

Hall effect in topologically trivial isolated flat-band systems

Raigo Nagashima^{1,*}, Masao Ogata^{1,2} and Naoto Tsuji^{1,3}¹Department of Physics, The University of Tokyo, Hongo, Tokyo 113-8656, Japan²Trans-Scale Quantum Science Institute, The University of Tokyo, Hongo, Tokyo 113-8656, Japan³RIKEN Center for Emergent Matter Science (CEMS), Wako 351-0198, Japan

(Received 22 July 2025; accepted 30 January 2026; published 23 February 2026)

We study the Hall effect in topologically trivial isolated flat-band systems (i.e., flat bands are separated from other bands and have zero Chern number) for a weak magnetic field. In a naive semiclassical picture, the Hall conductivity vanishes when dispersive bands are unoccupied, since there are no mobile carriers. To go beyond the semiclassical picture, we establish a fully quantum mechanical gauge-invariant formula for the Hall conductivity that can be applied to any lattice models. We apply the formula to a general $(N + M)$ -band model with N dispersive bands and M -fold degenerate isolated flat bands, and find that when the dispersive bands are unoccupied, the total conductivity takes a universal form consisting of the energy difference between the dispersive and flat bands, and the non-Abelian quantum geometric tensor of the flat bands, which can be nonzero in systems with vanishing Berry curvature. We numerically confirm the Hall effect for isolated flat-band lattice models on the honeycomb lattice ($N = M = 1$) and two different kagome lattices ($N = 2, M = 1$ and $N = 1, M = 2$).

DOI: 10.1103/5s6l-vc71

Introduction. Hall effect is a hallmark of transport phenomena [1] that can bring us essential information on charge carriers in materials, including their density, mass, the sign of charges, and scattering properties. At weak magnetic fields, one can often adopt a semiclassical transport theory, in which the Hall conductivity is given by [2,3]

$$\sigma_{xy} = \frac{\sigma_0 \omega_c \tau}{1 + (\omega_c \tau)^2}, \quad (1)$$

where $\sigma_0 = ne^2\tau/m$, $\omega_c = eB/m$ is the cyclotron frequency, n , e (>0), and m are the density, charge, and mass of carriers, B is the magnetic-field strength, and τ is the relaxation time. If one naively takes the large mass limit ($m \rightarrow \infty$), which corresponds to a flat-band limit, the Hall conductivity vanishes. A similar result is expected in a high magnetic field in two dimensions, since a Chern number (which determines the quantized Hall conductance) must be zero in exactly flat-band lattice systems [4]. Our natural question here is whether the Hall conductivity always vanishes in flat-band systems and, if not, whether we can find any fingerprints of genuine quantum effects in the Hall conductivity that can survive even in a weak magnetic-field regime.

Flat-band systems have attracted much attention for a long time, in part because the kinetic energy of carriers is suppressed, and the interaction dominates overall physics. In this context, flat-band systems have been discussed as a platform

of, e.g., ferromagnetism [5–9], fractional Chern insulators (for nearly flat bands) [10–14], quantum anomalous Hall effects on flat bands [15,16], quantum geometric effects with quadratic band touching [17], and flat-band superconductivity [18–23]. Flat-band systems have been realized not only in crystalline solids [24–26] such as kagome [27], pyrochlore [28], and moiré [29–32] materials, but also in magnon flat bands [33–35], photonic lattices [36,37], phononic crystals [38], and metal-organic frameworks [39] (see also Refs. [40,41]).

