left(\frac{k_{CA}^{u-}}{k_{AC}^{u+}} \right) \right] \\ &\quad + k_B \left[\Gamma_{AB}^{r+} \ln \left(\frac{k_{AB}^{r+} k_{BA}^{l-}}{k_{BA}^{r-} k_{AB}^{l+}} \right) + \Gamma_{DC}^{r-} \ln \left(\frac{k_{DC}^{r-} k_{CD}^{l+}}{k_{CD}^{r+} k_{DC}^{l-}} \right) \right] \\ &= k_B \Gamma_{\odot} \ln \left(\frac{k_{AB}^{l+} k_{BD}^{u+} k_{DC}^{l-} k_{CA}^{u-}}{k_{BA}^{l-} k_{DB}^{u-} k_{CD}^{l+} k_{AC}^{u+}} \right) + k_B (\Gamma_{AB}^{r+} - \Gamma_{DC}^{r-}) \ln \left(\frac{k_{AB}^{r+} k_{BA}^{l-}}{k_{BA}^{r-} k_{AB}^{l+}} \right) + k_B \Gamma_{DC}^{r-} \ln \left(\frac{k_{DC}^{r-} k_{CD}^{l+} k_{AB}^{r+} k_{BA}^{l-}}{k_{BA}^{r-} k_{AB}^{l+} k_{CD}^{r+} k_{DC}^{l-}} \right) \\ &= \kappa \Gamma_{\odot} \left[\left(\frac{k_B}{\kappa} \right) \ln \left(\frac{k_{AB}^{l+} k_{BD}^{u+} k_{DC}^{l-} k_{CA}^{u-}}{k_{BA}^{l-} k_{DB}^{u-} k_{CD}^{l+} k_{AC}^{u+}} \right) \right] + \{ \varepsilon_b \Gamma_{AB}^{r+} - (\varepsilon_b + \kappa) \Gamma_{DC}^{r-} \} \left[\left(\frac{k_B}{\kappa} \right) \ln \left(\frac{k_{BA}^{r+} k_{AB}^{l+} k_{CD}^{r+} k_{DC}^{l-}}{k_{DC}^{r-} k_{CD}^{l+} k_{AB}^{r+} k_{BA}^{l-}} \right) \right] \\ &\quad + (\Gamma_{AB}^{r+} - \Gamma_{DC}^{r-}) \left[k_B (1 + \theta) \ln \left(\frac{k_{AB}^{r+} k_{BA}^{l-}}{k_{BA}^{r-} k_{AB}^{l+}} \right) + k_B \theta \ln \left(\frac{k_{DC}^{r-} k_{CD}^{l+}}{k_{CD}^{r+} k_{DC}^{l-}} \right) \right]. \end{aligned} \quad (E5)$$

APPENDIX F: DETERMINATION OF THE SIGN OF $\Gamma_{\odot}$ FROM $\mathcal{PQ}^{-1}$ AND CRITERIA OF PSEUDO-ICC

1. Expression of $\mathcal{PQ}^{-1}$

From Eq. (33a), the condition $\mathcal{F}_E^u = 0$ implies

$$\left(\frac{k_{AB}^{a+} k_{BD}^{r+} k_{DC}^{a-} k_{CA}^{r-}}{k_{BA}^{a-} k_{DB}^{r-} k_{CD}^{a+} k_{AC}^{r+}} \right) = 1. \quad (F1)$$

We can rewrite the above condition as

$$\left(\frac{k_{AB}^{ab+} k_{BD}^{r+} k_{DC}^{ab-} k_{CA}^{r-}}{k_{BA}^{ab-} k_{DB}^{r-} k_{CD}^{ab+} k_{AC}^{r+}} \right) = \left(\frac{k_{BA}^{a+} k_{AB}^{ab+} k_{CD}^{a+} k_{DC}^{ab-}}{k_{AB}^{a-} k_{BA}^{ab-} k_{DC}^{a-} k_{CD}^{ab+}} \right). \quad (F2)$$

