

# Liouville-Fock-state lattices: From environment-induced frustration to synthetic lattice dynamics

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We introduce Liouville-Fock-state lattices (LFSs) as a framework for visualizing open quantum systems through matrix representations of the Lindblad master equation. These generalize Hamiltonian Fock-state lattices, which have been considered for realizing synthetic dimensions in quantum simulators. By vectorizing the Lindblad master equation, the dynamics map to a doubled Hilbert space, naturally forming a synthetic lattice. Unlike the unitary evolution of pure states, LFSs feature non-Hermitian dynamics with drifts, sources, and sinks—paralleling stochastic classical lattices. As such, we suggest that these systems can serve as potential quantum simulators of classical lattice models, with a computational advantage arising from large state spaces encoded in quantum models. For this purpose, we analyze both the Fock representation and alternative positive-semidefinite representations that yield real, non-negative populations, akin to classical probability distributions. Within this formalism, we also demonstrate a phenomenon, which we term environment-induced frustration, in which competing terms in the LFSL give rise to geometric frustration and a massively degenerate steady-state manifold.

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## I. INTRODUCTION

Lattice models are essential tools for describing a wide range of physical phenomena across various branches of physics, including statistical mechanics, optics, condensed matter, and high-energy physics. A notable example is quantum simulators, which often rely on discretized lattice frameworks [1,2]. The precise control achievable in ultracold atomic and ionic systems has made these simulators a practical reality. Over the past two decades, quantum simulators have enabled groundbreaking experiments, from realizing canonical condensed matter models [3–5] to exploring quantum thermalization and many-body localization [6–8]. Recently, quantum simulation has expanded further to incorporate photonic systems and address applications in high-energy physics and quantum chemistry [9].

Building on the flexibility of these quantum systems, methods for creating synthetic dimensions by exploiting internal degrees of freedom have been proposed [10–14]. These synthetic dimensions allow for the study of phenomena unique to higher-dimensional systems, which are typically absent in traditional three-dimensional setups [15–19]. Synthetic dimensions have been proposed and experimentally realized in various platforms, including cold atoms

[10,11,20–25], trapped ions [26,27], magnons [28], plasmonics [29], cavity and circuit quantum electrodynamic systems [12,14,30–34], electrical circuits [35], and optical systems [13,36–42].

Typically, these synthetic dimensions correspond to internal quantum states—one for each continuous degree of freedom—giving rise to so-called Fock state lattices (FSLs). An FSL is constructed in occupation number space, where Fock states are interpreted as lattice sites arranged according to the allowed transitions dictated by the system’s Hamiltonian. The corresponding transition amplitudes are naturally interpreted as tunneling amplitudes between sites. Importantly, “Fock state” is here used in a broad sense, encompassing any number state.

In many systems, particularly in quantum optics, FSLs lack translational invariance due to state-dependent tunneling rates—stemming from the normalization factors of creation and annihilation operators on Fock states—and from lattice boundaries set by the vacuum state. Nevertheless, essential features such as topology are determined by the global lattice geometry rather than translational symmetry [14,30–32,42,43]. This makes FSLs a powerful platform for exploring physical phenomena in state space rather than real space.

In this work, we extend the concept of FSLs to open quantum systems by introducing Liouville-Fock-state lattices (LFSs) derived from the Lindblad master equation (LME). Extending phenomena from closed to open quantum systems presents distinct challenges, particularly due to the role of quantum coherences in mixed states [44]. Such challenges are well recognized in studies of entanglement and the topological properties of mixed states [45–48], in the coherent control of open systems [49], and in the definition of thermodynamic quantities [50]. The difficulty arises from the fact that general density matrices  $\hat{\rho}$  span a much larger state space than

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pure states  $|\psi\rangle$ . Unlike traditional FSLs, where vectors are represented in terms of probability amplitudes, vectors in LFSs require a fundamentally different physical interpretation. While these differences introduce additional complexity, they also provide opportunities to explore phenomena beyond Hermitian systems—phenomena typically encountered in classical systems—such as anomalous transport [51,52] and lattices or networks with biased complex weights [53–56]. In these cases, one studies the evolution of classical probability distributions on lattices or graphs. Although the LFSL formalism yields complex “distributions,” we discuss alternative representations that are real and more closely resemble proper distributions. Equipped with directed tunneling, sources, and sinks, we therefore propose using LFSs as simulators of classical models. The novelty lies in utilizing only a few degrees of freedom while still accessing exponentially large state spaces. Similar reasoning has recently been suggested in Hamiltonian systems [57,58], which also achieve exponentially large state spaces; however, for open quantum systems, the accessible space includes both pure and mixed states, making it even larger.

Using the properties of the LFSs, we also demonstrate how the enlarged state space enables a different type of frustration—environment-induced frustration. In the LFSL, one may encounter competing tunneling terms that prevent the buildup of local order, similar to Hamiltonian frustration in square lattices with competing nearest- and next-nearest-neighbor tunnelings. This effect is unique to open systems; it can occur even when the system Hamiltonian is trivial. Specifically, it arises from environment-induced fluctuations and, therefore, does not appear in a simple non-Hermitian Hamiltonian description. Furthermore, as we argue, it should be relatively straightforward to detect it with current state-of-the-art experimental techniques in quantum optics.

Presenting the outline of our work, in Sec. II we introduce and provide a detailed description of LFSs. In particular, we address the Liouville vectorized representation in Sec. II A, which forms the basis of the LFSL framework. The role of symmetries and conserved quantities in shaping the lattices is discussed in Sec. II B, while in Sec. II C we use paradigmatic models from quantum optics to exemplify the construction of LFSs. To demonstrate the power of our approach, in Sec. III we show how geometrical frustration can emerge in connection with the existence of an infinite number of steady states. Since the choice of basis is a fundamental aspect of constructing synthetic lattices, in Sec. IV we propose two alternative representations that have the distinct feature of describing open-system states using real numbers. In Sec. V, we discuss how LFSs and these alternative representations can serve as platforms for simulating classical lattice dynamics. These lattices manifest in state space rather than in real space, and Sec. VI outlines how to experimentally detect the relevant quantities in typical quantum optical systems. Finally, we conclude in Sec. VII, and the Appendix provides a more detailed description of the alternative representations.

## II. LIOUVILLE-FOCK-STATE LATTICES

In this section, we introduce the concept of LFSs, which provide a graphical representation of the evolution governed

by the LME. While alternative representations of master equations in light-matter systems have been employed in the past [59,60], the explicit identification of the corresponding lattices and their characterization in terms of LFSs is, to the best of our knowledge, new. To establish the LFSL framework, we first review the vectorization of the LME and discuss how symmetries relate to conserved quantities—key ingredients for understanding the construction of LFSs. As demonstrations, we consider two examples: the open Jaynes-Cummings (JC) model and a harmonic oscillator undergoing two-photon loss. The latter provides a particularly clear illustration of how symmetries can be used to reduce the dimensionality of LFSs.

### A. The Liouville representation

Our approach is based on the LME [61,62], given by

$$\frac{d\hat{\rho}}{dt} = \mathcal{L}[\hat{\rho}] \equiv -i[\hat{H}, \hat{\rho}] + \sum_j \gamma_j \left( \hat{L}_j \hat{\rho} \hat{L}_j^\dagger - \frac{1}{2} \{ \hat{L}_j^\dagger \hat{L}_j, \hat{\rho} \} \right), \quad (1)$$

where  $\mathcal{L}$  is the Liouvillian superoperator,  $\hat{H}$  is the system Hamiltonian,  $\hat{L}_j$  are the Lindblad jump operators, and  $\gamma_j$  are the corresponding decay rates (we use hats on normal operators and calligraphic font for superoperators). The Liouvillian, being a superoperator that maps operators to other operators, makes it challenging to interpret the LME in terms of an FSL. To address this, we adopt the vectorized form of the LME [63], the so-called Liouville representation.

The Liouville representation of the density operator  $\hat{\rho}$  is constructed using the bra-flipper isomorphism, denoted here by the operator  $\hat{\mathcal{U}}$ . This isomorphism maps any density matrix  $\hat{\rho}$  to a corresponding superstate, or *superket*,  $|\rho\rangle\rangle$ , which resides in the Liouville space  $\mathcal{L} = \mathfrak{H} \otimes \mathfrak{H}^*$  [64,65]. Consider a Hilbert space  $\mathfrak{H}$  with an orthonormal basis  $\{|\psi_n\rangle, n \in \mathbb{N}\}$ . The space of bounded operators  $\mathcal{B}(\mathfrak{H})$  can then be expanded using the basis elements  $\{|\psi_n\rangle\langle\psi_m|, n, m \in \mathbb{N}\}$ . Under vectorization, these elements transform according to the rule

$$|\psi_n\rangle\langle\psi_m| \rightarrow |\psi_n\rangle \otimes |\psi_m\rangle^* = |\psi_n, \psi_m\rangle\rangle, \quad (2)$$

which effectively maps  $\mathfrak{H}$  to  $\mathfrak{H} \otimes \mathfrak{H}^*$ . Importantly, the state  $|\psi_n, \psi_m\rangle\rangle$  is generally distinct from the standard tensor product state  $|\psi_n, \psi_m\rangle \in \mathfrak{H} \otimes \mathfrak{H}$ .

A vectorized matrix  $\hat{M} \in \mathcal{B}(\mathfrak{H})$  is referred to as a *superket* and is denoted by  $|M\rangle\rangle \in \mathcal{L}$ . Similarly, superoperators  $S \in \mathcal{S}(\mathfrak{H})$ , which act on operators in  $\mathcal{B}(\mathfrak{H})$ , are represented as matrices in this framework. The precise form of a vectorized superoperator depends on its action, but a key identity that facilitates calculations is the *superket triple product identity* [65], given by

$$\begin{aligned} \hat{\mathcal{U}}(\hat{A}\hat{B}\hat{C}) &= (\hat{A} \otimes \hat{C}^T)|B\rangle\rangle = (\hat{A}\hat{B} \otimes \hat{I})|C\rangle\rangle \\ &= (\hat{I} \otimes (\hat{B}\hat{C})^T)|A\rangle\rangle, \end{aligned}$$

where  $\hat{A}^T$  denotes the matrix transpose of  $\hat{A}$  and  $\hat{I}$  is the identity matrix. Applying this identity, the Liouvillian  $\mathcal{L}$  can be rewritten as the non-Hermitian matrix

$$\tilde{\mathcal{L}} = -i[[\hat{H}, \hat{I}]]_- + \sum_j \gamma_j \left( \hat{L}_j \otimes \hat{L}_j^* - \frac{1}{2} [[\hat{L}_j^\dagger \hat{L}_j, \hat{I}]]_+ \right), \quad (3)$$

where the *super (anti)commutator* is defined as

$$[[\hat{A}, \hat{B}]]_{\pm} = \hat{A} \otimes \hat{B}^T \pm \hat{B} \otimes \hat{A}^T. \quad (4)$$

In this way, the vectorized density matrix evolves according to the following vector equation:

$$\frac{d}{dt}|\rho\rangle\rangle = \tilde{\mathcal{L}}|\rho\rangle\rangle. \quad (5)$$

Throughout, we denote the vectorized form of a superoperator using an overbar, i.e.,  $\mathcal{S} \rightarrow \tilde{\mathcal{S}}$ .

Although the state vector  $|\rho\rangle\rangle$  is formally a vector in a Hilbert space, its interpretation differs fundamentally from that of a standard quantum state vector  $|\psi\rangle \in \mathfrak{H}$ . Its norm, defined via the Hilbert-Schmidt inner product as

$$P \equiv \langle\langle \rho | \rho \rangle\rangle = \text{Tr}(\hat{\rho}^2) \leq 1, \quad (6)$$

corresponds to the *purity* of the state, which quantifies the degree of mixedness. Unlike the norm of a wavefunction  $\langle\psi|\psi\rangle$ , which remains invariant under unitary evolution, the purity  $P$  evolves dynamically under a given LME. However, the preservation of the trace of  $\hat{\rho}$  can be expressed in the vectorized formalism as

$$\frac{d}{dt}\langle\langle I | \rho \rangle\rangle = \frac{d}{dt}\text{Tr}(\hat{\rho}) = 0, \quad (7)$$

where  $|I\rangle\rangle$  denotes the vectorized identity operator. This equation implies that  $|I\rangle\rangle$  is a left-eigenvector of the Liouvillian with eigenvalue zero, i.e.,

$$\langle\langle I | \tilde{\mathcal{L}} = 0. \quad (8)$$

The components of a pure state  $|\psi\rangle$  in a given basis represent probability amplitudes, whereas the elements of  $|\rho\rangle\rangle$  serve a different purpose. Since  $|\rho\rangle\rangle$  is a vectorized form of  $\hat{\rho}$ , they contain the same elements, meaning that the components of  $|\rho\rangle\rangle$  encode both populations (the diagonal elements of  $\hat{\rho}$ ) and coherences (the off-diagonal elements of  $\hat{\rho}$ ).

