

Interacting type-II semi-Dirac quasiparticles

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Type-II semi-Dirac fermions in two dimensions have been proposed to describe topologically nontrivial low-energy excitations in titanium/vanadium oxide heterostructures. These quasiparticles appear at the merger of three Dirac cones, resulting in a nonzero Berry phase. We find, by employing Hartree-Fock, renormalization group, and random phase approximation techniques, that the spectrum is very sensitive to long-range electron-electron interactions and can undergo a profound transformation. Our results indicate that at the topological phase boundary, long-range correlations stabilize a hybrid electronic phase displaying both Dirac and type-II semi-Dirac qualities, with physical characteristics exhibiting continuously varying critical exponents as a function of the Fermi energy; for example, Landau levels in a magnetic field vary with the energy scale: $|\varepsilon_n(B)| \sim (nB)^{1/2} \rightarrow (nB)^{3/4}$, $n \in \mathbb{N}_0$. The quasiparticle spectrum evolves, driven by interactions, from anisotropic Dirac dispersion at the lowest energies toward the characteristic type-II semi-Dirac boomerang shape as the energy increases. The corresponding density of states concomitantly varies between linear and power one-third ($\rho(\varepsilon) \sim |\varepsilon| \rightarrow |\varepsilon|^{1/3}$). The crossover scale is controlled by the interaction strength $\alpha = e^2/(\hbar v)$ and the specifics of the effective interacting Hamiltonian.

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I. INTRODUCTION

Semi-Dirac fermions are a hybrid low-energy excitation, exhibiting the qualities of both massless Dirac particles and massive Galilean particles at once. More precisely, they disperse linearly in a given direction in momentum space and quadratically in the orthogonal direction. This is most easily understood as resulting from the merger of two Dirac cones at the critical point of a semimetal-insulator transition, whereby the Berry phases annihilate and the critical semi-Dirac spectrum emerges [1,2]. The fundamental anisotropy gives rise to unconventional phenomena in a variety of contexts, such as charge [3,4] and thermoelectric [5,6] transport; behavior in strong magnetic fields [7,8]; optical effects and surface interaction with light [9,10]; edge states and topological behavior in modified models with broken symmetries [11,12]; and anisotropic collective excitations such as plasmons, excitons, and spinons [13–16]. Furthermore, there are other possible flavors of semi-Dirac fermions. Higher-order generalizations have attracted theoretical study, where the quadratic dispersion is generalized to an arbitrary even power law [17,18]. Recently, there has been a proposed lattice model realization for the most basic nontrivial case in this class of Hamiltonians, namely, the quartic semi-Dirac fermions [19].

Moreover, there is an even more exotic, and less explored, species of semi-Dirac fermions that are not topologically

trivial. These excitations emerge at a critical point where not two but three Dirac cones coalesce, producing a finite Berry phase and what has been dubbed a type-II semi-Dirac spectrum [20]. The dispersion retains its semi-Dirac character in the sense that it is still linear and quadratic along the given axes, but exhibits a different admixture of momentum components for an arbitrary direction of propagation [20,21]. This symmetry-reduced model was first proposed to reconcile the finite Chern number and semi-Dirac behavior of quasiparticles in titanium/vanadium oxide heterostructures [20]. However, it may also be realized in more conventional platforms such as bilayer graphene, by leveraging interlayer sliding [22,23]. Such systems have been shown to exhibit a rich variety of topological phases that are accessible via fine tuning with light-matter interactions [24]. In addition, the peculiar anisotropy leads to highly unusual optical and transport properties [22,25]. Topological semimetallic phases have also been analyzed in detail both experimentally and theoretically in ZrSiS, reporting a measurement of the signature Landau level scaling with magnetic field [26].

Electron-electron interactions are known to have pronounced effects in ordinary semi-Dirac fermions [1,27–30]. Short-range interactions have been shown to stabilize modulated ordered phases in the charge, spin, and superconducting sectors around a quantum critical point [31,32]. On the other hand, the long-range Coulomb interaction produces two low-energy regimes in which there is unusually strong spectrum renormalization [17,33]. Motivated by these results, we study the spectrum renormalization of the topological type-II semi-Dirac fermions due to the long range, and random phase approximation (RPA)-screened, Coulomb interaction. We have found that, remarkably, the interacting system hosts massless Dirac quasiparticles at the lowest energies, smoothly

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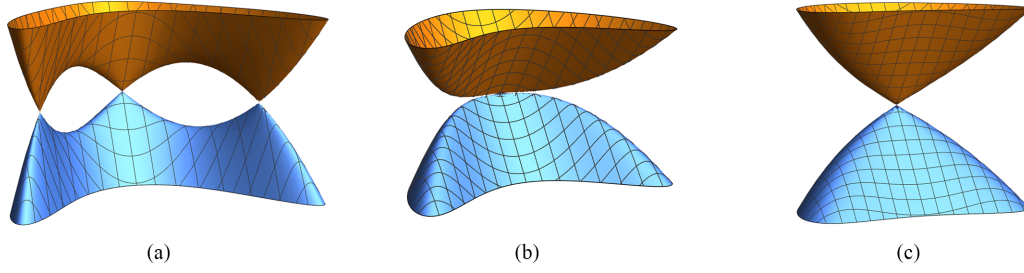


FIG. 1. Low-energy electronic dispersion corresponding to the Hamiltonian in Eq. (1) under a small perturbation Δ , such that $\mathcal{H} = (g_1 k_x^2 - v k_y + \Delta)\hat{\sigma}_x + g_2 k_x k_y \hat{\sigma}_y$. (a) $\Delta < 0$, leading to the presence of three Dirac cones located at $(0, \Delta/v)$, $(\pm\sqrt{-\Delta/g_1}, 0)$. (b) $\Delta = 0$: The three cones coalesce at the origin, giving rise to a type-II semi-Dirac point with a finite Berry phase. (c) $\Delta > 0$: The band crossing is shifted to $k_y = \Delta/v$. At the smallest momenta, the spectrum is linear, but exhibits type-II semi-Dirac behavior at larger energies.

transitioning to the free fermionic behavior with increasing energy. This is captured in the exponent η of the density of states ($\rho(\varepsilon) \sim \varepsilon^{\eta(\varepsilon)}$) as a function of energy varying from 1 to 1/3 on a sigmoid. The Fermi surface geometry evolves from convex to concave in accordance with the changing η . As a consequence, many physical properties and observables will have a continuously evolving dependence on the energy scale of interest. Given that Coulomb interactions are invariably present in systems of charged particles, our results can explain the appearance of the low-energy linear behavior in Ref. [20]. Moreover, our study implies that a measurement or calculation of the coefficient of the linear term in the Hamiltonian along the massive direction can be used as a diagnostic tool to estimate the effective interaction strength in such systems. As an illustration, we model the quasiparticles in the candidate material $(\text{TiO}_2)_5(\text{VO}_2)_3$ as interacting type-II fermions at the topological phase boundary, using parameter values from Ref. [20]. By comparing coefficients of the low-energy linear term along the massive direction, we make a rough estimate of the effective interaction strength in the material, which indeed lies in the weak-coupling regime. Intriguingly, a similar change in Fermi surface shape (driven by temperature) in the two-dimensional (2D) semimetal WTe_2 has been observed to lead to a spectacular reversal in the sign of the nonlinear current response under crossed electric and magnetic fields [34]. Even though our work is concerned with zero-temperature, interaction-driven effects, the qualitatively identical crossover prompts us to expect similarly novel magnetotransport and possibly other unusual phenomena.

