

Measurement-induced phase transition in state estimation of chaotic systems and the directed polymer

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We introduce a solvable model of a measurement-induced phase transition (MIPT) in a deterministic but chaotic dynamical system with a positive Lyapunov exponent. In this setup, an observer only has a probabilistic description of the system but mitigates chaos-induced uncertainty through repeated measurements. Using a minimal representation via a branching tree, we map this problem to the directed polymer (DP) model on the Cayley tree, although in a regime dominated by rare events. By studying the Shannon entropy of the probability distribution estimated by the observer, we demonstrate a phase transition distinguishing a chaotic phase with a reduced Lyapunov exponent from a strong-measurement phase where uncertainty remains bounded. Remarkably, the location of the MIPT transition coincides with the freezing transition of the DP, although the critical properties differ. We provide an exact, universal scaling function describing the entropy growth in the critical regime. Numerical simulations confirm our theoretical predictions, highlighting a simple yet powerful framework to explore measurement-induced transitions in classical chaotic systems.

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I. INTRODUCTION

In recent years, much attention has been devoted to the interplay between unitary evolution and external noise [1–9], particularly the one induced by the action of quantum measurements [10]. While a closed quantum system tends to encode local information into nonlocal degrees of freedom, resulting in the production of entanglement entropy [11–15], measurements of local quantities interfere by extracting information from the quantum state. As a result of such competition, two phases can emerge [16–18]: at weak measurements, nonlocal quantum correlations grow (volume law) [19,20], and the system thus exhibits a spontaneous error-correcting capacity [21–23]; conversely, at strong measurements, information remains confined to local degrees of freedom and entanglement growth saturates (area law), allowing for an effective matrix-product-states description [24,25]. Experiments have confirmed this phenomenology [26–28], although the observability of this transition in extended systems has been questioned because of the burden of postselection [29,30].

Due to the inherently random outcome of quantum measures, monitored systems can be seen as disordered with time being an extra spatial dimension. However, the distribution of outcomes is not fixed *a priori* but is determined by the

state itself (according to Born's rule). This implies that the results of measurements at different times are strongly correlated (up to the extreme case of the Zeno effect where they do not change) [29,31], in contrast with usual disordered systems where short-range disorder correlations are generally assumed [32,33]. Nonetheless, the interpretation as disordered systems for monitored noninteracting fermions [34–41] has revealed similarities with the Anderson transition [42] and nonlinear sigma models [43,44]. In general, approaches based on annealed averages [45–50], mean field [51–54], random matrices [55,56], and extensive numerics [25,57–59] provided the general picture that interacting quantum systems in $d + 1$ dimensions undergo a MIPT, with a critical point described by a yet unknown nonunitary conformal field theory [60–63].

A similar protocol can also be considered in a purely classical context, where the stochastic description of a system is continuously updated by measurements according to Bayes' theorem [64,65]; in contrast with quantum MIPTs, there is no postselection barrier since the measurements do not affect the actual state of the system. For a one-dimensional (1D) diffusive particle undergoing Bayesian monitoring, a short-time KPZ behavior has been suggested [66,67]. In this paper, we present a simple, solvable model exhibiting a monitoring-induced phase transition. Let us consider a generic chaotic classical dynamical system with positive Lyapunov exponents. Although it is a deterministic system, at the practical level, the exponential growth of any finite uncertainty about the initial conditions allows only a probabilistic description. To mitigate this uncertainty, an observer makes measurements in order to update their estimate of the state of the system. A minimal description of this setup is given by a branching tree,

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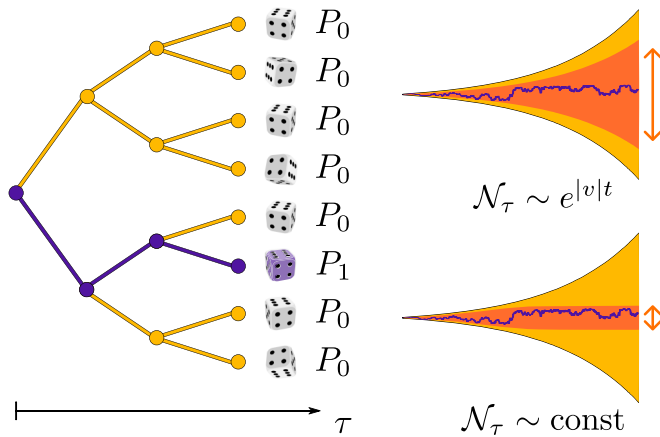


FIG. 1. Left: Cayley tree ($K = 2$), modeling the exponential growth $\Delta x_{\max} \sim e^{\lambda t}$ of the uncertainty on the position of a particle. The purple branch depicts the true trajectory x_τ of the particle: at each time-step τ , a finite-precision measurement on each site is performed, whose outcomes a_j are distributed with $P_{n_j}(a_j)$ depending on the occupancy of the site j , $n_j = 0, 1$. Right: sketch of the two phases of the model. The orange region represents a subregion of size $N_\tau \sim e^{S_\tau}$ in which the observer can be reasonably sure the particle is located. The effectiveness of measurements determines whether $N_\tau \sim \text{const}$ or N_τ still scale exponentially, but with a reduced Lyapunov exponent $|v| < \lambda$.

in which the branching ratio K is related to the (maximum) Lyapunov exponent; see Fig. 1.