In this Letter, we study the Hall effect in isolated topologically trivial flat-band lattice systems [i.e., systems in which exactly flat bands are separated from other bands and have zero Chern number; see Figs. 1(d)–1(f)] in a weak magnetic field. We assume that the system has time-reversal symmetry when there is no magnetic field. While the Hall conductivity can be evaluated quantum mechanically from the standard Kubo formula, it is not trivial to extract its weak magnetic-field limit on a lattice in a gauge-invariant way [44]. Previously, the weak-field Hall conductivity has been obtained in a continuum space [45–47]. Here, we derive a fully quantum mechanical gauge-invariant formula for the Hall conductivity that is applicable to any lattice models. On the other hand, the formula for the orbital magnetic susceptibility has been discussed since Landau and Peierls for a continuum space [48–51], and recently an exact relation between the Hall conductivity and the orbital magnetic susceptibility has been found for a continuum space [52]. In this Letter, we establish the corresponding exact relation for lattice systems. We note that the longitudinal conductivity of flat-band systems has been explored (see, e.g., Refs. [53–55]).

We apply the formula to a general $(N + M)$ -band model with N dispersive and M -fold degenerate isolated flat bands. The flat bands are topologically trivial (the Chern number is zero). When the chemical potential is close to the isolated

*Contact author: raigo.nagashima@kit.edu

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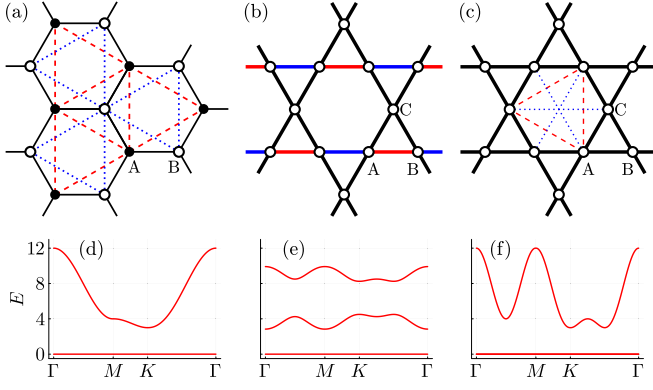


FIG. 1. (a) Gapped honeycomb lattice model of nearest-neighbor and next-nearest-neighbor hoppings (black, red dashed, and blue dotted lines correspond to the hopping strength $\sqrt{3}t$, t , and 0), having one isolated flat band at $E = 0$ [18]. (b) Gapped kagome lattice model with nearest-neighbor hoppings of different strengths (black, red, and blue lines correspond to the hopping strengths t , αt , t/α ; $\alpha = 2$), having one isolated flat band at $E = 0$ [42]. (c) Kagome-3 lattice model with nearest-neighbor, next-nearest-neighbor, and next-next-nearest-neighbor hoppings with the same amplitude t , having twofold degenerate isolated flat bands at $E = 0$ [43]. (d)–(f) Band structures of the models of panels (a), (b), and (c), respectively. Energy is measured in units of t .

flat band(s), we find that the Hall conductivity takes a finite value proportional to the inverse of the relaxation rate. This response takes a universal form consisting of the energy difference between the dispersive and isolated flat bands and the non-Abelian quantum geometric tensor [56,57] of the degenerate isolated flat bands. The finite Hall conductivity is attributed to the effect of interband transitions, as expressed through the non-Abelian quantum geometric tensor. These results are demonstrated for a gapped honeycomb lattice [$N = M = 1$: Fig. 1(a)] [18], gapped kagome lattice [$N = 2$, $M = 1$: Fig. 1(b)] [42], and kagome-3 lattice [$N = 1$, $M = 2$: Fig. 1(c)] [43] as concrete examples. The last example has zero Berry curvature at any point in the Brillouin zone. We also study the behavior of the Hall conductivity of the gapped honeycomb lattice when the isolated flat band is turned into a dispersive band.

Derivation of the formula. We consider a general electron system in the presence of a magnetic field B along the z axis. We neglect spin-orbit couplings and Zeeman effects [58]. Our argument can be applied to weakly disordered systems, where Anderson localization does not occur. The Hall conductivity σ_{xy} can be calculated from the Kubo formula as

$$\sigma_{xy} = \lim_{\omega \rightarrow 0} \frac{e^2}{i\omega} (\Pi_{xy}^R(\omega) - \Pi_{xy}^R(0)), \quad (2)$$

where $\Pi_{xy}^R(\omega)$ is an analytically continued ($i\omega_\lambda \rightarrow \hbar\omega + i0^+$) current-current correlation function $\Pi_{xy}(i\omega_\lambda)$ with $\omega_\lambda = 2\pi\lambda k_B T$ being the Matsubara frequency:

$$\Pi_{xy}(i\omega_\lambda) = \frac{1}{\beta V} \sum_{n,k,\sigma} \text{Tr}[v_{k,Bx} G_{k,B+} v_{k,Bx} G_{k,B-}]. \quad (3)$$