We consider the Hamiltonian describing type-II semi-Dirac fermions [20]:

$$\mathcal{H}(\mathbf{k}) = (g_1 k_x^2 - v k_y)\hat{\sigma}_x + g_2 k_x k_y \hat{\sigma}_y, \quad (1)$$

where the coupling g_1 is the inverse effective mass, v is the velocity in the massless direction, and g_2 is the cross anisotropy that characterizes the type-II nature of the particles. Such excitations can arise at the critical point of the topological transition shown in Fig. 1. We rescale the momenta, defining $p_{x,y} = (g_1/v)k_{x,y}$, and introduce the energy scale $\varepsilon_0 = v^2/g_1$ to arrive at the electronic dispersion

$$\varepsilon(\mathbf{p}) = \pm \varepsilon_0 \sqrt{(p_x^2 - p_y)^2 + (g p_x p_y)^2}, \quad (2)$$

where we define $g \equiv g_2/g_1$. A plot of the spectrum is shown in Fig. 2 using a representative $g = 1/2$, similarly to the value of $g = 0.41$ in $(\text{TiO}_2)_5(\text{VO}_2)_3$ [20].

II. ENERGY VARIABLES AND DENSITY OF STATES

When calculating diagrams and other quantities of interest in systems with such anisotropy, it is useful to recast into energy-angle variables which, we define as follows:

$$\begin{aligned} p_x^2 - p_y &= (\varepsilon/\varepsilon_0) \cos \varphi, \\ g p_x p_y &= (\varepsilon/\varepsilon_0) \sin \varphi. \end{aligned} \quad (3)$$

The solution of these equations in the low-energy limit ($\varepsilon/\varepsilon_0 \ll 1$) is

$$\begin{aligned} p_x &= \text{sgn}(\sin \varphi) \left(\frac{E |\sin \varphi|}{g} \right)^{1/3} + \mathcal{O}(E^{2/3}), \\ p_y &= \left(\frac{E |\sin \varphi|}{g} \right)^{2/3} - E \cos \varphi + \mathcal{O}(E^{4/3}), \end{aligned} \quad (4)$$

where $0 \leq \varphi \leq 2\pi$, $E \equiv \frac{\varepsilon}{\varepsilon_0}$.

In this limit, we calculate the Jacobian of the transformation, immediately yielding the density of states as a function of energy ε :

$$\rho(\varepsilon) = \iint \frac{d^2 k}{(2\pi)^2} \delta(\varepsilon - \varepsilon(\mathbf{k})) = \frac{1}{6\pi^{3/2}} \frac{\Gamma(1/6)}{\Gamma(2/3)} \left(\frac{E}{g_1^2 g_2} \right)^{1/3}, \quad (5)$$

with the numerical coefficient amounting to 0.123. We find the nonlinear transformation in Eq. (4) to be particularly effective in rendering the problem analytically tractable in the low-energy limit.

III. PERTURBATION THEORY AND RENORMALIZATION GROUP

We take into account the nonretarded Coulomb interaction $V(\mathbf{k}) = 2\pi e^2/|\mathbf{k}|$ to first order in perturbation theory. The effective dimensionless coupling that controls the perturbative expansion is $\alpha = e^2/v$ ($\hbar = 1$). The self-energy is $\hat{\Sigma}(\mathbf{k}) = i/(2\pi)^3 \int_{-\infty}^{\infty} d\nu \int d^2 k' \hat{G}(\mathbf{k}', \nu) V(\mathbf{k} - \mathbf{k}')$, where $\hat{G}^{-1}(\mathbf{k}, \nu) = \nu - \mathcal{H}(\mathbf{k}) + i0^+ \text{sign}(v)$ defines the fermionic Green's function. The frequency integral can be easily evaluated:

$$\hat{\Sigma}(\mathbf{k}) = \frac{1}{2} \int \frac{d^2 k'}{(2\pi)^2} \frac{2\pi e^2}{|\mathbf{k} - \mathbf{k}'|} \frac{1}{|\varepsilon(\mathbf{k}')|} \mathcal{H}(\mathbf{k}'). \quad (6)$$

We calculate corrections: Σ_v to the quasiparticle velocity, Σ_{g_1} to the inverse mass, and Σ_{g_2} to the cross anisotropy, which

result in the following universal Hamiltonian:

$$\tilde{\mathcal{H}}(\mathbf{k}) = [g_1(1 + \Sigma_{g_1})k_x^2 - v(1 + \Sigma_v)k_y]\hat{\sigma}_x + g_2(1 + \Sigma_{g_2})k_x k_y \hat{\sigma}_y, \quad (7)$$

omitting terms dependent on the high-energy cutoff Λ . For example, the correction Σ_v may be extracted by considering the asymptotic low-energy behavior of

$$\Sigma_v(\omega) = \frac{\alpha}{12\pi} \int_{\omega}^{\Lambda} \frac{dE}{E} I_v(E),$$

$$I_v(E) = 2 \int_0^{\pi} d\varphi \frac{E^{1/3}}{[\sin \varphi]^{2/3}} \times \frac{(E^{2/3}[\sin \varphi]^{2/3} - E \cos \varphi)^2}{(E^{2/3}[\sin \varphi]^{2/3} + (E^{2/3}[\sin \varphi]^{2/3} - E \cos \varphi)^2)^{3/2}},$$

which produces a leading logarithmic divergence $\Sigma_v = (\alpha/\pi) \ln(\Lambda/\omega)$. Details of the computation for the parameters $g_{1,2}, v$ are outlined in the Appendix. We have the following one-loop perturbation theory results:

$$v(\omega) = vL(\omega), \quad g_{1,2}(\omega) = g_{1,2}L(\omega),$$

$$L(\omega) = \left(1 + \frac{\alpha}{\pi} \ln(\Lambda/\omega)\right) = (1 + \Sigma_{v,g_1,g_2}), \quad (8)$$

where the infrared ω will be taken to be on-shell $\omega \equiv \varepsilon(\mathbf{k})$ in subsequent calculations, i.e., follows the form of the bare dispersion.

Next, we write down and integrate the corresponding renormalization group (RG) equations, where we introduce $\ell = \ln(\Lambda/\omega)$: $dv/d\ell = v(\ell)\alpha(\ell)/\pi = e^2/\pi$, $dg_1/d\ell = \alpha(\ell)g_1(\ell)/\pi$, $dg_2/d\ell = \alpha(\ell)g_2(\ell)/\pi$. From the first equation, it follows, since the electric charge does not renormalize, that the solution for $v(\ell)$ is trivial returning the perturbative result in Eq. (8), and α only renormalizes via its dependence on v : $\alpha(\omega) = \alpha/[1 + \frac{\alpha}{\pi} \ln(\Lambda/\omega)]$. This indicates that there are no higher-order logarithmic divergences for the RG to resum. It is easy to see that the same applies for $g_{1,2}$, and the equations integrate to what are now our final results in Eq. (8). Similarly to graphene [35] and ordinary semi-Dirac fermions, the coupling constant is marginally irrelevant. However, in contrast to type-I semi-Dirac fermions where the presence of a characteristic $\log^2$ divergence in g_1 at first order generates a higher-order resummation [17,30], the type-II couplings remain unchanged under RG.