This formulation allows a mapping to the famous DP problem in this geometry [68]. In [67], within this setup, the existence of two distinct phases has been demonstrated when assuming knowledge of the exact trajectory. These two phases correspond to a time-integrated overlap between the estimated and exact trajectory being 0 or 1. Here, however, consistent with the idea that the true trajectory is not knowable, we study the problem exclusively from the observer's point of view. This has a direct relation with the quantum case with Bayes' theorem replacing Born's rule: the aforementioned correlation between measurement outcomes requires us to analyze the DP problem in a regime of rare events [69]. We study the Shannon entropy of the observer-estimated probability distribution: we demonstrate the existence of a phase transition, between (i) a regime in which chaos persists albeit with a reduction in the effective Lyapunov exponent, and (ii) a regime in which the uncertainty saturates with time. Interestingly, the location of the transition coincides with that of the freezing transition of the DP. In addition, the Shannon entropy displays universal critical properties in proximity to the transition that we are able to thoroughly characterize by providing an explicit, exact, and universal scaling function describing the entropy growth at long times.

II. THE MODEL

We consider an abstract setting with an external observer looking at a dynamical system: in order to simulate the chaotic growth of uncertainty, we assume the state of the system to be one of the nodes of a Cayley tree branching with ratio K along the effective discrete-time direction $t = \tau \Delta t$, $\tau \in \mathbb{N}$. While

the evolution is deterministic, i.e., the system follows a specific path connecting the root to a leaf, because of their finite knowledge about the initial condition, the observer can only rely on a probabilistic description. We thus have a particle undergoing “directed” random walk: a particle in $x_\tau \in \{j\}_{j=1}^{K^\tau}$ at time τ hops to one of the K possible states $K(x_\tau - 1) + m$ at level $\tau + 1$, with $m \in \{1, \dots, K\}$. Let us notice that, by setting $K = e^{\lambda \Delta t}$, the maximum separation between two trajectories at time t is $\Delta x_{\max} \sim e^{\lambda t}$ so that λ can be identified with the maximal Lyapunov exponent.

We denote as $p_j^{(\tau)}$ the probability for the observer that the system is located in j at time-step τ . Right after the random-walk step, the observer performs instantaneous measurements of each site in order to mitigate the growth of uncertainty. To measure site j at time-step τ , a measuring apparatus returns a random variable $a_j^{(\tau)}$ distributed with probability density $P_1(a)$ if the site is occupied at time-step τ , and $P_0(a)$ otherwise. For clarity, although the discussion is generic, throughout this paper we will keep the Gaussian case as a yardstick where P_0 and P_1 are Gaussian distributions with unit variance and mean 0 and μ , respectively. While the outcome is not deterministic, the measurement is classical, so that the system is unaffected by it. As measurements from different sites are statistically independent, the probability of the outcomes $\mathbf{a}^{(\tau)} \equiv \{a_j^{(\tau)}\}_{j=1}^{K^\tau}$ conditioned to the particle being in site j is

$$P(\mathbf{a}^{(\tau)} | x_\tau = j) = \frac{P_1(a_j^{(\tau)})}{P_0(a_j^{(\tau)})} \prod_{j'=1}^{K^\tau} P_0(a_{j'}^{(\tau)}). \quad (1)$$

From the knowledge of the measurement outcomes $\mathbf{a}^{(\tau+1)}$ at level $\tau + 1$, the observer can update the estimate of the probabilities from $p_j^{(\tau)}$ to $p_j^{(\tau+1)}$ using Bayes' theorem,

$$p_j^{(\tau+1)} = \frac{P(\mathbf{a}^{(\tau+1)} | j) p_j^{(\tau+1,-)}}{\sum_{j'} P(\mathbf{a}^{(\tau+1)} | j') p_{j'}^{(\tau+1,-)}}, \quad (2)$$

where $p_j^{(\tau+1,-)} = p_{\lceil j/K \rceil}^{(\tau)} / K$ denotes the probabilities right before the measurements, and $\lceil \dots \rceil$ is the ceiling function. We will assume that the process is repeated several times, with the true trajectory chosen uniformly among all paths on the tree. For a given true trajectory, Eq. (1) determines the distribution of outcomes; then, knowing the outcomes, Eq. (2) gives the distribution inferred by the observer. To determine how effective the observer's measurements are, we will be interested in analyzing typical properties of the inferred distribution.

III. MAPPING TO A DIRECTED POLYMER

To simplify the dynamics induced by Eqs. (1) and (2), we make two observations. First, since the denominator in Eq. (2) serves only to impose normalization for a known set of measurement outcomes $\{\mathbf{a}^{(\tau')}\}_{\tau' \leq \tau}$, we introduce a set of non-normalized variables $\mathbf{z}^{(\tau)} := \{z_j^{(\tau)}\}_{j=1}^{K^\tau}$ evolving with $z_j^{(\tau+1)} = B(a_j^{(\tau+1)}) z_{\lceil j/K \rceil}^{(\tau)}$, where $B(a) := P_1(a)/[K P_0(a)]$. As at each step $p_j^{(\tau)} \propto z_j^{(\tau)}$, we can recover $p_j^{(\tau)} = z_j^{(\tau)} / Z^{(\tau)}$, with $Z^{(\tau)} := \sum_{j=1}^{K^\tau} z_j^{(\tau)} = \sum_{\text{paths}} \prod_{x \in \text{path}} B(a_x)$. Interpreting $B(a)$ as a Boltzmann weight associated with the nodes of the tree, $Z^{(\tau)}$ can be seen as the partition function of a DP on the tree [70].