So far, we have not specified a basis in which to analyze the vectorized Liouvillian  $\tilde{\mathcal{L}}$ . Let us now introduce a natural number basis  $\{|n\rangle\rangle\}$ , which we refer to as Fock states, for the Hilbert space  $\mathfrak{H}$ , which may, for example, correspond to bosonic or fermionic Fock states. In the bosonic case, these states satisfy  $\hat{a}|n\rangle\rangle = \sqrt{n}|n-1\rangle\rangle$ , while for fermions, the possible states are  $|0\rangle\rangle$  and  $|1\rangle\rangle$ . For spin systems, we can use the eigenstates of  $\hat{S}_z$ , given by  $\hat{S}_z|S, m\rangle\rangle = m|S, m\rangle\rangle$  as our number states.

In the Liouville space  $\mathfrak{L}$ , the corresponding Fock states are given by  $\{|n, m\rangle\rangle \mid n, m \in \mathbb{N}\}$ . Here, it should be understood that  $n$  and  $m$  can represent multiple modes and may include combinations of different particle types (bosonic, fermionic, or spin). Once we establish the Fock state basis  $\{|n, m\rangle\rangle\}$ , the matrix representation of  $\tilde{\mathcal{L}}$  in this basis defines the LFSL.

For a finite-dimensional Hilbert space  $\mathfrak{H}$  of dimension  $d$ , the corresponding Liouville space has dimension  $D(\mathfrak{L}) = d^2$ . Furthermore, the number of degrees of freedom is effectively doubled in Liouville space, meaning that the LFSL typically has twice the dimension of the corresponding Hamiltonian FSL—a feature that will be illustrated through several examples in the following sections. However, similar to FSLs, symmetries play a major role in determining the effective dimension and geometry of the lattice. Owing to the system's openness, however, the correspondence between symmetries

and conserved quantities does not always hold, which gives rise to unique features of the LFSL. This aspect will be discussed in detail in the following section.

## B. Symmetries and conserved quantities

For Hamiltonian systems, Noether's theorem establishes a fundamental connection between conserved quantities and continuous symmetries. In the context of FSLs, symmetries play a crucial role in determining the structure of the lattice, as explored in detail in Ref. [31]. However, in open quantum systems, the direct correspondence implied by Noether's theorem does not generally hold [66]. Nonetheless, symmetries still have a significant impact on the geometry of the LFSL, as we will see, and we therefore devote this subsection to summarizing their role in LMEs following the discussion of Ref. [66].

According to the definitions in Ref. [66], symmetries in open systems can be classified into two categories: weak and strong symmetries. The reason behind the classification of symmetries in open systems in these two types resides on the fact that not all the symmetries of the Liouvillian superoperator correspond to conserved quantity. Moreover, the existence of conserved quantities not related to symmetries is also possible.

To formalize this, let  $\mathcal{S} \in \mathfrak{H} \otimes \mathfrak{H}$  denote the Hermitian generator of a continuous symmetry of the Liouvillian. A *weak symmetry* is then characterized by the relation

$$\mathcal{U}^\dagger \mathcal{L} \mathcal{U} = \mathcal{L}, \quad (9)$$

where  $\mathcal{U} = \exp(i\phi\mathcal{S})$ . This kind of symmetry does not necessarily corresponds to a conserved quantity, as we are going to see, hence the name *weak*.

Now, let  $\mathcal{S}$  be the superoperator version of  $\hat{S} \in \mathfrak{H}$  constructed according to the relation

$$\mathcal{S} = \mathcal{S} \otimes I - I \otimes \mathcal{S}^*. \quad (10)$$

A strong symmetry is then generated by the  $\mathcal{S}$  for which  $\hat{S}$  in Eq. (10) commutes with every operator in the Liouvillian, i.e.,

$$[\hat{S}, \hat{H}] = [\hat{S}, \hat{L}_i] = 0. \quad (11)$$

The discussion of conserved quantities is most naturally framed in the Heisenberg picture, where the time evolution of an operator  $\hat{S}$  is governed by  $d\hat{S}/dt = \mathcal{L}^\dagger(\hat{S})$ . Hence, a conserved quantity in open-system dynamics corresponds to an operator  $\hat{S}$  that satisfies  $\mathcal{L}^\dagger|\hat{S}\rangle\rangle = 0$ , or equivalently, in vectorized form,

$$\frac{d}{dt}|\hat{S}\rangle\rangle = \tilde{\mathcal{L}}^\dagger|\hat{S}\rangle\rangle = 0. \quad (12)$$

This corresponds to a right eigenvector of  $\tilde{\mathcal{L}}^\dagger$  with eigenvalue zero, or equivalently, to a left eigenvector of  $\tilde{\mathcal{L}}$  with zero eigenvalue.

By assuming the aforementioned relation between  $\hat{S}$  and its superoperator version  $\mathcal{S}$  in Eq. (10), the presence of a weak symmetry can be formulated by the following identities [66]:

$$[\hat{S}, \hat{H}] = 0, \quad (13)$$

$$[\hat{S}, \hat{L}_k] = \alpha_k \hat{L}_k, \quad \alpha_k \in \mathbb{R}. \quad (14)$$

This includes the special case of  $\alpha_k = 0$ , which reproduces the condition (11) for a strong symmetry. It then follows that a strong symmetry implies in the existence of a conserved quantity

$$\mathcal{L}^\dagger(\hat{S}) = i[\hat{H}, \hat{S}] + \sum_k \frac{\gamma_k}{2} (2\hat{L}_k^\dagger \hat{S} \hat{L}_k - \{\hat{L}_k^\dagger \hat{L}_k, \hat{S}\}) = 0. \quad (15)$$

We note that if  $\alpha_k \neq 0$ , we have a (weak) symmetry but not a conserved quantity since the last term in Eq. (15) is nonzero. Moreover, the existence of conserved quantities that are not related to any symmetry is also possible since Eq. (15) can vanish without  $\hat{S}$  satisfying Eqs. (13) and (14).

To understand the role of symmetries and conserved quantities in LFSL dynamics, we need to clarify what a conserved quantity represents. The non-Hermiticity of  $\tilde{\mathcal{L}}$  implies that it may not be diagonalizable. In this case, its expansion must be performed in terms of generalized eigenvectors [65]. However, assuming diagonalizability, the sets of left and right eigenvectors are generally distinct. Suppose there are  $d$  linearly independent steady states, i.e.,  $d$  right eigenvectors of  $\tilde{\mathcal{L}}$  with zero eigenvalues:  $\tilde{\mathcal{L}}|\psi_i\rangle = 0$ ,  $i = 1, \dots, d$ . This implies that there exist at least  $d$  linearly independent conserved quantities satisfying  $\tilde{\mathcal{L}}^\dagger|J_i\rangle = 0$ ,  $i = 1, \dots, d$ . This is known as the *steady-state conserved quantity correspondence* [66]. Hence, conserved quantities represent information retained from the initial state [67] that persists throughout the dynamics. In the LFSL framework, this persistence manifests as nonvanishing populations. If  $|\rho_{\text{in}}\rangle$  is the initial state and  $|\rho_{\text{ss}}\rangle = \lim_{t \rightarrow \infty} e^{\tilde{\mathcal{L}}t}|\rho_{\text{in}}\rangle$  is the steady state, then

$$\begin{aligned} \langle\langle J_i | \rho_{\text{ss}} \rangle\rangle &= \lim_{t \rightarrow \infty} \langle\langle J_i | e^{\tilde{\mathcal{L}}t} \rho_{\text{in}} \rangle\rangle \\ &= \lim_{t \rightarrow \infty} \langle\langle e^{\tilde{\mathcal{L}}^\dagger t} J_i | \rho_{\text{in}} \rangle\rangle = \langle\langle J_i | \rho_{\text{in}} \rangle\rangle, \end{aligned}$$

where in the last equality we used the definition of a conserved quantity [Eq. (12)].

In the LFSL, a continuous symmetry, as in the FSL, leads to a splitting of the lattice by reducing its dimension by one, while a discrete symmetry corresponds to breaking the lattice without reducing its dimension. However, in contrast to the FSL, here it is possible to have sublattices whose corresponding amplitudes vanish. These lattices represent decaying subspaces and emerge due to weak symmetries. Consequently, strong symmetries only partition the lattice into components containing nonvanishing amplitudes, as they necessarily imply conserved quantities. In the following section, we provide examples of LFSLS and analyze their associated symmetries and conserved quantities.

In addition to symmetries, which may or may not be present, there are constraints that always play an important role in determining the LFSL. The LME describes the most general Markovian evolution that is completely positive and trace- and Hermiticity-preserving (CPTP). This ensures that any physical initial state remains physical throughout the evolution, i.e.,  $\rho(t)^\dagger = \rho(t) \geq 0$  and  $\text{Tr}[\rho(t)] = 1$ . The Hermiticity-preserving property of  $\tilde{\mathcal{L}}$  can be interpreted as a form of symmetry—though not a conventional real-valued one—as discussed in Ref. [68]. Expanding the density matrix

in the Fock basis as

$$|\rho\rangle\rangle = \sum_{n,m} \rho_{n,m} |n, m\rangle\rangle, \quad (16)$$

this condition enforces the relation  $\rho_{n,m} = \rho_{m,n}^*$ . As a consequence, the distribution on the lattice exhibits a symmetry, up to complex conjugation, about the sites whose populations correspond to the diagonal entries of the density matrix. We note that we refer to the coefficients  $\rho_{n,m}$  as populations, even though they are, in general, complex.

Additionally, trace preservation manifests dynamically in the LFSL as a form of detailed balance for the sites corresponding to populations. Expressing the Liouvillian in the Fock basis, we write

$$\tilde{\mathcal{L}} = \sum_{n,m,j,k} \tilde{\mathcal{L}}_{j,k}^{n,m} |n, m\rangle\rangle \langle\langle j, k|. \quad (17)$$

Using Eq. (7), we obtain

$$\sum_n \left( \frac{d}{dt} \rho_{n,n} - \sum_{j,k} \tilde{\mathcal{L}}_{j,k}^{n,n} \rho_{j,k} \right) = 0, \quad (18)$$

which expresses that the total change in populations is always balanced by the action of the Liouvillian.

### C. Examples

To build intuition for the LFSLS, we consider a few illustrative examples. Although simple, they demonstrate the general ideas and highlight the roles of strong and weak symmetries.

#### 1. Open Jaynes-Cummings model

Before introducing LFSLS, we begin with a brief recapitulation of FSLs to provide a smooth transition to the construction of LFSLS. To this end, we consider the paradigmatic JC model, introduced in 1963 to describe coherent light-matter interactions [69,70]. In this model, the atom is represented as a pseudo spin-1/2 particle, while the light field is described by a single bosonic mode. The total Hilbert space is given by  $\mathcal{H}_{\text{JC}} = \text{span}\{|n, e(g)\rangle, n \in \mathbb{N}\}$ , where  $|n, e(g)\rangle = |n\rangle \otimes |e(g)\rangle$ . Here,  $|n\rangle$  denotes the state of the bosonic mode with  $n$  excitations and energy  $\omega \hat{n}|n\rangle = \omega n|n\rangle$ , while  $|e(g)\rangle$  represents the excited (ground) state of the atom with energy difference  $\Omega$ .

When considering finite Hilbert spaces, such as spins, the Fock or number states are taken to be those in which  $\hat{\sigma}_z$  or  $\hat{S}_z$  are diagonal. In the present example, our Fock states are therefore  $|n, e(g)\rangle$ . Finally, the dynamics of the JC model is governed by the Hamiltonian, written in the rotating frame with respect to the total number of bosons operator,

$$\hat{H}_{\text{JC}} = \frac{\Delta}{2} \hat{\sigma}_z + g(\hat{\sigma}^+ \hat{a} + \hat{a}^\dagger \hat{\sigma}^-), \quad (19)$$

where  $\Delta = \Omega - \omega$  is the atom-field detuning and  $g$  is the light-matter coupling strength.

The interaction term describes the coupling between the two-level system and the number of bosonic excitation states, where  $\hat{\sigma}_\pm |g(e)\rangle = |e(g)\rangle$  and  $\hat{\sigma}_+ |e\rangle = \hat{\sigma}_- |g\rangle = 0$ , and the action of the annihilation and creation operators follows the bosonic algebra  $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ ,  $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ .

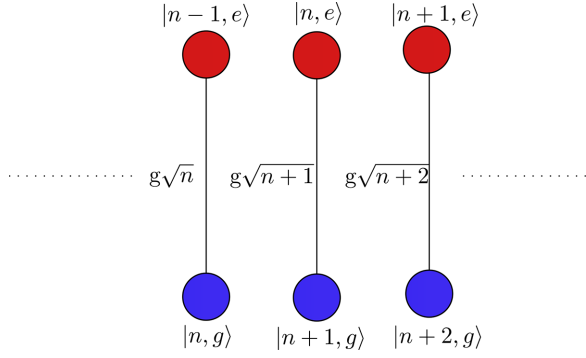


FIG. 1. The FSL of the JC model [Eq. (19)]. The red circles represent sites where the atom is in the excited state, while the blue circles correspond to the atom in the ground state. The solid lines along the rungs indicate tunneling with rates  $g\sqrt{n}$ . Due to the assumed RWA, no transitions occur along the legs of the ladder. The single disconnected site, corresponding to the state  $|0, g\rangle$ , is not shown.

To visualize the emerging FSL, we focus solely on the interaction term, as it is responsible for inducing transitions between the different basis states, which we take to represent the lattice sites. Given that the JC model consists of a bosonic degree of freedom and a two-level system, one might expect the corresponding lattice to be a composition of two semibounded (due to the vacuum state) one-dimensional lattices, i.e., a quasi-two-dimensional lattice or ladder lattice. However, in the JC model, the total number of excitations,

$$\hat{N} = \hat{n} + \frac{1}{2}(\hat{\sigma}_z + 1), \quad (20)$$

is a conserved quantity since  $[\hat{H}_{JC}, \hat{N}] = 0$ . This imposes the restriction that only the transitions  $|n, g\rangle \leftrightarrow |n-1, e\rangle$ , with coupling amplitude  $g\sqrt{n}$ , are allowed. As a result, the FSL of the JC model forms a lattice as shown in Fig. 1, with state-dependent tunneling along the rungs but no tunneling along the legs. Consequently, the FSL consists of an infinite set of two-site lattices (akin to double wells). Note that a single site,  $|0, g\rangle$ , remains completely disconnected from the rest of the lattice.

Due to Noether's theorem, this dimensionality reduction—from a quasi-two-dimensional lattice to a set of quasi-one-dimensional lattices—can be attributed to the existence of a continuous symmetry. In this case, since the total excitation number is conserved, it acts as the generator of a continuous symmetry. Therefore, we can conclude that the existence of a continuous symmetry implies a reduction in dimensionality by unity. A discrete symmetry, on the other hand, corresponds to a splitting of the lattice into different symmetry sectors [31].

Another simple example of FSL can be derived by adding to the JC Hamiltonian a driving term,

$$\hat{H}_\eta = \eta(\hat{a} + \hat{a}^\dagger), \quad (21)$$

which couples states  $|n, e(g)\rangle \leftrightarrow |n \pm 1, e(g)\rangle$ . This drive induces tunneling along the legs of the ladder in Fig. 1, as illustrated in Fig. 2.

To account for losses in the JC model, we move to the Liouville space representation. In this representation, density matrices are transformed into vectors, so the set of states

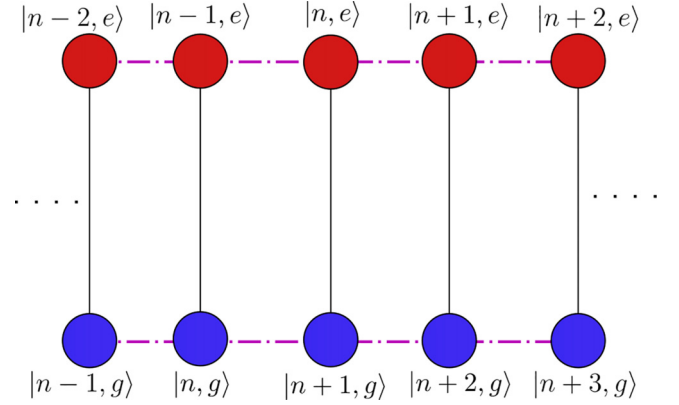


FIG. 2. The FSL of the driven JC model,  $\hat{H}_{JC} = \hat{H}_{JC} + \hat{H}_\eta$ , exhibits tunneling processes compared to the regular JC model. The tunnelings induced by  $\hat{H}_\eta$  [see Eq. (21)], indicated by purple dash-dotted lines, couple sites along the legs of the ladder. Note that the lattice is no longer separable, which follows from the fact that the drive breaks the continuous symmetry associated with excitation conservation in the JC model. Adding a drive on the atom instead, i.e.,  $\hat{H}_v = v\hat{\sigma}_x$ , would result in diagonal tunneling rates.

spanning the space takes the form

$$|n, e; m, e\rangle, \quad |n, e; m, g\rangle, \quad |n, g; m, e\rangle, \quad |n, g; m, g\rangle, \quad (22)$$

with  $n, m \in \mathbb{Z}_+$ . Once the Liouvillian superoperator is specified, the resulting LFSL follows directly. For the closed JC model [Eq. (19)], the vectorized Liouvillian reads

$$\begin{aligned} \tilde{\mathcal{L}}_{JC} = & -i\left(\frac{\Delta}{2}(\hat{\sigma}^z - \tilde{\sigma}^z) + g(\hat{a}^\dagger \hat{\sigma}^- - \tilde{a}^\dagger \tilde{\sigma}^-) \right. \\ & \left. + g(\hat{\sigma}^+ \hat{a} - \tilde{\sigma}^+ \tilde{a})\right), \end{aligned} \quad (23)$$

where, for simplicity, we omit the tensor products with identities. Operators with a tilde belong to the doubled Hilbert space, i.e.,  $\tilde{A} \in \text{Op}(\mathcal{H}^*)$ .

The LFSL corresponding to the Liouvillian in Eq. (23) consists of an infinite set of four-site squares, as illustrated in Fig. 3. However, not all lattices are squares: For instance, the state  $|0, g; 0, g\rangle$  is disconnected from all other sites, and states of the form  $|0, g; m, g\rangle$  or  $|n, g; 0, g\rangle$  form only two-site lattices. As in Hilbert space, the LFSL of the JC model also lacks translational invariance.

We now introduce interactions with an external environment. First, consider the JC model with the atom coupled to a dissipative bath. The simplest nontrivial jump operator is atomic decay,  $\hat{L} = \hat{\sigma}^-$ . The Liouvillian then reads

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_{JC} + \frac{\gamma}{2}(2\hat{\sigma}^- \otimes \tilde{\sigma}^- - \hat{\sigma}^+ \hat{\sigma}^- \otimes \mathbb{I} - \mathbb{I} \otimes \tilde{\sigma}^+ \tilde{\sigma}^-), \quad (24)$$

where  $\tilde{\mathcal{L}}_{JC}$  is given by Eq. (23).

By applying the Liouvillian to the basis states, we see that the unitary part of the dynamics again generates squares, as in Fig. 3. However, the jump terms now couple each square. In Fig. 4(a), we depict the LFSL of this open JC model.

Without losses, the excitation number is conserved,  $[\hat{N}, \hat{H}_{JC}] = 0$ , and consequently generates a continuous symmetry of the lattice. To determine whether this remains the

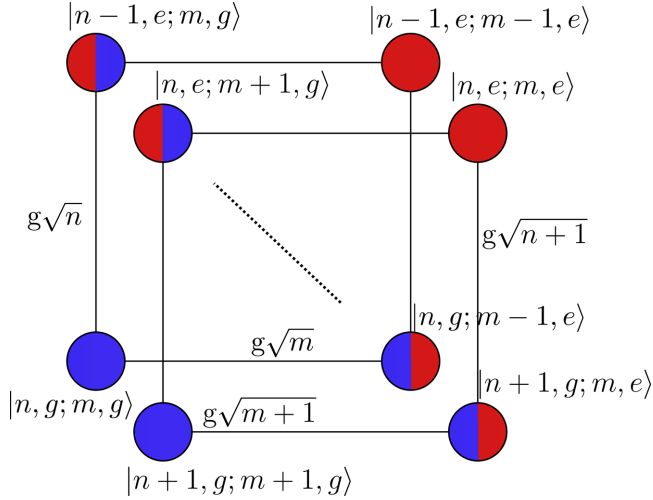


FIG. 3. The LFSL of the JC model. Each site is associated with four elements corresponding to the atom's density matrix:  $|g, g\rangle$ ,  $|g, e\rangle$ ,  $|e, g\rangle$ , and  $|e, e\rangle$ . Blue indicates that the atom is in the ground state,  $|g\rangle \in \mathcal{H}$  on the left side and  $|g\rangle \in \mathcal{H}^*$  on the right, while red indicates the excited state,  $|e\rangle \in \mathcal{H}$  on the left and  $|e\rangle \in \mathcal{H}^*$  on the right. The transition amplitudes along the horizontal edges are identical within each square, and the same holds for the vertical edges. The dotted line indicates that the lattice extends infinitely in that direction.

case in the presence of atomic decay, we verify the condition given by Eq. (14):

$$\begin{aligned} [\hat{N}, \hat{L}] &= \left[ \frac{1}{2}(\hat{\sigma}^z + \mathbb{I}) + \hat{a}^\dagger \hat{a}, \hat{\sigma}^- \right] \\ &= \frac{1}{2}[\hat{\sigma}^z, \hat{\sigma}^-] = -\hat{\sigma}^-. \end{aligned} \quad (25)$$

Therefore, Eq. (14) is satisfied with  $\alpha = -1$ , yielding a weak symmetry rather than a strong one, as expected since the total number of excitations is no longer conserved.

As previously discussed, weak symmetries break up the lattice into sublattices with, in general, nonsurviving amplitudes. Spontaneous emission described by  $\hat{L} = \hat{\sigma}^-$  couples states  $|e, n; e, m\rangle$  to states  $|g, n; g, m\rangle$ . We note that the differ-

ence between the total number of excitations for each space  $\mathfrak{H}$  and  $\mathfrak{H}^*$  remains constant despite this decay, i.e.,  $\hat{N} - \hat{N}^* = C$  for some integer constant  $C$ .

This model does not support any strong symmetry, which suggests that there are no conserved quantities. In this context, the LFSL is particularly useful for visualizing whether this is indeed the case. From Fig. 4(a), we see that states of the form  $|n, g; m, g\rangle$  are dark states, i.e., states that are unaffected by the action of the jump operators. To correspond to eigenstates of the Liouvillian, these states must also be eigenstates of the Hamiltonian. For a set of Hamiltonian eigenstates that are also dark states,  $\{|\psi_n\rangle\}$ , with corresponding eigenvalues  $\lambda_n$ , the corresponding eigenstates of the Liouvillian are given by

$$|\rho_{n,m}\rangle = |\psi_n\rangle \otimes |\psi_m\rangle^*, \quad (26)$$

with eigenvalues

$$\tilde{\mathcal{L}}|\rho_{n,m}\rangle = -i(\lambda_n - \lambda_m)|\rho_{n,m}\rangle. \quad (27)$$

In this model, not all of the identified dark states are eigenstates of the Hamiltonian, nor can linear combinations of them be. This can be seen from the fact that they are not connected to each other through the Hamiltonian interaction. However, as noted before, there is one state that is both a Hamiltonian eigenstate and a dark state:  $|0, g; 0, g\rangle$ . This state corresponds to the steady state of the model, which is evident from the LFSL, as it is the unique state that cannot transition to any decaying state due to the directional coupling.

Another option is to introduce single-photon loss, i.e.,  $\hat{L} = \hat{a}$ . The corresponding Liouvillian is given by

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_{JC} + \frac{\gamma}{2}(2\hat{a} \otimes \hat{a} - \hat{a}^\dagger \hat{a} \otimes \mathbb{I} - \mathbb{I} \otimes \hat{a}^\dagger \hat{a}). \quad (28)$$

Once again, this model exhibits a weak symmetry generated by the total excitation number operator, as shown by

$$[\hat{N}, \hat{L}] = \left[ \frac{1}{2}(\hat{\sigma}^z + \mathbb{I}) + \hat{a}^\dagger \hat{a}, \hat{a} \right] = [\hat{a}^\dagger \hat{a}, \hat{a}] = -\hat{a}. \quad (29)$$

Similar to the case of atomic decay, this weak symmetry results in a partitioning of the lattice such that the difference between the total excitation number in  $\mathfrak{H}$  and  $\mathfrak{H}^*$  remains constant. However, in this case, the squares are connected

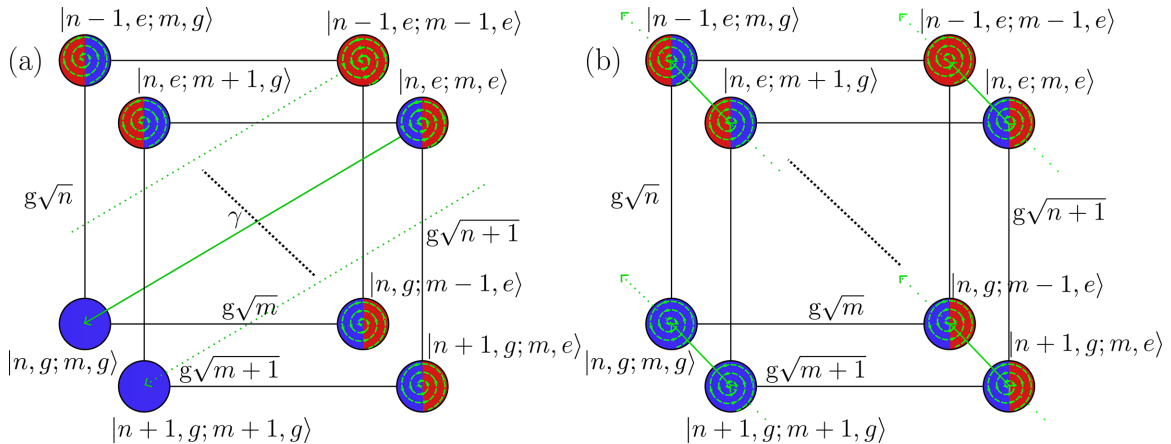


FIG. 4. LFSLs for the open JC model with atomic decay,  $\hat{L} = \hat{\sigma}^-$  (a), and single-photon loss,  $\hat{L} = \hat{a}$  (b). Green arrows indicate state transitions induced by the jump operator, while green spirals represent onsite amplitude decay. At sites marked with a green spiral, the population decreases in addition to tunneling to other sites.

through all four types of states since the jump operator acts only on the bosonic mode, as illustrated in Fig. 4(b).