IV. EFFECTIVE HAMILTONIAN

In addition to the logarithmic corrections, there appear two crucial cutoff-dependent terms in the self-energy. The zero-momentum contribution reads

$$\hat{\Sigma}(0) = \alpha \left(\frac{\varepsilon_0}{4\pi} \int \frac{d^2 p}{\sqrt{p_x^2 + p_y^2}} \frac{p_x^2 - p_y}{\sqrt{(p_x^2 - p_y)^2 + (gp_x p_y)^2}} \right) \hat{\sigma}_x$$

$$\equiv \alpha \Delta \hat{\sigma}_x, \quad (9)$$

and the corresponding $\hat{\sigma}_y$ component vanishes by parity. The generation of a constant, zero-momentum, term induced by interactions in the effective Hamiltonian is a characteristic

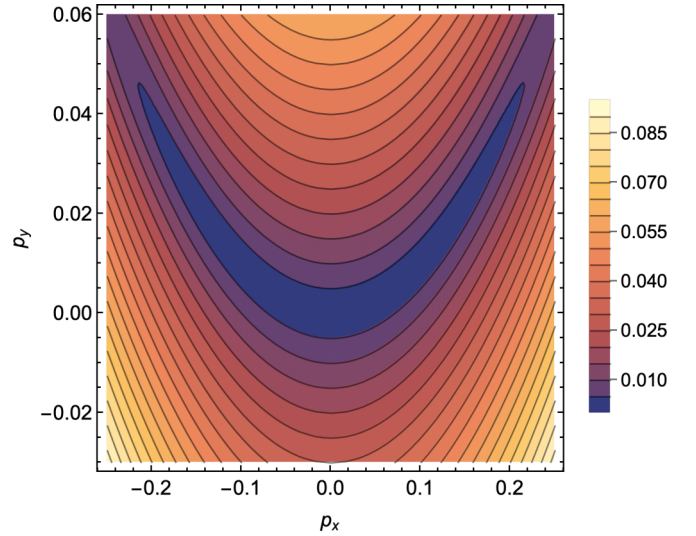


FIG. 2. Contour plot of the critical low-energy spectrum [Eq. (2)], illustrated schematically in Fig. 1(b). Energies are measured in units of ε_0 and momenta in units of v/g_1 . We take $g = 1/2$, similar to the cross anisotropy in the candidate material. The isoenergy curves have a characteristic boomerang shape.

feature of semi-Dirac fermions of all types. Moreover, expanding the Coulomb potential for small external momenta $V(\mathbf{k} - \mathbf{k}') \approx \frac{2\pi e^2}{k} (1 + \frac{\mathbf{k} \cdot \mathbf{k}'}{k^2})$, we find that the only (unaccounted for) term from Eq. (6) that does not vanish by parity is the one $\propto k_x \hat{\sigma}_y$, generating the following contribution:

$$\alpha \left(\frac{v}{4\pi} \int \frac{d^2 p}{(p_x^2 + p_y^2)^{3/2}} \frac{gp_x^2 p_y}{\sqrt{(p_x^2 - p_y)^2 + (gp_x p_y)^2}} \right) k_x \hat{\sigma}_y$$

$$\equiv \alpha c k_x \hat{\sigma}_y. \quad (10)$$

Assembling the corrections together, we arrive at the interacting effective Hamiltonian:

$$\mathcal{H}(\mathbf{k}) = (g_1 L(\omega) k_x^2 - v L(\omega) k_y + \alpha \Delta) \hat{\sigma}_x + (g_2 L(\omega) k_x k_y + \alpha c k_x) \hat{\sigma}_y. \quad (11)$$

Henceforth, v, g_1, g_2, α will denote the bare couplings. We find $\Delta > 0$, which implies a shift of the dispersion in the k_y direction, meaning that the band crossing is now at $k_y = \alpha \Delta / v$ (whereas $\Delta < 0$ would have meant the presence of three Dirac cones, see Fig. 1). Measuring momenta from the new semi-Dirac point, the Hamiltonian assumes the same form as Eq. (1), but with the logarithmically enhanced parameters in Eq. (8), and an additional interaction-driven linear correction

$$\alpha \left(\frac{g_2}{v} \Delta + c \right) k_x \hat{\sigma}_y, \quad (12)$$

which indicates that the dispersion is effectively anisotropic Dirac at small momenta. As shown in Fig. 3, the spectrum smoothly transitions from linear Dirac to type-II semi-Dirac behavior with increasing energy. To capture this crossover, we model the density of states for the interacting fermions as $\rho(\varepsilon) \sim \varepsilon^{\eta(\varepsilon)}$. In order to compute η , we numerically evaluate the area enclosed by cyclotron orbits in momentum space as a function of energy. A discretized differentiation returns the density of states, from which the exponent can be easily

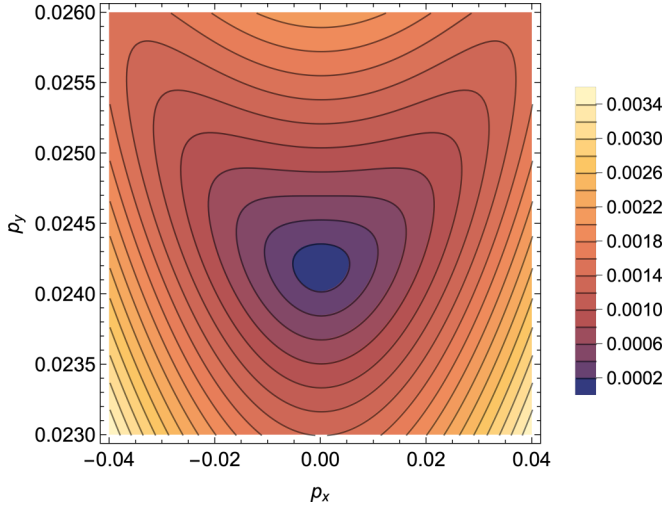


FIG. 3. Type-II semi-Dirac spectrum renormalized by the bare Coulomb interaction at the one-loop Hartree-Fock (HF) level for $\alpha = 0.2$ and $g = 1/2$. The convex lowest-energy contours indicate an anisotropic Dirac cone, with concave semi-Dirac behavior appearing at higher energies corresponding to Eq. (13). Noting the relatively narrow range of p_y considered in the plot, it is clear that p_x controls the crossover energy scale at which the Fermi surface changes geometry and the qualitative nature of quasiparticles transforms. An ultraviolet momentum cutoff $k_\Lambda = 1.1$ on a disk region, in units of v/g_1 , is used to produce a numerical fit of Eq. (6) that returns the values $\Delta/\varepsilon_0 = 1.88/4\pi$ and $c/v = 1.39/4\pi$.

extracted. Results for varying interaction strength are shown in Fig. 4. The crossover from linear to semi-Dirac excitations is clearly reflected in the decrease of η from 1 to the saturation at $1/3$. The onset is controlled by an interaction-dependent energy scale corresponding to

$$|k_x^*| \approx \frac{\alpha[(g_2/v)\Delta + c]}{g_1 L(\omega(k_x^*))}, \quad (13)$$

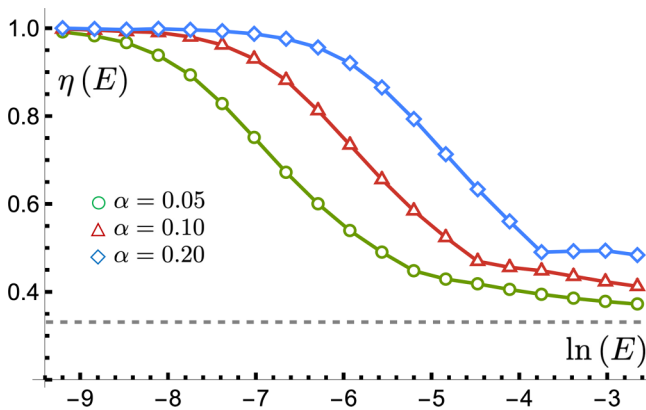


FIG. 4. Evolution of the power $\eta(E)$ in the density of states $\rho(E) \sim E^{\eta(E)}$ for $g = 1/2$ under different interaction strengths $\alpha = 0.05, 0.10, 0.20$ in the Hartree-Fock approximation. The smooth transition from linear Dirac behavior $\eta = 1$ at the smallest energies, approaching type-II semi-Dirac behavior $\eta = 1/3$ (dashed line) with increasing energy, captures the variation in the interacting spectrum shown in Fig. 3.

the regime where the quadratic term $\sim k_x^2$ in Eq. (11) begins to dominate over the linear k_x term in Eq. (12). At small energies, the renormalization (increase) of g_1 favors the semi-Dirac spectrum and suppresses the linear regime. In contrast, the running of g_1 under Coulomb interactions in ordinary semi-Dirac fermions gives rise to a linear spectrum for a large window of intermediate energies, with the renormalized semi-Dirac behavior present at the lowest energies [17].