Let us denote as $\mathbf{p}^{(\tau)} := \{p_j^{(\tau)}\}_{j=1}^{K^\tau}$ the full set of probabilities for a given true trajectory x_τ and outcomes $\{\mathbf{a}^{(\tau')}\}_{\tau' \leq \tau}$. Consider an arbitrary functional $F[\mathbf{p}^{(\tau)}]$. Then, its average over the uniform distribution of x_τ and the corresponding distribution of outcomes $\{\mathbf{a}^{(\tau')}\}_{\tau' \leq \tau}$ reads

$$\langle F[\mathbf{p}^{(\tau)}] \rangle := \frac{1}{K^\tau} \sum_{x_\tau=1}^{K^\tau} \int d\mathbf{a} F[\mathbf{p}^{(\tau)}] \prod_{\tau'=1}^{\tau} P(\mathbf{a}^{(\tau')} | x_{\tau'}), \quad (3)$$

where $d\mathbf{a} = \prod_{j,\tau'} da_j^{(\tau')}$ and we implicitly used that on the tree, the end point x_τ also determines all the intermediate points $x_{\tau'}$ for $\tau' = 1, \dots, \tau$. For clarity, the $\mathbf{p}^{(\tau)}$'s on the right-hand side (r.h.s.) of Eq. (3) depend implicitly on $\{\mathbf{a}^{(\tau')}\}_{\tau'}$ through Eq. (2). Using Eq. (1), we can identify in Eq. (3) a factor $\prod_{\tau',j} P_0(a_j^{(\tau')})$, which we interpret as the measure for the variables $a_j^{(\tau')}$. From the definition of $B(a)$, the remaining factor can be recognized as the normalization $Z^{(\tau)}$, leading to (see also [71])

$$\langle F[\mathbf{p}^{(\tau)}] \rangle = \left\langle F \left[\left\{ \frac{z_j^{(\tau)}}{Z^{(\tau)}} \right\}_j \right] Z^{(\tau)} \right\rangle_0, \quad (4)$$

where $\langle \dots \rangle_0$ denotes the process where all $a_j^{(\tau')}$ are i.i.d. with distribution $P_0(a)$. Note that, by definition, $\langle B(a) \rangle_0 = 1/K$, which consistently implies $\langle Z^{(\tau)} \rangle_0 = 1$.

IV. ENTROPY DYNAMICS

The more the two $P_{0,1}(a)$ distributions differ, the more the observer will be able to discern where the particle is, typically resulting in a set of $p_j^{(\tau)}$ peaked around a few j 's. A possible quantifier of the “distance” between $P_{0,1}(a)$, and therefore of the effectiveness of the measurement protocol, is the so-called Kullback-Leibler divergence of $P_1(a)$ with respect to $P_0(a)$, $D_{\text{KL}}(P_1 \| P_0) := \langle \ln[P_1(a)/P_0(a)] \rangle_1 \geq 0$, where $\langle \dots \rangle_1 = \int da \dots P_1(a)$, which measures the surprise of an observer to find that a is distributed according to $P_1(a)$, while $P_0(a)$ is expected. If $P_{0,1}$ are taken to be Gaussian with a relative shift μ , one has $D_{\text{KL}}(P_1 \| P_0) = \mu^2/2$.

Our goal is now to determine whether the observer is able to effectively locate the particle. To this aim, we evaluate the growth in time of the average Shannon entropy $\langle S_t \rangle = \langle -\sum_{j=1}^{K^\tau} p_j^{(\tau)} \ln p_j^{(\tau)} \rangle$, which quantifies the degree of uncertainty on the particle location. More specifically, one can think of $\mathcal{N}_\tau \sim e^{S_t}$ as an estimate of the number of sites in which the probability of finding the particle is significantly different from zero; see Fig. 1. According to Eq. (4), it can be expressed as

$$\langle S_t \rangle = - \sum_{j=1}^{K^\tau} \langle z_j^{(\tau)} \ln z_j^{(\tau)} \rangle_0 + \langle Z^{(\tau)} \ln Z^{(\tau)} \rangle_0. \quad (5)$$

The first term in this expression can be promptly computed as each $z_j^{(\tau)}$ is expressed as the product of independent factors leading to $\sum_{j=1}^{K^\tau} \langle z_j^{(\tau)} \ln z_j^{(\tau)} \rangle_0 = vt$ (see the Appendixes). Here, we introduced

$$v \equiv \frac{1}{\Delta t} D_{\text{KL}}(P_1 \| P_0) - \lambda, \quad (6)$$

which can be seen as a control parameter accounting for the competition between the measurement precision, quantified by D_{KL} , and the chaotic spreading of the trajectories characterized by the Lyapunov exponent λ .

On the other hand, the second term on the r.h.s. of Eq. (5) is more involved. A standard method in disordered systems to cope with this kind of averaging is to calculate integer moments $\langle Z^n \rangle$ for generic n and subsequently consider an analytic continuation in n , the so-called replica trick. The typical value of the free energy $\propto \ln Z$ in quenched-disorder problems amounts to considering the $n \rightarrow 0$ expansion. In contrast, in monitoring problems, as we anticipated, the fact that the inference is based on Bayes' theorem (2) and (4) means that the entropy (5) is controlled by $Z \ln Z = \partial_n Z^n|_{n=1}$. It follows that the entropy is written as the difference between the $n \rightarrow 1$ replicated polymer partition function with “point-to-line” and “point-to-point” boundary conditions. Here, however, we avoid replicas and consider the distribution of the DP partition function $Z^{(\tau)}$. In general, this is a difficult problem since the configurations entering $Z^{(\tau)}$ have different degrees of correlation based on the overlap between different paths. For the tree, this difficulty can be solved using self-similarity as in Ref. [68]: we observe that a tree of level $\tau + 1$ can be obtained by juxtaposing K independent trees of level τ and connecting their vertices with a branching point. In terms of the $Z^{(\tau)}$, this leads to the recurrence relation

$$Z^{(\tau+1)} \stackrel{\text{in law}}{=} B(a) \sum_{\kappa=1}^K Z_\kappa^{(\tau)}, \quad (7)$$

where the $Z_\kappa^{(\tau)}$'s are K independent realizations of $Z^{(\tau)}$, a is drawn from $P_0(a)$, and the equality is meant in law for probability distributions. Equation (7) can be turned into a deterministic recursive equation for the Laplace transform $G_\tau(y) := \langle \exp(-e^{-y} Z^{(\tau)}) \rangle_0$,