Here, $\beta = 1/k_B T$ (T is the temperature), V is the volume of the system, and $v_{k,Bv}$ ($v = x, y$) and $G_{k,B(\pm)}$ are the

velocity operator and the Matsubara Green's function in the presence of the magnetic field B , respectively. We take the summation over n for the fermionic Matsubara frequency $\varepsilon_n = (2n + 1)\pi k_B T$ ($n \in \mathbb{Z}$), and the trace is taken over all the Bloch bands. We abbreviate $G_B(\mathbf{k}, i\varepsilon_n)$ and $G_B(\mathbf{k}, i\varepsilon_n + i\omega_\lambda)$ as $G_{k,B}$ and $G_{k,B+}$, respectively. Since we are interested in the weak field limit ($B \rightarrow 0$), we take the terms in the order of B^1 in the velocity operator and the Green's function [59].

We begin with the real-space one-body Hamiltonian H without a magnetic field: $H = \sum_{jk} t_{jk} |j\rangle \langle k|$, where j and k are site indices and t_{jk} is assumed to be real. The system preserves the time-reversal symmetry before applying the magnetic field B . We include the effect of B by the Peierls substitution: $H \rightarrow H_B = \sum_{jk} t_{jk} e^{i\varphi_{jk}} |j\rangle \langle k|$ with $\varphi_{jk} = -(e/\hbar) \int_j^k \mathbf{A}(\mathbf{r}) \cdot d\mathbf{l}$, where $\mathbf{A}$ is a vector potential of a static magnetic field. To guarantee the gauge invariance of the formula, we expand the correlation function in terms of B (not $\mathbf{A}$) with a trick used in Refs. [60,61]. This is achieved by using the fact that the trace of the correlation function always yields a flux around sites j, k, l : $\Phi_{jkl} = \varphi_{jk} + \varphi_{kl} + \varphi_{lj} = -(e/\hbar) \iint_{jkl} \mathbf{B} \cdot d\mathbf{S}$, which is manifestly gauge invariant. By defining the Green's function G_B and the velocity operator v_{Bv} ($v = x, y$) in the presence of the magnetic field as $G_B = (i\varepsilon_n - H_B)^{-1}$ and $v_{Bv} = (1/i\hbar)[v, H_B]$ with the position operator $v = \sum_l v_l |l\rangle \langle l|$, we can calculate the correlation function while preserving the gauge invariance. After performing the standard contour integral and analytic continuation and taking the limit of $\omega \rightarrow 0$ in Eq. (2), we obtain a gauge-invariant formula for the Hall conductivity: $\sigma_{xy}^{\text{Total}} = \sigma_{xy}^{\text{TR}} + \sigma_{xy}^{\text{TH}}$, $\sigma_{xy}^X = \int_{-\infty}^{\infty} d\varepsilon (-f'(\varepsilon)) \sigma_{xy}^X(\varepsilon)$ ($X = \text{TR}, \text{TH}$), with

$$\begin{aligned} \frac{\sigma_{xy}^{\text{TR}}(\varepsilon)}{\sigma_{xy0}} = & -\frac{1}{N} \sum_k \text{Im Tr} [H_k^x G_k^R H_k^y G_k^R H_k^x G_k^R H_k^y G_k^A \\ & + H_k^y G_k^R H_k^x G_k^R H_k^y G_k^R H_k^x G_k^A \\ & + \frac{1}{2} H_k^{xy} (G_k^R H_k^x G_k^R H_k^y G_k^A + G_k^R H_k^y G_k^R H_k^x G_k^A \\ & + G_k^R H_k^x G_k^A H_k^y G_k^R + G_k^R H_k^y G_k^A H_k^x G_k^R \\ & + G_k^A H_k^x G_k^R H_k^y G_k^R + G_k^A H_k^y G_k^R H_k^x G_k^R)], \quad (4) \end{aligned}$$