To showcase the predictive power of our results, we will attempt a rough estimate of the effective interaction strength in $(\text{TiO}_2)_5(\text{VO}_2)_3$, for which the band structure is well known to be type-II semi-Dirac with a small low-energy linear regime. Modeling the low-energy excitations of the material as interacting fermions at the critical point, the aim is to deduce the effective interaction strength that would reproduce the coefficient of the linear term found in the physical system. We assume parameters $g_1 = 0.256$, $g_2 = 0.1044$, and $|v| = 0.3929$, identical to the material [20], and numerically evaluate $\Delta = [\varepsilon_0/(4\pi)]2.244$ and $c = [v/(4\pi)]0.892$ on a square region with a high-energy cutoff $|p_\Lambda^{x,y}| = 1.1$. Comparing Eq. (12) to the term present in the $(\text{TiO}_2)_5(\text{VO}_2)_3$ Hamiltonian: $-0.02044k_x \hat{\sigma}_y$, we arrive at an effective interaction strength of $\alpha_{\text{eff}} = 0.36$, which is well within the weak-coupling regime. In light of the conclusion in Ref. [20] that the material is an “anisotropic linear Dirac system that is so close to a type-II semi-Dirac behavior that it can hardly be distinguished from it,” we feel that our characterization of the quasiparticles as weakly interacting fermions at criticality is justified.

V. BEYOND HARTREE-FOCK: RANDOM PHASE APPROXIMATION AND STATIC POLARIZATION FUNCTION

Going beyond the first-order perturbative level, we introduce static screening via the RPA. We have evaluated the zero-frequency polarization function numerically, as outlined in the Appendix, and have found that it is well approximated by the form

$$|\Pi(\mathbf{p})| = [(c_1|p_x|^{1/2})^w + (c_2|p_y|^{1/3})^w]^{1/w}, \quad w \gg 1 \quad (14)$$

equivalent to the following structure:

$$|\Pi(\mathbf{p})| = \begin{cases} c_1|p_x|^{1/2}, & \frac{c_2|p_y|^{1/3}}{c_1|p_x|^{1/2}} \leq 1, \\ c_2|p_y|^{1/3}, & \frac{c_2|p_y|^{1/3}}{c_1|p_x|^{1/2}} \geq 1. \end{cases} \quad (15)$$

The number of fermionic species is taken to be $N = 4$. The boundary between the two regions is visible in the plot of the polarization in Fig. 5. We are pleasantly surprised to find that it is possible to evaluate the polarization along the linear direction $\Pi(0, p_y)$ analytically (by making use of Feynman parametrization), as described in the Appendix. The analytical and numerical results for the coefficient c_2 and $p_y^{1/3}$ dependence are in precise agreement.

The self-energy is then calculated as in Eq. (6), but instead using the dressed Coulomb potential $V_{\text{RPA}}(\mathbf{p}) = V(\mathbf{p})/(1 - V(\mathbf{p})\Pi(\mathbf{p}))$, and the variation in the exponent of the renormalized density of states is summarized in Fig. 6. While results are qualitatively similar to their HF counterparts, it is evident that introducing screening naturally inhibits

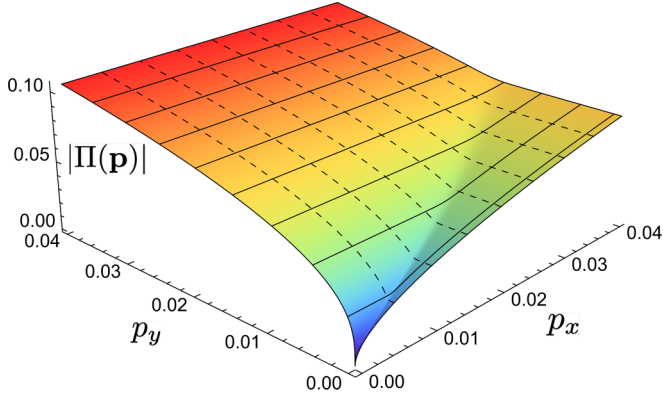


FIG. 5. Static polarization function for type-II semi-Dirac fermions. The ravine residing on $p_y = (c_1/c_2)^3 p_x^{3/2}$ demarcates two regions where the polarization is dominated by $\sqrt{p_x}$ and $p_y^{1/3}$ dependence, and flat in the other direction. The dashed and solid mesh lines at constant p_x , p_y exhibit this structure. Values of $c_1 = N(0.098) = 0.392$ and $c_2 = N(0.077) = 0.308$ were calculated, and $w = 50$ was used for illustration.

the bare interaction effects, suppressing enhancement of the spectrum. The logarithmic corrections to all couplings that appear prominently within HF disappear quickly in RPA, as α increases, reflecting the lifting of the bare-Coulomb-interaction-generated infrared divergence. The RPA potential is highly anisotropic and exhibits fractional powers in both momentum directions, which dominate over the bare Coulomb potential and generally lead to the generation of nondivergent terms with different symmetries, as outlined in the Appendix. Note that Δ and c are now functions of α , and in fact decay with increasing interaction strength, partially neutralizing the growth of the crossover energy scale with

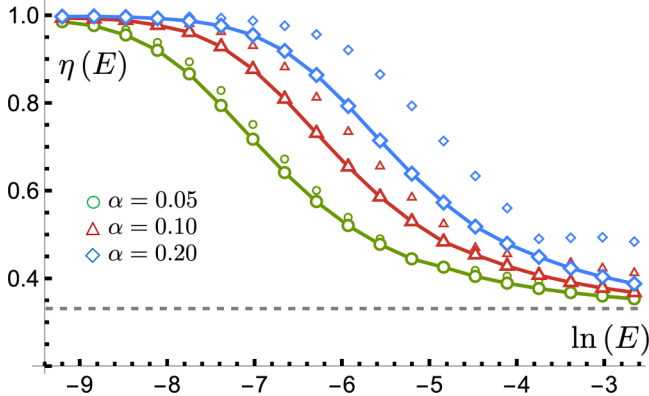


FIG. 6. Evolution of the density of states scaling $\eta(E)$ under the RPA-screened Coulomb interaction (solid curves) for $g = 1$. We have included the corresponding Hartree-Fock results, given by the smaller unconnected markers, for comparison. For the interaction strengths $\alpha = 0.05, 0.10, 0.20$, values of $\Delta/\varepsilon_0 = 1.83/(4\pi), 1.55/(4\pi), 1.19/(4\pi)$ and $c/v = 1.39/(4\pi), 1.19/(4\pi), 0.93/(4\pi)$ are calculated, respectively, in the RPA. At the HF level, the values $\Delta/\varepsilon_0 = 1.89/(4\pi)$ and $c/v = 1.71/(4\pi)$ ($g = 1$) are independent of α . Results are qualitatively similar, with the smaller values of η under the RPA indicating a weaker renormalization as expected.

α . Details of the numerical self-energy fits are given in the Appendix.