$$G_{\tau+1}(y) = \langle G_\tau[y - \ln B(a)]^K \rangle_0. \quad (8)$$

The evolution equation (8) belongs to a wide class of nonlinear reaction-diffusion equations including the famous Kolmogorov-Petrovsky-Piskunov (KPP) equation valid for continuous space and time (see below). Quite generally, the solution $G_\tau(y)$ behaves as a ballistically moving stationary wavefront, monotonically interpolating between $G_\tau(-\infty) = 0$ and $G_\tau(\infty) = 1$. This traveling stationary solution characterizes the distribution of $\ln Z^{(\tau)}$ for large τ around its typical value. In our case, however, it is more convenient to rewrite $G_\tau(y) = 1 - e^{-y} u_\tau(y)$. Because of the normalization condition $\langle Z^{(\tau)} \rangle_0 = 1$, $G_\tau(y) = 1 - e^{-y} + O(e^{-2y})$ and $u_\tau(+\infty) = 1$, while from $G_\tau(y \rightarrow -\infty) \rightarrow 0$, we deduce $u_\tau(y \rightarrow -\infty) \sim e^y \rightarrow 0$. In terms of this function, we express (see the Appendixes)

$$\langle Z^{(\tau)} \ln Z^{(\tau)} \rangle_0 = \int_{-\infty}^{\infty} dy [u_0(y) - u_\tau(y)]. \quad (9)$$

V. ENTROPY GROWTH RATE

While our model is formulated for $K = e^{\lambda \Delta t} \in \mathbb{N}$, Eq. (8) allows us to take the continuous-time limit $\Delta t \rightarrow 0$ while $\lambda = O(1)$. Consistently, as suggested by the control parameter v defined by Eq. (6), the accuracy of the measurements has

to be scaled choosing $P_1(a) = P_0(a) + O(\sqrt{\Delta t})$. As shown in the Appendixes, this corresponds to setting $D_{\text{KL}}(P_1 \| P_0) = \sigma^2 \Delta t / 2$, which is the only residual parameter of the finer structure of $P_{0,1}(a)$. Note that for the illustrative Gaussian case, this simply corresponds to setting $\mu = \sigma \sqrt{\Delta t}$. This limit simplifies the discussion from a technical point of view, although it is not strictly necessary as the same phenomenology can be obtained in discrete time (see the Appendixes). Setting $G_\tau(y) = 1 - h_t(y)$, in the $\Delta t \rightarrow 0$ limit, Eq. (8) assumes the more familiar form of the KPP equation

$$\partial_t h = \frac{\sigma^2}{2} \partial_y^2 h + \left(\lambda + \frac{\sigma^2}{2} \right) \partial_y h + \lambda F(h), \quad (10)$$

where $F(h) = -(1-h)\ln(1-h)$. Beyond this specific form, the results will be universal, given some general properties such as $F(0) = F(1) = 0$, with $F'(0) = 1$ and $F''(h) < 0$, implying that $h = 1$ and $h = 0$ are fixed points, respectively, stable/unstable. At long times there are two cases: for $\sigma \geq \sqrt{2\lambda}$ the solution behaves as a traveling wave, $h_t(y) \simeq \bar{h}(y - y_t)$ with $y_t \simeq v_{\text{KPP}} t - \frac{2\alpha\sigma}{\sqrt{2\lambda}} \ln t + o(1)$ and the translation speed $v_{\text{KPP}} = -(\frac{\sigma}{\sqrt{2}} - \sqrt{\lambda})^2$; for $\sigma < \sqrt{2\lambda}$, it converges to a limit $h_t(y) \rightarrow h_\infty(y)$ and $v_{\text{KPP}} = 0$ [72]. However, it is more relevant for us to take the $\Delta t \rightarrow 0$ limit of Eq. (9) setting $u_t(y) := e^y h_t(y) = u_\tau(y)$ and considering the corresponding partial differential equation

$$\partial_t u_t(y) = -v \partial_y u_t(y) + \frac{\sigma^2}{2} \partial_y^2 u_t(y) - \lambda e^y \tilde{F}(e^{-y} u), \quad (11)$$

where $\tilde{F}(h) := h - F(h)$ and $v = \sigma^2/2 - \lambda$ comes from the continuum limit of Eq. (6). In the limit of noninformative measurements $\sigma \rightarrow 0$, the entropy grows linearly as $\langle S_t \rangle = \lambda t$, hence we first discuss the asymptotic rate of entropy production $s := \lim_{t \rightarrow \infty} \langle S_t \rangle / t$. In this perspective, since $u_t(y) e^{-y} \xrightarrow{y \rightarrow -\infty} 1$, for any fixed y_0 , the contribution for $y < y_0$ to the integral on the r.h.s. of Eq. (9) is $O(1)$ in time. It is thus inessential to the calculation of the rate which is instead controlled by large positive y . In this regime, $h(y) \ll 1$, and since $\tilde{F}(h) = O(h^2)$ at small h , we can neglect the nonlinear part in Eq. (11). As u_t grows from 0 to 1, we can interpret $u_t(y)$ as the cumulative probability distribution of a Wiener process with drift v and diffusion constant $\sigma^2/2$. Thus, in this linearized approximation, $u_t(y)$ translates at velocity v while broadening diffusively. For $v \neq 0$, the drift is the dominant factor (see the inset in Fig. 2): for $v > 0$, the integral in Eq. (9) is $\sim vt$; conversely, for $v < 0$ the wavefront exits the domain of integration $y > 0$, and $\langle Z^{(\tau)} \ln Z^{(\tau)} \rangle_0 \rightarrow O(1)$ (the precise value can be computed in an expansion at small σ^2 , see [71]). From these considerations, we deduce the exact growth rate

$$s := \lim_{t \rightarrow \infty} \frac{\langle S_t \rangle}{t} = \begin{cases} |v|, & v \leq 0, \\ 0, & v > 0. \end{cases} \quad (12)$$

