

Entropic witness for quantum memory in open system dynamics

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The dynamics of open quantum systems are often modeled by non-Markovian processes that account for memory effects arising from interactions with the environment. It is well known that the memory provided by the environment can be classical or quantum in nature. Remarkably, the quantumness of the memory can be witnessed locally by measurements on the open system alone, without requiring access to the environment. However, existing witnesses are computationally challenging for systems beyond qubits, which severely limits their practical applicability. In this work, we present a tractable criterion for quantum memory based on the von Neumann entropy, which is easily computable for systems of any dimension. Using this witness, we investigate the nature of memory in a class of physically motivated finite-dimensional qudit dynamics. Moreover, we demonstrate that this criterion is also suitable for detecting quantum memory in continuous-variable systems. As an illustrative example, we analyze non-Markovian Gaussian dynamics of a damped harmonic oscillator.

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I. INTRODUCTION

Non-Markovian dynamics in open quantum systems have attracted significant attention in recent decades. While many open quantum systems are adequately described by Markovian dynamics, particularly through the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation, numerous theoretical and experimental advances have underscored the need to study the rich quantum dynamics that go beyond these frameworks [1–16]. Non-Markovian behavior always relies on some sort of memory effect. In the typical open quantum system picture, this memory is provided by the environment the system interacts with. Due to the memory effects in the bath, the time evolution of the system can depend on its state in the past, leading to a non-Markovian dynamics. Indeed, in microscopic models including the open system and its environment, the emergence of non-Markovian dynamics is closely related to slowly decaying bath correlation functions, i.e., the fact that the system's influence on the bath persists long enough to impact the future evolution. However, in particular in master equation approaches, the origin of the memory effects is not as obvious. Moreover, it is known that non-Markovian quantum dynamics can also emerge from classical memory effects, and the scenario may not even involve a quantum environment at all [17–22]. Thus,

non-Markovianity in quantum dynamics is not necessarily a genuine quantum memory effect. For an arbitrary given quantum dynamics, this leads to the question of whether it is actually quantum information that needs to be stored in the environment or whether a classical memory would suffice to achieve the same dynamics. Crucially, standard definitions and witnesses of quantum non-Markovianity, such as nondivisibility or increases in trace distance, merely indicate that a given dynamics is non-Markovian and thus involves some form of memory effect [23–26]. However, these measures provide no insight into the quantumness of this memory.

The term *quantum memory* is ambiguous in itself, and various incommensurable definitions have been used in the literature [27–35]. The definitions mainly differ in their assumptions about what is known to an experimenter who wants to assess the quantumness of the memory in a non-Markovian system. If a full description of the global dynamics of the system and environment is available, the storage of quantum information in the environmental degrees of freedom could be directly evaluated. However, in practice, experimental access to the properties of the environment is usually very limited and the experimenter may try to base their evaluation of the memory's quantumness only on information measurable on the system alone.

Under this constraint, the most complete picture an experimenter can obtain is given by the so-called process tensor [8,36,37]. This object contains all information about the possible outcomes of arbitrary interventions (e.g., measurements or quantum channels) the experimenter can perform on the system at predefined times [8,38–42]. In particular, the process tensor contains all possible influences of the environment, and it has been shown that the quantumness of the memory is directly related to the process tensor being entangled [29,32,43,44]. While the process tensor is in principle accessible by measurements on the system, its experimen-

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tal determination can be very challenging as it contains all multi-time correlations in the dynamics [45]. Furthermore, the verification of entanglement in the process tensor itself can be difficult for all but the smallest dimensions [29].

However, we have shown in Ref. [31] that the need for quantum memory effects in an environment can be verified even when much less information than the full process tensor is available. The criterion in Ref. [31] is based only on the reduced dynamics in the system, which not only is easier to measure in experiments but also makes the approach directly applicable to master equations, which also only describe the reduced open system dynamics.

In this paper, we build on the work described in Ref. [31] and overcome one of its main obstacles for a practical implementation. We present an entropic witness for the verification of quantum memory in open system dynamics, whose great advantage over previous work is its practical applicability to systems of arbitrary dimension. The original theorem, despite being general in scope, was based on entanglement measures that are often practically intractable for all but the smallest quantum systems like qubits. In this work, we present examples beyond the qubit dynamics for which the quantum memory is detectable with the entropic witness. In particular, we apply it to Gaussian dynamics of a continuous-variable system, demonstrating that this criterion is able to detect quantum memory for dynamics in infinite-dimensional Hilbert spaces.

The paper is structured as follows. In the next section, we outline the scenario and define what we mean by classical and quantum memory in this context. In Sec. III, we review the quantum memory criterion from Ref. [31] in a slightly generalized form before we derive the entropic witness in Sec. IV. We demonstrate the usefulness of this approach in Sec. V by means of a qudit example, for which the original witness would be hard to compute. In Sec. VI, we then turn to the case of Gaussian dynamics, showing that the non-Markovian damping of a harmonic oscillator cannot be realized with classical memory, in general.

II. SCENARIO

The central object of interest in our study is the *dynamics* $\mathcal{D}$ of an open quantum system $\mathcal{S}$. We define $\mathcal{D}$ to be a discrete or continuous family of completely positive trace-preserving (CPT) maps $\mathcal{D} = (\mathcal{E}_t)$ mapping the system state from the initial time t_0 to time $t > t_0$. Assuming that system and environment are uncorrelated at time t_0 , this framework covers any physically valid time evolution of $\mathcal{S}$. Master equation approaches and many microscopic derivations of system-bath dynamics are based on this assumption, and we will stick to it here as well.

The simplest dynamics that can possibly show memory effects consists of two CPT maps, i.e., $\mathcal{D} = (\mathcal{E}_{t_1}, \mathcal{E}_{t_2})$. Such a dynamics can be regarded as two snapshots of an underlying time-continuous dynamics.