$$\begin{aligned} \frac{\sigma_{xy}^{\text{TH}}(\varepsilon)}{\sigma_{xy0}} = & \frac{1}{N} \sum_k \text{Im Tr} [H_k^x G_k^R H_k^y G_k^R H_k^x G_k^R H_k^y G_k^R \\ & + \frac{1}{2} H_k^{xy} G_k^R (H_k^x G_k^R H_k^y G_k^R + H_k^y G_k^R H_k^x G_k^R)]. \quad (5) \end{aligned}$$

Here, H_k is the one-body Hamiltonian in $\mathbf{k}$ space without the magnetic field, $\partial_v = \partial/\partial(k_v a)$, a is the lattice constant, $l_B = \sqrt{\hbar/eB}$ is the magnetic length, $\sigma_{xy0} = 2(e^2/h) \cdot (a/l_B)^2 \propto B^1$, N is the number of unit cells, $H_k^v = \partial_v H_k$, $H_k^{xy} = \partial_x \partial_y H_k$, $G_k^R = (\varepsilon^R - H_k)^{-1}$, and $G_k^A = (\varepsilon^A - H_k)^{-1}$ are the retarded and advanced Green's functions, $\varepsilon^R = \varepsilon + i\Gamma$ and $\varepsilon^A = \varepsilon - i\Gamma$ with Γ being the relaxation rate, and $f(\varepsilon) = 1/(e^{\beta(\varepsilon - \mu)} + 1)$ is the Fermi distribution function. σ_{xy}^{TR} corresponds to the transport contribution of the Hall conductivity, which includes the results of the Boltzmann equation and reproduces the classical Drude results when considering the free electrons, having both the retarded and advanced Green's functions. σ_{xy}^{TH} is the thermodynamic contribution of the Hall

conductivity arising from the current induced by the magnetic field [52], which only has the retarded Green's functions. Only electrons close to the Fermi surface can contribute to the transport and thermodynamic components.

Some technical comments are in order. Although the scattering rate Γ is phenomenologically introduced, it is dominantly determined by impurity scattering at low temperature, which is generally momentum independent. This does not alter the band dispersion because it lacks momentum dependence. We note that the H_k^{xy} terms have been absent in the previous research based on the continuous model [45,52,62]. These terms naively describe the effect of anisotropy of the energy dispersions. We also note that the formulation in Ref. [62] (which is similar to the one in Refs. [45,52]) does not properly account for the Peierls phase terms when calculating the noninteracting current vertex.

We can show that the thermodynamic contribution of the Hall effect σ_{xy}^{TH} satisfies the following exact relation with the orbital magnetic susceptibility χ_{orb} for lattice systems:

$$\frac{\partial}{\partial \mu} \chi_{\text{orb}} = -\frac{\sigma_{\text{orb}}^{\text{TH}}}{eB}. \quad (6)$$

Here, the expression of χ_{orb} for lattice systems has been shown in Refs. [59,63]. The relation (6) is a straightforward extension of the one derived for a continuum space [52] (also known as the Strěda formula [64–66]) to lattice systems. Since our formula for $\sigma_{xy}^{\text{Total}}$ [Eqs. (4) and (5)] is gauge invariant, and the gauge invariance of the orbital magnetic susceptibility χ_{orb} guarantees the gauge invariance of σ_{xy}^{TH} through Eq. (6), the gauge invariance of σ_{xy}^{TR} is assured. We note that the Hall response is zero when the external magnetic field vanishes, since both transport σ_{xy}^{TR} and thermodynamic σ_{xy}^{TH} contributions are proportional to the constant $\sigma_{xy0} = 2(e^2/h) \times (a/l_B)^2 \propto B^1$.