As anticipated, the velocity v tunes a continuous phase transition of the rate of entropy production. Note that the diffusive front described by Eq. (11) should not be confused with the traveling wave $h_t(y) = \bar{h}(y - y_t)$ from (10): in $h_t(y)$, the front $u_t(y)$ is visible only as an exponentially suppressed far tail at very large y [71]. In particular, for $\sigma > \sqrt{2\lambda}$, $v_{\text{KPP}} < 0$ and $v > 0$ so that the two fronts move in opposite directions, a

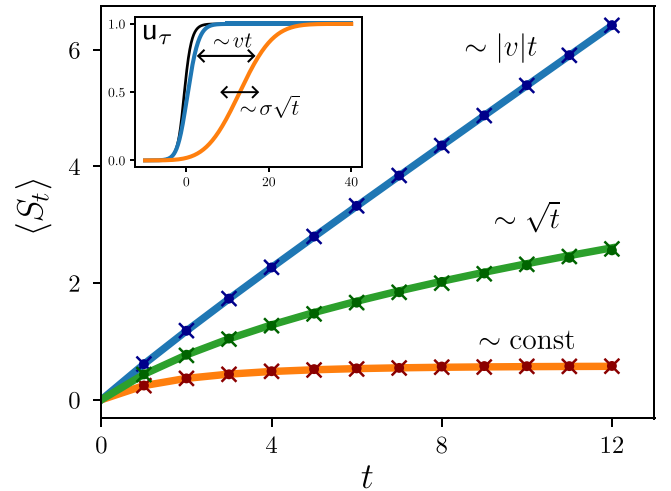


FIG. 2. Numerical analysis with $K = 2$, $P_0(a) \sim \mathcal{N}(0, 1)$ a zero-centered Gaussian, and $P_1(a) \sim \mathcal{N}(\mu, 1)$, so that $D(P_1 \| P_0) = \text{Var}(P_1 \| P_0)/2 = \mu^2/2$ [71]. For $v < 0$, $\langle S_t \rangle \sim |v|t$ grows linearly in time (blue, $v = -0.51$); for $v > 0$, $\langle S_t \rangle$ reaches a constant finite value (orange, $v = 0.93$). For $v = 0$, $\langle S_t \rangle \propto \sqrt{t}$ (green). Dots and crosses display $\langle S_t \rangle$ from trajectories of the particle x_t and of the evolved probabilities (2), respectively [71]. Full lines show the entropy (5) obtained numerically solving Eqs. (8) and (9) for $u_t(y)$. Inset: solutions of Eq. (11) are compared to the initial condition (black). For $v > 0$ (orange), the drift and broadening are clearly visible. For $v < 0$ (blue), $u_t(y)$ converges to a limiting form $e^y h_\infty(y)$.

manifestation of the fact that $\langle Z^{(\tau)} \ln Z^{(\tau)} \rangle_0$ is controlled by rare instances of $\ln Z^{(\tau)}$. Instead, when $\sigma < \sqrt{2\lambda}$, $v_{\text{KPP}} = 0$ and $v < 0$: this indicates that the propagation to the left of $u_t(y) = e^y h_t(y)$ must eventually stop due to nonlinearity (see the inset in Fig. 2), although this has no effect on the growth rate s . With the Kullback-Leibler divergence being positive, in this phase $|v| \leq \lambda$: as $\mathcal{N}_\tau \sim e^{|v|t}$, $|v|$ plays the role of a reduced effective Lyapunov exponent (see Fig. 1). At $\sigma = \sqrt{2\lambda}$, $v = 0$, the front (11) broadens diffusively, so one expects $\langle S_t \rangle = O(\sqrt{t})$. However, in the critical case, a more careful analysis of nonlinearity is needed, as we explain below.

VI. CRITICAL REGIME

For small $v < 0$, equating $|v|t_v \sim \sqrt{t_v}$, one needs $t \gtrsim t_v = |v|^{-2}$ to distinguish the critical behavior from the linear growth. This suggests that we consider the limit $v \rightarrow 0$, $t \rightarrow \infty$ while keeping the rescaled time $T = v^2 t / \sigma^2$ fixed. From the diffusive part in Eq. (11), one sees that we also have to scale the space variable as $Y = y|v|/\sigma^2$ and consider $U_T(Y) = u_t(y)$. In the $v \rightarrow 0$ limit, for $Y > 0$, the nonlinear term becomes negligible as $e^{\sigma^2 Y / |v|} \tilde{F}(e^{-\sigma^2 Y / |v|} u) \sim e^{-\sigma^2 Y / |v|} u^2 \rightarrow 0$; on the other hand, for $Y < 0$, as $0 < h_t(y) < 1$, $U_T(Y) < e^{-\sigma^2 |Y| / |v|} \xrightarrow{v \rightarrow 0} 0$. In other words, $U_T(Y)$ satisfies drifted diffusion for $Y > 0$ but with a wall imposing $U_T(Y \leq 0) = 0$. Since furthermore $U_T(+\infty) = 1$, one can interpret $U_T(Y)$ as the cumulative probability of a drifted Wiener process with a reflecting wall at $Y = 0$. Its expression can be computed explicitly (see the Appendixes), leading to the asymptotic

