

Winding-control mechanism of non-Hermitian systems

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(Received 25 June 2025; accepted 22 April 2026; published 15 May 2026)

Non-Hermitian quantum systems exhibit various interesting and interconnected spectral, topological, and boundary-sensitive features. By introducing conditional boundary conditions (CBCs) for non-Hermitian quantum systems, we explore a winding-control mechanism that selectively collapses specific periodic boundary condition loop-type spectra onto their open boundary condition counterparts, guided by their specific winding numbers, together with a composite reconstruction of the Brillouin zone (BZ) and generalized Brillouin zone (GBZ). The corresponding eigenstates also manifest nontrivial skin effects or extended behaviors arising from the interplay between BZ and GBZ structures. Intuitively, the winding-control mechanism is tied to the residual imaginary velocity originating from the corresponding Fermi sea, establishing the CBCs as the transition boundaries between different non-Hermitian topology of spectral windings. Furthermore, we can generalize our control by incorporating similarity transformations and holomorphic mappings with the boundary controls. We demonstrate the winding control numerically within various models, which enriches our knowledge of non-Hermitian physics across the spectrum, topology, and bulk-boundary correspondence.

DOI: 10.1103/6n5b-18jv

Introduction. The burgeoning field of non-Hermitian physics has attracted significant attention [1–26], due to its capacity to effectively capture open quantum systems [27–31], optical (photonic) systems [32–37], electric circuit simulations [38–42], and classical mechanical platforms [43–48], among others. A key hallmark is their striking sensitivity to boundary conditions, such as the non-Hermitian skin effect (NHSE), where an extensive number of eigenstates accumulate at boundaries under open boundary conditions (OBCs), in stark contrast to their counterparts under periodic boundary conditions (PBCs) [10,12]. This theoretical cornerstone in noninteracting cases has prompted advances across various areas of condensed matter physics, including localization versus delocalization [49–54], quantum entanglement [55–62], and the NHSE in dynamical and many-body systems [63–71].

The anomalous boundary sensitivity has challenged conventional bulk-boundary correspondences and motivated the generalized Brillouin zone (GBZ) formalism, which precisely characterizes the bulk spectrum under OBCs by generalizing the Bloch momenta within the Brillouin zone (BZ) to the complex plane [10,12]. Fine-tuned couplings between the two ends of one-dimensional (1D) chains uncover various exotic behaviors, including the emergent scale-free localization [42,72–76]. However, the underlying mechanism and targeted control of such boundary sensitivity and fine-tuning,

especially in general scenarios such as multiple PBC spectral loops, remain obscure.

In this Letter, we propose a winding-based control theory and formalism, revealing and relying on the intimate connections between spectrum, GBZ, winding number, and boundary conditions in general. Throughout this work, the PBC spectra we refer to are those of non-Hermitian systems exhibiting the NHSE, i.e., loop-type PBC spectra [13,16]. By introducing conditional boundary conditions (CBCs) with exclusive hopping, we can selectively collapse pieces of the PBC spectrum to their OBC counterparts according to their respective winding numbers, accompanied by a composite reconstruction of corresponding segments of the BZ and GBZ between the Bloch points separating different winding domains. Such a hybrid spectral structure gives rise to eigenstates with nontrivial localization properties, which cannot be understood from the perspective of either the BZ or GBZ alone. Furthermore, through holomorphic mappings, including simple similarity transformations, we achieve further control over the relative figures of the BZ and GBZ before CBC implementations, paving the way for systematic control over non-Hermitian spectra and band structures. We demonstrate the accuracy and versatility of our control with multiple 1D non-Hermitian lattice models.

Winding-control mechanism via imaginary velocity and boundary conditions. Given a 1D noninteracting fermionic non-Hermitian Hamiltonian $\hat{H}$ [77], we are interested in the Fermi sea with all single-particle states with $\text{Re}(\epsilon) < \mu$ filled and $\text{Re}(\epsilon) > \mu$ empty [78]. For a band with two Fermi points $\text{Re}(\epsilon_{F,1}) = \text{Re}(\epsilon_{F,2}) = \mu$ [79], the velocity expectation value equals

$$\bar{v} = \sum_{\text{Re}(\epsilon_z) < \mu} v_z = (\epsilon_{F,2} - \epsilon_{F,1})L/2\pi, \quad (1)$$

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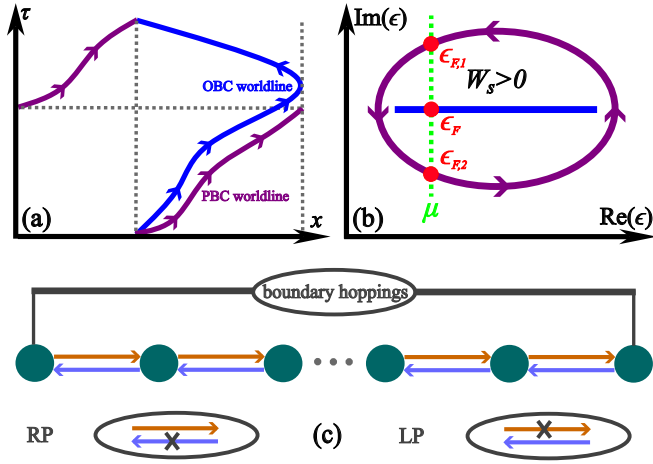


FIG. 1. (a) Schematically, a worldline may wind around the system under PBCs (purple), yet its winding number must be trivial under OBCs (blue). (b) The Fermi points $\epsilon_{F,1}$, $\epsilon_{F,2}$, and ϵ_F are intersections of the spectra (purple under PBCs and blue under OBCs) with the Fermi energy μ (green dashed line), resulting in an imaginary velocity $\text{Im}(\bar{v}) < 0$ and spectral winding $W_s > 0$ under PBCs, and vanishing $\text{Im}(\bar{v})$ and W_s under OBCs, following the spectrum-velocity-winding relations in Eqs. (1)–(3). (c) The RP (LP) CBC fully suppresses the left-bound (right-bound) hoppings across the boundary, admitting only non-negative (nonpositive) $\bar{v}$ and nonpositive (non-negative) W_s .

where the summation is over the BZ or GBZ between the Fermi points. While $\bar{v}$ vanishes for a Hermitian $\hat{H}$, it may obtain a residue imaginary part in non-Hermitian quantum systems, as $\text{Im}(\epsilon_{F,1})$ and $\text{Im}(\epsilon_{F,2})$ may differ [Fig. 1(b)]. Such an $\text{Im}(\bar{v}) = dx/d\tau$ is related to the collective displacements of the Fermi sea in imaginary time $\tau = it \in \mathbb{R}$, and may carry profound physical consequences, such as a mechanism and diagnosis for delocalization [54] and the circulation of path-integral worldlines [Fig. 1(a)] in space-imaginary-time ($x - \tau$) [54,80]. In particular, under the finite-temperature path-integral framework, the dominant worldlines representing real-space configurations (period L) versus the imaginary time (period β) witness a nonzero winding number W , which is semiclassically related to $\text{Im}(\bar{v})$ via

$$W = \frac{dx}{L} \frac{\beta}{d\tau} = \text{Im}(\bar{v})\beta/L, \quad (2)$$

where $\beta = 1/k_B T$ is the inverse temperature and L is the system size. Such a worldline winding W may become nonzero under PBCs but is guaranteed to vanish under OBCs [Fig. 1(a)]. These relations have been precisely confirmed in quantum Monte Carlo stochastic series expansion (QMC-SSE) calculations [54,80].