Definition 1. We call a dynamics $\mathcal{D} = (\mathcal{E}_{t_1}, \mathcal{E}_{t_2})$ *realizable with classical memory* if there exists a set of Kraus operators K_i , with $\sum_i K_i^\dagger K_i = \mathbb{1}$, and a set of CPT maps Φ_i such that

$$\mathcal{E}_{t_1}[\rho] = \sum_i K_i \rho K_i^\dagger, \quad \mathcal{E}_{t_2}[\rho] = \sum_i \Phi_i[K_i \rho K_i^\dagger]. \quad (1)$$

Otherwise, the dynamics is said to require quantum memory to be realized. This definition has been proposed in Ref. [31]; a similar construction can be found in Refs. [32,46]. The reasoning behind the construction is the following. Any CPT map $\mathcal{E}_{t_i}$ can be written in a Kraus representation. Each Kraus operator K_i can be seen as a measurement operator of a quantum measurement with outcome i . This outcome i is classical data and can be stored in a classical memory. The subsequent evolution in the system is given by a quantum channel Φ_i that is conditioned on the data i from the classical memory. Being a CPT map, each Φ_i can be thought of as being realized with a fresh, uncorrelated environment.

We note that other definitions of quantum memory or similar concepts have been proposed, for example, in Refs. [28,29,31–34,47]. They differ from the definition given here primarily in their assumptions on prior knowledge about the environment, for example, concerning its dimension. The definition used in Ref. [31] and in this paper is motivated by the question whether a given dynamics is *realizable* by means of classical memory. Thus, even if the actual physical environment that causes the dynamics $\mathcal{D}$ implements the memory in a quantum system rather than a classical one, an experimenter who only has access to the maps $\mathcal{E}_{t_1}, \mathcal{E}_{t_2}$ must conclude that a quantum memory is *not necessary for the realization* of $\mathcal{D}$ if it can be written in the form of Definition 1.

In general, the question of whether the non-Markovian dynamics can be realized with classical memory, as given in Eq. (1), is hard to answer, similarly to the separability problem of a mixed quantum state. However, we can verify the necessity of a quantum memory by employing suitable witnesses that rule out the existence of a classical realization in the sense of Definition 1. Such a sufficient criterion was proposed in Ref. [31]. In the following section, we will review it in a slightly modified form before we continue with the main result of this paper: an entropic witness that is computationally more tractable and allows for the investigation of a much broader class of dynamics.

III. LOCAL WITNESS FOR QUANTUM MEMORY

Besides the system of interest $\mathcal{S}$ whose dynamics $\mathcal{D}$ we would like to analyze, we consider an ancilla quantum system $\mathcal{A}$ that remains unaffected by the dynamics. Initially, the system $\mathcal{S}$ and the ancilla $\mathcal{A}$ are prepared in a joint state $\rho_0^{\mathcal{SA}}$. The system undergoes the dynamics $\mathcal{D}$ while the ancilla is untouched. At time t , the joint state of system and ancilla is given by

$$\rho_t^{\mathcal{SA}} = (\mathcal{E}_t \otimes \mathbb{1}_{\mathcal{A}})[\rho_0^{\mathcal{SA}}]. \quad (2)$$

Furthermore, we define the following functions:

$$\begin{aligned} F_f[\rho^{\mathcal{SA}}] &= \min_{\{p_k, \rho_k\}} \sum_k p_k f(\rho_k), \\ F_f^\#[\rho^{\mathcal{SA}}] &= \max_{\{p_k, \rho_k\}} \sum_k p_k f(\rho_k), \end{aligned} \quad (3)$$

where the minimization and maximization run over all decompositions of $\rho^{\mathcal{SA}}$, i.e., $\sum_k p_k \rho_k = \rho^{\mathcal{SA}}$, and f is some non-negative function that is nonincreasing under local CPT maps on $\mathcal{S}$. The necessity of quantum memory for the real-

ization of a dynamics $\mathcal{D}$ on $\mathcal{S}$ can then be verified by the following theorem.

Theorem 1. Let $\mathcal{E}_{t_1}$ and $\mathcal{E}_{t_2}$ be two CPT maps on system $\mathcal{S}$, and $\rho_0^{\mathcal{SA}}$ an initial joint state of $\mathcal{S}$ with an ancilla $\mathcal{A}$. Let $\rho_{t_1}^{\mathcal{SA}}$ and $\rho_{t_2}^{\mathcal{SA}}$ be the joint states at times t_1, t_2 as defined in Eq. (2), and F_f and $F_f^\sharp$ as in Eq. (3). If we observe

$$F_f^\sharp[\rho_{t_1}^{\mathcal{SA}}] < F_f[\rho_{t_2}^{\mathcal{SA}}] \quad (4)$$

for some f , the dynamics $\mathcal{D} = (\mathcal{E}_{t_1}, \mathcal{E}_{t_2})$ is not realizable with classical memory.

In Ref. [31], this criterion was proposed for the special case of $\rho_0^{\mathcal{SA}}$ being initially a maximally entangled state and f being an entanglement monotone. In this case, $\rho_t^{\mathcal{SA}}$ is the Choi state of the channel $\mathcal{E}_t$, F becomes its entanglement of formation, and $F^\sharp$ its entanglement of assistance [48,49]. The proof of Theorem 1 is analogous to the one in Ref. [31] and can be found in the Appendix.

It should be noted that the physical implementation of the ancilla system is not inevitably necessary. From Eq. (2), it can be seen that all quantities used in the theorem above can also be calculated from the knowledge of the system dynamics $\mathcal{E}_t$ applied to a fictitious system-ancilla state $\rho_0^{\mathcal{SA}}$. Whether a direct channel tomography of the system dynamics $\mathcal{E}_{t_{1,2}}$ or the preparation and measurement of an extended state including an ancilla is easier to realize will strongly depend on the specific experimental situation [50]. Crucially, a physically implemented ancilla $\mathcal{A}$ must not couple either to the system $\mathcal{S}$ or to the system's environment as this could create unwanted correlations that invalidate the results (however, a coupling of $\mathcal{A}$ to an independent Markovian environment would not alter the validity of the theorem). On the other hand, a full channel tomography provides more information than what is needed to verify quantum memory in certain cases. Notably, the initial system-ancilla state need not be a maximally entangled state, and especially for large systems, it may be beneficial to use the fact that the quantum memory witness Theorem 1 holds for any size of the ancilla.