Additionally, it follows that the dark states, i.e., states that are unaffected by losses, are those in which no bosons are present, i.e.,  $|0, e(g); 0, e(g)\rangle$ . Since  $|0, g; 0, g\rangle$  is the only such state that is also an eigenstate of the Hamiltonian, it follows that this is the unique steady state [71].

Pure dephasing in the bare Fock basis, e.g.,  $\hat{L} = \hat{a}^\dagger \hat{a}$  or  $\hat{\sigma}_z$ , does not induce hopping in the LFSL. Nevertheless, such terms lead to onsite decay, as expected, since the coherences of the density operator decay over time.

## 2. Two-photon loss cavity mode

As a second example, we consider a single cavity mode with Hamiltonian  $\hat{H} = \omega \hat{a}^\dagger \hat{a}$  undergoing two-photon loss, described by the jump operator  $\hat{L} = \hat{a}^2$ . This example is particularly instructive, as it illustrates the role of a strong symmetry. Since the Liouville space is given by the direct product of two bosonic Hilbert spaces, the LFSL is two-dimensional, with sites labeled by the Fock numbers  $n$  and  $m$ , i.e.,  $|n, m\rangle$ .

This system possesses a weak symmetry generated by the number operator  $\hat{N} = \hat{a}^\dagger \hat{a}$  [66], which leads to a splitting of the lattice into one-dimensional sublattices characterized by a fixed integer difference  $\Delta = n - m$ . Additionally, the system exhibits a discrete strong symmetry corresponding to the parity transformation  $\hat{a} \rightarrow -\hat{a}$ . This symmetry further divides the one-dimensional sublattices into two sets, each associated with a parity sector of even or odd of  $\mathfrak{H}$  and  $\mathfrak{H}^*$ .

Figure 5 illustrates this structure. Blue and red arrows indicate parity and hopping direction, while sites marked with green spirals represent onsite decay. The four decay-free sites span the steady-state subspace. Among these,  $|0, 0\rangle$  and  $|1, 1\rangle$  correspond to the zero- and one-photon Fock states, representing the conserved quantities in the even and odd parity sectors. The remaining two sites represent their coherences and provide an example of conserved quantities that do not arise from a symmetry, as discussed in Ref. [66].

Introducing a coherent drive,  $\eta(\hat{a}^\dagger + \hat{a})$ , in the Hamiltonian or incorporating single-photon loss couples the parity sectors, significantly altering the structure and dynamics of the LFSL and breaking the strong symmetry.

## III. ENVIRONMENT-INDUCED FRUSTATION IN LIOUVILLE SPACE

In this section by employing the LFSL framework, we show how the existence of an infinite number of steady states is related to geometrical frustration in Liouville space. To fulfill this goal, we consider two models: the open tight-binding model and an incoherently driven harmonic oscillator.

### A. Open tight-binding model

Let us consider a model that results in a translationally invariant LFSL, namely, a single-particle hopping, coherently or incoherently, between neighboring sites in a one-dimensional lattice. Hence, contrary to the examples of the previous section, the occupation states  $|n\rangle$  of this model do not represent Fock states in the traditional sense. What makes this model

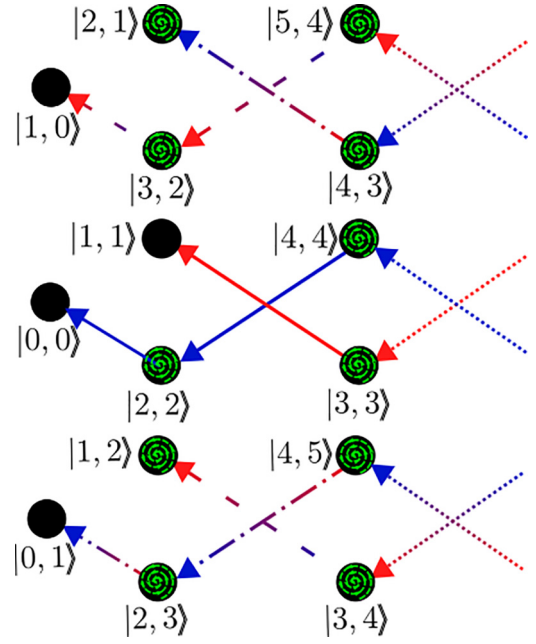


FIG. 5. The LFSL corresponding to a harmonic oscillator undergoing two-photon loss. The depicted lattice is restricted to the diagonal and the first two off-diagonal elements of the density matrix. Solid blue and red arrows indicate transitions between states with even and odd parities in  $\mathfrak{H} \otimes \mathfrak{H}^*$ , respectively. Dot-dashed lines represent transitions between even states in  $\mathfrak{H}$  and odd states in  $\mathfrak{H}^*$ , while long-dashed lines correspond to transitions between odd states in  $\mathfrak{H}$  and even states in  $\mathfrak{H}^*$ . Onsite green spiral arrows indicate decaying states.

interesting is that while the Hamiltonian is trivial, in the Liouville space the model is frustrated. The Hamiltonian is given by

$$\hat{H} = \eta \sum_n (|n\rangle \langle n+1| + \text{H.c.}), \quad (30)$$

and we introduce the jump operators

$$L_1 = \sum_n |n\rangle \langle n+1|, \quad L_2 = \sum_n |n+1\rangle \langle n|, \quad (31)$$

with corresponding rates  $\gamma_1$  and  $\gamma_2$ , and  $|n\rangle$  the state for a particle at site  $n$ . Since the jump operators commute with the Hamiltonian,  $[\hat{L}_1, \hat{H}] = [\hat{L}_2, \hat{H}] = 0$ , we expect an energy dephasing, but the fact that  $\hat{L}_{1,2}$  are non-Hermitian makes the model nontrivial. The resulting LFSL forms a square lattice structure, as illustrated in Fig. 6, with red and blue lines representing incoherent and coherent tunnelings.

Using the translational invariance of the LFSL, we can construct the eigenvectors of the Liouvillian using Bloch states  $|\theta\rangle$ , characterized by the quasimomentum  $\theta \in [-\pi/2, \pi/2)$ . Specifically, the eigenstates take the form  $|\hat{\rho}_{\theta\tilde{\theta}}\rangle = |\theta, \tilde{\theta}\rangle$  as we can see through the action of the Liouvillian on momentum states

$$\begin{aligned} \tilde{\mathcal{L}}|\theta, \tilde{\theta}\rangle &= [-2i\eta(\cos\theta - \cos\tilde{\theta}) \\ &+ \gamma_1 e^{i(\theta-\tilde{\theta})} + \gamma_2 e^{-i(\theta-\tilde{\theta})} - (\gamma_1 + \gamma_2)]|\theta, \tilde{\theta}\rangle. \end{aligned} \quad (32)$$

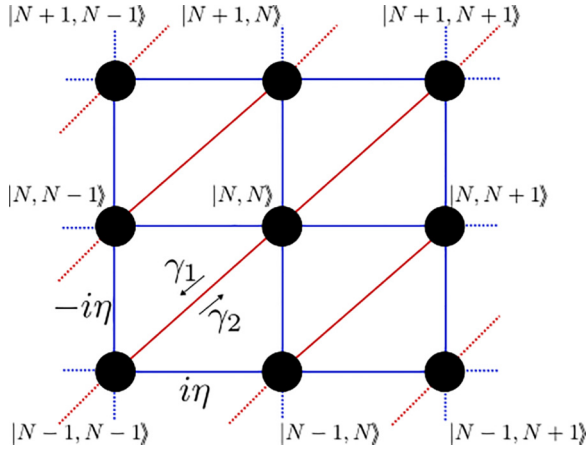


FIG. 6. LFSL of the tight-binding model [Eq. (30)] with incoherent tunneling processes [Eq. (31)]. Horizontal and vertical blue lines arise from coherent tunneling, while diagonal red hoppings stem from the incoherent jumps.

Hence, the Liouvillian spectrum is simply given by

$$\begin{aligned} \mu(\theta, \tilde{\theta}) = & -(\gamma_1 + \gamma_2)[1 - \cos(\theta - \tilde{\theta})] \\ & + i[(\gamma_1 - \gamma_2)\sin(\theta - \tilde{\theta}) - 2\eta(\cos\theta - \cos\tilde{\theta})]. \end{aligned} \quad (33)$$

Equation (33) reveals that the stationary states are diagonal in the quasimomentum basis, meaning  $\theta = \tilde{\theta}$ . Furthermore, since  $\lambda(\theta, \theta) = 0$ , there are no oscillating coherences, implying that a general steady state can be expressed as

$$\rho_{ss} = \int_{-\pi/2}^{\pi/2} \rho(\theta, \theta) |\theta\rangle \langle \theta| d\theta. \quad (34)$$

Thus, the open evolution effectively suppresses coherences between states in the quasimomentum basis. Interestingly, this structure of the stationary states holds for all values of  $\gamma_1$  and  $\gamma_2$ .

This behavior contrasts with what would be the corresponding bosonic model, where the Hamiltonian is given by  $\hat{H} = \eta(\hat{a} + \hat{a}^\dagger)$  and the jump operators are  $\hat{L}_1 = \hat{a}$  and  $\hat{L}_2 = \hat{a}^\dagger$ . Stability constraints in the bosonic case require  $\gamma_1 \geq \gamma_2$ , which restricts the steady-state properties. This highlights the crucial role of bosonic algebra in shaping the LFSL model: While an infinite number of steady states always exist in the former model, the latter only permits such behavior in the balanced case  $\gamma_1 = \gamma_2$ .

We now explore the connection between the presence of an infinite number of steady states and geometrical frustration in the LFSL. In closed lattice systems, geometrical frustration occurs when highly degenerate ground states arise from the underlying lattice geometry, preventing the system from further minimizing its energy [72,73]. In an open setting, the natural analog of ground states is the set of steady states of the Liouvillian, i.e., the eigenstates corresponding to zero eigenvalues. Therefore, geometrical frustration in an open lattice model implies the existence of an infinite number of steady states, although the reverse need not be true. Hence, a large

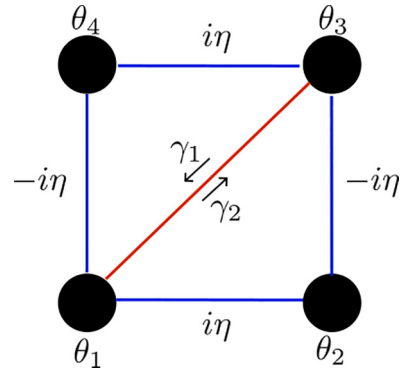


FIG. 7. Unit cell of the LFSL corresponding to the open tight-binding model [Eq. (30)]. Each site is assigned a phase  $\theta_i$ , and the relevant tunneling rates are explicitly indicated.

steady-state degeneracy alone is not sufficient to classify a system as frustrated.

To determine whether the LFSL of the above model exhibits frustration, we recast the system as a classical lattice model. In many-body settings (bosons), this typically emerges from a mean-field approximation, where onsite operators are replaced by complex numbers, transforming the lattice Hamiltonian into a classical energy functional. Specifically, we assign a complex variable

$$\psi_i = \sqrt{n_i} e^{i\theta_i} \quad (35)$$

to each lattice site  $\mathbf{i} = (i, j)$  and optimize the functional  $\tilde{\mathcal{L}}[n_i, \theta_i]$ . In the traditional framework,  $\psi_i$  represents the local order parameter, where  $n_i$  corresponds to the particle number/density at site  $\mathbf{i}$ , and  $\theta_i$  denotes its phase. In our case, these quantities correspond to the complex elements of the density operator.