Crucially, the worldline winding number W is closely connected to the spectral winding number W_s of the PBC spectrum, as exemplified by a single-band model with energy dispersion ϵ_z :

$$W_s = \frac{1}{2\pi} \oint_{z \in \text{BZ}} d \arg(\epsilon_z - \epsilon_0), \quad (3)$$

where ϵ_0 is an arbitrary reference energy inside the spectral loop and the BZ is the unit circle in the complex plane. Given

a specific Fermi energy μ , a positive $\text{Im}(\bar{v})$ corresponds to a positive W_s , right-bound quasiparticles and worldlines, and a negative W_s , and vice versa [80]. Notably, the counterclockwise contour integration [Fig. 1(b)] introduces a sign inversion for W_s in our convention. We note that boundary conditions can significantly influence such directional flows. The simplest and most straightforward case is the OBC, which forces $\bar{v}$ to vanish and constrains the spectrum: Except for spectral endpoints, all energy contours must be doubly degenerate under OBCs, irrespective of whether $\hat{H}$ is Hermitian or non-Hermitian, single band or multiband; otherwise, there always exists a traversing Fermi energy μ leading to a contradictory nonzero $\bar{v}$; see the Supplemental Material [81] for details on the origin and examples of such double degeneracy. Simultaneously, the open boundaries halt the semiclassical current $\bar{v}$ that would have been present under PBCs and cause the states to accumulate on the boundary—a defining signature of the NHSE and generalizable to higher-dimensional or many-body NHSE [71,82].

Similarly, if we can selectively manipulate the winding numbers through boundary conditions or alternative schemes, we can piecewise modify the corresponding spectra and GBZ, which underpins our winding-based control mechanism. For instance, consider a segment of a PBC spectrum and BZ with $W_s > 0$ like in Fig. 1(b), which supports left-moving worldlines in imaginary time. If we implement a boundary condition that affects its left-moving behavior, this piece of the spectrum, along with its BZ segment, will deviate from its PBC counterpart. One possibility is the OBC and another is a right-permissible (RP) CBC, a unidirectional version of the OBC, where the leftward-permissible (LP) hopping across the boundary is completely suppressed [Fig. 1(c)]. Such a CBC fully suppresses W_s and $\text{Im}(\bar{v})$ and collapses the corresponding spectrum and BZ segment onto its OBC spectrum and GBZ counterpart; at the same time, their eigenstates will manifest the left-localized NHSE, where the existing RP hopping poses little interference. Unlike the OBCs, on the other hand, $W_s < 0$ is compatible with the RP CBCs; thus, the related spectrum and BZ segments remain intact, and their eigenstates remain extended [54]. Analogously, vice versa for the LP CBCs.

Importantly, the intersections between the PBC BZ and the OBC GBZ, known as the Bloch points, mark the boundaries between segments with winding numbers of opposite signs; also, they separate the right-localized and left-localized skin modes under OBCs. Specifically, for a single-band model $H(z) = \epsilon_z$ with $z \in \text{BZ}$, the winding number in Eq. (3) equals $W_s = \mathcal{N}_0 - \mathcal{N}_p$, where $\mathcal{N}_0$ and $\mathcal{N}_p$ are the number of the zeros and poles of $H(z) - \epsilon_0$ inside the BZ, respectively. Given a right-hopping range of q , thus q -fold pole at $z = 0$, and whether ϵ_0 lies on the OBC spectrum associated with the GBZ segment inside or outside the BZ, $\mathcal{N}_0$ is either $q + 1$ or $q - 1$ and thus W_s is either positive or negative. This conclusion follows from the GBZ condition, which mandates that the zeros with the q th and $(q + 1)$ th largest moduli lie on the GBZ [13] for a common 2-bifurcation point on OBC spectrum [83], yet generalizable to arbitrary n -bifurcation points or even systems with multiple bands. On the complex energy plane, these Bloch points separate the inside and outside segments of the GBZ and guarantee degeneracy between the PBC and

OBC spectra [84]. Under the CBCs, the Bloch points serve as junctions, allowing for smooth switches between the PBC and OBC segments, simultaneously on the spectrum and the BZ/GBZ, as we will later discuss and demonstrate in the examples.

The spectral loops separated by the Bloch points are governed primarily by either LP or RP hoppings, depending on their spectral winding numbers under PBCs—non-Hermitian point-gap topological indices. As the LR and RP hoppings on the boundary decrease continuously to zero, the system evolves toward full OBCs. A spectral loop entirely collapses onto an OBC spectral portion with zero spectral winding number when and only when the corresponding boundary hopping vanishes; otherwise, its spectral winding number survives, as long as finite boundary hoppings and thus the area enclosed by the loop remain. Therefore, in single-band systems, the PBC spectra on either side of the Bloch points with opposite winding numbers are controlled by the LP or RP hoppings, respectively; correspondingly, with the LP and RP CBCs, we can independently control their topological characters and transitions. Besides, such control of spectral winding numbers and topological transitions holds in multiband systems. For instance, the PBC-to-OBC evolution of a two-band model under an increasing imaginary flux in Ref. [85] can be regarded as a simple case of our winding-control scheme with a single spectral winding number and no Bloch points, where the GBZ lies entirely inside or outside the BZ. In addition, the modified PBC, employed in a previous work focusing on the non-Hermitian bulk-boundary correspondence [86], reduces to one of our CBCs in the thermodynamic limit.

Furthermore, we can enhance our control by applying additional transformations that modify the relative standings between the BZ and GBZ, including similarity transformations and, more broadly, holomorphic mappings. Indeed, the interplay between the imaginary velocity $\text{Im}(\bar{v})$ and the worldline winding W is encoded in Eqs. (1) and (2), irrespective of the boundary conditions or the mappings and transformations. On one hand, for any Fermi sea under CBCs or after holomorphic mappings [87], the imaginary velocity is explicitly obtainable via Eq. (1); on the other hand, the worldline winding, e.g., directly realized in QMC-SSE simulations under the corresponding (inverse) temperature β , is theoretically comparable via Eq. (2) [54,80]. In the following, we will demonstrate such formalism on several model examples.