General $(N + M)$ -band model. We apply the formula to an $(N + M)$ -band model with N dispersive (with eigenfunctions u_{nk} and energy E_{nk} for $n \in D = \{1, \dots, N\}$) and M -fold degenerate isolated flat bands (u_{lk} and $E_{lk} = E_{\text{flat}} = \text{const}$ for $l \in F = \{N + 1, \dots, N + M\}$), satisfying $|E_{n \in D, k} - E_{\text{flat}}| \gg \Gamma > 0$ for any k points in the Brillouin zone. The degenerate isolated flat bands are topologically trivial in the sense that the Chern numbers are zero (where we use the non-Abelian Berry curvature for degenerate flat bands) [67], and their degeneracy does not arise from spin. We take the band basis in which the Green's functions are diagonal: $G_{nn'}^{\text{R/A}} = (\varepsilon^{\text{R/A}} - E_{nk})^{-1} \delta_{nn'}$. After performing some calculations with the explicit expressions of H_k^v and H_k^{xy} , we arrive at a simple description for the Hall conductivity around the degenerate isolated flat bands ($|\mu - E_{\text{flat}}| \ll \min(|E_{n \in D, k} - E_{\text{flat}}|)$) at $T = 0$,

$$\begin{aligned} \frac{\sigma_{xy}^{\text{Total}}(\mu)}{\sigma_{xy0}} &= \frac{\Gamma^3}{[(\mu - E_{\text{flat}})^2 + \Gamma^2]^2} \cdot \Lambda, \\ \Lambda &:= -4 \cdot \frac{1}{N} \sum_k \sum_{n \in D} \sum_{l, m \in F} (E_{nk} - E_{\text{flat}}) \\ &\quad \times \text{Re}[(Q_{xyk, n}^{lm} - Q_{yxk, n}^{lm})Q_{xyk}^{ml}], \end{aligned} \quad (7)$$

neglecting the terms that become zero at $\mu = E_{\text{flat}}$ when $\Gamma \rightarrow 0$. Here, Q_{xyk}^{ab} and $Q_{xyk, n}^{ab}$ are the full and the n th component

non-Abelian quantum geometric tensor [56]:

$$\begin{aligned} Q_{xyk}^{ab} &= \sum_{j \in D} \langle \partial_x u_{ak} | u_{jk} \rangle \langle u_{jk} | \partial_y u_{bk} \rangle \\ &= \langle \partial_x u_{ak} | \left(1 - \sum_{j \in F} |u_{jk}\rangle \langle u_{jk}| \right) | \partial_y u_{bk} \rangle, \end{aligned} \quad (8)$$

$$Q_{xyk, n}^{ab} = \langle \partial_x u_{ak} | u_{nk} \rangle \langle u_{nk} | \partial_y u_{bk} \rangle, \quad (9)$$

satisfying $\sum_{n \in D} Q_{xyk, n}^{ab} = Q_{xyk}^{ab}$. The real part of the (non-Abelian) quantum geometric tensor describes how the state parametrized by the parameter (here k) changes as the parameter infinitesimally changes, and its imaginary part gives the (non-Abelian) Berry curvature, corresponding to the (non-Abelian) gauge field. For the finite temperature $T > 0$ and exact clean limit ($\Gamma \rightarrow 0$), we replace $\Gamma^3/[(\mu - E_{\text{flat}})^2 + \Gamma^2]^2$ with $(\pi/2)(\beta/4)\text{sech}^2(\beta(\mu - E_{\text{flat}})/2)$. See Supplemental Material for the derivation [68]. We also show the individual expressions of the transport and thermodynamic contributions in the Supplemental Material [68]. Λ does not diverge as the band gap ($E_{n \in D, k} - E_{\text{flat}}$) grows, since the geometric part is suppressed as the band gap increases. Importantly, Λ can be nonzero even in the system preserving both inversion and time-reversal symmetry before applying an external magnetic field, where the Berry curvature is zero, as we see below. We note that the non-Abelian quantum geometric tensor returns to the Abelian one when the isolated flat band does not have a degeneracy. For the two-band case with one isolated flat band ($N = M = 1$), the geometric part of the formula $\text{Re}[(Q_{xyk, n}^{lm} - Q_{yxk, n}^{lm})Q_{xyk}^{ml}]$ is reduced to the square of the Berry curvature of the isolated flat band, and its integral is nonvanishing even in the topologically trivial system.