The right-hand side (rhs) of the above equation can be expressed as

$$\begin{aligned} \text{rhs} &= \left(\frac{k_{\text{BA}}^{a+} k_{\text{AB}}^{ab+} k_{\text{CD}}^{a+} k_{\text{DC}}^{ab-}}{k_{\text{AB}}^{a-} k_{\text{BA}}^{ab-} k_{\text{DC}}^{a-} k_{\text{CD}}^{ab+}} \right) \\ &= \left\{ \frac{1 + (k_{\text{AB}}^{b+}/k_{\text{AB}}^{a+})}{1 + (k_{\text{BA}}^{b-}/k_{\text{BA}}^{a-})} \right\} \left\{ \frac{1 + (k_{\text{CD}}^{b+}/k_{\text{CD}}^{a+})}{1 + (k_{\text{DC}}^{b-}/k_{\text{DC}}^{a-})} \right\}^{-1} \equiv \mathcal{P}\mathcal{Q}^{-1}. \end{aligned} \quad (\text{F3})$$

One can determine the sign of $\Gamma_{\odot}$ from $\mathcal{P}\mathcal{Q}^{-1}$.

2. Determination of the sign of $\Gamma_{\odot}$ from $\mathcal{P}\mathcal{Q}^{-1}$

When, $\mathcal{P} = \mathcal{Q}$, applying Eq. (F3) to Eq. (F2), we find the left-hand side (lhs) of Eq. (F2) as

$$\text{lhs} = \left(\frac{k_{\text{AB}}^{ab+} k_{\text{BD}}^{r+} k_{\text{DC}}^{ab-} k_{\text{CA}}^{r-}}{k_{\text{BA}}^{ab-} k_{\text{DB}}^{r-} k_{\text{CD}}^{ab+} k_{\text{AC}}^{r+}} \right) = 1, \quad (\text{F4})$$

which can be rewritten as

$$\left(\frac{k_{\text{AB}}^{ab+} \rho_{\text{A}}}{k_{\text{BA}}^{ab-} \rho_{\text{B}}} \right) \left(\frac{k_{\text{BD}}^{r+} \rho_{\text{B}}}{k_{\text{DB}}^{r-} \rho_{\text{D}}} \right) \left(\frac{k_{\text{DC}}^{ab-} \rho_{\text{D}}}{k_{\text{CD}}^{ab+} \rho_{\text{C}}} \right) \left(\frac{k_{\text{CA}}^{r-} \rho_{\text{C}}}{k_{\text{AC}}^{r+} \rho_{\text{A}}} \right) = 1. \quad (\text{F5})$$

To satisfy this condition at the steady state, following Eq. (16), each term within the parentheses must be equal to 1. When combined with Eq. (7), this in turn leads to $\Gamma_{\odot} = \Gamma_{\odot} = 0$.

On the other hand, following the same procedure, we can verify that for $\mathcal{P}\mathcal{Q}^{-1} \geq 1$, we have

$$\left(\frac{k_{\text{AB}}^{ab+} \rho_{\text{A}}}{k_{\text{BA}}^{ab-} \rho_{\text{B}}} \right) \left(\frac{k_{\text{BD}}^{r+} \rho_{\text{B}}}{k_{\text{DB}}^{r-} \rho_{\text{D}}} \right) \left(\frac{k_{\text{DC}}^{ab-} \rho_{\text{D}}}{k_{\text{CD}}^{ab+} \rho_{\text{C}}} \right) \left(\frac{k_{\text{CA}}^{r-} \rho_{\text{C}}}{k_{\text{AC}}^{r+} \rho_{\text{A}}} \right) \geq 1. \quad (\text{F6})$$

At steady state, when each term in parentheses is the same, the application of Eqs. (7) and (16) yields the result $\Gamma_{\odot} \leq 0$.

a. Consistency check: zero force-zero current

As a simple consistency check, let us consider the trivial case where both forces vanish, $(\mathcal{F}_{\text{E}}^r, \mathcal{F}_{\text{N}}^r) = 0$. In this limit, both currents ($J_{\text{E}}^r, J_{\text{N}}^r$) must be zero.