A paradigmatic model. To demonstrate winding control, we consider the following 1D single-band non-Hermitian model:

$$\begin{aligned} \hat{H}_p = \sum_x & \left[i(c_{x+1}^\dagger c_x - c_x^\dagger c_{x+1}) + \frac{1}{2}(c_{x+2}^\dagger c_x - c_x^\dagger c_{x+2}) \right] \\ & + \lambda_R \left[i c_1^\dagger c_L + \frac{1}{2}(c_1^\dagger c_{L-1} + c_2^\dagger c_L) \right] \\ & - \lambda_L \left[i c_L^\dagger c_1 + \frac{1}{2}(c_{L-1}^\dagger c_1 + c_L^\dagger c_2) \right], \end{aligned} \quad (4)$$

where c_x ($c_x^\dagger$) denotes the fermion annihilation (creation) operator at site $x \in [1, L]$ and $\lambda_L, \lambda_R \in [0, 1]$ parametrize the left- and right-bound hoppings between the boundaries, respectively. We stress that the physical mechanism does not rely on the specific choice of hopping parameters, as

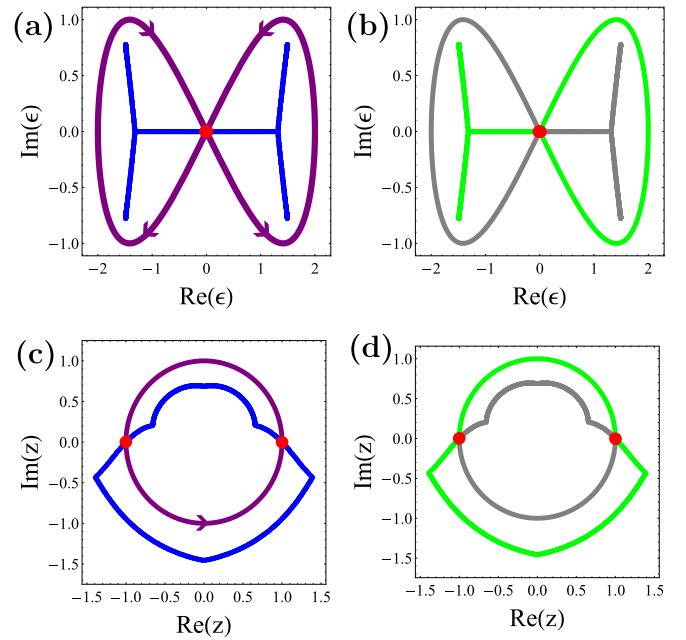


FIG. 2. The model in Eq. (4) serves as a typical example of winding-control spectrum and GBZ'. (a) The spectrum and (c) BZ under PBCs (purple) evolve to the spectrum and GBZ under OBCs (blue). In comparison, under the LP or RP CBC, (b) the spectrum and (d) GBZ' become a composite of certain PBC and OBC pieces, as the green or gray curves, respectively. The arrows indicate the spectral loops under PBCs, enclosing a positive (negative) winding number W_s region on the right (left) side. The red dots denote the Bloch points.

we adopt for Eq. (4) for simple display and without loss of generality. The PBC corresponds to $\lambda_L = \lambda_R = 1$, where the BZ is simply the unit circle in the complex plane [purple in Fig. 2(c)], and the spectrum $H_p(z) = i(z^{-1} - z) + 1/2(z^{-2} - z^2)$, $z \in \text{BZ}$, traces a self-intersecting loop in the complex energy plane [purple in Fig. 2(a)]. For $\lambda_L = \lambda_R = 0$, on the other hand, the OBC and its corresponding GBZ [blue in Fig. 2(c)] take effect. Indeed, the contour of the resulting spectrum $H_p(z)$, $z \in \text{GBZ}$ [blue arc in Fig. 2(a)], expels any simple loops and stands doubly degenerate in general. The BZ and GBZ intersect at two Bloch points [red dots in Fig. 2(c)], which precisely separate the parts clinging to left-localized NHSE (GBZ inside BZ) and right-localized NHSE (GBZ outside BZ); consistently [88], these Bloch points also correspond to the self-intersection point of the PBC spectrum, partitioning segments with positive and negative winding numbers W_s .

Now, we impose the LP CBCs $\lambda_L = 1$ and $\lambda_R = 0$, which turn off right-bound hopping across the boundary and block right-moving quasiparticles from circulating the 1D chain. This constraint in winding number influences the BZ and spectrum, but only the segments with negative winding numbers W_s —the left segment of the PBC spectrum collapses into the OBC spectrum [green curves in Fig. 2(b)], while the right segment remains intact; their respective (upper portion of) BZ and (lower portion of) GBZ merge at the Bloch points into a composite GBZ' [green loop in Fig. 2(d)]. Likewise, by imposing the RP CBCs with $\lambda_L = 0$ and $\lambda_R = 1$, a

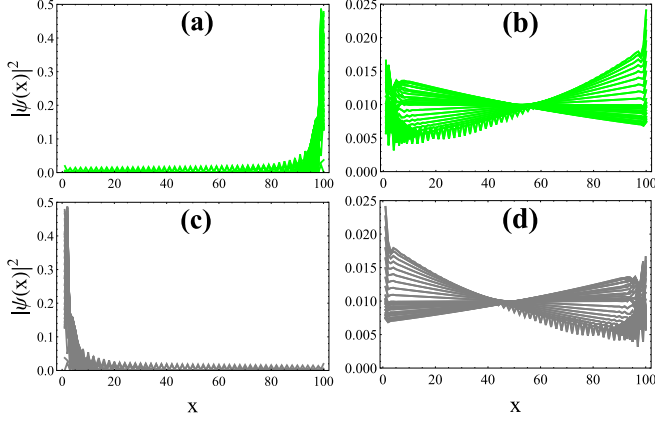


FIG. 3. Under LP CBC, (a) the eigenstates resembling the OBC spectrum and GBZ display right-localized NHSE, while (b) the eigenstates resembling the PBC spectrum and BZ show extended behaviors; see the green pieces in Fig. 2. Likewise, under RP CBC, (c) the eigenstates representing the OBC spectrum and GBZ display left-localized NHSE, while (d) the eigenstates representing the PBC spectrum and BZ show extended behaviors; see the gray pieces in Fig. 2.

nonpositiveness requirement is enforced upon the winding number W_s , resulting in a combined spectrum and GBZ' (gray curves in Fig. 2) as joining the respective PBC and OBC segments.