For a Hamiltonian system, the optimization process amounts to finding the global minimum of the energy. The key idea is that if a minimum can be identified within a unit cell; i.e., locally, it also uniquely determines the global minimum. Due to translational invariance, one typically assumes constant densities,  $n_i \equiv n$ , and focuses on the “phase locking” induced by tunneling terms and their phases. Frustration occurs when it is impossible to minimize the different terms in the energy functional individually. The antiferromagnetic triangular lattice serves as the paradigmatic example, where it is not possible to arrange the spins such that all nearest neighbors are perfectly antiparallel.

In contrast to locally minimize the energy, for our Liouvillian system, we search for a solution satisfying  $\tilde{\mathcal{L}}[n_i, \theta_i] = 0$ , which corresponds to a steady state. As previously discussed, the CPTP property of the Lindblad master equation and the requirement that  $|\rho\rangle\rangle$  represents a physical state impose constraints on the complex numbers  $\psi_i$ . However, to gain insight, we analyze a single unit cell of the LFSL, treat these numbers as independent variables, and assume constant density  $n_i \equiv n$ . Hence, we search for a solution of the functional equation with respect to the phase angles  $\theta_i$  only.

The unit cell with its tunneling rates is depicted in Fig. 7. The “unit cell mean-field” Liouvillian functional becomes, up

to trivial constants,

$$\begin{aligned}\tilde{\mathcal{L}}[\psi_i] = & i\eta(\psi_1^*\psi_2 + \psi_2^*\psi_1) - i\eta(\psi_2^*\psi_3 + \psi_3^*\psi_2) \\ & + i\eta(\psi_3^*\psi_4 + \psi_4^*\psi_3) - i\eta(\psi_4^*\psi_1 + \psi_1^*\psi_4) \\ & + \gamma_1\psi_1^*\psi_3 + \gamma_2\psi_3^*\psi_1.\end{aligned}$$

Using the assumption  $n_i \equiv n$ , and setting  $\tilde{\mathcal{L}} = 0$ , we arrive at the following two equations after separating real and imaginary parts:

$$\begin{aligned}\cos(\theta_1 - \theta_2) - \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_4) \\ - \cos(\theta_1 - \theta_4) + (\alpha_2 - \alpha_1)\sin(\theta_1 - \theta_3) = 0,\end{aligned}\quad (36)$$

$$(\alpha_1 + \alpha_2)\cos(\theta_1 - \theta_3) = 0,\quad (37)$$

where  $\alpha_{1,2} = \gamma_{1,2}/2\eta$ .

If the high degeneracy of steady states stems from frustration, we expect that we cannot independently identify the different terms of Eq. (36) to be zero. Without loss of generality, we can set  $\theta_1 = 0$ . From the second equation, we obtain, for example,  $\theta_3 = \pi/2$ . However, one sees that each term of the first equation cannot be made to be zero due to the term  $(\alpha_2 - \alpha_1)\sin(\theta_1 - \theta_3) = -(\alpha_2 - \alpha_1)$ . In the balanced case  $\gamma_1 = \gamma_2$ , we still have that each term of the first equation cannot be made zero simultaneously. Another way to see that is by checking the case: For the first term to vanish, we require  $\theta_1 - \theta_2 = \pi/2$  (up to additional multiples of  $\pi$ ), and similarly, for the second term,  $\theta_2 - \theta_3 = \pi/2$ . This implies that  $\theta_1 - \theta_3$  must be a multiple of  $\pi$ , leading to  $\cos(\theta_1 - \theta_3) \neq 0$ , and the second equation of Eq. (37) is not fulfilled. Thus, there is no choice of  $(\theta_1, \theta_2, \theta_3, \theta_4)$  that simultaneously causes every term in the functional to vanish. We therefore conclude that the steady-state degeneracy is a result of frustration.

Giving up the constraint that every term of Eq. (37) should simultaneously vanish, we note that for certain parameters  $\alpha_{1,2}$ , Eq. (37) does not permit real solutions  $\theta_i$ . One may expect that in such cases it is not legitimate to assume  $n_i \equiv n$ , and global (classical) solutions, beyond the unit cell, will break the translational invariance of the LFSL.

### B. Incoherently driven oscillator

The previous model demonstrates frustration, but even though it could in principle be realized experimentally [74], implementation is not straightforward. It is therefore of interest to present a more experimentally accessible model that also exhibits frustration. To this end, we consider an incoherently driven oscillator, i.e., a quantum oscillator that, in addition to being driven coherently by a Hamiltonian term, is also subject to an incoherent drive through a jump operator:

$$\hat{H}_{\text{dHo}} = \Delta\hat{a}^\dagger\hat{a} + \eta(\hat{a}^\dagger + \hat{a}),\quad (38)$$

$$\hat{L} = \hat{a}^\dagger + \hat{a}.\quad (39)$$

Since the FSL of the driven oscillator is a one-dimensional chain, the LFSL forms a square lattice. As in the case of a one-photon gain or loss, i.e.,  $\hat{L} = \hat{a}^\dagger$  or  $\hat{L} = \hat{a}$ , the incoherent drive induces diagonal hopping in the square lattice, arising from terms like  $\hat{a}^\dagger\hat{a}$ . Consequently, the lattice is not simply a tight-binding, or nearest-neighbor, lattice. Moreover, the

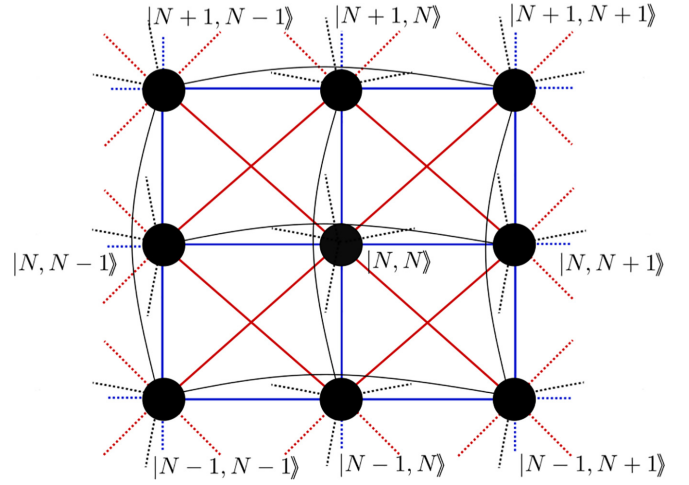


FIG. 8. LFSL of the incoherently driven oscillator [Eq. (38)]. Here, we identify three types of tunneling: solid blue lines represent those arising from coherent driving (Hamiltonian term  $\eta(\hat{a} + \hat{a}^\dagger)$ ); solid red lines correspond to tunneling induced by  $\hat{L}\hat{L}^\dagger$  and hence terms like  $\hat{a}^\dagger\hat{a}$ ; finally, solid black lines represent tunneling due to  $(\hat{a}^\dagger)^2$  and  $\hat{a}^2$ , which stem from  $\hat{L}^\dagger\hat{L}$  (and similarly for  $\hat{L}\hat{L}^\dagger$ ).

terms in the Liouvillian resulting from  $\hat{L}^\dagger\hat{L}$  introduce next-nearest-neighbor hopping due to  $(\hat{a}^\dagger)^2$  and  $\hat{a}^2$ . We show the LFSL of this model, along with its different tunneling terms, in Fig. 8.

This model is somewhat special since there is no biased flow in the LFSL; the tunneling rates are equally strong in both directions. Nevertheless, it is possible to find the steady state analytically by noting that we can write  $\hat{L} = \sqrt{2}\hat{x}$  in the quadrature representation of the boson operators, i.e.,  $\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$  and  $\hat{p} = \frac{i}{\sqrt{2}}(\hat{a} - \hat{a}^\dagger)$ . The jump operator has the effect of localizing the particle in the  $x$ -quadrature representation. Recognizing this, the most general steady state can be expressed as

$$\hat{\rho}_{\text{ss}} = \sum_n p_n \hat{\rho}_n^{\text{ss}}, \quad \text{with} \quad (40)$$

$$\hat{\rho}_n^{\text{ss}} = \int dx |\psi_n(x)|^2 |x\rangle\langle x|, \quad (41)$$

where  $\psi_n(x)$  are the eigenfunctions of the Hamiltonian, and the probabilities  $p_n$  depend on the system's initial state. This is somewhat analogous to the steady state [Eq. (34)] of the previous model, and importantly, both are not diagonal in the Fock basis.

To find the spectrum of the Liouvillian, we resort to *third quantization* [75], a method for solving quadratic Liouvillians. Originally applied to fermionic Lindbladians, it has recently been extended to the bosonic case [68,76]. The resulting spectrum is purely imaginary and equidistant, with the spacing determined by the effective frequency  $\Delta$ , i.e.,

$$\mu_{n,m} = i(n - m)\Delta, \quad n, m \in \mathbb{N}. \quad (42)$$

Note that since  $n$  and  $m$  are natural numbers, the system possesses an infinite number of conserved quantities, and hence also steady states. Note, especially, that the real parts are zero, and hence there is no relaxation.

In the FSL of the Hamiltonian, the term  $\Delta\hat{a}^\dagger\hat{a}$  introduces a linear onsite energy offset, while the drive induces nearest-neighbor tunneling in the FSL, reminiscent of the Wannier-Stark problem. Similarly, in the Wannier-Stark Hamiltonian, the spectrum is also independent of the tunneling rate, while the eigenstates remain dependent on it [77].

In order to study frustration, we follow the approach described previously; i.e., we assign a local order parameter [Eq. (35)] to every lattice site. For large photon numbers, we take a constant  $n_i$ , reducing the problem to the functional  $\tilde{\mathcal{L}}[\theta_i] = 0$ . In a nonfrustrated case, this functional should be solvable locally on a single lattice cell. Since the model contains next-next-nearest neighbor couplings, we consider a  $3 \times 3$  cell.

Labeling the corresponding phases as  $\theta_i$  with  $i = 1, 2, \dots, 9$ , and separating real and imaginary parts, the local constraints become

$$\sum_{i=1}^8 \cos(\theta_i - \theta_{i+1}) - \left( \sum_{j=1}^4 \cos(\theta_j - \theta_{j+4}) \right) = 0, \quad (43)$$

$$\begin{aligned} & \cos(\theta_2 - \theta_6) + \cos(\theta_6 - \theta_8) + \cos(\theta_8 - \theta_4) + \cos(\theta_4 - \theta_2) \\ & - \frac{1}{2} \cos(\theta_2 - \theta_8) - \frac{1}{2} \cos(\theta_4 - \theta_6) - 2 = 0. \end{aligned} \quad (44)$$

The first constraint arises from the Hamiltonian drive, which generates nearest-neighbor couplings. The second constraint comes from the dissipator. The sites involved are those centered on the edges of the  $3 \times 3$  cell of Fig. 8, forming a square with nearest- (first four terms) and next-nearest neighbor couplings. The constant term 2 originates from contributions such as  $\hat{a}\hat{a}^\dagger$ ,  $\hat{b}^\dagger\hat{b}$ , etc., i.e., terms that do not induce hopping but provide a constant onsite energy shift.

We seek a solution that satisfies all 12 terms of the first constraint (43) locally, e.g.,  $\cos(\theta_1 - \theta_2) = 0$ . Given such a solution, we then ask whether it is consistent with the second constraint (44). If a solution exists that satisfies this, the system would be considered nonfrustrated. However, a closer examination shows that it is impossible to satisfy Eq. (44) while also fulfilling Eq. (43) locally. Therefore, we conclude that this model is frustrated as well.

#### IV. ALTERNATIVE REPRESENTATIONS

In the design of synthetic lattices, the choice of representation used to vectorize the system is crucial, as it directly determines both the lattice structure and the ensuing dynamics. While Fock states are a natural choice due to their experimental relevance, they pose challenges for simulating, for example, classical lattice dynamics, since the resulting populations are generally complex. Considering the simulation of classical lattice or network models is relevant because effects such as directed tunneling may appear more naturally in these systems [51–56]. Analogous to phase-space methods [78], alternative representations of  $\hat{\rho}$  exist that guarantee real and non-negative populations. In such cases, however, the resulting lattices are not constructed from Fock states, and thus referring to them as LFSLs would be misleading. Nevertheless, it is worth mentioning these possibilities, both for

the reasons outlined above and because the resulting lattices exhibit interesting properties in their own right.