This is evident since $\mathcal{F}_{\text{E}}^r = 0$, Eq. (33c) gives $\mathcal{M} = \mathcal{N}$. Likewise, $\mathcal{F}_{\text{N}}^r = 0$ implies $\mathcal{M} = 1$ from Eq. (33e). Thus, the condition $(\mathcal{F}_{\text{E}}^r, \mathcal{F}_{\text{N}}^r) = 0$ is equivalent to $\mathcal{M} = \mathcal{N} = 1$, and hence $\mathcal{X} = \mathcal{Y} = 0$. Similarly, for $\mathcal{F}_{\text{E}}^r = 0$, the Fermi functions further imply $\mathcal{P}\mathcal{Q}^{-1} = 1$, consistent with $\mathcal{M} = 1$. This yields $\Gamma_{\odot} = 0$.

Therefore, $\mathcal{X} = \mathcal{Y} = \Gamma_{\odot} = 0$ and Eq. (38) gives $\Gamma_{\text{AB}}^{r+} = \Gamma_{\text{CD}}^{r+} = 0$. So, both currents ($J_{\text{E}}^r, J_{\text{N}}^r$) vanish, as expected.

b. Setting $\mathcal{F}_{\text{E}}^r = 0$: pseudo-ICC in J_{E}^r

In this case, $\mathcal{F}_{\text{E}}^r = 0$ and $\mathcal{F}_{\text{N}}^r > 0$. From Eq. (33c), the condition $\mathcal{F}_{\text{E}}^r = 0$ implies $\mathcal{M} = \mathcal{N}$, while Eq. (33e) shows that $\mathcal{F}_{\text{N}}^r > 0$ requires $\mathcal{M} > 1$. Thus, we obtain $\mathcal{M} = \mathcal{N} > 1$, which means $\mathcal{X} > 0$ and $\mathcal{Y} > 0$, and consequently $J_{\text{N}}^r = \frac{1}{2}(\mathcal{X} + \mathcal{Y}) > 0$. This agrees with the non-negativity of the entropy production rate, $\dot{\Sigma} = J_{\text{N}}^r \mathcal{F}_{\text{N}}^r$. Hence, the spin-polarized particle current remains positive [dotted red line in Fig. 5(a)].

Since the particle current is positive in this case, the energy current satisfies $J_{\text{E}}^r > \kappa \Gamma_{\text{CD}}^{r+}$ [cf. Eq. (32)]. With $\mathcal{F}_{\text{E}}^r = 0$ and $\mathcal{M} > 1$, it follows that $\Gamma_{\odot} \leq 0$ for $\kappa \geq 0$, which in turn implies [cf. Eq. (38)] that $\Gamma_{\text{CD}}^{r+} > 0$ for $\kappa > 0$, and $\Gamma_{\text{CD}}^{r+} \geq 0$ for $\kappa < 0$. Thus, for $\kappa > 0$, we immediately obtain $J_{\text{E}}^r > 0$, while for $\kappa < 0$ we must examine two subcases: (1) $\varepsilon_{\text{b}} + \kappa > 0$ and (2) $\varepsilon_{\text{b}} + \kappa < 0$.

(1) For case (1), it is straightforward to show that $J_{\text{E}}^r = \varepsilon_{\text{b}} \Gamma_{\text{AB}}^{r+} + (\varepsilon_{\text{b}} + \kappa) \Gamma_{\text{CD}}^{r+} > 0$, when $\Gamma_{\text{CD}}^{r+} > 0$, and that $J_{\text{E}}^r > \kappa \Gamma_{\text{CD}}^{r+} > 0$, when $\Gamma_{\text{CD}}^{r+} < 0$.

(2) In contrast, for case (2), J_{E}^r can be either positive or negative [cf. Eq. (32)], indicating the possibility of pseudo-ICC in J_{E}^r [dashed blue-green line Fig. 5(a)].

c. Setting $\mathcal{F}_{\text{N}}^r = 0$: pseudo-ICC in J_{N}^r

In this case, $\mathcal{F}_{\text{N}}^r = 0$ and $\mathcal{F}_{\text{E}}^r > 0$. From Eq. (33e), these conditions immediately give $\mathcal{M} > 1$. Moreover, using $\mathcal{F}_{\text{N}}^r = 0$ in Eq. (33d), we obtain

$$(\varepsilon_{\text{b}} + \kappa) \ln \mathcal{M} = \varepsilon_{\text{b}} \ln \mathcal{N}. \quad (\text{F7})$$

Substituting $\mathcal{M} > 1$ into this relation leads to different outcomes depending on the sign of κ , as summarized below:

(1) For $\kappa > 0$ and case (1) with $\kappa < 0$, but $\varepsilon_b + \kappa > 0$, Eq. (F7) gives $\mathcal{N} > 1$. Thus, the microscopic condition becomes $\mathcal{M} > 1$ and $\mathcal{N} > 1$, which implies $\mathcal{X} > 0$ and $\mathcal{Y} > 0$ [cf. Eq. (37)], leading to $J_N^r = \frac{1}{2}(\mathcal{X} + \mathcal{Y}) > 0$.