The criterion in Eq. (4) is sufficient but not necessary. It resembles non-Markovianity witnesses based on the entanglement increase with an ancilla [24], the crucial difference being the function $F^\sharp$ on the left-hand side of Eq. (4). The quantities F and $F^\sharp$ are generally hard to compute [51]. In Ref. [31], examples have been given for which F and $F^\sharp$ can be calculated if f is chosen to be the concurrence, but this approach is essentially limited to qubit dynamics. To overcome this problem of computability, we propose a witness based on entropic bounds that are straightforwardly computable in any dimension.

IV. ENTROPIC WITNESS FOR QUANTUM MEMORY

The left- and the right-hand side of Eq. (4) can be bounded by entropic quantities if we choose f to be the entanglement of formation E , defined by

$$E[\rho^{\mathcal{SA}}] = \min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k S_S(|\psi_k\rangle\langle\psi_k|), \quad (5)$$

where $\sum_k p_k |\psi_k\rangle\langle\psi_k| = \rho^{\mathcal{SA}}$ and S_S is the von Neumann entropy of the reduced state of system $\mathcal{S}$, i.e., $S_S(\rho^{\mathcal{SA}}) =$

$-\text{tr}[\text{tr}_{\mathcal{A}}(\rho^{\mathcal{SA}}) \ln \text{tr}_{\mathcal{A}}(\rho^{\mathcal{SA}})]$. Likewise, $S_{\mathcal{A}}$ denotes the von Neumann entropy of the reduced state of the ancilla $\mathcal{A}$. It was shown in Ref. [48] that¹

$$F_E^\sharp[\rho^{\mathcal{SA}}] \leq S_S[\rho^{\mathcal{SA}}]. \quad (6)$$

The right-hand side of Eq. (4) can be bounded by [52]

$$F_E[\rho^{\mathcal{SA}}] \geq \max\{-S_{\mathcal{S}|\mathcal{A}}[\rho^{\mathcal{SA}}], -S_{\mathcal{A}|\mathcal{S}}[\rho^{\mathcal{SA}}]\}, \quad (7)$$

where

$$S_{ij}[\rho^{\mathcal{SA}}] := S[\rho^{\mathcal{SA}}] - S_j[\rho^{\mathcal{SA}}] \quad (8)$$

is the conditional quantum entropy, and $S[\rho^{\mathcal{SA}}]$ denotes the von Neumann entropy of the joint state. Invoking Eqs. (4), (6), and (7), we arrive at an inequality for witnessing quantum memory.

Theorem 2. Let $\mathcal{E}_{t_1}$ and $\mathcal{E}_{t_2}$ be two CPT maps on system $\mathcal{S}$, and $\rho_0^{\mathcal{SA}}$ an initial joint state of $\mathcal{S}$ with an ancilla $\mathcal{A}$. Let $\rho_{t_1}^{\mathcal{SA}}$ and $\rho_{t_2}^{\mathcal{SA}}$ be the joint states at times t_1, t_2 as defined in Eq. (2). If we observe

$$S_S[\rho_{t_1}^{\mathcal{SA}}] < \max\{-S_{\mathcal{S}|\mathcal{A}}[\rho_{t_2}^{\mathcal{SA}}], -S_{\mathcal{A}|\mathcal{S}}[\rho_{t_2}^{\mathcal{SA}}]\}, \quad (9)$$

the dynamics $\mathcal{D} = (\mathcal{E}_{t_1}, \mathcal{E}_{t_2})$ is not realizable with classical memory.

The great advance compared to previous work [31] is the computationally tractable witness in this theorem. So far, only qubit systems could be treated, whereas these entropic quantities can easily be computed for finite quantum systems of any dimension. We will give an example in the next section. We then extend the approach to the infinite-dimensional case and consider Gaussian dynamics whose quantum memory can be verified with the entropic criterion, too.

V. EXAMPLE I: QUDIT DYNAMICS

In order to illustrate the concept, we first apply the quantum memory criterion to a toy model of dynamics describing non-Markovian damping of a qudit system $\mathcal{S}$ of dimension d [53]. The system is coupled to a single memory qubit $\mathcal{M}$, which in turn is damped by a Markovian bath at zero temperature. Tracing out the memory qubit, we are interested in the extent to which the entropic criterion (9) applied to the local dynamics of $\mathcal{S}$ alone can detect the quantumness of the memory in the environment. For $d = 2$, this model reduces to the well-studied non-Markovian qubit amplitude damping master equation [54–57], which has been shown to require quantum memory [31].

The interaction Hamiltonian between the system $\mathcal{S}$ and the memory $\mathcal{M}$ of this toy model is given by

$$H_{\mathcal{SM}} = \omega(J_-^{(d)} \otimes \sigma_+ + J_+^{(d)} \otimes \sigma_-), \quad (10)$$

where $J_\pm^{(d)}$ are the d -dimensional ladder operators and $\sigma_\pm = J_\pm^{(2)}$. The damping of $\mathcal{M}$ is given by a single Lindblad dissipator with constant rate γ . Thus, the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation for the joint state of

¹A tighter bound is given by $F_E^\sharp[\rho^{\mathcal{SA}}] \leq \min\{S_S[\rho^{\mathcal{SA}}], S_{\mathcal{A}}[\rho^{\mathcal{SA}}]\}$. However, this bound does not lead to a tighter inequality (9), so we do not use it here.