VI. CONCLUSIONS AND IMPLICATIONS FOR OBSERVABLES

The change of electronic dispersion has profound implications for the behavior of various characteristics at low temperatures/energies. As the density of states evolves between powers $\eta = 1$ at the lowest energies and $\eta = 1/3$ for higher energies (noninteracting quasiparticles), the specific heat evolves from $C_V(T) \sim T^2$ ($\eta = 1$) to $C_V(T) \sim T^{4/3}$ ($\eta = 1/3$). In general, $C_V(T) \sim T^{\eta+1}$.

The electronic compressibility κ is defined as $\kappa^{-1} = n^2(\partial\mu/\partial n)$ and can be probed experimentally [35–38]. Here, μ is the chemical potential, n is the electronic density ($n \sim \mu^{\eta+1}$), and $\partial\mu/\partial n$ is the inverse density of states. The charge response is typically determined by $\partial\mu/\partial n$, which is related to the inverse capacitance. One finds $\partial\mu/\partial n \sim 1/\mu \sim 1/\sqrt{n}$ ($\eta = 1$) and $\partial\mu/\partial n \sim 1/\mu^{1/3} \sim 1/n^{1/4}$ ($\eta = 1/3$). In general, $\partial\mu/\partial n \sim 1/\mu^\eta \sim 1/n^{\eta/(\eta+1)}$.

Landau level quantization [28,29,36,39,40] is governed by the semiclassical formula $A(\varepsilon_n(B)) = \int_0^{\varepsilon_n(B)} \rho(\omega) d\omega \sim nB$, where $A(\varepsilon) = \int_{\varepsilon(\mathbf{k}) \leq \varepsilon} d^2\mathbf{k}/(2\pi)^2$ is the area enclosed by the energy dispersion in momentum space. Thus, we find $\varepsilon_n(B) \sim \pm[nB]^{1/(\eta+1)}$. In particular for the noninteracting fermions, $\varepsilon_n(B) \sim \pm[nB]^{3/4}$ presents a unique signature for experimental detection. The scaling is distinct from the conventional $B^{2/3}$ in type-I semi-Dirac systems, and there is a zero energy Landau level due to the finite Berry phase. Note that the conventional magnetic field dependence may be mimicked by interacting type-II fermions under suitable conditions.

Transport coefficients, such as dynamical conductivity, can be sensitive to the geometry of the Fermi surface [41]. In our case, the shape evolves from convex to concave (boomerang) with increasing Fermi energy (Fig. 3), with an expected associated change in the conductivity's frequency dependence. Moreover, it is of direct relevance that nonlinear magnetotransport has been observed to be very sensitive to changes in Fermi surface geometry in 2D semimetals such as WTe_2 [34]. The change of convexity with Fermi level induces a sign inversion of the nonlinear magnetoresistance, and accompanying current density response. In the context of our study, the features of the band structure are driven by interactions, and the chemical potential can easily be manipulated by doping, whereas in the magnetotransport experiments the pertinent features are electron-hole pockets, and the Fermi level is controlled via temperature. However, given the common underlying fermiology, it is reasonable to expect that peculiar magnetotransport and other potentially unusual phenomena may be observed in type-II semi-Dirac systems.

We expect that the conclusions reached in this work will remain valid for finite, small particle density (i.e., small filling of the upper band). Similarly to graphene, the charge is not renormalized, as reflected in the nonsingular behavior of the polarization function. Topological characteristics, such as the Chern number and the anomalous Hall conductivity, are not affected by the interactions. The interacting Dirac liquid exhibits remarkable evolution of the Fermi surface shape as a function of the long-range interaction and Fermi energy. The

system hosts hybrid quasiparticles with both Dirac and semi-Dirac qualities as reflected in the density of states evolution from linear to power one-third, with accompanying changes to the scaling of associated observables.

Can these effects be detected and exploited in real materials? We would suggest, for example, that the modification of the interaction strength α by using dielectric screening (i.e., $e^2 \rightarrow e^2/\kappa$, where κ is the dielectric constant) can control the density of states exponent presented in Figs. 4 and 6; thus presenting a pathway to tuning the Fermi level at which the surface changes geometry. Given the striking reversal of the current density response in WTe_2 , for instance, we expect that such tunability might lend itself well to a variety of applications. Furthermore, our results can explain the small low-energy linear regime in $(\text{TiO}_2)_5(\text{VO}_2)_3$ and, more generally, may be used to determine the effective interaction strength in such systems. As an illustration, we calculate an estimate of α_{eff} that lies in the weak-coupling regime and is consistent with the literature on the material.

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DATA AVAILABILITY

The Appendix provides sufficient information how the data are generated. We make use of conventional techniques. Any additional data are available from the authors upon reasonable request.

APPENDIX: COMPUTATIONAL TECHNIQUES

In this Appendix, we provide additional information and details. We have used the conventional many-body techniques, namely, the Hartree-Fock and random phase approximations, as described in Refs. [42,36] for example.

$$I_v(E) = 2 \int_0^\pi d\varphi \frac{E^{1/3}}{[\sin \varphi]^{2/3}} \frac{(E^{2/3}[\sin \varphi]^{2/3} - E \cos \varphi)^2}{(E^{2/3}[\sin \varphi]^{2/3} + (E^{2/3}[\sin \varphi]^{2/3} - E \cos \varphi)^2)^{3/2}}. \quad (\text{A3})$$

We need the behavior of this function as $E \rightarrow 0$. First, we observe that the integration over angles from 0 to 2π is exactly twice the integral from 0 to π by parity of the integrand. Next, it is clear that the value of the integral is accumulated near $\varphi \approx 0, \pi$, with the result being identical near those two values due to the symmetry of the integrand as $E \rightarrow 0$. Consequently, the asymptotic behavior is determined by the following expression, for φ near the lower limit:

$$\begin{aligned} I_v(E) &= 4 \int_{\sqrt{E}} d\varphi \frac{E^{5/3} \varphi^{2/3}}{(E^{2/3} \varphi^{2/3} + E^{4/3} \varphi^{4/3})^{3/2}} \\ &= 12/\sqrt{1+E} = 12, \quad E \rightarrow 0. \end{aligned} \quad (\text{A4})$$

1. Hartree-Fock approximation

In the HF approximation (i.e., to first order perturbatively in the interaction strength α), the self-energy is given by Eq. (6), leading to the logarithmic renormalization of the couplings described in Eqs. (7) and (8). We have set the dimensionless anisotropy $g = 1$ in our formulas below, in order to avoid cumbersome notation.

There are several ways to extract the leading behavior. For example, for small external momenta $k \rightarrow 0$, we can expand $\frac{1}{|\mathbf{k}-\mathbf{k}'|} = \frac{1}{k'} \{1 + \frac{\mathbf{k} \cdot \mathbf{k}'}{k'^2} - \frac{k^2}{2k'^2} + \frac{3(\mathbf{k} \cdot \mathbf{k}')^2}{2k'^4}\} + O(k^3)$. From here, we can extract the leading infrared behavior at small momenta relative to the ultraviolet momentum cutoff Λ ($k \ll \Lambda$):

$$\begin{aligned} \Sigma_v(k) &\approx \frac{e^2}{4\pi} \int_{k'>k} d^2k' \frac{k_y'^2}{k'^3 |\varepsilon(\mathbf{k}')|} = \frac{e^2}{4\pi} \int_{k'>k} d^2k' \frac{k_y'^2}{k'^3 |vk_y'|} \\ &= \frac{\alpha}{4\pi} \int_k^\Lambda \frac{dk'}{k'} \int_0^{2\pi} d\varphi \frac{\sin^2 \varphi}{|\sin \varphi|} = \frac{\alpha}{\pi} \ln(\Lambda/k). \end{aligned} \quad (\text{A1})$$

In the intermediate integration, we have used the small momentum approximation $|\varepsilon(\mathbf{k}')| \approx |vk_y'|$. Since the integral is highly anisotropic, the above expression should only be viewed as a way of intermediate momentum power counting, leading to an expected logarithmic structure in the final result.