(2) In contrast, for case (2) (i.e., $\varepsilon_b + \kappa < 0$), Eq. (F7) yields $\mathcal{N} < 1$. Here, the eigenstates $|\mathbb{C}\rangle$ and $|\mathbb{D}\rangle$ flip together, giving $\mathcal{M} > 1$ and $\mathcal{N} < 1$, and hence $\mathcal{X} > 0$ and $\mathcal{Y} < 0$. Consequently, $J_N^r = \frac{1}{2}(\mathcal{X} + \mathcal{Y}) \geq 0$, meaning the spin-polarized particle current can be positive or negative, depending on the balance between the positive $\mathcal{X}$ and negative $\mathcal{Y}$. This allows for pseudo-ICC in J_N^r [dashed red line in Fig. 5(b)].

Again, since $\mathcal{F}_E^r > 0$ and $\mathcal{M} > 1$, it follows that $\Gamma_{\mathbb{O}} \leq 0$ for $\kappa \geq 0$, leading to the following casewise outcomes:

(1) For $\kappa > 0$: The particle current is positive, allowing us to rewrite the energy current as $J_E^r > \kappa \Gamma_{\mathbb{CD}}^{r+}$. Because $\mathcal{X} > 0$ and $\mathcal{Y} > 0$, Eq. (38) gives $\Gamma_{\mathbb{CD}}^{r+} > 0$ (as $\Gamma_{\mathbb{O}} < 0$). Hence, $J_E^r > \kappa \Gamma_{\mathbb{CD}}^{r+} > 0$.

(2) Case (1): $\varepsilon_b + \kappa > 0$ with $\kappa < 0$: Here, $\Gamma_{\mathbb{CD}}^{r+} \geq 0$ (because $\Gamma_{\mathbb{O}} > 0$ and $\mathcal{Y} > 0$). Consequently, $J_E^r = \varepsilon_b \Gamma_{\mathbb{AB}}^{r+} + (\varepsilon_b + \kappa) \Gamma_{\mathbb{CD}}^{r+} > 0$, whenever $\Gamma_{\mathbb{CD}}^{r+} > 0$, and $J_E^r > \kappa \Gamma_{\mathbb{CD}}^{r+} > 0$, when $\Gamma_{\mathbb{CD}}^{r+} < 0$.

(3) Case (2): $\varepsilon_b + \kappa < 0$: With $\Gamma_{\mathbb{O}} > 0$, $\mathcal{X} > 0$, and $\mathcal{Y} < 0$, Eq. (38) yields $\Gamma_{\mathbb{AB}}^{r+} > 0$, $\Gamma_{\mathbb{CD}}^{r+} < 0$. Therefore, $J_E^r = \varepsilon_b \Gamma_{\mathbb{AB}}^{r+} + (\varepsilon_b + \kappa) \Gamma_{\mathbb{CD}}^{r+} > 0$.