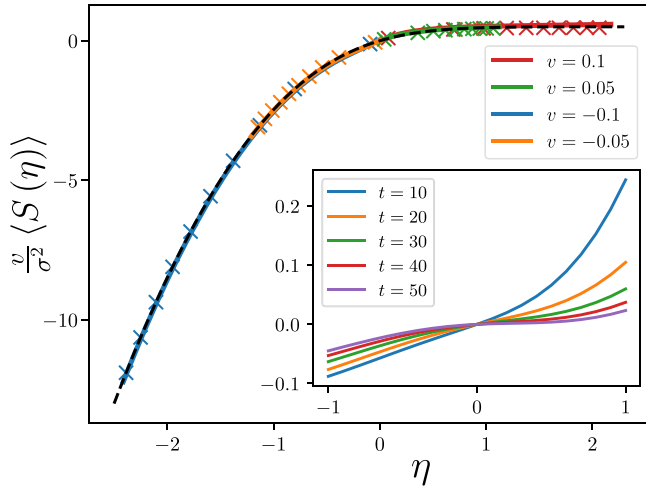


FIG. 3. Scaling limit for the discrete model (see the caption of Fig. 2) by solving Eq. (8) (crosses) and for the continuous-time model (full lines) by solving Eq. (11). In both cases, we compute $v/\sigma^2\langle S \rangle$ as a function of $\eta = v/\sigma\sqrt{t/2}$, at times up to $t = 10^3$, for various values of v . Numerical results are compared to the theoretical scaling function $S(\eta)$ of Eq. (14) (black dashed line). Inset: The difference $v/\sigma^2\langle S \rangle - S$ is shown as a function of η for increasing times t .

large time behavior close to criticality,

$$\langle S_t \rangle \simeq \frac{\sigma^2}{v} S\left(v\sqrt{\frac{t}{2\sigma^2}}\right) \quad (13)$$

with the scaling function

$$S(\eta) = \left(\frac{1}{2} + \eta^2\right) \text{erf } \eta - \eta^2 + \frac{\eta}{\sqrt{\pi}} e^{-\eta^2}. \quad (14)$$

For $v = 0$ this gives $\langle S_t \rangle = \sigma\sqrt{2t/\pi}$, while in the regimes $\eta \rightarrow \pm\infty$, we recover Eq. (12) for $v > 0$ and $v < 0$, respectively. Moreover, we can now compute the critical scaling of $\langle S_t \rangle$, namely $\langle S_t \rangle \sim \sigma^2/(2v)$ for $v \rightarrow 0^+$ and $\langle S_t \rangle \sim |v|t + \sigma^2/(2|v|)$ for $v \rightarrow 0^-$ consistently with Eq. (12). Note that Eq. (13) and the scaling form (14) are completely universal in our protocol, and they also apply to the discrete case where v is given by Eq. (6), while $\sigma^2 = \text{Var}(P_1||P_0)/\Delta t$ with $\text{Var}(P_1||P_0) := \langle \ln^2[P_1(a)/P_0(a)] \rangle_1 - \langle \ln[P_1(a)/P_0(a)] \rangle_1^2$. Indeed, a comparison with the numerics performed on the discrete model shows perfect agreement (see Fig. 3).

VII. CONCLUSIONS

In this paper, we considered the competition between exponential spreading of uncertainty and Bayesian updating of information by repeated measures. We introduced a toy model in terms of the directed random walk of a particle on a tree and obtained a connection to the DP on the Cayley tree. We could then employ the many tools available for this framework, but with significant differences due to the reweighting of polymer configurations due to Bayes' theorem. Our results can be regarded both as a simple and solvable example of a MIPT for a classical particle on the tree and as a fascinating transition in chaos mitigation. Interestingly, we find that the

critical point for the MIPT discussed here coincides with the well-known freezing transition of the DP, even though the MIPT is dominated by rare events and the critical properties are indeed different.

Several perspectives open up. From the practical standpoint of chaos mitigation, it would be of interest to analyze more optimized strategies where one tries to minimize the number of measurements to be taken while still pinpointing the state of the system. From the perspective of the DP, it would be interesting to assess whether the coincidence of the two critical points (MIPT and freezing) observed on the tree is a more general property. Additionally, while no transition is expected for a lattice in dimension $d = 1$ [66], the tree provides a good qualitative description for sufficiently high d (presumably for $d > 2$ when the polymer shows a high-temperature phase and self-averaging properties [73,74]).

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: ONE-POINT CONTRIBUTION TO $\langle S_t \rangle$

We will now compute the one-point contribution to $\langle S_t \rangle$, i.e., the second term on the r.h.s. of Eq. (5). This can be rewritten in terms of replicas as

$$\sum_{j=1}^{K^\tau} \langle z_j^{(\tau)} \ln z_j^{(\tau)} \rangle_0 = \partial_n \sum_{j=1}^{K^\tau} \langle (z_j^{(\tau)})^n \rangle_0 \Big|_{n=1}, \quad (A1)$$

while, according to our definition of the z_j ,

$$z_j^{(\tau)} = \prod_{p \in \text{branch}} \frac{P_1(a_p)}{K P_0(a_p)}, \quad (A2)$$

where the product runs over the tree branch that connects site j to the origin. As all the a_p are independent, one has

$$\sum_{j=1}^{K^\tau} \langle (z_j^{(\tau)})^n \rangle_0 = K^{-(n-1)\tau} \left\langle \left(\frac{P_1(a)}{P_0(a)} \right)^{n-1} \right\rangle_1, \quad (A3)$$

where we made use of the fact that $\langle [P_1(a)/P_0(a)]^n \rangle_0 = \langle [P_1(a)/P_0(a)]^{n-1} \rangle_1$. Finally, by taking the derivative, we find

$$\sum_{j=1}^{K^\tau} \langle z_j^{(\tau)} \ln z_j^{(\tau)} \rangle_0 = \tau [D_{\text{KL}}(P_1 || P_0) - \ln K]. \quad (A4)$$

APPENDIX B: PROOF OF EQ. (9)