Further, we illustrate the eigenstate distributions in Fig. 3. When the right-bound (left-bound) hopping is turned off at the boundary with the LP (RP) CBCs, right-localized (left-localized) NHSE emerges [see Fig. 3(a) [Fig. 3(c)]]; these eigenstates correspond to the portions of spectrum and GBZ' inherited from the OBC counterparts. In contrast, the eigenstates associated with the PBC segments remain fully extended [Figs. 3(b) and 3(d)], as in translation-invariant systems under PBCs. Furthermore, we provide in the Supplemental Material additional rigorous explanations of spectrum and eigenstate behaviors under CBCs, continuous interpolations between the PBC and CBC, and winding control of a model with three PBC spectral loops [81]. Importantly, the double loop under PBC, the single-loop arc under the two CBCs, and the arc under OBC exhibit distinct topological winding numbers, as shown in Fig. 2, with the CBCs defining the transition boundaries between these spectral configurations. As a general statement of the winding-control mechanism, the spectral configurations transition according to the winding signs of individual loops, a behavior that applies universally to the models discussed below and in the Supplemental Material [81].

Winding control with similarity transformation and holomorphic mapping. The above CBC has limitations, as it only controls the presence or absence of sections between existing Bloch points in a binary fashion. To achieve further control over the existence and location of the Bloch points, we employ a similarity transformation in conjunction with the CBCs. For simplicity, let us first consider $H_{p,\varrho} = S^{-1}H_pS$, where $S = \text{diag}\{1, \varrho, \dots, \varrho^L\}$ and $\varrho \in (0, +\infty)$, which is trivial for OBC spectrum. However, with $\lambda_L = \lambda_R = 1$ as before, the PBC spectrum $H_p(z)$ follows a transformed BZ, a circle of radius

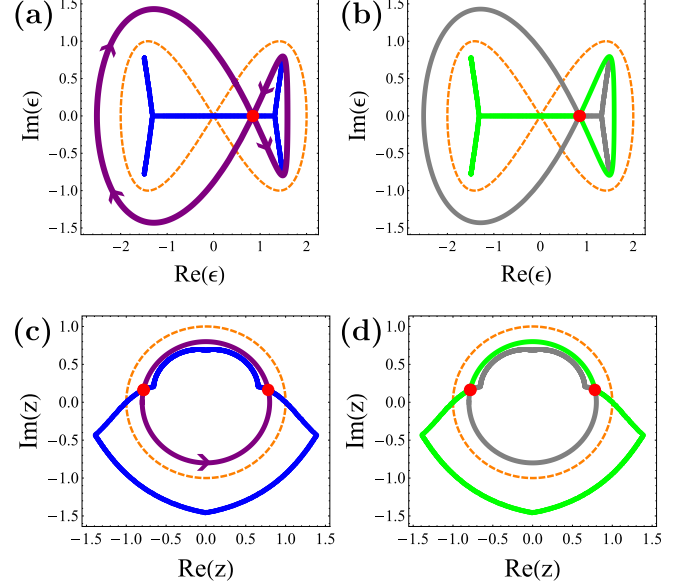


FIG. 4. The spectrum and GBZ (BZ) of the model in Eq. (4) following a similarity transformation with $\varrho = 0.8$ showcase (a) an identical OBC spectrum and (c) GBZ to the original model as in Fig. 2, yet (a) a modified PBC spectrum, effectively according to (c) a transformed BZ—the circle of radius $\varrho = 0.8$. The original PBC spectrum and BZ (with $\varrho = 1$) are shown as the orange dashed loops for comparison. Similar to Fig. 2, with the LP (green) or RP (gray) CBC, (b) the spectrum and (d) GBZ' become a composite of certain PBC and OBC pieces. Note that varying ϱ changes the BZ and, thus, the location of the Bloch points.

ϱ in the complex plane. The Bloch points, the intersections between the OBC and PBC spectra, also displace depending on ϱ ; see Fig. 4 for illustrations [89].

With CBC implemented, the PBC spectral loops with the positive or negative winding number W_s (purple), or the OBC spectral lines with zero W_s (blue), located to the right or left of the Bloch point (red dot), along with the corresponding (transformed) BZ and GBZ, can be selectively chosen and combined in a manner analogous to the case in Eq. (4) [see Figs. 4(b) and 4(d)]. Accordingly, the CBC eigenstates exhibit distinct skin effects or extended behaviors similar to Fig. 3, depending on their position relative to the Bloch points as governed by the winding-control mechanism.

Beyond the specific similarity transformation discussed above, we can extend our framework to more general holomorphic mappings $\mathcal{F}$, which cast the original unit-circle BZ to $\mathcal{F}[\text{BZ}]$ in the complex plane [90]. Equally, the effect of $\mathcal{F}$ on the original lattice model Hamiltonian $\hat{H}$ is obtainable from its transformation on the model's non-Bloch Hamiltonian $H(z) \rightarrow H(\mathcal{F}(z))$. For example, the similarity transformation S is a straightforward holomorphic mapping $\mathcal{F}(z) = \varrho z$, which results in a contraction or dilatation of the BZ along the radial direction as in Fig. 4. Importantly, additional constraints are necessary to ensure the physical meaning of such a holomorphic mapping; for simplicity, we require that $\mathcal{F}[\text{BZ}]$ remains a single loop and consider case-specific constraints as appropriate.

Next, we consider another holomorphic mapping $\mathcal{F}'(z) = z - \eta$, $\eta \in \mathbb{C}$, which translates the BZ to $\mathcal{F}'[\text{BZ}]$ via a

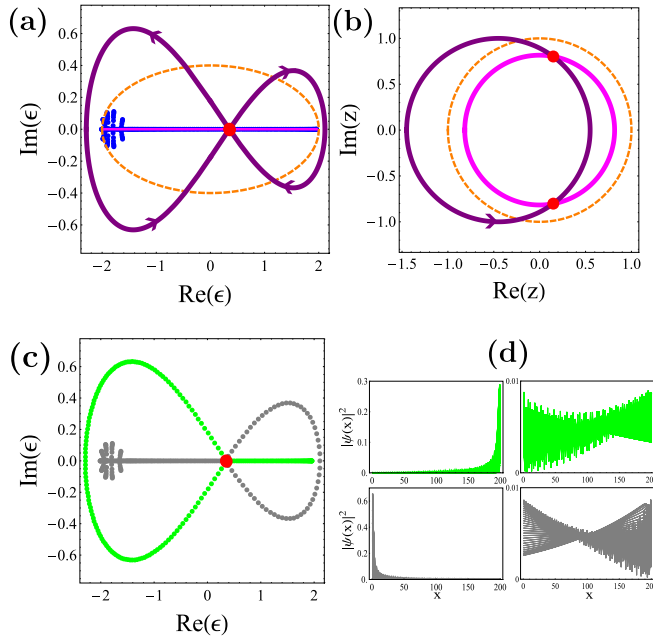


FIG. 5. The spectrum and GBZ (BZ) of the HN models showcase the winding control we possess with a holomorphic mapping $\mathcal{F}'(z) = z - \eta$ and CBC. (a) The OBC (magenta, blue) and PBC (orange, purple) spectra of the original (magenta, orange) and transformed (blue, purple) HN models consistently follow their corresponding PBC BZ (orange), transformed BZ (purple), and OBC GBZ (magenta) in panel (b). Together with the LP (green) and RP (gray) CBCs, (c) the spectra and (d) eigenstates of the transformed model follow the winding-control mechanism, combining features (winding number W_s , skin effects, or extended wave functions) of the respective pieces of OBC and transformed PBC between the Bloch points (red dots), as we demonstrated in the previous examples. We set $\eta = 0.45$, $\gamma = 0.2$, and truncation $n_{\max} = 20$ on a 1D chain with $L = 200$ sites.