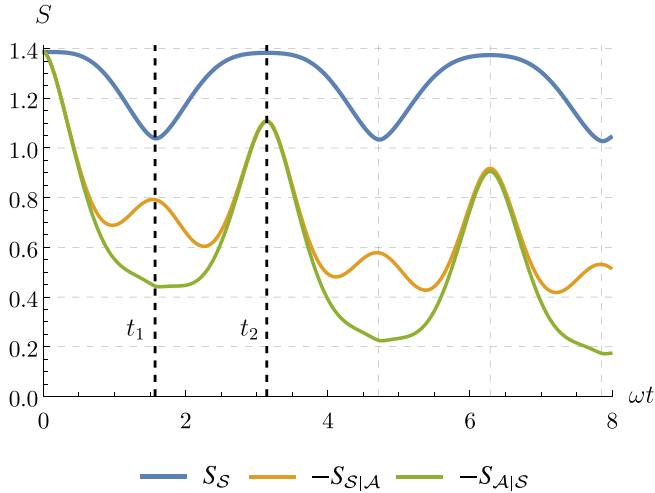


FIG. 1. Time evolution of the entropic quantities used in Eq. (9) for a system of dimension $d = 4$ and $\gamma/\omega = 0.05$. The initial state ρ_0^{SA} of system and ancilla in Eq. (2) is chosen to be the maximally entangled state $|\Phi_+\rangle = \sum_{l=0}^{d-1} |ll\rangle/\sqrt{d}$. The minimum of S_S at time t_1 is smaller than the subsequent maximum of $-S_{S|A}$ and $-S_{A|S}$ at time t_2 . Thus, quantum memory is demonstrated by the entropic criterion in Eq. (9). The Markovian damping of the memory qubit [see Eq. (11)] prevents us from revealing the quantum memory at later times. The second revival of the conditional entropies is already too small to satisfy Eq. (9).

system and memory is given by

$$\begin{aligned} \dot{\rho}_{SM} &= -i[H_{SM}, \rho_{SM}] + \gamma \mathcal{D}[\mathbb{1}_S \otimes \sigma_-] \rho_{SM} \\ &= \mathcal{L} \rho_{SM}, \end{aligned} \quad (11)$$

where $\mathcal{D}[X]\rho = X\rho X^\dagger - (X^\dagger X\rho + \rho X^\dagger X)/2$. The dynamics $\mathcal{E}_t$ of the system S is obtained by integrating this master equation for an initial memory state $\rho_M = |0\rangle\langle 0|$ up to time t and tracing over $\mathcal{M}$

$$\mathcal{E}_t[\rho_S] = \text{tr}_{\mathcal{M}}[e^{\mathcal{L}t} \rho_S \otimes \rho_M]. \quad (12)$$

For $d > 2$, the system dynamics $\mathcal{E}_t$ does not have a simple closed form. The following results were obtained numerically. In Fig. 1, we plot the entropic quantities involved in inequality (9) over time for a system of dimension $d = 4$ and a ratio between damping and coupling of $\gamma/\omega = 0.05$. The ancilla $\mathcal{A}$ has the same dimension as the system, and the initial system-ancilla state is chosen to be the maximally entangled state $\rho_0^{SA} = |\Phi_+\rangle\langle\Phi_+|$, with $|\Phi_+\rangle = \sum_{l=0}^{d-1} |ll\rangle/\sqrt{d}$.

The optimal times t_1 and t_2 for the detection of quantum memory in this particular example are marked. At time t_2 , the right-hand side of Eq. (9) is greater than the left-hand side at time t_1 . The second revival in the plot is already too small to witness quantum memory with the entropic inequality [see Eq. (9)].

Generally, the satisfiability of Eq. (9) sensitively depends on the ratio ω/γ . A stronger damping washes out the signature, and quantum memory becomes undetectable. For the maximally entangled initial state chosen here, we always have $-S_{S|A}[\rho_t^{SA}] \geq -S_{A|S}[\rho_t^{SA}]$. Thus, by using Eq. (9) and

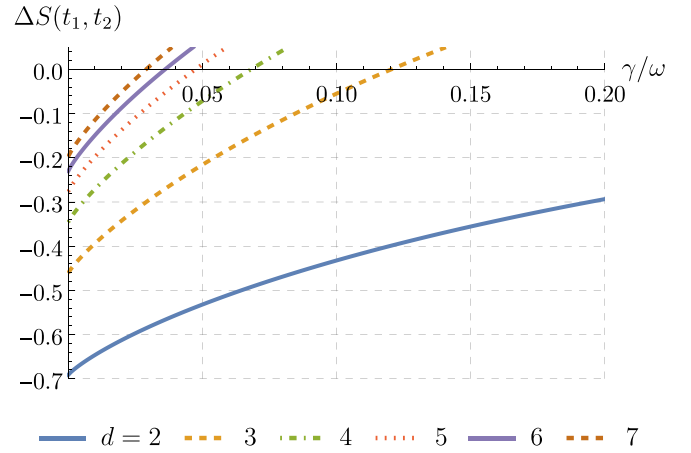


FIG. 2. We plot ΔS from Eq. (13) as a function of γ/ω for different dimensions d of the qudit system S . The times t_1 and t_2 in Eq. (13) are chosen to minimize ΔS for the respective parameters (see main text and Fig. 1). For $d > 2$, there is a critical γ/ω above which the verification of the quantum memory by means of the entropic witness becomes impossible for the given dynamics.

defining the witness

$$\Delta S(t_1, t_2) = S_S[\rho_{t_1}^{SA}] + S_{S|A}[\rho_{t_2}^{SA}], \quad (13)$$

quantum memory is demonstrated for $\Delta S < 0$. In Fig. 2, we plot the witness $\Delta S(t_1, t_2)$ as a function of the damping-to-coupling ratio γ/ω for different dimensions d . Similarly to Fig. 1, t_1 is always chosen to be the first minimum of S_S , and t_2 the first subsequent maximum of $-S_{S|A}$ for the given d and γ/ω . We see that the critical γ/ω , below which quantum memory can be witnessed, decreases with increasing dimension. Only in the qubit case $d = 2$ can the quantumness of the memory be demonstrated for arbitrary finite γ/ω .

Due to the inequalities in Eqs. (6) and (7), the entropic witness is in general looser than the entanglement witness in Ref. [31]. However, it must be emphasized that the entropic witness is able to verify the quantumness of the memory for system dimensions for which the entanglement quantities in the witness in Ref. [31] could hardly be calculated efficiently, making it practically applicable to systems beyond the qubit.

VI. GAUSSIAN DYNAMICS

The entropic quantities used in Theorem 2 are well defined also for an infinite (separable) Hilbert space [52,58,59]. Thus, this form of the quantum memory criterion can also be applied to continuous-variable (CV) systems. We demonstrate this power here using non-Markovian Gaussian channels. Non-Markovian dynamics in CV systems have been studied extensively, e.g., in Refs. [7,11,33,60–66]. In the following, we show how the necessity of quantum memory for the realization of such a dynamics can be witnessed by means of Eq. (9) and provide illustrative examples.