We want now to estimate the collective contribution to $\langle S_t \rangle$, i.e., the first term on the r.h.s. of Eq. (5). First, we express it in terms of $G_\tau(y)$. To do so, we notice that, integrating twice in

Z both sides of the identity

$$Z^{-1} = \int_0^{+\infty} ds e^{-sZ}, \quad (\text{B1})$$

we get

$$Z \ln Z = \int_0^{+\infty} \frac{ds}{s^2} (e^{-sZ} - e^{-s} + (Z-1)e^{-s}s(s+1)). \quad (\text{B2})$$

Let us now set $Z = Z^{(\tau)}$, $s = e^{-y}$ and take the average of both sides: by taking into account the fact that $\langle Z^{(\tau)} \rangle_0 = 1$ and $G_0(y) = e^{-e^{-y}}$, we get

$$\langle Z^{(\tau)} \ln Z^{(\tau)} \rangle_0 = \int_{-\infty}^{\infty} dy e^y [G_\tau(y) - G_0(y)], \quad (\text{B3})$$

which, expressed in terms of $u_\tau(y) = e^{-y}[1 - G_\tau(y)]$, gives Eq. (9).

APPENDIX C: DISCRETE-TIME CASE

We will show that we can recover the linearized form of Eq. (11) without the assumption of small Δt in the regime $\tau \gg 1$. Indeed, by expressing the discrete recursion relation Eq. (8) in terms of $u_\tau(y) = e^y[1 - G_\tau(y)]$ and expanding to the leading order in $u_\tau(y)e^{-y}$, we get

$$u_{\tau+1}(y) = \left\langle u_\tau \left(y + \ln K - \ln \frac{P_1(a)}{P_0(a)} \right) \right\rangle_1, \quad (\text{C1})$$

where now $u_0(y) = e^y(1 - e^{-e^{-y}})$. The solution can thus be expressed as

$$u_\tau(y) = \int dy' \mathcal{U}_\tau(y') u_0(y+y'), \quad (\text{C2})$$

where $\mathcal{U}_\tau(y)$ is the Green's function associated with Eq. (C1). The latter can in turn be expressed in terms of Fourier modes as

$$\mathcal{U}_\tau(q) = \int \frac{dq}{2\pi} e^{iqy + \omega(q)\tau} \quad (\text{C3})$$

with $\omega(q) = iq \ln K + \ln \langle e^{-iq \ln P_1(a)/P_0(a)} \rangle_1$. We now want to evaluate the latter expression in the large-time limit. Setting $q \rightarrow q/\sqrt{\tau}$, only the first two orders in q of $\omega(q)$ contribute: as $\omega(q) = -iqv\Delta t - \text{Var}(P_1 \parallel P_0)q^2/2 + O(q^3)$ we see that, for any $t = \tau\Delta t$,

$$\mathcal{U}_\tau(y) = \frac{e^{-(y-vt)^2/(2\sigma^2t)}}{\sqrt{2\pi\sigma^2t}} + O(\tau^{-1}) \quad (\text{C4})$$

and

$$u_\tau(y) = \int \frac{dy'}{\sqrt{2\pi\sigma^2t}} e^{-(y-vt)^2/(2\sigma^2t)} u_0(y+y'), \quad (\text{C5})$$

which provides the general solution of Eq. (11) in the linear regime.

APPENDIX D: CONTINUUM LIMIT

To properly define the continuum limit, one has to take $P_0(a) - P_1(a) \sim O(\sqrt{\Delta t})$, namely

$$P_0(a) = P_1(a) + \pi_1(a)\sqrt{\Delta t} + \pi_2(a)\Delta t + O(\Delta t)^{3/2}, \quad (\text{D1})$$

where the normalization requires $\int \pi_{1,2}(a) = 0$. From this one has, up to terms $O(\Delta t)^{3/2}$,

$$\ln \frac{P_1(a)}{P_0(a)} = -\frac{\pi_1(a)}{P_1(a)}\sqrt{\Delta t} + \left(\frac{1}{2} \frac{\pi_1^2(a)}{P_1^2(a)} - \frac{\pi_2(a)}{P_1(a)} \right) \Delta t. \quad (\text{D2})$$

Taking the average of (D2) with respect to P_1 gives, at leading order,

$$D_{\text{KL}}(P_1 \parallel P_0) \simeq \frac{\Delta t}{2} \int da \frac{\pi_1^2(a)}{P_1(a)} := \frac{\sigma^2}{2} \Delta t, \quad (\text{D3})$$

while taking the variance of (D2) with respect to P_1 gives

$$\begin{aligned} \text{Var}(P_1 \parallel P_0) &= \text{Var}(P_0 \parallel P_1) \\ &= \langle \ln^2[P_1(a)/P_0(a)] \rangle_1 - \langle \ln[P_1(a)/P_0(a)] \rangle_1^2 \\ &= \Delta t \int da \frac{\pi_1^2(a)}{P_1(a)} + o(\Delta t) \\ &= 2D_{\text{KL}}(P_1 \parallel P_0) + o(\Delta t), \end{aligned} \quad (\text{D4})$$

which is the relation presented in the main text, with $\text{Var}(P_1 \parallel P_0) = \sigma^2 \Delta t$.