displacement of η . For example, the non-Bloch Hamiltonian of the Hatano-Nelson (HN) model is $H_{\text{HN}}(z) = (1 + \gamma)z + (1 - \gamma)z^{-1}$ [12,91,92], yielding the PBC and OBC spectra [orange dashed loop and magenta line in Fig. 5(a)] following $z \in \text{BZ}$ and $z \in \text{GBZ}$ [orange dashed and magenta solid loops in Fig. 5(b)], respectively. Further, $\mathcal{F}'$ with $\eta = 0.45$ shifts the original BZ to $\mathcal{F}'[\text{BZ}]$ as the purple loop, which now intersects the OBC GBZ (magenta loop) at two emergent Bloch points (red dots) as in Fig. 5(b). According to $z \in \mathcal{F}'[\text{BZ}]$, the PBC spectrum $H_{\text{HN}}(z)$ transforms into two loops with opposite winding numbers demarcated by the Bloch point [purple curves and red dots in Fig. 5(a)]. Nevertheless, we expect the OBC spectrum to remain mainly invariant following the holomorphic mapping. In analogy with the previous examples, we may introduce further winding control via LP or RP CBCs. The LP (RP) CBC should impose a $W_s > 0$ ($W_s < 0$) constraint and collapse the right (left) loop with $W_s < 0$ ($W_s > 0$) from the transformed PBC spectrum onto its OBC counterpart, while the left (right) loop with $W_s > 0$ ($W_s < 0$) remains intact; their GBZ' should also combine the corresponding BZ and GBZ pieces between the Bloch points.

In practice, following the effect of $\mathcal{F}'$ on the non-Bloch Hamiltonian $H_{\text{HN}}(z)$, we transform the original HN model into

the following tight-binding lattice model:

$$\hat{H}' = \sum_x \left[(1 + \gamma)(c_x^\dagger c_{x+1} - \eta c_x^\dagger c_x) + (1 - \gamma) \sum_{n=0}^{\infty} \eta^n c_{x+n}^\dagger c_{x-1} \right], \quad (5)$$

where we have essentially expanded $H_{\text{HN}}(z - \eta)$ in Laurent series, which remains valid when $|\eta/z| < 1$ for all $z \in \mathcal{F}'[\text{BZ}]$ (see the Supplemental Material [81] for details). We take $\eta = 0.45$ and truncate the hopping range n at a sufficient $n_{\max} = 20$, and summarize the results in Fig. 5.

Under OBCs, the transformed model yields a spectrum, represented by the blue points in Fig. 5(a), which provides a good approximation of the original OBC spectrum (magenta solid line) [93]. Also, its PBC spectrum's consistency with our theoretical expectation is self-evident from the $H_{\text{HN}}(z - \eta)$ nature of Eq. (5). Further, as we employ CBCs for the transformed model, the resulting spectra and eigenstates combine the corresponding pieces from the transformed PBC and OBC spectra and eigenstates [Figs. 5(c) and 5(d)], as we have argued in the previous paragraphs. Therefore, we have demonstrated that we can obtain further control and design over the spectrum and GBZ' through winding-control boundary conditions and general holomorphic mappings, and we may incorporate other holomorphic mappings for further control targets.

Conclusions and discussions. In summary, we have proposed and explored a winding-control mechanism rooted in the interplay between the spectral winding number, the imaginary velocity, and the boundary conditions. With CBCs of unilateral boundary hoppings, we selectively combine segments of PBC and OBC spectra as well as corresponding BZ and GBZ, separated by Bloch points and dictated by their winding characteristics. As a result, the localization behaviors of the eigenstates are determined not by individual BZ or GBZ but instead from their combined intersections under boundary control. Additionally, similarity transformations or holomorphic mappings extend this control, providing a versatile and systematic framework for manipulating non-Hermitian spectral topology. We have validated this winding control through various 1D lattice model examples.

It should be emphasized that Fig. 1 provides an elementary yet fundamental illustration—a single PBC spectral loop—that elucidates the core mechanism of winding control. In particular, the non-negative (or nonpositive) spectral winding number either aligns or antialigns with the LP or RP CBCs, thereby determining whether the corresponding spectral portions form a closed PBC loop with an enclosed area or an OBC arc or even loop with a net zero winding number. Furthermore, this underlying mechanism remains valid for more complex single-band models, e.g., the three-loop PBC spectrum presented in the Supplemental Material [81] and the PBC spectrum enclosing a non-simply-connected OBC spectrum corresponding to disconnected GBZs in Ref. [94]. Indeed, as the imaginary velocity, the semiclassical bridge connecting the spectral winding number and the CBCs, receives contributions from the entire Fermi sea in an additive

and cumulative manner, its controllable degrees of freedom and physical consequences can be attributed more locally for each spectral loop, instead of globally for the entire spectrum, forming ground for our winding-control mechanism of spectrum and topology.

Our winding-control mechanism also holds broader application potential beyond CBCs. For example, by joining different models, such as the HN model and the non-Hermitian Su-Schrieffer-Heeger model, the winding control can hybridize their spectra [81]. Furthermore, cases with multiple bands provide sublattice (spin)-dependent or summation-based CBC generalizations of the aforementioned scenarios. Remarkably, the present work does not account for symmetry effects. In the context of the $\mathbb{Z}_2$ skin effect, exemplified by a two-band model endowed with anomalous time-reversal symmetry [16], the PBC spectra consist of a pair of counterrotating loops, thereby rendering the net spectral winding identically zero. Accordingly, for such multiband systems, our winding-control mechanism applies to a band-resolved

basis rather than to the aggregate, a fact we have verified numerically. Notably, the imaginary velocity and worldline winding numbers remain physical in quantum many-body representations, validating many-body extensions of winding control and CBCs into interacting systems beyond single-particle spectra and GBZs, an interesting topic for future studies. A potential experimental scheme for realizing the required CBCs, employing directional amplifiers to tailor boundary hoppings in an acoustic crystal [95], is presented in the Supplemental Material [81] as a proposal for future investigation.

Acknowledgments. Y.F. was supported by a startup grant from Zhejiang Normal University. Y.Z. was supported by the National Natural Science Foundation of China (Grants No. 92270102 and No. 12174008) and the National Key R&D Program of China (Grant No. 2022YFA1403700).