Let us first review relevant properties of Gaussian quantum states, following the notation in Ref. [67]. We consider an N -mode CV system with Hilbert space $\mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}_i$. Each infinite-dimensional Hilbert space $\mathcal{H}_i$ is equipped with creation and annihilation operators obeying the usual bosonic

commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$ and $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$. It is common to introduce position and momentum quadratures,

$$q_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger), \quad p_i = \frac{i}{\sqrt{2}}(a_i^\dagger - a_i), \quad (14)$$

and combine them to a $2N$ -dimensional phase space vector $\vec{X} = (q_1, p_1, \dots, q_N, p_N)$ such that the canonical commutation relations take the form

$$[X_i, X_j] = i\Omega_{ij}, \quad i, j = 1, \dots, 2N, \quad \Omega = \bigoplus_{i=1}^N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (15)$$

Gaussian quantum states ρ are fully determined by their position-momentum mean value $\langle \vec{X} \rangle = \text{tr}[\vec{X}\rho]$, and the corresponding (symmetrized) covariance matrix

$$\sigma_{ij} = \frac{1}{2}\langle X_i X_j + X_j X_i \rangle - \langle X_i \rangle \langle X_j \rangle. \quad (16)$$

As a shift of the mean value amounts to a local unitary transformation that leaves entropic quantities invariant, we may assume, without loss of generality, that it is zero in the following, $\langle \vec{X} \rangle = 0$. A valid density matrix satisfies $\sigma + i\Omega/2 \geq 0$.

A Gaussian channel $\mathcal{E} : \rho \rightarrow \rho' = \mathcal{E}[\rho]$ maps Gaussian quantum states onto themselves, and can thus be expressed as a linear map of covariance matrices that takes the form [67,68]

$$\sigma \mapsto \sigma' = M^\top \sigma M + N. \quad (17)$$

Here, the matrices M and N have to satisfy

$$N + \frac{i}{2}\Omega - \frac{i}{2}M^\top \Omega M \geq 0 \quad (18)$$

in order to describe a completely positive Gaussian channel [68].

We can now adapt the inequality (9) for Gaussian dynamics $\mathcal{D}$ of a system $\mathcal{S}$, i.e., a family of Gaussian quantum channels $\mathcal{E}_t$ described by matrices M_t, N_t . We need an initial joint Gaussian state $\rho_0^{\mathcal{S}\mathcal{A}}$ between the system $\mathcal{S}$ and an ancilla $\mathcal{A}$ [see Eq. (2)]. Its covariance matrix can be written as

$$\sigma_0 = \begin{pmatrix} \alpha_0 & \gamma_0 \\ \gamma_0^\top & \beta_0 \end{pmatrix}, \quad (19)$$

where α_0 and β_0 are the covariance matrices of the reduced states of $\mathcal{S}$ and $\mathcal{A}$, respectively, and γ_0 describes the correlations between the two. Applying the Gaussian channel $\mathcal{E}_t$ to the subsystem $\mathcal{S}$, we obtain a joint state $\rho_t^{\mathcal{S}\mathcal{A}}$ whose covariance matrix reads

$$\sigma_t = \begin{pmatrix} \alpha_t & \gamma_t \\ \gamma_t^\top & \beta_t \end{pmatrix} = \begin{pmatrix} M_t^\top \alpha_0 M_t + N_t & M_t^\top \gamma_0 \\ \gamma_0^\top M_t & \beta_0 \end{pmatrix}. \quad (20)$$

The entropies of the joint and the reduced states in Eq. (9) of Theorem 2 can then be calculated directly from the symplectic eigenvalues of the matrices σ_t, α_t , and β_t at times t_1 and t_2 (see Refs. [69,70] for details), and with Eq. (8), the criterion for quantum memory reads

$$\Delta S(t_1, t_2) = S[\alpha_{t_1}] + S[\sigma_{t_2}] - \max \{S[\alpha_{t_2}], S[\beta_{t_2}]\} < 0. \quad (21)$$

VII. EXAMPLE II: SINGLE-MODE DYNAMICS

Let us focus on the special case of both system $\mathcal{S}$ and ancilla $\mathcal{A}$ being single modes for which the quantum memory criterion becomes particularly simple. The von Neumann entropy of the single-mode Gaussian system state with covariance matrix α (and analogously for the reduced ancilla state described by β) is given by [67]

$$S_1(\alpha) = h(\sqrt{\det \alpha}), \quad (22)$$

with

$$h(x) = (x + \frac{1}{2}) \ln(x + \frac{1}{2}) - (x - \frac{1}{2}) \ln(x - \frac{1}{2}). \quad (23)$$

The entropy of the joint two-mode Gaussian state reads [67]

$$S_2(\sigma) = h[n_-(\sigma)] + h[n_+(\sigma)], \quad (24)$$

with

$$n_{\mp}(\sigma) = \sqrt{\frac{\Delta(\sigma) \mp \sqrt{\Delta(\sigma)^2 - 4 \det(\sigma)}}{2}} \quad (25)$$

and $\Delta(\sigma) = \det(\alpha) + \det(\beta) + 2 \det(\gamma)$.

For the sake of concreteness, let us consider an initial two-mode squeezed state of system and ancilla whose covariance matrix is of the form

$$\sigma_0 = \frac{1}{2} \begin{pmatrix} A_r & B_r \\ B_r & A_r \end{pmatrix}, \quad (26)$$

with

$$A_r = \cosh r \mathbb{1}, \quad B_r = \sinh r \sigma_z, \quad (27)$$

and $r > 0$ the squeezing parameter. The CV equivalent of a maximally entangled Bell state is obtained in the limit $r \rightarrow \infty$. The time-evolved state is then described by

$$\alpha_t = \frac{1}{2} M_t^\top A_r M_t + N_t, \quad \beta_t = \frac{A_r}{2}, \quad \gamma_t = \frac{M_t^\top B_r}{2}. \quad (28)$$

In the following, we evaluate the quantum memory witness for Gaussian dynamics describing energy loss of a single mode.