A more accurate way to extract the logarithmic behavior would be to use the energy-angle variables introduced in Eq. (4). This mapping fully preserves the correct anisotropic structure of the energy dispersion at low energies. We calculate the Jacobian of the transformation in the low-energy limit $|J(E, \varphi)| = \frac{E^{1/3}}{3|\sin \varphi|^{2/3}}$ and obtain

$$\Sigma_v(\omega) = \frac{\alpha}{12\pi} \int_\omega^\Lambda \frac{dE}{E} I_v(E). \quad (\text{A2})$$

Here, E, ω, Λ are measured in units of ε_0 [Eq. (4)]. The function $I_v(E)$ is defined as

Here, we have used, for low energy and $\varphi \sim 0$, the approximation $E^{2/3}(\sin \varphi)^{2/3} \gg E \cos \varphi$, leading to the condition for the angle $\varphi > \sqrt{E}$. From Eq. (A2), we finally obtain, in the limit $\omega/\Lambda \ll 1$,

$$\Sigma_v(\omega) = \frac{\alpha}{\pi} \ln\left(\frac{\Lambda}{\omega}\right), \quad \omega = \varepsilon(\mathbf{k}), \quad (\text{A5})$$

where the energy dependence is to be taken “on-shell,” i.e., at the noninteracting energy dispersion, naturally capturing the anisotropy in the system.

Proceeding to the renormalization of the coupling g_2 , we obtain

$$\Sigma_{g_2} \approx \frac{3e^2}{4\pi} \int_{k'>k} d^2k' \frac{k_x'^2 k_y'^2}{k'^5 |\varepsilon(\mathbf{k}')|} = \frac{\alpha}{4\pi} \int_\omega^\Lambda \frac{dE}{E} I_{g_2}(E), \quad (\text{A6})$$

where

$$I_{g_2}(E) = 2 \int_0^\pi d\varphi \frac{E^{1/3}}{[\sin \varphi]^{2/3}} \frac{E^{2/3}[\sin \varphi]^{2/3} (E^{2/3}[\sin \varphi]^{2/3} - E \cos \varphi)^2}{(E^{2/3}[\sin \varphi]^{2/3} + (E^{2/3}[\sin \varphi]^{2/3} - E \cos \varphi)^2)^{5/2}}. \quad (\text{A7})$$

From here, following the procedure we used in the calculation of $I_v(E)$ above, we find for $E \rightarrow 0$:

$$I_{g_2}(E) = 4 \int_{\sqrt{E}} d\varphi \frac{E^{7/3} \varphi^{4/3}}{(E^{2/3} \varphi^{2/3} + E^{4/3} \varphi^{4/3})^{5/2}} = 4/(1+E)^{3/2} = 4, \quad E \rightarrow 0. \quad (\text{A8})$$

Consequently,

$$\Sigma_{g_2}(\omega) = \frac{\alpha}{\pi} \ln \left(\frac{\Lambda}{\omega} \right), \quad \omega = \varepsilon(\mathbf{k}). \quad (\text{A9})$$

We have implemented a full, discretized numerical integration of the self-energy by using the following representation [equivalent, via change of variables, to Eq. (6)]: $\hat{\Sigma}(\mathbf{k}) = \frac{1}{2} \int \frac{d^2 k'}{(2\pi)^2} V(\mathbf{k}') \frac{\mathcal{H}(\mathbf{k}+\mathbf{k}')}{[\varepsilon(\mathbf{k}+\mathbf{k}')]}$. Here, $V(\mathbf{k}) = 2\pi e^2/k$. This has confirmed the asymptotic results [Eqs. (A5) and (A9)] and in addition has determined similar behavior for the coupling g_1 ,

$$\Sigma_{g_1}(\omega) = \frac{\alpha}{\pi} \ln \left(\frac{\Lambda}{\omega} \right). \quad (\text{A10})$$

Thus, in the HF approximation the leading low-energy correction to the three couplings is the divergent logarithmic renormalization with identical coefficients. The rest of the self-energy contains two important nonuniversal (i.e., ultra-violet cutoff Λ -dependent) contributions: $\hat{\Sigma}(0) = \alpha \Delta \hat{\sigma}_x$ and a leading linear term $\alpha c k_x \hat{\sigma}_y$ generated by the lowest order in the expansion of the potential, which we extract numerically. We find that Δ grows rapidly with Λ , whereas c exhibits a much weaker increase. Both quantities show a mild dependence on g . The values of Λ and g used in the HF results in the main text are $\Lambda = 1.1$ in units of ε_0 and $g = 1/2$.

$$\Pi(\mathbf{p}) = \frac{-N}{8\pi^2} \int_0^1 dt \int d^2 \mathbf{k} \frac{t[\varepsilon^2(\mathbf{k}+\mathbf{p}) - \varepsilon^2(\mathbf{k})] + \varepsilon^2(\mathbf{k}) - \varepsilon_x(\mathbf{k})\varepsilon_x(\mathbf{k}+\mathbf{p}) - \varepsilon_y(\mathbf{k})\varepsilon_y(\mathbf{k}+\mathbf{p})}{[t[\varepsilon^2(\mathbf{k}+\mathbf{p}) - \varepsilon^2(\mathbf{k})] + \varepsilon^2(\mathbf{k})]^{3/2}}. \quad (\text{A13})$$

Performing the integral over k_y exactly, we have

$$\Pi(0, p_y) = \frac{-N}{8\pi^2} \int_0^1 dt \int_{-\infty}^{\infty} dk_x \frac{4p_y^2 t(1-t)(1+k_x^2)^{3/2}}{k_x^6 + p_y^2 t(1-t)(1+k_x^2)^2}. \quad (\text{A14})$$

Let us introduce the notation $z = p_y^2 t(1-t)$. We evaluate the integral for $z \ll 1$ with the result

$$\begin{aligned} & 2 \int_0^{\infty} dk_x \frac{4z(1+k_x^2)^{3/2}}{k_x^6 + z(1+k_x^2)^2} \\ &= 8z^{1/6} \int_0^{\infty} dk_x \frac{(1+z^{1/3}k_x^2)^{3/2}}{k_x^6 + (1+z^{1/3}k_x^2)^2} \\ &\approx \frac{8\pi}{3} z^{1/6} + O(\sqrt{z}). \end{aligned} \quad (\text{A15})$$

2. Polarization function

We calculate the static polarization function by using the conventional formula for the one-loop integral:

$$\Pi(\mathbf{p}) = -iN \iint \frac{d^2 k}{(2\pi)^2} \frac{d\omega}{2\pi} \text{Tr}(\hat{G}(\mathbf{k}, \omega) \hat{G}(\mathbf{k}+\mathbf{p}, \omega)). \quad (\text{A11})$$

The integer N represents the number of fermionic species contributing to the vacuum polarization. We have used $N = 4$ appropriate for the material from Ref. [20]. From here, we obtain

$$\begin{aligned} \Pi(\mathbf{p}) = & -N \int \frac{d^2 k}{(2\pi)^2} \frac{1}{\varepsilon(\mathbf{k}) + \varepsilon(\mathbf{k}+\mathbf{p})} \\ & \times \left(1 - \frac{\varepsilon_x(\mathbf{k})\varepsilon_x(\mathbf{k}+\mathbf{p}) + \varepsilon_y(\mathbf{k})\varepsilon_y(\mathbf{k}+\mathbf{p})}{\varepsilon(\mathbf{k})\varepsilon(\mathbf{k}+\mathbf{p})} \right), \end{aligned} \quad (\text{A12})$$

with the functions $\varepsilon_x(\mathbf{k}) = k_x^2 - k_y$, $\varepsilon_y(\mathbf{k}) = k_x k_y$, and $\varepsilon(\mathbf{k}) = \sqrt{\varepsilon_x(\mathbf{k})^2 + \varepsilon_y(\mathbf{k})^2}$. (We have taken all the couplings to one for simplicity of notation.) We have evaluated the function in Eq. (A12) numerically, which leads to the result in Eqs. (11) and (12) and Fig. 5.