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- [1] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, 2nd ed. (John Wiley & Sons, Inc., New York, 1985).
- [2] D. Kondepudi and I. Prigogine, *Modern Thermodynamics*, 2nd ed. (John Wiley & Sons Ltd, Chichester, 2015).
- [3] J. Wang, G. Casati, and G. Benenti, Inverse currents in Hamiltonian coupled transport, *Phys. Rev. Lett.* **124**, 110607 (2020).
- [4] Y. Zhang and Z. Xie, Inverse currents in Coulomb-coupled quantum dots, *Phys. Rev. E* **104**, 064142 (2021).
- [5] Y. Zhang, S. Wang, W. Li, and M. Yue, Inverse current induced thermoelectric conversion in a parallel-coupled double quantum dot system, *Phys. Scr.* **98**, 105245 (2023).
- [6] N. Gupta, S. Ghosh, and A. Ghosh, Graph theoretic analysis of three-terminal quantum dot thermocouples: Onsager relations and spin-thermoelectric effects, *Phys. Rev. B* **109**, 125124 (2024).
- [7] S. Ghosh, N. Gupta, and A. Ghosh, Universal behavior of the Coulomb-coupled fermionic thermal diode, *Entropy* **24**, 1810 (2022).
- [8] P. Strasberg, *Quantum Stochastic Thermodynamics: Foundations and Selected Applications* (Oxford University Press, Great Clarendon Street, 2022).
- [9] U. Harbola, M. Esposito, and S. Mukamel, Quantum master equation for electron transport through quantum dots and single molecules, *Phys. Rev. B* **74**, 235309 (2006).
- [10] R. Sánchez and M. Büttiker, Optimal energy quanta to current conversion, *Phys. Rev. B* **83**, 085428 (2011).
- [11] L. Wang, Z. Wang, C. Wang, and J. Ren, Cycle flux ranking of network analysis in quantum thermal devices, *Phys. Rev. Lett.* **128**, 067701 (2022).
- [12] R. S. Whitney, R. Sánchez, and J. Splettstoesser, Quantum thermodynamics of nanoscale thermoelectrics and electronic devices, in *Thermodynamics in the Quantum Regime: Fundamental Aspects and New Directions*, edited by F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso (Springer International Publishing, Cham, 2018), pp. 175–206.
- [13] N. Gupta, S. Ghosh, and A. Ghosh, Top-ranked cycle flux network analysis of molecular photocells, *Phys. Rev. E* **108**, 034305 (2023).
- [14] D. Perez Daroca, P. Roura-Bas, and A. A. Aligia, Role of asymmetry in thermoelectric properties of a double quantum dot out of equilibrium, *Phys. Rev. B* **111**, 045134 (2025).
- [15] A. Hamo, A. Benyamini, I. Shapir, I. Khivrich, J. Waissman, K. Kaasbjerg, Y. Oreg, F. von Oppen, and S. Ilani, Electron attraction mediated by Coulomb repulsion, *Nature (London)* **535**, 395 (2016).
- [16] S. M. Tabatabaei, Charge-Kondo effect mediated by repulsive interactions, *Phys. Rev. B* **97**, 235131 (2018).
- [17] W. A. Little, Possibility of synthesizing an organic superconductor, *Phys. Rev.* **134**, A1416 (1964).
- [18] G. E. D. K. Prawiroatmodjo, M. Leijnse, F. Trier, Y. Chen, D. V. Christensen, M. von Soosten, N. Pryds, and T. S. Jespersen, Transport and excitations in a negative- U quantum dot at the LaAlO₃/SrTiO₃ interface, *Nat. Commun.* **8**, 395 (2017).
- [19] N. Gupta, S. Bhattacharyya, and A. Ghosh, Statistical generalization of regenerative bosonic and fermionic stirling cycles, *Phys. Rev. E* **104**, 054130 (2021).
- [20] S. Kumar, N. Gupta, S. Ghosh, and A. Ghosh, Introduction to quantum thermodynamic cycles, *J. Chem. Sci.* **135**, 36 (2023).
- [21] A. Ghosh, S. S. Sinha, and D. S. Ray, Fermionic oscillator in a fermionic bath, *Phys. Rev. E* **86**, 011138 (2012).
- [22] S. S. Sinha, A. Ghosh, and D. S. Ray, Decay of a metastable state induced by a spin bath, *Phys. Rev. E* **84**, 041113 (2011).
- [23] S. S. Sinha, A. Ghosh, and D. S. Ray, Quantum Smoluchowski equation for a spin bath, *Phys. Rev. E* **84**, 031118 (2011).
- [24] G. G. Damas, R. J. de Assis, and N. G. de Almeida, Cooling with fermionic thermal reservoirs, *Phys. Rev. E* **107**, 034128 (2023).
- [25] B. Dutta, J. T. Peltonen, D. S. Antonenko, M. Meschke, M. A. Skvortsov, B. Kubala, J. König, C. B. Winkelmann, H. Courtois, and J. P. Pekola, Thermal conductance of a single-electron transistor, *Phys. Rev. Lett.* **119**, 077701 (2017).
- [26] B. Dutta, D. Majidi, N. W. Talarico, N. Lo Gullo, H. Courtois, and C. B. Winkelmann, Single-quantum-dot heat valve, *Phys. Rev. Lett.* **125**, 237701 (2020).
- [27] H. Thierschmann, R. Sánchez, B. Sothmann, F. Arnold, C. Heyn, W. Hansen, H. Buhmann, and L. W. Molenkamp, Three-terminal energy harvester with coupled quantum dots, *Nat. Nanotechnol.* **10**, 854 (2015).
- [28] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Great Clarendon Street, 2007).
- [29] A. Levy and R. Kosloff, The local approach to quantum transport may violate the second law of thermodynamics, *Europhys. Lett.* **107**, 20004 (2014).