Note that to obtain Eq. (10), we also used that, to leading order,

$$\left\langle \ln \frac{P_1(a)}{P_0(a)} \right\rangle_0 \simeq -\Delta t \int da \frac{\pi_1^2(a)}{P_1(a)} = -\frac{\sigma^2}{2} \Delta t. \quad (\text{D5})$$

APPENDIX E: RATE OF ENTROPY PRODUCTION

From Eqs. (5) and (9), one has the exact relation

$$\partial_t \langle S_t \rangle = -v - \int_{-\infty}^{+\infty} dy \partial_y u_t(y). \quad (\text{E1})$$

Using (11) and integrating over $y \in (-\infty, \infty)$ using that $u_t(y)$ vanishes at $y = -\infty$ and tends to 1 at $y = +\infty$, one finds another exact relation for the entropy production at time t ,

$$\partial_t \langle S_t \rangle = \lambda \int_{-\infty}^{+\infty} dy e^y \tilde{F}[e^{-y} u_t(y)]. \quad (\text{E2})$$

One can check that $h^2/2 < \tilde{F}(h) < h^2$. Hence one needs to evaluate $A_t = \int_{-\infty}^{+\infty} dy e^{-y} u_t(y)^2 = \int_{-\infty}^{+\infty} dy e^y h_t(y)^2$. This is always a convergent integral since $h_t(y) \rightarrow 1$ for $y \rightarrow -\infty$ and $h_t(y) \sim e^{-y}$ for $y \rightarrow +\infty$. For $v > 0$, the KPP front solution $h_t(y) = \tilde{h}(y - y_t)$ has a strictly negative velocity, which implies that on $[y_0, +\infty)$ for any y_0 one has that $h_t(y) \rightarrow 0$. Hence for $v > 0$, $A_t \rightarrow 0$, and the rate of entropy production vanishes, $s = 0$. More precisely, one finds that for $\sqrt{2\lambda} < \sigma < 2\sqrt{2\lambda}$ the decay of A_t is dominated by the KPP front with $A_t \rightarrow e^{y_t} \int dz e^z \tilde{h}(z)^2 \sim e^{-|v_{\text{KPP}}|t}$, while for $\sigma > 2\sqrt{2\lambda}$ it is dominated by the far tail of $h_t(y)$. For $v = 0$ the KPP front moves more slowly to the left, $y_t \simeq -\frac{1}{2} \ln t$, leading to slower decay of $A_t \sim 1/\sqrt{t}$, consistent with the results in the main text.

APPENDIX F: CRITICAL SCALING

By setting $y = \sigma^2 Y/|v|$, $t = \sigma^2 T/v^2$, Eq. (11) becomes, for $U_T(Y) = u_t(y)$ with $Y, T = O(1)$,

$$\partial_T U_T(Y) = -\xi \partial_Y U_T(Y) + \frac{1}{2} \partial_Y^2 U_T(Y) + \dots, \quad (\text{F1})$$

with $\xi = \text{sgn}(v)$ and $U_T(+\infty) = 1$. The nonlinear part ... reads $\frac{\sigma^2}{v^2} e^{\sigma^2 Y/|v|} \tilde{F}[e^{-\sigma^2 Y/|v|} U_T(Y)]$. As explained in the text, it can be neglected for $Y > 0$, while it acts as a wall imposing $U_T(Y \leq 0) = 0$. Thus, Eq. (F1) is the evolution equation for the cumulative probability density of a Wiener process on $Y > 0$, with drift velocity ξ and a reflecting boundary wall at $Y = 0$. Taking as initial condition $U_{T=0}(Y) = \theta(Y)$, the solution of this problem can be written as a Galilean transformation of the $\xi = 0$ solution obtained through the reflection principle [75], namely [76]

$$U_T(Y) = \Phi\left(\frac{Y - \xi T}{\sqrt{2T}}\right) - e^{2\xi Y} \left[1 - \Phi\left(\frac{Y + \xi T}{\sqrt{2T}}\right)\right], \quad (\text{F2})$$

where $\Phi(x) = [1 + \text{erf}(x)]/2$ is the cumulative of the Gaussian distribution with variance 1/2.

To compute the entropy, we use (E1). Recalling that $t = \sigma^2 T/v^2$ and $y = \sigma^2 Y/|v|$, this leads to

$$\frac{|v|}{\sigma^2} \partial_T \langle S_T \rangle = -\xi - \int_0^{+\infty} dY \partial_T U_T(Y) = \frac{1}{2} \partial_Y U_T(Y)|_{Y=0}$$

implying that

$$\frac{v}{\sigma^2} \partial_T \langle S_T \rangle = \frac{e^{-\frac{T}{2}}}{\sqrt{2\pi\xi\sqrt{T}}} - \frac{1}{2} \text{erfc}\left(\frac{\xi\sqrt{T}}{\sqrt{2}}\right), \quad (\text{F3})$$

which, integrated over time, gives Eq. (13) with the scaling function (14) with $\eta = \xi\sqrt{T/2}$, which is analytic in η .

Although this derivation was carried out in the continuous model, let us notice that it is possible to retrieve the same result in the discrete-time case $\Delta t = O(1)$ as well. Indeed, rescaling $t = \tau \Delta t = T \sigma^2/v^2$, the variable $T = O(1)$ is naturally continuous via the simultaneous limit $\tau \rightarrow \infty$, $v \rightarrow 0$. Namely, finite increments $\tau \rightarrow \tau + 1$ correspond to infinitesimal increments $T \rightarrow T + v^2/\sigma^2$.

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- [71] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/6375-8ncz> for additional details.
- [72] As obtained by linearizing (10) for $h_t(y) \ll 1$ [68]. Here $v_{\text{KPP}} = \lim_{t \rightarrow +\infty} \frac{1}{t} \log Z^{(\tau)}$ is the intensive free energy associated with typical polymer paths (i.e., $n = 0$). $v_{\text{KPP}} = 0$ corresponds to the high-T phase with $\bar{h}(z) \sim e^{-z}$, $\alpha = 0$, and $\sigma > \sqrt{2\lambda}$ to the low-T phase with $\bar{h}(z) \sim e^{-\sqrt{2\lambda}z/\sigma}$ for $z \gg 1$ and $\alpha = 3/4$. At the transition, $\alpha = 1/4$.
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