It is also instructive to evaluate certain limits analytically, if possible. It turns out that we can perform an asymptotically exact calculation of the polarization function in the p_y direction. Using the Feynman parametrization, $\frac{1}{AB} = \int_0^1 \frac{dt}{[t(A-B)+B]^2}$, we obtain

Along the way, we have changed variables $k_x/z^{1/6} \rightarrow k_x$ and used the integral $\int_0^\infty dk_x 1/(k_x^6 + 1) = \pi/3$. The final result is

$$\begin{aligned} \Pi(0, p_y) = & \frac{-N}{8\pi^2} \frac{8\pi}{3} p_y^{1/3} \int_0^1 [t(1-t)]^{1/6} dt \\ = & \frac{-N}{24} \frac{\Gamma(1/6)}{\Gamma(5/6)\Gamma(1/3)} p_y^{1/3} \\ = & -N 0.077 p_y^{1/3}. \end{aligned} \quad (\text{A16})$$

This result is in complete agreement with the numerical expression [Eqs. (11) and (12)].

3. Random phase approximation

The random phase approximation takes into account the effective screening of the bare Coulomb potential. Within the

RPA [42], we have for the self-energy

$$\hat{\Sigma}(\mathbf{k}) = \frac{1}{2} \int \frac{d^2 k'}{(2\pi)^2} V_{\text{RPA}}(\mathbf{k}') \frac{\mathcal{H}(\mathbf{k}' + \mathbf{k})}{|\varepsilon(\mathbf{k}' + \mathbf{k})|}, \quad (\text{A17})$$

where the statically screened potential is given by $V_{\text{RPA}}(\mathbf{k}') = V(\mathbf{k}')/(1 - V(\mathbf{k}')\Pi(\mathbf{k}'))$. The zero-momentum self-energy

$$\hat{\Sigma}(0) = \frac{1}{2} \int \frac{d^2 k'}{(2\pi)^2} V_{\text{RPA}}(\mathbf{k}') \frac{k_x^2}{|\varepsilon(\mathbf{k}')|} \hat{\sigma}_x = \alpha \Delta(\alpha) \hat{\sigma}_x \quad (\text{A18})$$

now acquires a nonmultiplicative dependence on α via V_{RPA} . We find that the same applies to the constant c generated by the linear term $\alpha c(\alpha)k_x \hat{\sigma}_y$ in the small momentum expansion.

All calculations within RPA were performed numerically because the anisotropic form of the polarization does not allow for any analytical calculation. The computation was performed on a disk in momentum space, $|\mathbf{k}'| \leq \Lambda$, and was challenging because of the presence of the small quantity $|\mathbf{k}|$ in Eq. (A17), which led to particularly nonuniform behavior of the integrand in both the radial and azimuthal directions of $\mathbf{k}'$. We then implemented a highly accurate empirical fit, which depended on both $|\mathbf{k}|$ and $\arg(\mathbf{k}')$, to the numerically computed self-energy for a wide range of α and Λ . We find that it takes an astonishing 36 terms to achieve a visually satisfactory fit for both Σ_x and Σ_y ($\hat{\Sigma} = \Sigma_x \hat{\sigma}_x + \Sigma_y \hat{\sigma}_y$). The main effect of the accounting for polarization in $V_{\text{RPA}}(\mathbf{k})$ that we have observed is a very rapid decrease of the leading logarithmic contributions, given by Eqs. (A5), (A9), and (A10), to the self-energy. For example, already for α as small as 0.025, these contributions have decreased more than tenfold

compared to the HF approximation. This lifting of the infrared divergence is due to the highly anisotropic RPA potential with fractional powers in both momentum directions dominating over the bare Coulomb potential. Therefore, for the purposes of describing the crossover and evolution of the density of states, most of the aforementioned terms in the empirical fits can safely be neglected, and we may adequately represent the self-energy by retaining only few dominant terms. For example, at the value of the Coulomb coupling $\alpha = 0.2$, we find that the most important terms have the form

$$\hat{\Sigma}_x = \alpha(0.57k_y + \Delta(\alpha))\hat{\sigma}_x, \quad (\text{A19})$$

$$\hat{\Sigma}_y = \alpha(0.56k_x k_y + c(\alpha)k_x)\hat{\sigma}_y. \quad (\text{A20})$$

Note that the fully precise fits including all terms were used in generating the results in Fig. 6.

4. Scaling of density of states

In order to compute the energy-dependent scaling η of the interacting density of states $\rho(E) \sim E^{\eta(E)}$, we first discretize the isoenergy contours into a set of points and calculate the area enclosed by using the relation $A = (1/2) \sum_{i=1}^n (x_i y_{i+1} - x_{i+1} y_i)$, which gives the area A bounded by a polygon with vertices (x_i, y_i) . We then perform a numerical differentiation as we increment E in the interval $[10^{-4}, 10^{-1}]$, with equal spacing on a logarithmic scale, which returns a quantity proportional to the density of states. By taking logarithms, the exponent $\eta(E)$ is easily extracted.

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- [1] G. Montambaux, F. Piéchon, J.-N. Fuchs, and M. O. Goerbig, Merging of Dirac points in a two-dimensional crystal, *Phys. Rev. B* **80**, 153412 (2009).
 - [2] G. Montambaux, F. Piéchon, J.-N. Fuchs, and M. Goerbig, A universal Hamiltonian for motion and merging of Dirac points in a two-dimensional crystal, *Eur. Phys. J. B* **72**, 509 (2009).
 - [3] J. P. Carbotte, K. R. Bryenton, and E. J. Nicol, Optical properties of a semi-Dirac material, *Phys. Rev. B* **99**, 115406 (2019).
 - [4] D. O. Oriekhov and V. P. Gusynin, Optical conductivity of semi-Dirac and pseudospin-1 models: Zitterbewegung approach, *Phys. Rev. B* **106**, 115143 (2022).
 - [5] A. Mawrie and B. Muralidharan, Quantum thermoelectrics based on two-dimensional semi-Dirac materials, *Phys. Rev. B* **100**, 081403(R) (2019).
 - [6] V. Vargiamidis, P. Vasilopoulos, and N. Neophytou, Off-resonant light-induced topological phase transition and thermoelectric transport in semi-Dirac materials, *Phys. Rev. B* **112**, 045406 (2025).
 - [7] P. Sinha, S. Murakami, and S. Basu, Landau levels and magneto-optical transport properties of a semi-Dirac system, *Phys. Rev. B* **105**, 205407 (2022).
 - [8] D. Asafov and I. Pavlov, Collapse of Landau levels in semi-Dirac materials, *Phys. Rev. B* **110**, 125126 (2024).
 - [9] X. Zhou, W. Chen, and X. Zhu, Anisotropic magneto-optical absorption and linear dichroism in two-dimensional semi-Dirac electron systems, *Phys. Rev. B* **104**, 235403 (2021).
 - [10] H.-L. Cheng and Y.-M. Zhang, Goos-Hänchen shift and photonic spin Hall effect in semi-Dirac material heterostructures, *Phys. Rev. A* **112**, 013508 (2025).
 - [11] S. Mondal and S. Basu, Topological phases of a semi-Dirac Chern insulator in the presence of extended range hopping, *Phys. Rev. B* **105**, 235441 (2022).
 - [12] M. García Olmos, Y. Baba, A. López, M. Amado, and R. A. Molina, Spin-dependent transport through edge states in 2D semi-Dirac materials with Rashba spin-orbit coupling and band inversion, *2D Mater.* **12**, 045019 (2025).
 - [13] A. N. Arafat, O. L. Berman, and G. Gumbs, Superfluidity of indirect momentum space dark dipolar excitons in a double layer with anisotropic tilted semi-Dirac bands, *Phys. Rev. B* **109**, 224506 (2024).
 - [14] G. Ross-Harvey, A. Iurov, L. Zhemchuzhna, G. Gumbs, D. Huang, and P. Fekete, Dynamical polarization function, anisotropic plasmon modes, and dephasing rates for interacting electrons in semi-Dirac bands, *Phys. Rev. B* **111**, 045413 (2025).
 - [15] S. Chatterjee, A. Maity, J. Potten, T. Müller, A. Feuerpfel, R. Thomale, K. Penc, H. O. Jeschke, R. Samajdar, and Y. Iqbal, Semi-Dirac spin liquids and frustrated quantum magnetism on the trellis lattice, *Phys. Rev. Res.* **8**, 013191 (2026).
 - [16] D. Giri, J. Schliemann, R. A. Molina, and A. Lopez, Tunable plasmon modes and topological transitions in monolayer and bilayer semi-Dirac materials, *Phys. Rev. B* **112**, 165420 (2025).