One such alternative is the Bloch representation, where the state of a finite-dimensional system is expressed as [79,80]

$$\hat{\rho} = \frac{1}{d} \left( \hat{I} + \sqrt{\frac{d(d-1)}{2}} \mathbf{R} \cdot \vec{\lambda} \right), \quad (45)$$

where  $d$  is the Hilbert space dimension,  $\mathbf{R} = (R_1, R_2, \dots, R_{d^2-1})$  is the Bloch vector, and  $\vec{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_{d^2-1})$  is a vector composed of the generalized Gell-Mann matrices [81].

Although the Bloch vector satisfies  $|\mathbf{R}| \leq 1$  and remains real, its components can take negative values. Moreover, for  $d > 2$ , only a subset of the Bloch sphere corresponds to physically valid quantum states [79,80]. By extending the Bloch vector to include  $R_0 = 1$  and defining an augmented basis with  $\hat{\lambda}_0 = \hat{I}$ , the LME can be written in the compact homogeneous form

$$\frac{d\mathbf{R}}{dt} = \hat{\mathcal{L}}\mathbf{R}, \quad (46)$$

where the generator  $\hat{\mathcal{L}}$  is unitarily related to  $\tilde{\mathcal{L}}$ , i.e.,  $\hat{\mathcal{L}} = \hat{V}\tilde{\mathcal{L}}\hat{V}^\dagger$  for a unitary  $\hat{V}$ . The Bloch representation effectively separates the real and imaginary parts of elements such as  $\rho_{nm}$ , which is possible due to the hermiticity of  $\hat{\rho}$ , i.e.,  $\rho_{nm} = \rho_{mn}^*$ . For a more detailed discussion, see Appendix 1.

A representation that ensures positive semidefiniteness is based on symmetric informationally complete positive operator-valued measures (SIC-POVMs) [82]. To construct this representation, we define a complete set of projectors  $\hat{\Pi}_i = (1/d)|\psi_i\rangle\langle\psi_i|$ , which satisfy the constant overlap condition  $\text{Tr}[\hat{\Pi}_i\hat{\Pi}_j] = \frac{1}{d^2} \frac{d\delta_{i,j}+1}{d+1}$ . Using this basis, the density matrix can be expanded as [82]

$$|\rho\rangle\rangle = \sum_{i=1}^{d^2} (d(d+1)p_i - 1) |\Pi_i\rangle\rangle, \quad (47)$$

where  $|\Pi_i\rangle\rangle$  are the vectorized elements of the POVM and  $p_i = \text{Tr}[\hat{\rho}\hat{\Pi}_i] \geq 0$  represents the probability of finding the system in the state  $|\Pi_i\rangle\rangle$ . Within this representation, the probability vector  $\vec{p}$  evolves as

$$\frac{d\vec{p}}{dt} = M\vec{p} - \vec{m}, \quad (48)$$

where the matrix elements  $M_{i,j}$  are given by  $M_{i,j} = d(d+1)\langle\langle\Pi_i|\hat{\mathcal{L}}|\Pi_j\rangle\rangle$  and the components of  $\vec{m}$  are defined as  $m_i = \sum_j M_{i,j}/d(d+1)$ .

Interpreting this as a lattice model, the inhomogeneous term  $\vec{m}$  acts as a source/sink term, reflecting the openness of the system. This becomes evident from the evolution equation for the components,  $\dot{p}_i = \sum_j M_{i,j}(p_j - 1)$ , where the inhomogeneous term behaves as a constant, nonlocal background field that can create or annihilate population. In analogy with classical stochastic master equations, probability conservation in the homogeneous case is ensured by the constraint  $\sum_i M_{i,j} = 0$  [83]. In such cases, the negative terms of  $M$  typically reside on its diagonal, representing transitions out of these sites. However, in our models negative terms can also appear on off-diagonal elements, arising from prohibited

transitions—potentially due to quantum interference—that induce a preferred drift in the lattice. For an example of the lattice structure generated by  $M$ , see Appendix 2. There we present in some detail an example and of how a lattice is constructed and how the state evolves within it.

## V. CLASSICAL SIMULATIONS

The lattice models obtained by vectorizing the LME typically feature drifts, sources, and sinks, structures more commonly associated with classical models. This suggests potential applications of open quantum systems for simulating classical lattice and network dynamics. A key advantage is that very large state spaces, possibly comprising an enormous number of states, can be encoded using only a few physical degrees of freedom. This has already been experimentally demonstrated for simulating quantum models with FSLs [57], which operate within the Hilbert space  $\mathfrak{H}$ , in contrast to the doubled space  $\mathfrak{L} = \mathfrak{H} \otimes \mathfrak{H}^*$  naturally arising in LFSLs. Similar reasoning has recently been employed to propose efficient approaches for solving classical optimization problems on a digital quantum computer [58].

While the Fock and Bloch representations do not directly yield classical probability distributions, they provide powerful frameworks for investigating generic properties of complex dynamics [56]. A hallmark of such models is anomalous diffusion and transport [51,52], phenomena that remain under active investigation due to their broad relevance, ranging from biological processes [84] to environmental studies such as pollution spreading [85].

The SIC-POVM representation of the density matrix is positive semidefinite; however, as mentioned above, its lattice evolution may involve negative tunneling rates. These negative rates typically give rise to anomalous transport in the lattice, once again linking the system to the classical models discussed earlier. Interestingly, similar to the phase-space representation of quantum mechanics—where it is impossible to construct a distribution that is both positive semidefinite and reproduces the correct marginal distributions [86]—we encounter a comparable no-go property here. In a positive-semidefinite representation of  $\hat{\rho}$ , the cost is the appearance of negative tunneling rates.

## VI. DETECTION SCHEMES

Regardless of the specific application of LFSLs—whether for simulating classical lattice dynamics or studying frustration—we need to extract relevant information from  $\hat{\rho}(t)$ . Keeping in mind, we not only employ synthetic dimensions but also consider the evolution in Liouville space, i.e., the doubled Hilbert space. Full state tomography—now a standard tool in many quantum optics laboratories—would provide complete access to the lattice dynamics. However, such measurements are experimentally costly, and in many cases full information is not required to explore the properties of interest. In this section, we outline how experimental measurements can access LFSL observables such as site populations, correlations, and currents. For concreteness, we focus on a single bosonic mode, relevant for cavity or cir-

cuit QED and trapped-ion systems, although generalization to higher-dimensional systems is straightforward.

At time  $t$ , the system is assumed to be in the state  $\hat{\rho}(t)$ . Full quantum-state tomography [87]—demonstrated in cavity QED [88–91], circuit QED [92–95], nonlinear optics [96], and trapped ions [97–100]—provides complete information, such as the Wigner function, from which lattice “populations” can be extracted. Here, we focus only on the partial information required to extract the quantities of interest.

Working in the Fock basis, access to the coefficient  $\rho_{n,m} = \langle n | \hat{\rho} | m \rangle = \langle n, m | \rho \rangle$  for lattice site  $(n, m)$  corresponds to measuring

$$\hat{O}_{nm}(\phi) = e^{i\phi} |n\rangle\langle m| + e^{-i\phi} |m\rangle\langle n|, \quad (49)$$

which yields  $\text{Re}(\rho_{n,m})$  for  $\phi = 0$  and  $\text{Im}(\rho_{n,m})$  for  $\phi = \pi/2$ . The operator  $\hat{O}_{nm}$  has the structure of a Pauli  $\hat{\sigma}_x$  acting in the  $\{|n\rangle, |m\rangle\}$  subspace, suggesting a measurement strategy in which this subspace is mapped onto an ancilla qubit, enabling efficient tomography.

A straightforward way to achieve this is by driving a sideband transition. Specifically, the ancilla qubit, with internal states  $|g\rangle$  and  $|e\rangle$ , is driven resonantly on the  $n \leftrightarrow m$  multiphoton transition [101]. In this way, the two target Fock states are coupled via the qubit in a Rabi-like manner,

$$\hat{H}_{\text{int}} = \frac{\Omega_{nm}}{2} (|n, e\rangle\langle m, g| + |m, g\rangle\langle n, e|), \quad (50)$$

where  $\Omega_{nm}$  is the effective Rabi frequency. The sideband drive assumes that all other transitions are largely detuned and therefore do not affect the ancilla qubit. If the interaction time is chosen appropriately, this Hamiltonian realizes the unitary

$$\hat{U} = \exp \left[ -i \frac{\pi}{4} (|n, e\rangle\langle m, g| + \text{H.c.}) \right], \quad (51)$$

which performs a  $\pi/2$  rotation, swapping the amplitudes onto the qubit. Measuring the qubit’s Bloch vector then yields the desired populations via

$$\langle \hat{\sigma}_x \rangle = \text{Re}(\rho_{n,m}), \quad \langle \hat{\sigma}_y \rangle = \text{Im}(\rho_{n,m}).$$

In the LFSL picture, this corresponds to probing the “population” of the lattice site  $(m, n)$ . Furthermore, tracking the motion of the Bloch vector directly reveals the current through the corresponding lattice link. Given  $\rho_{n,m}$ , one can define the local “order parameter”  $\theta_{n,m}$  at that site, which allows construction of measures of frustration, for example,

$$F(t) = \sum_{\text{loops}} \left| \sum_{i,j \in \text{loop}} \cos(\theta_i(t) - \theta_j(t)) \right|. \quad (52)$$

A large value of  $F(t)$  indicates that the system struggles to locally satisfy the constraints and, moreover, it is expected to be sensitive to fluctuations in the initial states.

For nonlocal observables, measurements of operators such as  $\hat{a}$  or  $\hat{n}$  give weighted sums over lattice currents, for example,

$$\langle \hat{a} \rangle = \sum_n \sqrt{n+1} \rho_{n,n+1}, \quad \langle \hat{n} \rangle = \sum_n n \rho_{n,n}.$$

The above represents just one possible realization. One can also borrow ideas from full state tomography and im-

plement only partial reconstructions when full information is not required. Alternatives include *quantum nondemolition* schemes [102–104], or the use of the *Selective Number-dependent Arbitrary Phase* gate [105] combined with displacements (as in conventional tomography).

## VII. CONCLUSION

In summary, we have introduced LFSs as an extension of FSLs to open quantum systems. By vectorizing the Lindblad master equation, we map the dynamics onto an extended Hilbert space that naturally forms a synthetic lattice with distinct structural and dynamical properties. Unlike conventional FSLs, LFSs are governed by non-Hermitian dynamics, leading to population drifts, sources, and sinks, and requiring a reinterpretation of state vectors beyond probability amplitudes. Moreover, the role of symmetries in LFSs deviates from Noether's theorem in closed systems, fundamentally altering their geometric structure.

Given these unique features, we suggest that LFSs provide a promising framework for exploring nonequilibrium phenomena such as anomalous transport in stochastic classical lattice models. A detailed analysis of such applications is left for future work. Instead, we have demonstrated two examples of a phenomenon we term environment-induced frustration. Geometric frustration arises from competing tunneling processes around closed loops in the lattice. While one-dimensional tight-binding lattices lack such loops, their Liouville-space counterparts are effectively two-dimensional, allowing these frustrated configurations to appear. In contrast to Hamiltonian systems, where frustration manifests as a massively degenerate ground state, the Liouvillian case leads to a highly degenerate steady-state manifold. Exploring frustration in quantum-optical systems with synthetic dimensions thus offers an experimentally accessible platform for studying such effects. We have outlined how relevant observables, such as site populations and currents, can be extracted experimentally.

Beyond these examples, LFSs open several promising directions. They provide a framework for studying open quantum systems and, akin to FSLs, allow the incorporation of topological concepts in the extended space. The physical meaning of the state vectors  $|\rho\rangle\rangle$ —and the possible interpretation of interference at the Liouville-space level—remains an open question. Another compelling avenue concerns universality in nonequilibrium critical models: Just as dimensionality governs universal properties in Hamiltonian systems, the dimensionality of LFSs may play an analogous role for criticality in open quantum dynamics.

## ACKNOWLEDGMENT

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## DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

## APPENDIX: ALTERNATIVE REPRESENTATIONS

The vectorized representation within a Fock basis offers several advantages, such as its experimental relevance and the ease with which LFSs can be extracted. However, it also presents disadvantages, particularly in the simulation of other lattice models. Since the elements  $\rho_{n,m}$  of  $|\rho\rangle\rangle$  represent populations in the lattice, it would be preferable if they were both real and non-negative. Here, we give more details on the alternative representations that fulfill one of these two requirements, as described in the main text, that is, the Bloch vector and SIC-POVM representations

### 1. The Bloch vector representation

A straightforward way to transition from the Liouville-Fock representation to a real-valued representation is by utilizing the hermiticity of  $\hat{\rho}$ . Specifically, the combinations

$$\rho_{n,m} + \rho_{m,n} \quad \text{and} \quad i(\rho_{n,m} - \rho_{m,n}) \quad (\text{A1})$$

are both real quantities. This transformation defines the *Bloch representation*, which ensures that the density matrix elements are expressed in terms of real numbers.