Data availability. The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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- [1] Y. Ashida, Z. Gong, and M. Ueda, Non-Hermitian physics, *Adv. Phys.* **69**, 249 (2020).
 - [2] E. J. Bergholtz, J. C. Budich, and F. K. Kunst, Exceptional topology of non-Hermitian systems, *Rev. Mod. Phys.* **93**, 015005 (2021).
 - [3] T. E. Lee, Anomalous edge state in a non-Hermitian lattice, *Phys. Rev. Lett.* **116**, 133903 (2016).
 - [4] D. Leykam, K. Y. Bliokh, C. Huang, Y. D. Chong, and F. Nori, Edge modes, degeneracies, and topological numbers in non-Hermitian systems, *Phys. Rev. Lett.* **118**, 040401 (2017).
 - [5] V. M. Martinez Alvarez, J. E. Barrios Vargas, and L. E. F. Foa Torres, Non-Hermitian robust edge states in one dimension: Anomalous localization and eigenspace condensation at exceptional points, *Phys. Rev. B* **97**, 121401(R) (2018).
 - [6] H. Shen, B. Zhen, and L. Fu, Topological band theory for non-Hermitian Hamiltonians, *Phys. Rev. Lett.* **120**, 146402 (2018).
 - [7] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Topological phases of non-Hermitian systems, *Phys. Rev. X* **8**, 031079 (2018).
 - [8] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Symmetry and topology in non-Hermitian physics, *Phys. Rev. X* **9**, 041015 (2019).
 - [9] F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, Biorthogonal bulk-boundary correspondence in non-Hermitian systems, *Phys. Rev. Lett.* **121**, 026808 (2018).
 - [10] S. Yao and Z. Wang, Edge states and topological invariants of non-Hermitian systems, *Phys. Rev. Lett.* **121**, 086803 (2018).
 - [11] S. Yao, F. Song, and Z. Wang, Non-Hermitian Chern bands, *Phys. Rev. Lett.* **121**, 136802 (2018).
 - [12] K. Yokomizo and S. Murakami, Non-Bloch band theory of non-Hermitian systems, *Phys. Rev. Lett.* **123**, 066404 (2019).
 - [13] K. Zhang, Z. Yang, and C. Fang, Correspondence between winding numbers and skin modes in non-Hermitian systems, *Phys. Rev. Lett.* **125**, 126402 (2020).
 - [14] D. S. Borgnia, A. J. Kruchkov, and R.-J. Slager, Non-Hermitian boundary modes and topology, *Phys. Rev. Lett.* **124**, 056802 (2020).
 - [15] Z. Yang, K. Zhang, C. Fang, and J. Hu, Non-Hermitian bulk-boundary correspondence and auxiliary generalized Brillouin zone theory, *Phys. Rev. Lett.* **125**, 226402 (2020).
 - [16] N. Okuma, K. Kawabata, K. Shiozaki, and M. Sato, Topological origin of non-Hermitian skin effects, *Phys. Rev. Lett.* **124**, 086801 (2020).
 - [17] H.-Y. Wang, F. Song, and Z. Wang, Amoeba formulation of non-Bloch band theory in arbitrary dimensions, *Phys. Rev. X* **14**, 021011 (2024).
 - [18] C. H. Lee, L. Li, and J. Gong, Hybrid higher-order skin-topological modes in nonreciprocal systems, *Phys. Rev. Lett.* **123**, 016805 (2019).
 - [19] K. Kawabata, M. Sato, and K. Shiozaki, Higher-order non-Hermitian skin effect, *Phys. Rev. B* **102**, 205118 (2020).
 - [20] R. Okugawa, R. Takahashi, and K. Yokomizo, Second-order topological non-Hermitian skin effects, *Phys. Rev. B* **102**, 241202(R) (2020).
 - [21] Y. Fu, J. Hu, and S. Wan, Non-Hermitian second-order skin and topological modes, *Phys. Rev. B* **103**, 045420 (2021).
 - [22] H. Hu and E. Zhao, Knots and non-Hermitian Bloch bands, *Phys. Rev. Lett.* **126**, 010401 (2021).
 - [23] Y. Li, X. Ji, Y. Chen, X. Yan, and X. Yang, Topological energy braiding of non-Bloch bands, *Phys. Rev. B* **106**, 195425 (2022).
 - [24] Y. Fu and Y. Zhang, Braiding topology of non-Hermitian open-boundary bands, *Phys. Rev. B* **110**, L121401 (2024).
 - [25] X. Ji, H. Geng, N. Akhtar, and X. Yang, Floquet engineering of point-gapped topological superconductors, *Phys. Rev. B* **111**, 195419 (2025).
 - [26] Y.-M. Hu and Z. Wang, Green's functions of multiband non-Hermitian systems, *Phys. Rev. Res.* **5**, 043073 (2023).
 - [27] F. Song, S. Yao, and Z. Wang, Non-Hermitian skin effect and chiral damping in open quantum systems, *Phys. Rev. Lett.* **123**, 170401 (2019).