- [30] L. Tesser, B. Bhandari, P. A. Erdman, E. Paladino, R. Fazio, and F. Taddei, Heat rectification through single and coupled quantum dots, *New J. Phys.* **24**, 035001 (2022).
- [31] M. Esposito, K. Lindenberg, and C. V. den Broeck, Entropy production as correlation between system and reservoir, *New J. Phys.* **12**, 013013 (2010).
- [32] G. T. Landi and M. Paternostro, Irreversible entropy production: From classical to quantum, *Rev. Mod. Phys.* **93**, 035008 (2021).
- [33] D. Gelbwaser-Klimovsky, W. Niedenzu, and G. Kurizki, Thermodynamics of quantum systems under dynamical control, *Adv. At. Mol. Opt. Phys.* **64**, 329 (2015).
- [34] G. Kurizki and A. G. Kofman, *Thermodynamics and Control of Open Quantum Systems* (Cambridge University Press, Cambridge, UK, 2022).
- [35] J. Schnakenberg, Network theory of microscopic and macroscopic behavior of master equation systems, *Rev. Mod. Phys.* **48**, 571 (1976).
- [36] R. Guan and J. Liu, Anomalous flow in correlated quantum systems: No-go result and multiple-charge scenario, *Phys. Rev. A* **112**, 022220 (2025).
- [37] S. Donsa, S. Andergassen, and K. Held, Double quantum dot as a minimal thermoelectric generator, *Phys. Rev. B* **89**, 125103 (2014).
- [38] P. A. Erdman, F. Mazza, R. Bosisio, G. Benenti, R. Fazio, and F. Taddei, Thermoelectric properties of an interacting quantum dot based heat engine, *Phys. Rev. B* **95**, 245432 (2017).
- [39] T. Ruokola and T. Ojanen, Single-electron heat diode: Asymmetric heat transport between electronic reservoirs through Coulomb islands, *Phys. Rev. B* **83**, 241404(R) (2011).
- [40] Y. Zhang, X. Zhang, Z. Ye, G. Lin, and J. Chen, Three-terminal quantum-dot thermal management devices, *Appl. Phys. Lett.* **110**, 153501 (2017).
- [41] A. A. Aligia, D. P. Daroca, L. Arrachea, and P. Roura-Bas, Heat current across a capacitively coupled double quantum dot, *Phys. Rev. B* **101**, 075417 (2020).
- [42] J. Yang, C. Elouard, J. Splettstoesser, B. Sothmann, R. Sánchez, and A. N. Jordan, Thermal transistor and thermometer based on Coulomb-coupled conductors, *Phys. Rev. B* **100**, 045418 (2019).
- [43] A. Kutvonen, T. Sagawa, and T. Ala-Nissila, Thermodynamics of information exchange between two coupled quantum dots, *Phys. Rev. E* **93**, 032147 (2016).
- [44] S. Ghosh and A. Ghosh, Thermodynamics of quantum coupled transport, *arXiv:2603.06336*.
- [45] G. Benenti, G. Casati, F. Marchesoni, and J. Wang, Autonomous circular heat engine, *Phys. Rev. E* **106**, 044104 (2022).
- [46] N. Gupta, S. Bhattacharyya, B. Das, S. Datta, V. Mukherjee, and A. Ghosh, Floquet quantum thermal transistor, *Phys. Rev. E* **106**, 024110 (2022).