A. Lossy Gaussian channels

A simple dynamics modeling damping in a Gaussian mode is given by

$$\mathcal{E}_t : \quad M_t = \sqrt{1 - \eta_t} \mathbb{1}, \quad N_t = \eta_t \frac{\mathbb{1}}{2}, \quad (29)$$

where $0 \leq \eta \leq 1$ is the loss parameter [71]. Full loss is reached for $\eta = 1$, whereas $\eta = 0$ is the identity channel. A microscopic realization of such a dynamics is presented in the subsequent section. We do not further specify the time evolution of η_t at this point, but only assume that the losses at time t_1 and t_2 are described by η_1 and η_2 , respectively. For $\eta_2 \geq \eta_1$, i.e., a situation where the loss grows over time, the dynamics is Markovian and therefore does not require memory at all. For the opposite case of a partially reversed loss $\eta_2 < \eta_1$, the dynamics not only is non-Markovian but also requires quantum memory as can be shown with our entropic witness. Plugging Eq. (29) into Eq. (28), we find for ΔS in

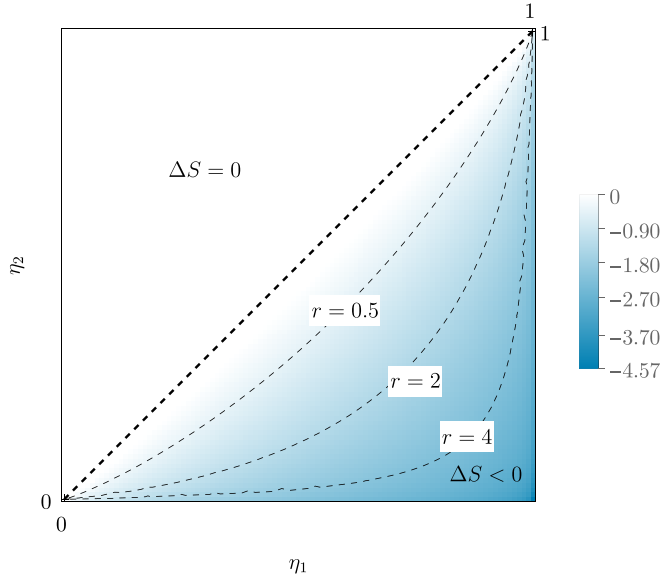


FIG. 3. Minimal value of ΔS obtained by varying $r > 0$ for every combination of η_1 and η_2 . The diagonal dashed line describes the case that $\eta_1 = \eta_2$, and every combination of η_1 and η_2 lying beneath this line corresponds to dynamical maps $\mathcal{E}_{t_1}$ and $\mathcal{E}_{t_2}$, which can only be realized using quantum memory, since there is an arbitrary small $r > 0$ such that $\Delta S < 0$. In general, the detection of quantum memory gets easier for larger differences of η_1 and η_2 . We also plot boundaries $\Delta S = 0$ for three different fixed values of the squeezing parameter r (dashed lines). Stronger squeezing shrinks the parameter region for which quantum memory can be detected. This is interesting since it tells us that greater entanglement in the initial state $\rho_0^{\mathcal{S}\mathcal{A}}$ diminishes the ability of the entropic criterion to witness quantum memory.

Eq. (21),

$$\Delta S(\eta_1, \eta_2) = h\left(\frac{\eta_1 + (1 - \eta_1) \cosh r}{2}\right) + h\left(\frac{1 - \eta_2 + \eta_2 \cosh r}{2}\right) - h\left(\frac{\cosh r}{2}\right). \quad (30)$$

The ability to demonstrate the quantumness of the memory depends on the squeezing parameter r of the initial system-ancilla state in Eq. (26). We plot ΔS in Fig. 3. For each pair (η_1, η_2) , ΔS was minimized over the squeezing parameter r , and we see that quantum memory can indeed be witnessed for any choice $\eta_2 < \eta_1$. Intuitively, the detection of quantum memory becomes easier for greater differences between the loss parameters η_1 and η_2 .

We also plot the boundary $\Delta S = 0$ for several fixed choices of r . Interestingly, stronger squeezing, and therefore stronger initial entanglement between $\mathcal{S}$ and $\mathcal{A}$, decreases the parameter region for which the entropic witness can verify quantum memory.

B. Non-Markovian damped harmonic oscillator

Next, let us take a look at a common, time-continuous open quantum system dynamics that essentially leads to the lossy channel described above [72]: a damped harmonic oscillator

(operators $a, a^\dagger$), whose dynamics is given by the master equation

$$\dot{\rho} = -i[\omega_t a^\dagger a, \rho] + \frac{\gamma_t}{2}([a, \rho a^\dagger] + [a\rho, a^\dagger]). \quad (31)$$

For constant positive parameters $\omega_t = \text{const.} > 0$ and $\gamma_t = \text{const.} > 0$, this is the well-known GKSL generator for a Markovian semigroup dynamics. If, however, the damping rate is negative during certain time intervals ($\gamma_t < 0$), the master equation describes non-Markovian dynamics [62]. In order to generate a proper CPT dynamical map, restrictions on the function γ_t apply. Below, we will construct admissible functions ω_t and γ_t from an explicit microscopic model.