It is known that the density matrix of any finite-dimensional system can be expanded in terms of the generalized Gell-Mann matrices  $\hat{\lambda}$  [80]. As already given in the main text, this expansion takes the form

$$\rho = \frac{1}{d} \left( \mathbb{I} + \sqrt{\frac{d(d-1)}{2}} \vec{R} \cdot \vec{\lambda} \right), \quad (\text{A2})$$

where  $\vec{R} = (R_1, \dots, R_{d^2-1})$  is the Bloch vector. Applying the bra-flipper operator, one obtains the vectorized form

$$|\rho\rangle\rangle = \sum_{j=0}^{d^2-1} \tilde{R}_j |\Lambda_j\rangle\rangle, \quad (\text{A3})$$

with the coefficients

$$\tilde{R}_0 = \frac{1}{\sqrt{d}}, \quad \tilde{R}_j = \frac{\beta}{d} R_j \quad \text{for } j \neq 0. \quad (\text{A4})$$

Here, we define the superkets as  $|\Lambda_0\rangle\rangle = |\lambda_0\rangle\rangle/\sqrt{d}$  and  $|\Lambda_i\rangle\rangle = |\lambda_i\rangle\rangle/\sqrt{2}$  for  $i \neq 0$ , where  $\lambda_0 = \mathbb{I}$  is the identity matrix. The (generalized) Bloch vector  $\vec{R} = (\tilde{R}_0, \dots, \tilde{R}_{d^2-1})^T$  evolves in time according to

$$\frac{d\vec{R}}{dt} = \tilde{\mathcal{L}}^A \vec{R}, \quad (\text{A5})$$

where the transformed Liouvillian is given by

$$\tilde{\mathcal{L}}^A = V \tilde{\mathcal{L}} V^\dagger, \quad \text{with} \quad V = \sum_{i=0}^{d^2-1} |i\rangle\rangle \langle\langle \Lambda_i|. \quad (\text{A6})$$

That is,  $\tilde{\mathcal{L}}^A$  is obtained via a basis transformation of the Liouvillian matrix. Consequently, given the Liouvillian  $\tilde{\mathcal{L}}$  in the Fock basis, it is, in principle, straightforward to compute  $\tilde{\mathcal{L}}^A$ , from which the Bloch-representation lattice can be extracted.

## 2. The Urgleichung—SIC-POVM representation

The third representation addressed by this work is the SIC-POVMs [82]. This approach has the distinct advantage of being both real and non-negative, thereby possessing the desired properties of a probability distribution. However, as noted in the main text, the matrix governing time evolution exhibits properties that are not typically found in classical stochastic master equations—most notably, the presence of negative transition rates. Moreover, in higher dimensions  $d$ , extracting the distribution and corresponding lattice become extremely difficult [82].

SIC-POVMs are defined through a complete set of  $d^2$  projectors

$$\hat{\Pi}_i = (1/d)|\psi_i\rangle\langle\psi_i|, \quad (\text{A7})$$

such that  $\sum_{i=1}^{d^2} \hat{\Pi}_i = \mathbb{I}$ , with the distinct feature of having a constant overlap  $\text{Tr}(\hat{\Pi}_i \hat{\Pi}_j) = (1/d^2)(d \delta_{i,j} + 1)/(d + 1)$ . This property makes them interesting for quantum state tomography [106,107] as it optimizes the number of measurements needed for the task. One can show that the density matrix can be expanded as [82]

$$|\rho\rangle\rangle = \sum_{i=1}^{d^2} (d(d+1)p_i - 1)|\Pi_i\rangle\rangle, \quad (\text{A8})$$

with  $|\Pi_i\rangle\rangle$  being the vectorized elements of the POVM and  $p_i = \text{Tr}(\rho \Pi_i) \geq 0$  the probability (i.e.,  $\sum_i p_i = 1$ ) to find the system in the state  $|\Pi_i\rangle\rangle$ . By using this expansion and projecting the evolution equation (5) onto the SIC-POVM basis, we derive an inhomogeneous differential evolution equation for the probability vector  $\vec{p} = (p_1, \dots, p_{d^2})$

$$\frac{d\vec{p}}{dt} = d(d+1)M\vec{p} - \vec{m}, \quad (\text{A9})$$

where  $M$  is a matrix whose elements are  $M_{i,j} = \langle\langle \Pi_i | \tilde{\mathcal{L}} | \Pi_j \rangle\rangle$  and the components of the vector  $\vec{m}$ ,  $m_i = \sum_j M_{i,j}$ . The similarity to a classical stochastic evolution equation with transition matrix  $M$  and a source/sink term  $\vec{m}$  is noticeable; however, there are crucial differences that we discuss in the following.

In a classical stochastic equation, probability conservation is ensured by the constraint on the time-evolution generator,  $\sum_i M_{i,j} = 0$ , where  $M_{i,j}$  are the matrix elements of  $M$ . That this property holds also here becomes evident by rewriting Eq. (A9) in terms of the components of  $\vec{p}$

$$\frac{dp_i}{dt} = \sum_{j=0}^{d^2} [d(d+1)p_j - 1]M_{i,j}. \quad (\text{A10})$$

Summing over  $i$  confirms that  $\sum_i M_{i,j} = 0$ . An alternative way to prove this can be done through the completeness of the projectors in the vectorized formalism. Noting that the identity operator is always conserved under Liouvillian evolution,  $\tilde{\mathcal{L}}^\dagger|\mathbb{I}\rangle\rangle = 0$  [66], it follows that

$$\sum_i M_{i,j} = \sum_i \langle\langle \Pi_i | \tilde{\mathcal{L}} | \Pi_j \rangle\rangle = \langle\langle \mathbb{I} | \tilde{\mathcal{L}} | \Pi_j \rangle\rangle = 0. \quad (\text{A11})$$

The crucial difference between  $M$  and a stochastic matrix is the fact that it can have negative off-diagonal entries. This is

an essential feature for an evolution that displays a periodic time-asymptotic behavior rather than a steady-state one, as we will exemplify later.

The inhomogeneous term, represented by the vector  $\vec{m}$ , given that the elements of  $M$  can be negative, acts as a source or sink term for the components of the probability vector. Thus, it plays a crucial role in shaping the time-asymptotic behavior of the system. If the matrix  $M$  is invertible, a unique steady-state solution exists and is given by

$$\vec{p}_{ss} = \frac{M^{-1}\vec{m}}{d(d+1)}. \quad (\text{A12})$$

However, invertibility is not guaranteed, since  $M$  is not an Hermitian matrix in general

$$M_{i,j}^* = (\langle\langle \Pi_i | \tilde{\mathcal{L}} | \Pi_j \rangle\rangle)^\dagger = \langle\langle \Pi_j | \tilde{\mathcal{L}}^\dagger | \Pi_i \rangle\rangle \neq M_{j,i}. \quad (\text{A13})$$

Then, one has that the steady-state probability distribution has to satisfy

$$p_i^{ss} = \frac{m_i - \sum_{j \neq i} d(d+1)M_{i,j}p_j^{ss}}{d(d+1)M_{i,i}}. \quad (\text{A14})$$

We can envision the case where  $\vec{m} = 0$ . For that to be true, we have that

$$m_i = \sum_j M_{i,j} = \langle\langle \Pi_i | \tilde{\mathcal{L}} | \Pi_j \rangle\rangle = \langle\langle \Pi_i | \tilde{\mathcal{L}} | \mathbb{I} \rangle\rangle = 0, \quad (\text{A15})$$

where we have used the completeness property of the SIC-POVMs. This equation implies that the source term is zero when the maximally mixed state is a steady state of the dynamics, since then  $\tilde{\mathcal{L}}|\mathbb{I}\rangle\rangle = 0$ . Interestingly, this aligns with the condition for a classical stochastic process to converge to a uniform probability distribution as its steady state [83], despite  $M$  not necessarily being symmetric. From the definition of probabilities in the SIC-POVM representation,  $p_i = \text{Tr}(\Pi_i \rho)$ , and using the fact that  $\text{Tr}(\Pi_i) = 1/d$ , it follows that for the maximally mixed state  $\rho = (1/d)\mathbb{I}$ , we have

$$p_i = \frac{1}{d^2}, \quad \forall i. \quad (\text{A16})$$

This demonstrates that the probability distribution associated with the maximally mixed state is uniformly distributed, similar to the classical case.

The above observation is merely a consequence of the correspondence between steady states in the present LME framework and those in SIC-POVM representation. Let  $\{|\rho_i\rangle\rangle\}$  be the set of right eigenvectors of the Liouvillian, i.e.,

$$\tilde{\mathcal{L}}|\rho_i\rangle\rangle = \lambda_i|\rho_i\rangle\rangle, \quad \lambda_i \in \mathbb{C}. \quad (\text{A17})$$

Furthermore, let each eigenvector be expanded in the SIC-POVM basis as

$$|\rho_i\rangle\rangle = \sum_j (d(d+1)p_j^{(i)} - 1)|\Pi_j\rangle\rangle. \quad (\text{A18})$$

Applying the Liouvillian to this expansion and using the eigenvalue equation, we obtain

$$\tilde{\mathcal{L}}|\rho_i\rangle\rangle = \sum_j (d(d+1)p_j^{(i)} - 1)\tilde{\mathcal{L}}|\Pi_j\rangle\rangle = \lambda_i|\rho_i\rangle\rangle$$

$\Rightarrow$

$$\frac{dp_k^{(i)}}{dt} = \sum_j (d(d+1)p_j^{(i)} - 1)M_{k,j} = \lambda_i p_k^{(i)}.$$

Thus, we see that if  $\lambda_i = 0$  the corresponding probability vector to the right eigenvector is a steady state of the SIC-POVM evolution.

It is also possible that one gets a periodic state in the time-asymptotic limit. This is easily seen by considering a Liouvillian whose spectrum has purely imaginary eigenvalues, that is,

$$\bar{\mathcal{L}}|\rho_i\rangle\rangle = i\lambda_i|\rho_i\rangle\rangle. \quad (\text{A19})$$

Hence, a general asymptotic state is given by

$$\begin{aligned} |\rho^{as}(t)\rangle\rangle &= \sum_n \rho_n(0)|\rho_n^{(0)}\rangle\rangle + \sum_i \rho_i(0)e^{i\lambda_i t}|\rho_i^+\rangle\rangle \\ &+ \rho_i(0)^*e^{-i\lambda_i t}|\rho_i^-\rangle\rangle, \end{aligned} \quad (\text{A20})$$

whose corresponding probability vector in the SIC-POVM representation evolves in time since  $(d/dt)\text{Tr}(\Pi_i\rho^{as}(t)) \neq 0$ .

As an example, we examine an SIC-POVM representation of the evolution of a qubit undergoing spontaneous decay. The Liouvillian consists of the following Hamiltonian and jump operator:

$$\hat{H} = \frac{\Omega}{2}\hat{\sigma}_z + g\hat{\sigma}_x, \quad \hat{L} = \hat{\sigma}^-, \quad (\text{A21})$$

where  $\hat{\sigma}_{x,z}$  are the Pauli matrices ( $\hat{\sigma}^-$  is the lowering operator),  $\Omega$  is the energy separation between the states  $|0\rangle$  and  $|1\rangle$ , and  $g$  is the coupling between them. Vectorizing the Liouvillian yields the matrix

$$\bar{\mathcal{L}} = \begin{pmatrix} -\gamma & ig & -ig & 0 \\ ig & (-\gamma/2 - i\Omega) & 0 & -ig \\ -ig & 0 & (-\gamma/2 + i\Omega) & ig \\ \gamma & -ig & ig & 0 \end{pmatrix}. \quad (\text{A22})$$

To proceed, we choose an SIC-POVM set. In this case, we select the one represented by the states

$$|\psi_1\rangle = |0\rangle, \quad |\psi_2\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle, \quad (\text{A23})$$

$$|\psi_3\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{i2\pi/3}|1\rangle, \quad (\text{A24})$$

$$|\psi_4\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{i4\pi/3}|1\rangle, \quad (\text{A25})$$

where the corresponding POVM elements are given by

$$\Pi_i = \frac{1}{2}|\psi_i\rangle\langle\psi_i|, \quad i = 1, 2, 3, 4. \quad (\text{A26})$$

Using the matrix elements  $\langle\langle\Pi_i|\bar{\mathcal{L}}|\Pi_j\rangle\rangle$ , we find, after some laborious calculations, that

$$M = \begin{pmatrix} -\gamma/4 & -\gamma/12 & \frac{-\gamma+\sqrt{6}g}{12} & \frac{-\gamma-\sqrt{6}g}{12} \\ \gamma/12 & -\gamma/36 & \frac{2\gamma-2\sqrt{3}\Omega-\sqrt{6}g}{36} & \frac{2\gamma+2\sqrt{3}\Omega+\sqrt{6}g}{36} \\ \frac{\gamma-\sqrt{6}g}{12} & \frac{2\gamma+2\sqrt{3}\Omega+\sqrt{6}g}{36} & -\gamma/36 & \frac{\gamma-\sqrt{3}\Omega+\sqrt{6}g}{18} \\ \frac{\gamma+\sqrt{6}g}{12} & \frac{2\gamma-2\sqrt{3}\Omega-\sqrt{6}g}{36} & \frac{\gamma+\sqrt{3}\Omega-\sqrt{6}g}{18} & -\gamma/36 \end{pmatrix} \quad (\text{A27})$$