- [28] A. McDonald, R. Hanai, and A. A. Clerk, Nonequilibrium stationary states of quantum non-Hermitian lattice models, *Phys. Rev. B* **105**, 064302 (2022).
- [29] A. Altland, M. Fleischhauer, and S. Diehl, Symmetry classes of open fermionic quantum matter, *Phys. Rev. X* **11**, 021037 (2021).
- [30] A. Gómez-León, T. Ramos, A. González-Tudela, and D. Porras, Bridging the gap between topological non-Hermitian physics and open quantum systems, *Phys. Rev. A* **106**, L011501 (2022).
- [31] B. Li, H.-R. Wang, F. Song, and Z. Wang, Non-Bloch dynamics and topology in a classical nonequilibrium process, *Phys. Rev. B* **109**, L201121 (2024).
- [32] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Observation of $\mathcal{PT}$ -symmetry breaking in complex optical potentials, *Phys. Rev. Lett.* **103**, 093902 (2009).
- [33] W. Chen, Ş. Kaya Özdemir, G. Zhao, J. Wiersig, and L. Yang, Exceptional points enhance sensing in an optical microcavity, *Nature (London)* **548**, 192 (2017).
- [34] L. Xiao, K. Wang, X. Zhan, Z. Bian, K. Kawabata, M. Ueda, W. Yi, and P. Xue, Observation of critical phenomena in parity-time-symmetric quantum dynamics, *Phys. Rev. Lett.* **123**, 230401 (2019).
- [35] L. Xiao, T. Deng, K. Wang, G. Zhu, Z. Wang, W. Yi, and P. Xue, Non-Hermitian bulk-boundary correspondence in quantum dynamics, *Nat. Phys.* **16**, 761 (2020).
- [36] L. Xiao, T. Deng, K. Wang, Z. Wang, W. Yi, and P. Xue, Observation of non-Bloch parity-time symmetry and exceptional points, *Phys. Rev. Lett.* **126**, 230402 (2021).
- [37] L. Xiao, W.-T. Xue, F. Song, Y.-M. Hu, W. Yi, Z. Wang, and P. Xue, Observation of non-Hermitian edge burst in quantum dynamics, *Phys. Rev. Lett.* **133**, 070801 (2024).
- [38] M. Ezawa, Non-Hermitian boundary and interface states in nonreciprocal higher-order topological metals and electrical circuits, *Phys. Rev. B* **99**, 121411(R) (2019).
- [39] M. Ezawa, Electric circuits for non-Hermitian Chern insulators, *Phys. Rev. B* **100**, 081401(R) (2019).
- [40] T. Hofmann, T. Helbig, F. Schindler, N. Salgo, M. Brzezińska, M. Greiter, T. Kiessling, D. Wolf, A. Vollhardt, A. Kabaš, C. H. Lee, A. Bilušić, R. Thomale, and T. Neupert, Reciprocal skin effect and its realization in a topoelectrical circuit, *Phys. Rev. Res.* **2**, 023265 (2020).
- [41] T. Helbig, T. Hofmann, S. Imhof, M. Abdelghany, T. Kiessling, L. W. Molenkamp, C. H. Lee, A. Szameit, M. Greiter, and R. Thomale, Generalized bulk-boundary correspondence in non-Hermitian topoelectrical circuits, *Nat. Phys.* **16**, 747 (2020).
- [42] L. Li, C. H. Lee, and J. Gong, Impurity induced scale-free localization, *Commun. Phys.* **4**, 42 (2021).
- [43] M. Brandenbourger, X. Locsin, E. Lerner, and C. Coulais, Non-reciprocal robotic metamaterials, *Nat. Commun.* **10**, 4608 (2019).
- [44] A. Ghatak, M. Brandenbourger, J. van Wezel, and C. Coulais, Observation of non-hermitian topology and its bulk-edge correspondence in an active mechanical metamaterial, *Proc. Natl. Acad. Sci. USA* **117**, 29561 (2020).
- [45] Y. Chen, X. Li, C. Scheibner, V. Vitelli, and G. Huang, Realization of active metamaterials with odd micropolar elasticity, *Nat. Commun.* **12**, 5935 (2021).
- [46] W. Wang, X. Wang, and G. Ma, Non-Hermitian morphing of topological modes, *Nature (London)* **608**, 50 (2022).
- [47] W. Wang, M. Hu, X. Wang, G. Ma, and K. Ding, Experimental realization of geometry-dependent skin effect in a reciprocal two-dimensional lattice, *Phys. Rev. Lett.* **131**, 207201 (2023).
- [48] Z. Li, L.-W. Wang, X. Wang, Z.-K. Lin, G. Ma, and J.-H. Jiang, Observation of dynamic non-Hermitian skin effects, *Nat. Commun.* **15**, 6544 (2024).
- [49] R. Hamazaki, K. Kawabata, and M. Ueda, Non-Hermitian many-body localization, *Phys. Rev. Lett.* **123**, 090603 (2019).
- [50] L.-Z. Tang, L.-F. Zhang, G.-Q. Zhang, and D.-W. Zhang, Topological Anderson insulators in two-dimensional non-Hermitian disordered systems, *Phys. Rev. A* **101**, 063612 (2020).
- [51] S. Schiffer, X.-J. Liu, H. Hu, and J. Wang, Anderson localization transition in a robust $\mathcal{PT}$ -symmetric phase of a generalized Aubry-André model, *Phys. Rev. A* **103**, L011302 (2021).
- [52] Y. Liu, Q. Zhou, and S. Chen, Localization transition, spectrum structure, and winding numbers for one-dimensional non-Hermitian quasicrystals, *Phys. Rev. B* **104**, 024201 (2021).
- [53] K. Kawabata and S. Ryu, Nonunitary scaling theory of non-Hermitian localization, *Phys. Rev. Lett.* **126**, 166801 (2021).
- [54] S.-X. Hu, Y. Fu, and Y. Zhang, Non-Hermitian delocalization induced by residue imaginary velocity, *Commun. Phys.* **8**, 269 (2025).
- [55] E. Lee, H. Lee, and B.-J. Yang, Many-body approach to non-Hermitian physics in fermionic systems, *Phys. Rev. B* **101**, 121109(R) (2020).
- [56] S. Mu, C. H. Lee, L. Li, and J. Gong, Emergent Fermi surface in a many-body non-Hermitian fermionic chain, *Phys. Rev. B* **102**, 081115(R) (2020).
- [57] P.-Y. Chang, J.-S. You, X. Wen, and S. Ryu, Entanglement spectrum and entropy in topological non-Hermitian systems and nonunitary conformal field theory, *Phys. Rev. Res.* **2**, 033069 (2020).
- [58] R. Shen and C. H. Lee, Non-Hermitian skin clusters from strong interactions, *Commun. Phys.* **5**, 238 (2022).
- [59] S.-B. Zhang, M. M. Denner, T. c. v. Bzdušek, M. A. Sentef, and T. Neupert, Symmetry breaking and spectral structure of the interacting Hatano-Nelson model, *Phys. Rev. B* **106**, L121102 (2022).
- [60] N. Matsumoto, K. Kawabata, Y. Ashida, S. Furukawa, and M. Ueda, Continuous phase transition without gap closing in non-Hermitian quantum many-body systems, *Phys. Rev. Lett.* **125**, 260601 (2020).
- [61] F. Alsallom, L. Herviou, O. V. Yazyev, and M. Brzezińska, Fate of the non-Hermitian skin effect in many-body fermionic systems, *Phys. Rev. Res.* **4**, 033122 (2022).
- [62] K. Kawabata, T. Numasawa, and S. Ryu, Entanglement phase transition induced by the non-Hermitian skin effect, *Phys. Rev. X* **13**, 021007 (2023).
- [63] T. Haga, M. Nakagawa, R. Hamazaki, and M. Ueda, Liouvillian skin effect: Slowing down of relaxation processes without gap closing, *Phys. Rev. Lett.* **127**, 070402 (2021).
- [64] F. Yang, Q.-D. Jiang, and E. J. Bergholtz, Liouvillian skin effect in an exactly solvable model, *Phys. Rev. Res.* **4**, 023160 (2022).
- [65] S. Guo, C. Dong, F. Zhang, J. Hu, and Z. Yang, Theoretical prediction of a non-Hermitian skin effect in ultracold-atom systems, *Phys. Rev. A* **106**, L061302 (2022).
- [66] H. Li and S. Wan, Dynamic skin effects in non-Hermitian systems, *Phys. Rev. B* **106**, L241112 (2022).