The generator being quadratic in annihilation and creation operators, we conclude that the master equation (31) induces a Gaussian dynamical map. For the second-order moments $\langle a^\dagger a \rangle_t$, $\langle aa \rangle_t$, and $\langle a^\dagger a^\dagger \rangle_t$, it is straightforward to determine the Heisenberg equations of motion from Eq. (31). Their solutions turn out to be simple time-dependent exponentials with

$$\begin{aligned} \langle a^\dagger a \rangle_t &= \langle a^\dagger a \rangle_0 e^{-\Gamma_t}, \\ \langle aa \rangle_t &= \langle aa \rangle_0 e^{-2i\Phi_t - \Gamma_t}, \\ \langle a^\dagger a^\dagger \rangle_t &= \langle a^\dagger a^\dagger \rangle_0 e^{+2i\Phi_t - \Gamma_t}. \end{aligned} \quad (32)$$

Here, the integrated phase and damping terms follow naturally from the parameters of the master equation and read

$$\Phi_t = \int_0^t ds \omega_s, \quad \Gamma_t = \int_0^t ds \gamma_s. \quad (33)$$

It is now straightforward to relate the covariance matrix σ_t in Eq. (16) expressed in terms of (q_t, p_t) to its initial matrix σ_0 making use of the usual relations in Eq. (14) and the results in Eq. (32). We find the Gaussian channel relation $\sigma_t = M_t^\top \sigma_0 M_t + N_t$ from Eq. (17) with [62]

$$\begin{aligned} M_t &= e^{-\Gamma_t/2} \begin{pmatrix} \cos \Phi_t & -\sin \Phi_t \\ \sin \Phi_t & \cos \Phi_t \end{pmatrix}, \\ N_t &= (1 - e^{-\Gamma_t}) \frac{\mathbb{1}}{2}. \end{aligned} \quad (34)$$

The time-dependent phase Φ_t appearing in the rotation matrix of M_t reflects the Hamiltonian contribution to the dynamics in Eq. (31). Comparing Eq. (34) to Eq. (29), we see that the damped harmonic oscillator model describes a time-continuous lossy channel with

$$\eta_t = 1 - e^{-\Gamma_t}, \quad (35)$$

which undergoes an additional unitary rotation in phase space described by Φ_t . Since the entropies in Eq. (21) are invariant under local unitaries on the system, the quantumness of the memory involved in the dynamics is solely determined by the function Γ_t . We know from Fig. 3 that whenever η_t is a nonmonotonic function, the corresponding dynamics requires quantum memory.

Let us now specify the non-Markovian Gaussian channel by fixing the functions ω_t and γ_t of its master equation. In order to ensure a CPT dynamical map, we determine those functions from the full, unitary system-environment dynamics of an oscillator (operators $a, a^\dagger$) coupled to a bath of

such oscillators (operators $b_\lambda, b_\lambda^\dagger$) in their ground state (zero temperature). The mode a could be a cavity mode, and then the b_λ -modes would correspond to the environmental electromagnetic modes, in their ground state. The model was also used to describe non-Markovian output coupling for an atom laser from a Bose-Einstein condensate [73]. The total system-environment Hamiltonian is chosen to be

$$H_{\text{tot}} = \omega a^\dagger a + \sum_\lambda (g_\lambda a b_\lambda^\dagger + g_\lambda^* a^\dagger b_\lambda) + \sum_\lambda \omega_\lambda b_\lambda^\dagger b_\lambda. \quad (36)$$

For this model, the reduced dynamics of the system oscillator follows an exact master equation [72],

$$\dot{\rho} = -i[\omega a^\dagger a, \rho] + G_t^*[a, \rho a^\dagger] + G_t[a\rho, a^\dagger]. \quad (37)$$

Crucially, setting

$$\omega_t := \omega + \text{Im}(G_t), \quad \gamma_t := 2 \text{Re}(G_t), \quad (38)$$

the exact master equation (37) takes the desired form of Eq. (31). Here, the complex, time-dependent coefficient G_t can be written in the form

$$G_t = \frac{\int_0^t ds \alpha(t-s) c_s}{c_t}, \quad (39)$$

where

$$\alpha(t-s) = \sum_\lambda |g_\lambda|^2 e^{-i\omega_\lambda(t-s)} \quad (40)$$

is the (zero temperature) bath correlation function of the underlying system-bath model (36). The complex amplitude c_t satisfies the linear equation of motion

$$\dot{c}_t = -i\omega c_t - \int_0^t ds \alpha(t-s) c_s. \quad (41)$$

In what follows, we set the initial condition to $c(0) = 1$ (G_t and therefore ρ_t is in fact independent of the initial condition). It can then be seen that

$$|c_t|^2 = e^{-\Gamma_t}, \quad (42)$$

providing a direct way to determine the channel parameter Γ_t directly from the evolution equation (41) for the amplitude c_t . Depending on the choice of the bath correlation function, the rate γ_t may indeed become negative at times, and thus the model can describe non-Markovian quantum dynamics.

We now fix our model by specifying the bath correlation function. We choose the exponentially decaying

$$\alpha(t-s) = |g|^2 e^{-\kappa(t-s) - i\Omega(t-s)}, \quad (43)$$

with constant parameters κ, Ω . Then, with Eq. (41), the relevant amplitude c_t follows from the time-local equation,

$$\ddot{c}_t + (\kappa + i\omega + i\Omega)\dot{c}_t + [|g|^2 + i\omega(\kappa + i\Omega)]c_t = 0, \quad (44)$$

with the initial conditions $c_0 = 1$ and $\dot{c}_0 = -i\omega$. Solving this ordinary differential equation (ODE), we can display $\eta_t = 1 - e^{-\Gamma_t} = 1 - |c_t|^2$ as depicted in Fig. 4.

In Sec. VII A, we saw that two consecutive lossy channels $\mathcal{E}_{t_1}$ and $\mathcal{E}_{t_2}$ with $\eta_2 < \eta_1$ cannot be connected via classical memory. Therefore, noting that η_t is a nonmonotonous function for our choice of the bath correlation function (see Fig. 4), we can conclude that the master equation (31) based on the

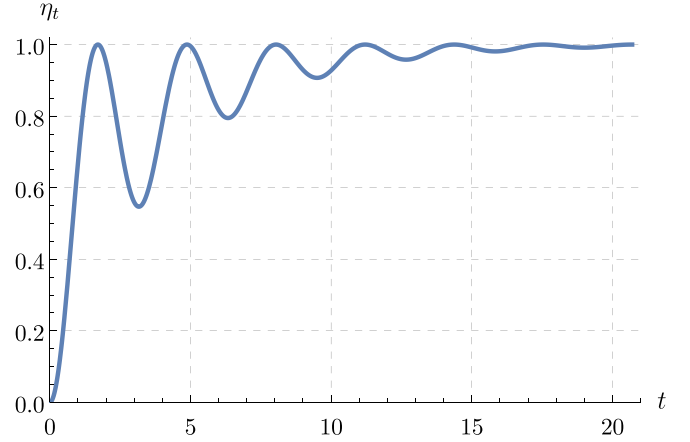


FIG. 4. The loss parameter $\eta_t = 1 - |c_t|^2$ for $|g|^2 = 1$, $\kappa = 1/4$, $\omega = 1$, and $\Omega = 1$. The nonmonotonous behavior cannot be explained by classical memory effects, as we can always find two times $t_2 > t_1$ for which $\eta_{t_2} < \eta_{t_1}$, a signature of quantum memory as shown in Sec. VII A.