- [17] M. M. Elsayed, B. Uchoa, and V. N. Kotov, Coulomb interactions in systems of generalized semi-Dirac fermions, *Phys. Rev. B* **111**, 165127 (2025).
- [18] B. Roy and M. S. Foster, Quantum multicriticality near the Dirac-semimetal to band-insulator critical point in two dimensions: A controlled ascent from one dimension, *Phys. Rev. X* **8**, 011049 (2018).
- [19] M. M. Elsayed and V. N. Kotov, Microscopic lattice model for quartic semi-Dirac fermions in two dimensions, *J. Phys.: Condens. Matter* **37**, 315501 (2025).
- [20] H. Huang, Z. Liu, H. Zhang, W. Duan, and D. Vanderbilt, Emergence of a Chern-insulating state from a semi-Dirac dispersion, *Phys. Rev. B* **92**, 161115(R) (2015).
- [21] V. Pardo and W. E. Pickett, Metal-insulator transition through a semi-Dirac point in oxide nanostructures: VO₂ (001) layers confined within TiO₂, *Phys. Rev. B* **81**, 035111 (2010).
- [22] Z. Liao, H. Zeng, E. Wang, and H. Huang, Berry curvature dipole and nonlinear Hall effect in type-II semi-Dirac systems, *Small* **21**, 2409691 (2025).
- [23] Y.-W. Son, S.-M. Choi, Y. P. Hong, S. Woo, and S.-H. Jhi, Electronic topological transition in sliding bilayer graphene, *Phys. Rev. B* **84**, 155410 (2011).
- [24] J.-N. Chen, Y.-Y. Yang, Y.-L. Zhou, Y.-J. Wu, H.-J. Duan, M.-X. Deng, and R.-Q. Wang, Photon-modulated linear and nonlinear anomalous Hall effects in type-II semi-Dirac semimetals, *Phys. Rev. B* **105**, 085124 (2022).
- [25] Q.-Y. Xiong, J.-Y. Ba, H.-J. Duan, M.-X. Deng, Y.-M. Wang, and R.-Q. Wang, Optical conductivity and polarization rotation of type-II semi-Dirac materials, *Phys. Rev. B* **107**, 155150 (2023).
- [26] Y. Shao, S. Moon, A. N. Rudenko, J. Wang, J. Herzog-Arbeitman, M. Ozerov, D. Graf, Z. Sun, R. Queiroz, S. H. Lee, Y. Zhu, Z. Mao, M. I. Katsnelson, B. A. Bernevig, D. Smirnov, A. J. Millis, and D. N. Basov, Semi-Dirac fermions in a topological metal, *Phys. Rev. X* **14**, 041057 (2024).
- [27] P. Dietl, F. Piéchon, and G. Montambaux, New magnetic field dependence of Landau levels in a graphenelike structure, *Phys. Rev. Lett.* **100**, 236405 (2008).
- [28] M. Goerbig and G. Montambaux, Dirac fermions in condensed matter and beyond, in *Dirac Matter*, edited by B. Duplantier, V. Rivasseau, and J.-N. Fuchs (Springer International Publishing, Berlin, 2017), pp. 25–53.
- [29] R. de Gail, J.-N. Fuchs, M. Goerbig, F. Piéchon, and G. Montambaux, Manipulation of Dirac points in graphene-like crystals, *Physica B* **407**, 1948 (2012).
- [30] V. N. Kotov, B. Uchoa, and O. P. Sushkov, Coulomb interactions and renormalization of semi-Dirac fermions near a topological Lifshitz transition, *Phys. Rev. B* **103**, 045403 (2021).
- [31] B. Uchoa and K. Seo, Superconducting states for semi-Dirac fermions at zero and finite magnetic fields, *Phys. Rev. B* **96**, 220503(R) (2017).
- [32] M. D. Uryszek, E. Christou, A. Jaefari, F. Krüger, and B. Uchoa, Quantum criticality of semi-Dirac fermions in 2 + 1 dimensions, *Phys. Rev. B* **100**, 155101 (2019).
- [33] H. Isobe, B.-J. Yang, A. Chubukov, J. Schmalian, and N. Nagaosa, Emergent non-Fermi-liquid at the quantum critical point of a topological phase transition in two dimensions, *Phys. Rev. Lett.* **116**, 076803 (2016).
- [34] P. He, C.-H. Hsu, S. Shi, K. Cai, J. Wang, Q. Wang, G. Eda, H. Lin, V. M. Pereira, and H. Yang, Nonlinear magnetotransport shaped by Fermi surface topology and convexity, *Nat. Commun.* **10**, 1290 (2019).
- [35] V. N. Kotov, B. Uchoa, V. M. Pereira, F. Guinea, and A. H. Castro Neto, Electron-electron interactions in graphene: Current status and perspectives, *Rev. Mod. Phys.* **84**, 1067 (2012).
- [36] S. M. Girvin and K. Yang, *Modern Condensed Matter Physics* (Cambridge University Press, Cambridge, 2019).
- [37] J. Martin, N. Akerman, G. Ulbricht, T. Lohmann, J. H. Smet, K. von Klitzing, and A. Yacoby, Observation of electron-hole puddles in graphene using a scanning single-electron transistor, *Nat. Phys.* **4**, 144 (2008).
- [38] D. E. Sheehy and J. Schmalian, Quantum critical scaling in graphene, *Phys. Rev. Lett.* **99**, 226803 (2007).
- [39] A. A. Abrikosov, *Fundamentals of the Theory of Metals* (North-Holland, Amsterdam, 1988).
- [40] G. P. Mikitik and Y. V. Sharlai, Manifestation of Berry's phase in metal physics, *Phys. Rev. Lett.* **82**, 2147 (1999).
- [41] D. L. Maslov and A. V. Chubukov, Optical response of correlated electron systems, *Rep. Prog. Phys.* **80**, 026503 (2017).
- [42] G. F. Giuliani and G. Vignale, *Quantum Theory of the Electron Liquid* (Cambridge University Press, Cambridge, 2010).