The formula in Eq. (7) is universal in any system with isolated flat band(s), yielding a divergent response in the clean limit. The form of Eq. (7) shows a clear separation of the geometric contribution described by the non-Abelian quantum geometric tensor and the contribution from the band structure. The Hall conductivity depends on the extrinsic parameter Γ around the degenerate flat bands $\mu \approx E_{\text{flat}}$ as $\sigma_{xy}^{\text{Total}} \propto \Gamma^{-1}$, which differs from that of the dispersive bands depending on Γ as $\sigma_{xy}^{\text{Total}} \propto \Gamma^{-2}$ [52], indicating that the relaxation of the Hall response is suppressed more than in the dispersive bands. The formula remains valid even when the isolated flat bands acquire finite dispersions and/or there is a small split between them, as long as they are smaller than Γ .

Since the longitudinal conductivity σ_{vv} under no magnetic field in the clean limit neglecting the dispersive terms reads as $\sigma_{vv} = 8(e^2/h)(1/N) \sum_{k, l \in F} Q_{vvk}^{ll}$, which is positive $Q_{vvk}^{ll} = \sum_{j \in D, l \in F} |\langle u_{jk} | \partial_v u_{lk} \rangle|^2 > 0$, the B dependence of the resistivity $\rho_{xy}(B) = \sigma_{xy}^{\text{Total}}(B)/(\sigma_{xx}\sigma_{yy} + (\sigma_{xy}^{\text{Total}}(B))^2)$ follows $\rho_{xy}(B) \propto B^1$ for small B .

The physical interpretation of the Hall conductivity is as follows: When the chemical potential is away from the isolated flat bands, there are no carriers, and the response is zero. When the chemical potential is close to the isolated flat band (typically in the width of 2Γ or $2k_B T$ for finite T), electrons can be virtually excited from the isolated flat bands, where electrons become mobile and contribute to the Hall conductivity through geometric effects. In the weak-field regime,

the contribution is determined by the non-Abelian quantum geometric tensor, which is in contrast to the strong-field case of the dispersive bands, where the Berry curvature gives the quantized Hall conductivity, when the chemical potential is located between the Landau levels.

Although the interaction effect is not included here, our formula will be valid as long as the energy scale of the interaction is much smaller than that of the relaxation (or temperature). For instance, when the isolated flat band is half filled, its eigenfunction is spatially overlapping, and the system has on-site repulsive interactions $U > 0$ at each site, one of the aligned spin configurations is chosen, and the ground state is ferromagnetic [69,70] with the energy gap proportional to U . However, when the relaxation energy scale is much larger than U , the splitting of the flat bands is negligible and thus buried, implying that the system is effectively noninteracting; our formula remains valid. This argument can be applied to the non-half-filling ferromagnetic case as well.

We note that our formalism can be applied to the system with “nearly” degenerate isolated flat bands, where the flat bands are slightly gapped from each other, as long as the gap between the two flat bands is sufficiently smaller than the relaxation rate Γ .