choice of bath correlation function (43) does indeed describe a non-Markovian evolution where quantum memory is fundamentally involved: There is no way to arrive at the general form of the non-Markovian master equation (31) from a model involving only classical memory. This is in contrast, for instance, to the master equation of eternal non-Markovianity, which, by construction, may be obtained from random unitary dynamics or the probabilistic mixing of Markovian evolutions (see Ref. [20]), and thus need not involve any quantum (environmental) memory.

VIII. CONCLUSION

Non-Markovian quantum dynamics can arise from the presence of either a classical or a quantum memory. A key question in this context is how to distinguish the nature of the memory using only local information about the dynamics. Existing criteria often rely on entanglement measures, which are notoriously challenging to compute for systems beyond qubits. In this article, we propose a tractable way to witness quantum memory in quantum systems extending beyond qubits. We start from the definition of quantum memory provided in Ref. [31] and generalize a sufficient criterion of quantum memory presented there. Based on this witness, an entropic criterion to verify the quantumness of the memory of the dynamics is derived. This criterion does not only hold in any dimension but is easily computable, which is not the case for the original criterion in Ref. [31].

We provide several examples of its application. First, the entropic witness is used to show that non-Markovian damping dynamics of a qudit requires quantum memory, in general. In order to illustrate that the entropic witness can also be used for dynamics of infinite-dimensional systems, we show how to witness the quantumness of the memory in Gaussian processes. As an example, a model of a lossy channel is investigated. A time-continuous realization of this dynamics is given by a damped harmonic oscillator. By considering

the corresponding master equation, we show that the memory involved in this process is of quantum nature.

The entropic witness is a sufficient but not a necessary criterion for quantum memory. While it is derived as a bound of the original criterion in Ref. [31] and is, thus, in principle, less sensitive to quantum memory than the original, we demonstrate in this contribution that it remains capable of detecting quantum memory across a broad class of open system dynamics. Moreover, its computational simplicity makes it a practical and effective tool for characterizing the nature of the memory in non-Markovian quantum dynamics. The entropic witness requires tomography of either the dynamics or a joint system-ancilla state; the choice will depend on experimental considerations. Recent proposals show that partial channel tomography can be sufficient to verify the quantumness of the memory in certain cases of finite-dimensional systems [43]. Whether a similar construction is also possible in the case of continuous-variable systems is left as an open question for future research.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX : PROOF OF THEOREM 1

Suppose we are given a dynamics $\mathcal{D} = (\mathcal{E}_{t_1}, \mathcal{E}_{t_2})$ on the system $\mathcal{S}$ that requires only classical memory. A joint state $\rho_0^{\mathcal{SA}}$ of the system with an otherwise untouched ancilla $\mathcal{A}$ evolves to a state $\rho_{t_1}^{\mathcal{SA}}$ as given in Eq. (2). According to Definition 1,

at time t_1 there is a decomposition $\{p_i, \rho_i\}$ of this state $\rho_{t_1}^{\mathcal{SA}}$ given by

$$\rho_i = \frac{1}{p_i} (K_i \otimes \mathbb{1}_{\mathcal{A}}) \rho_0^{\mathcal{SA}} (K_i^\dagger \otimes \mathbb{1}_{\mathcal{A}}), \quad (\text{A1})$$

where $\{K_i\}$ is a local measurement implementing the first map $\mathcal{E}_{t_1}$ on the system, $p_i = \text{tr}[(K_i^\dagger K_i \otimes \mathbb{1}_{\mathcal{A}}) \rho_0^{\mathcal{SA}}]$, and $\sum_i p_i \rho_i = \rho_{t_1}^{\mathcal{SA}}$. Using the quantities defined in Eq. (3), we can write

$$F_f^\#[\rho_{t_1}^{\mathcal{SA}}] = \max_{\{p_k, \rho_k\}} \sum_k p_k f(\rho_k) \geq \sum_i p_i f(\rho_i), \quad (\text{A2})$$

where $\sum_k p_k \rho_k = \rho_{t_1}^{\mathcal{SA}}$, and, by definition of the maximum, the decomposition labeled by i and given in Eq. (A1) leads to a smaller value. We now define

$$\rho'_i = (\Phi_i \otimes \mathbb{1}_{\mathcal{A}})[\rho_i]. \quad (\text{A3})$$

According to Definition 1, the ρ'_i decompose the system-ancilla state at time t_2 , i.e.,

$$\sum_i p_i \rho'_i = \rho_{t_2}^{\mathcal{SA}}. \quad (\text{A4})$$

Since f is a function that is nonincreasing under the local quantum channels Φ_i , we have

$$\sum_i p_i f(\rho_i) \geq \sum_i p_i f(\rho'_i). \quad (\text{A5})$$

The right-hand side of Eq. (A5) can further be lower bounded. By the definition of the minimum, we have

$$\sum_i p_i f(\rho'_i) \geq \min_{\{p_l, \rho_l\}} \sum_l p_l f(\rho_l) = F_f[\rho_{t_2}^{\mathcal{SA}}], \quad (\text{A6})$$

where $\sum_l p_l \rho_l = \rho_{t_2}^{\mathcal{SA}}$.

Thus, under the assumption of classical memory given in Definition 1, we obtain the inequality

$$F_f^\#[\rho_{t_1}^{\mathcal{SA}}] \geq F_f[\rho_{t_2}^{\mathcal{SA}}]. \quad (\text{A7})$$

Any violation of this inequality therefore demonstrates the necessity of quantum memory. This is Theorem 1. ■

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