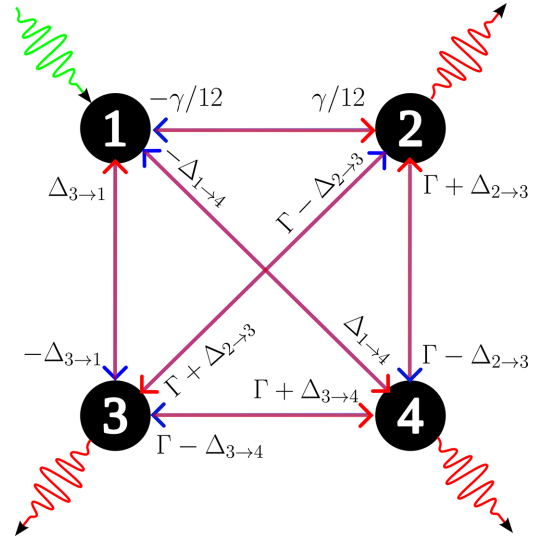


FIG. 9. A schematic plot of the sites associated with the probabilities in the SIC-POVM representation of a qubit and the transitions between them, as given by the matrix  $M$  [Eq. (A27)]. These transitions are generated by the Lindblad open evolution with the Hamiltonian and jump operator given in Eq. (A21). The green “photonlike” arrow represents a constant input at site 1, while the red arrows indicate constant output/decay for the remaining sites. The red arrows illustrate positive transition amplitudes, while blue represent negative ones. The transition amplitudes are given by  $\Gamma = \gamma/18$ ,  $\Delta_{1 \rightarrow 4} = (\gamma + \sqrt{6})/12$ ,  $\Delta_{3 \rightarrow 1} = (-\gamma + \sqrt{6}g)/12$ ,  $\Delta_{2 \rightarrow 3} = (2\sqrt{3} + \sqrt{6}g)/36$ , and  $\Delta_{3 \rightarrow 4} = \sqrt{3}\Omega - \sqrt{6}g$ .

and that  $\vec{m}$  is given by

$$\vec{m} = (-\gamma/2, \gamma/6, \gamma/6, \gamma/6)^T. \quad (\text{A28})$$

One can verify that  $M$  satisfies the probability conservation condition [Eq. (A11)].

As the generator of evolution, we observe that, in addition to having negative diagonal entries—similar to classical stochastic matrices—negative off-diagonal elements can also appear, depending on the relative magnitudes of  $\gamma$ ,  $\Omega$ , and  $g$ . This signifies a nonreciprocal interaction between the two states in the process, where one site can suppress transitions from other sites to itself. Furthermore, the components of  $\vec{m}$  represent an on-site pump for site 1 and on-site losses for the other sites. In Fig. 9, we provide a graphical representation of this process.

To understand the consequences of having negative transition amplitudes, let us first analyze the case where  $\gamma = 0$ . Due to the noncommuting terms in the Hamiltonian, one expects periodic evolution rather than a steady state in the long-time limit. That is, the probabilities on each site oscillate around an average value—manifesting as Rabi oscillations in this model.

First, we observe that the sites 1 and 2 are disconnected. Second, the negative transition rates  $\Delta_{4 \rightarrow 1} = \Delta_{1 \rightarrow 3} = -(\sqrt{6}/12)g$  and  $\Delta_{3 \rightarrow 2} = \Delta_{2 \rightarrow 4} = -(2\sqrt{3}\Omega + \sqrt{6}g)/36$  favor the cyclic path  $1 \rightarrow 4 \rightarrow 2 \rightarrow 3 \rightarrow 1$  over the reverse direction. Furthermore, if  $\Omega > \sqrt{2}g$ , another cyclic path emerges:  $2 \rightarrow 3 \rightarrow 4 \rightarrow 2$ . Conversely, if  $\Omega < \sqrt{2}g$ , the preferred cycle becomes  $1 \rightarrow 4 \rightarrow 3 \rightarrow 1$ . We emphasize that

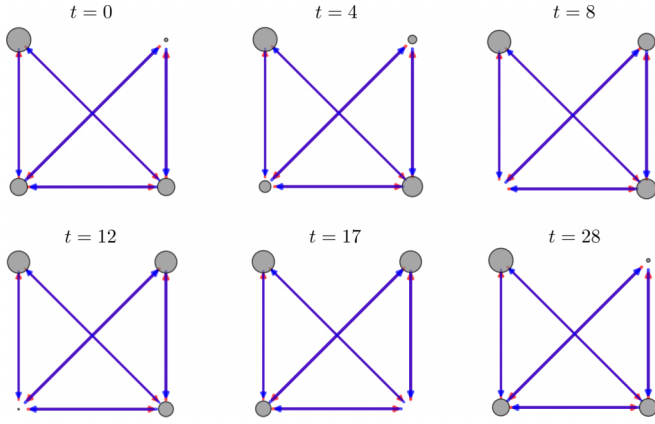


FIG. 10. Snapshots of the evolution of the SIC-POVM probability vector, generated by the Liouvillian in Eq. (A22), for  $\gamma = 0$  with the initial state  $|0\rangle$ . The red directed arrows represent edges with positive transition amplitudes, while the blue edges indicate those with negative transition amplitudes. The size of each site is proportional to the probability of the system occupying the corresponding state.

the existence of a periodic state is a direct consequence of negative transition amplitudes. Figure 10 presents snapshots of the probability evolution in the lattice.

If  $\gamma \neq 0$ , we expect the system to reach a steady state rather than exhibiting sustained periodic behavior. The steady-state distribution must satisfy Eq. (A14), meaning that the probability flow due to on-site terms is balanced by transitions between sites. Considering the case where  $\Delta_{3 \rightarrow 1} < 0$ , i.e.,  $\gamma > \sqrt{6}g$ , the transition amplitude from site 3 to site 1 is suppressed. Consequently, the only positive probability flow into site 1 comes from the on-site term. The balance equation then implies that, in general, a nonzero probability of occupying site 1—corresponding to the projector on state  $|0\rangle$ —is achieved. This aligns with the intuition that the jump operator tends to drive the spin down, while the Hamiltonian term counteracts by driving the spin up. However, in the limit where  $\gamma \gg g$ ,  $\Omega$ , dissipation dominates over coherent driving, effectively emptying state 1.

$$\tilde{\mathcal{L}}|\rho_{ss}\rangle\rangle = \begin{pmatrix} -\gamma & ig \\ ig & (-\gamma/2 - i\Omega) \\ -ig & 0 \\ \gamma & -ig \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0.$$

Solving this system and normalizing  $|\rho_{ss}\rangle\rangle$  with respect to the vectorized identity, we obtain

$$|\rho_{ss}\rangle\rangle = \frac{1}{2g^2 + \gamma^2/4 + \Omega^2} \begin{pmatrix} g^2 \\ (-\Omega + i\gamma/2)g \\ (-\Omega - i\gamma/2)g \\ g^2 + \gamma^2/4 + \Omega^2 \end{pmatrix}. \quad (\text{A31})$$

Notably, we confirm the expected result when  $g = 0$ , in which case the steady state simplifies to  $|\rho_{ss}\rangle\rangle = |1, 1\rangle$ . Now, by using the definition of the SIC-POVM representation prob-

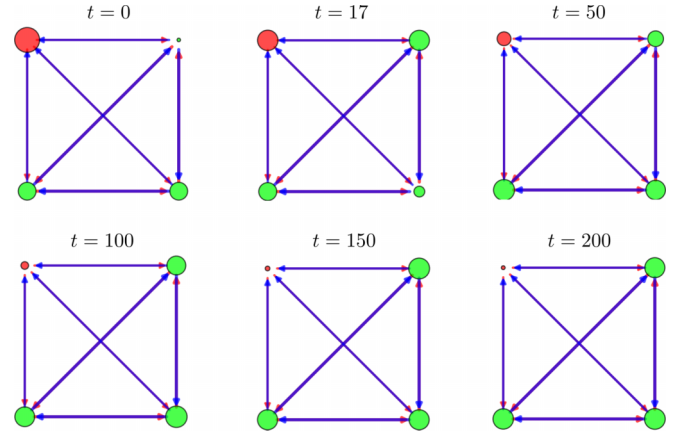


FIG. 11. Snapshots of the evolution of the SIC-POVM probability vector generated by the Liouvillian in Eq. (A22) with parameters  $\Omega = 1$ ,  $g = 10^{-2}$ ,  $\gamma = 10^{-1}$ , and initial state  $|0\rangle$ . Red directed arrows (partially hidden beneath the blue arrows) indicate edges with positive transition amplitudes, while blue edges indicate negative amplitudes. The size of each site is proportional to the probability of the system occupying that state. Red arrows represent an input of probability from the background, and green arrows represent an output to the background, with magnitudes determined by the components of  $\tilde{m}$ , as defined in Eq. (A28).

Using Eq. (A14) for  $i = 1$  and approximating  $-\gamma \pm \sqrt{6}g \approx -\gamma$ , we obtain

$$p_1^{ss} \approx \frac{\gamma/2 - \gamma/2(p_2^{ss} + p_3^{ss} + p_4^{ss})}{\gamma/4} \Leftrightarrow p_1^{ss} \approx 2p_1^{ss} \Leftrightarrow p_1^{ss} = 0. \quad (\text{A29})$$

Following a similar reasoning, we can estimate the remaining steady-state probabilities in this regime. A similar calculation shows that  $p_2^{ss} = p_3^{ss} = p_4^{ss} = 1/3$ .

The steady-state solution can be found analytically from the steady-state density matrix, as mentioned earlier. To obtain the steady-state density matrix, we seek the null space of the Liouvillian matrix given in Eq. (A22), namely,

$$\begin{pmatrix} -ig & 0 \\ 0 & -ig \\ (-\gamma/2 + i\Omega) & ig \\ ig & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0. \quad (\text{A30})$$

abilities we obtain  $p_i^{ss} = \text{Tr}(\Pi_i \rho_{ss})$ , and the steady-state solution of the SIC-POVM evolution becomes

$$p_1^{ss} = \frac{1}{2} \frac{g^2}{2g^2 + \gamma^2/4 + \Omega^2}, \quad (\text{A32})$$

$$p_2^{ss} = \frac{1}{6} \frac{(3g^2 + \gamma^2 + 2\Omega^2 - 2\sqrt{2}\Omega g)}{2g^2 + \gamma^2/4 + \Omega^2}, \quad (\text{A33})$$

$$p_3^{ss} = \frac{1}{6} \frac{(3g^2 + \gamma^2 + 2\Omega^2 + \sqrt{2}\Omega g - (\sqrt{6}/2)\gamma g)}{2g^2 + \gamma^2/4 + \Omega^2}, \quad (\text{A34})$$

$$p_4^{ss} = \frac{1}{6} \frac{(3g^2 + \gamma^2 + 2\Omega^2 + \sqrt{2}\Omega g + (\sqrt{6}/2)\gamma g)}{2g^2 + \gamma^2/4 + \Omega^2}. \quad (\text{A35})$$

To address how the relative magnitudes of the parameters affect the lattice dynamics when  $\gamma \neq 0$ , we first note that the cyclic path  $2 \rightarrow 3 \rightarrow 4 \rightarrow 2$  may appear in the transient regime if  $\gamma - (\sqrt{6}/2)g < \sqrt{3}\Omega$ . Next, considering the case where  $\Delta_{3 \rightarrow 1} > 0$ , the cyclic path  $1 \rightarrow 4 \rightarrow 3 \rightarrow 1$  exists in the transient regime if

$$-\gamma + \sqrt{6}g < \sqrt{3}\Omega < \gamma + \sqrt{6}g. \quad (\text{A36})$$

The cyclic path present under closed evolution,  $1 \rightarrow 4 \rightarrow 2 \rightarrow 3 \rightarrow 1$ , exists if  $\Delta_{2 \rightarrow 3} > \Gamma$ , which translates to the condition  $\gamma < (\sqrt{6}/2)g$ . Finally, the cyclic path  $2 \rightarrow 3 \rightarrow 4 \rightarrow 2$  is also possible if  $\sqrt{3}\Omega > \gamma + \sqrt{6}g$ .

In Fig. 11, we show the probability distribution evolution at selected time steps for the case  $\gamma > \sqrt{6}g$  and  $\sqrt{3}\Omega > \gamma + \sqrt{6}g$ .

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