Concrete examples. We consider the three concrete examples based on the lattice models in Fig. 1. We fix the hopping amplitude $t = 1$ as a unit of energy and consider the $T = 0$ case for simplicity. See Supplemental Material for the details of the models [68]. For the finite temperature case, we read Γ as $(8/\pi)k_B T \sim k_B T$. The first example is the gapped honeycomb lattice model with an isolated flat band [18] [Fig. 1(a)], consisting of the nearest-neighbor hopping $\sqrt{3}t$, the next-nearest-neighbor hopping t only from site A to A, and the on-site potential $mt = 3t$ at each site. The second example is the gapped kagome lattice model with an isolated flat band and two dispersive bands [42] [Fig. 1(b)]. This model has three kinds of nearest-neighbor hopping amplitude: t between sites A and C, and B and C; αt between sites A and B in one direction and t/α in the opposite, and the on-site potential $\alpha^2 + (1/\alpha)^2$ at each site. We choose $\alpha = 2$ in the calculation. The third example is the kagome-3 model with twofold degenerate isolated flat bands and one dispersive band [43] [Fig. 1(c)]. This model includes the nearest-neighbor, next-nearest-neighbor, and next-next-nearest-neighbor hoppings with the same amplitude t and the on-site potential $2t$ at each site. We note that the trace of the non-Abelian Berry curvature is zero, indicating the complete absence of the conventional topological effect. All the models have (an) isolated flat band(s) at $E = 0$. The results of the numerical calculation of the Hall conductivity with $\mu \approx E_{\text{flat}}$ directly by Eq. (7), and by Eqs. (4) and (5) in Fig. 2, are represented by the solid lines and the markers, respectively. The solid lines are calculated for $\Gamma = 0.2, 0.1$, and 0.05 from outside to inside. The red, green, and blue colors, and circle, triangle, and diamond markers correspond to the case of the gapped honeycomb ($\Lambda \approx 0.184$), gapped kagome ($\Lambda \approx 0.036$), and kagome-3 models ($\Lambda \approx 6.357$), and $\Gamma = 0.2, 0.1$, and 0.05 , respectively. Both methods yield results that agree with each other very well, so we can conclude that the total Hall conductivity at the isolated flat band(s) indicates the universal behavior described by the intertwined combination of the

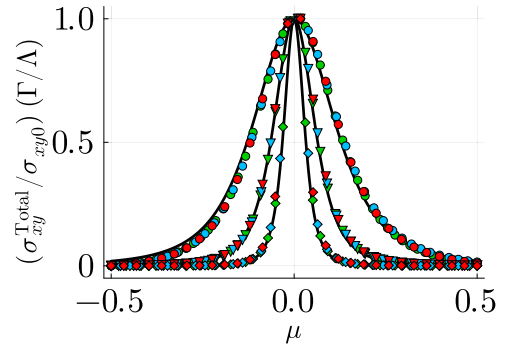


FIG. 2. The normalized total Hall conductivity $(\sigma_{xy}^{\text{Total}}/\sigma_{xy0}) \times (\Gamma/\Lambda)$ of the gapped honeycomb ($\Lambda \approx 0.184$, red), gapped kagome ($\Lambda \approx 0.036$, blue), and kagome-3 models ($\Lambda \approx 6.357$, green). Equation (7) gives the solid lines for $\Gamma = 0.2, 0.1$, and 0.05 from outside to inside. The markers are calculated by Eqs. (4) and (5) for $\Gamma = 0.2$ (circle), 0.1 (triangle), and 0.05 (diamond). A finite small value at $\mu = -0.5$ is subtracted in the markers.

(non-)Abelian quantum geometric tensor and band-structure contributions in Eq. (7). The greatly enhanced Hall response in the kagome-3 model ($\Lambda \approx 6.357$) suggests a significant contribution from the degenerate isolated flat bands via the non-Abelian quantum geometric tensor, which cannot be captured by the conventional topological characterization by the (non-Abelian) Berry curvature. See Supplemental Material for the plots of the geometric part of the formula of those models [68]. The results presented here can also be applied to the finite temperature case, provided that the energy scale of the temperature, $k_B T$, is smaller than the energy gap between the dispersive and isolated flat bands.

Crossover. To understand the role of the flatness of the isolated flat band, we here introduce a finite dispersion to the isolated flat band of the gapped honeycomb lattice model without changing the form of the dispersive band. The magnitude of the nearest-neighbor hopping and next-nearest-neighbor hopping from sites A to A and B to B are $\sqrt{3}(t - \delta t)$, t , and δt , respectively. The modified band structure with $\delta t = 0.2$ is shown in Fig. 3(a). The dispersive band shifts $-3\delta t$ and does not change the $\mathbf{k}$ dependence.