- [67] K. Shimomura and M. Sato, General criterion for non-Hermitian skin effects and application: Fock space skin effects in many-body systems, *Phys. Rev. Lett.* **133**, 136502 (2024).
- [68] R. Shen, F. Qin, J.-Y. Desaulles, Z. Papić, and C. H. Lee, Enhanced many-body quantum scars from the non-Hermitian Fock skin effect, *Phys. Rev. Lett.* **133**, 216601 (2024).
- [69] J. Gliozzi, G. De Tomasi, and T. L. Hughes, Many-body non-Hermitian skin effect for multipoles, *Phys. Rev. Lett.* **133**, 136503 (2024).
- [70] Y.-M. Hu, Z. Wang, B. Lian, and Z. Wang, Many-body non-Hermitian skin effect with exact steady states in the dissipative quantum link model, *Phys. Rev. Lett.* **135**, 260401 (2025).
- [71] J. Gliozzi, F. Balducci, T. L. Hughes, and G. D. Tomasi, Non-Hermitian multipole skin effects challenge localization *Phys. Rev. B* **113**, L100203 (2026).
- [72] C.-X. Guo, C.-H. Liu, X.-M. Zhao, Y. Liu, and S. Chen, Exact solution of non-Hermitian systems with generalized boundary conditions: Size-dependent boundary effect and fragility of the skin effect, *Phys. Rev. Lett.* **127**, 116801 (2021).
- [73] P. Molignini, O. Arandes, and E. J. Bergholtz, Anomalous skin effects in disordered systems with a single non-Hermitian impurity, *Phys. Rev. Res.* **5**, 033058 (2023).
- [74] C.-X. Guo, X. Wang, H. Hu, and S. Chen, Accumulation of scale-free localized states induced by local non-Hermiticity, *Phys. Rev. B* **107**, 134121 (2023).
- [75] Y. Fu and Y. Zhang, Hybrid scale-free skin effect in non-Hermitian systems: A transfer matrix approach, *Phys. Rev. B* **108**, 205423 (2023).
- [76] B. Li, H.-R. Wang, F. Song, and Z. Wang, Scale-free localization and $\mathcal{PT}$ symmetry breaking from local non-Hermiticity, *Phys. Rev. B* **108**, L161409 (2023).
- [77] Throughout the Letter, we adopt a unified notation convention for the Hamiltonians. The symbol with a hat, $\hat{H}$, explicitly denotes the Hamiltonian operator in the second-quantized formalism. In contrast, the symbol without a hat, H , refers to the Hamiltonian matrix under OBCs, with boundary hoppings excluded. Meanwhile, $H(z)$ represents the so-called non-Bloch Hamiltonian, obtained by analytically continuing the momentum-space Hamiltonian $H(k)$, $k \in \text{BZ}$ (defined under PBCs) through the mapping $e^{ik} \rightarrow z \in \mathbb{C}$.
- [78] This filling convention can be generalized to filled (empty) states with $\text{Re}(e^{i\theta}\epsilon) < \mu$ [$\text{Re}(e^{i\theta}\epsilon) > \mu$] via a mapping $\hat{H}' = \hat{H}e^{i\theta} - \mu\hat{N}$, where $\hat{N}$ is the fermion number operator, e.g., the Fermi sea $\text{Im}(\epsilon) > 0$ in terms of long-time steady-state dynamics [54]. Such rotations and (or) translations are unphysical, as they correspond to transformations of the Hamiltonian that leave the Fermi surface invariant and equivalent to modifications of the Fermi surface over the original Hamiltonian. In this Letter, we choose the Fermi surface such that the contribution from real velocities is eliminated while the imaginary velocity is retained. In this case, the quasiparticle worldlines reside in the space spanned by imaginary time and configuration coordinates, which is consistent with the physical interpretation in our previous QMC-SSE calculations [54,80], as further elaborated below.
- [79] Multiple bands, if they exist, need to be summed over, except in the $\mathbb{Z}_2$ skin scenario.
- [80] S.-X. Hu, Y. Fu, and Y. Zhang, Nontrivial worldline winding in non-Hermitian quantum systems, *Phys. Rev. B* **108**, 245114 (2023).
- [81] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/6n5b-l8jv> for technical details, additional figures or discussions, and further clarification of the main text.
- [82] K. Zhang, Z. Yang, and C. Fang, Universal non-Hermitian skin effect in two and higher dimensions, *Nat. Commun.* **13**, 2496 (2022).
- [83] An energy on the OBC spectrum that corresponds to n distinct points on the GBZ is referred to as an n -bifurcation point [88].
- [84] Without loss of clarity, we hereafter refer to such intersections on both the BZ/GBZ and the spectra as Bloch points.
- [85] C. H. Lee and R. Thomale, Anatomy of skin modes and topology in non-Hermitian systems, *Phys. Rev. B* **99**, 201103(R) (2019).
- [86] K.-I. Imura and Y. Takane, Generalized bulk-edge correspondence for non-Hermitian topological systems, *Phys. Rev. B* **100**, 165430 (2019).
- [87] An analogue of Eq. (1) holds for more generic cases with multiple Fermi points.
- [88] D. Wu, J. Xie, Y. Zhou, and J. An, Connections between the open-boundary spectrum and the generalized Brillouin zone in non-Hermitian systems, *Phys. Rev. B* **105**, 045422 (2022).
- [89] Equivalently, one can obtain the PBC spectrum $H_{p,\rho}(z)$ by evaluating it on standard BZ (unit circle), and the corresponding OBC spectrum by applying the transformed GBZ associated with $H_{p,\rho}$. Both schemes yield identical PBC and OBC spectra. In Fig. 4, we have adopted the former approach.
- [90] Here and below, we denote $\mathcal{F}(z)$ and $\mathcal{F}[\text{BZ}]$ as the images of the element z and the set BZ under the mapping $\mathcal{F}$, respectively.
- [91] N. Hatano and D. R. Nelson, Localization transitions in non-Hermitian quantum mechanics, *Phys. Rev. Lett.* **77**, 570 (1996).
- [92] N. Hatano and D. R. Nelson, Vortex pinning and non-Hermitian quantum mechanics, *Phys. Rev. B* **56**, 8651 (1997).
- [93] The truncation mainly influences the far-left edge of the OBC spectrum, leaving multiple protruding branches, but does not change the winding-control mechanism. A larger truncation n_{max} allows a better approximation to the model and spectrum in Eq. (5).
- [94] H. Wang, J. Zhong, and S. Fan, Topology of the generalized Brillouin zone of one-dimensional models, [arXiv:2510.07214](https://arxiv.org/abs/2510.07214).
- [95] L. Zhang, Y. Yang, Y. Ge, Y.-J. Guan, Q. Chen, Q. Yan, F. Chen, R. Xi, Y. Li, D. Jia, S.-Q. Yuan, H.-X. Sun, H. Chen, and B. Zhang, Acoustic non-Hermitian skin effect from twisted winding topology, *Nat. Commun.* **12**, 6297 (2021).