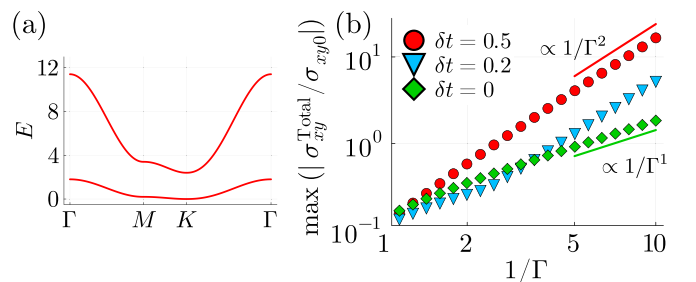


FIG. 3. (a) The modified band structure of the gapped honeycomb lattice model with $\delta t = 0.2$. (b) Γ dependence of the maximum absolute value of the total Hall conductivity $|\sigma_{xy}^{\text{Total}}/\sigma_{xy0}|$ around the isolated flat band ($\mu \in [-t, t]$). The red circle, blue triangle, and green diamond curves correspond to the cases of $\delta t = 0.5, 0.2$, and 0 , respectively.

We calculate the Hall conductivity for this model by sweeping the parameter Γ while fixing δt within $\mu \in [-t, t]$, and obtain the Γ dependence of the peak height of the total Hall conductivity ($\max(|\sigma_{xy}^{\text{Total}}/\sigma_{xy0}|)$) as shown in Fig. 3(b). The red circle, blue triangle, and green diamond curves correspond to the parameters $\delta t = 0.5, 0.2$, and 0 , respectively. We can see a clear crossover from the flat band ($\propto \Gamma^{-1}$) to the dispersive one ($\propto \Gamma^{-2}$) in the blue triangle curve. The crossover occurs around $\delta t/(\Gamma/2) \approx 1$, suggesting that the bandwidth should be smaller than $\Gamma/2$ to realize the Hall response of the isolated flat band. When δt is smaller than $\Gamma/2$, the band can be seen as a flat band, and the Hall response is proportional to Γ^{-1} , exhibiting the intrinsic nature of the Hall effect. On the other hand, when $2\delta t$ becomes larger than Γ and the lower band becomes dispersive, the transport contribution overwhelms the thermodynamic contribution (see Supplemental Material [68]). Moreover, the response becomes proportional to Γ^{-2} , a well-known property observed in the case of free electrons [52]. We note that $\sigma_{xy}^{\text{Total}}$ does not change its sign within $\mu \in [-t, t]$ when δt is small (in our case, $\delta t = 0, 0.2$) and it is always positive. When the dispersion effect becomes noticeable ($\delta t = 0.5$), $\sigma_{xy}^{\text{Total}}$ changes its sign near $\mu = 0$, which is consistent with the usual behavior of the Hall conductivity of the free electron [52]. This crossover behavior is somewhat similar to that of the longitudinal conductivity [53]. The modification of the formula [Eq. (7)] by introducing small dispersion to the isolated flat band for the two-band case ($N = M = 1$) is presented in the Supplemental Material [68].

Relation to experiments. Let us discuss the observability of the phenomena in experiments. In typical materials with the lattice constant $a \sim 1 \text{ \AA}$, the ratio $(a/l_B)^2$ is on the order of 10^{-5} for a typical magnetic-field strength $B \sim 1 \text{ T}$. Then, the typical scale of the Hall conductivity in isolated flat-band systems is on the order of $10^{-5} \times (e^2/h)$, which is within the reach of experimental observations. The observation will become much easier in clean samples because of the Γ^{-1} dependence. If one considers systems with a large lattice

constant (e.g., moiré materials with $a \sim 10^{-8} \text{ m}$), the value of the Hall conductivity can be enhanced. A challenging part is finding appropriate materials with isolated flat bands with a sufficiently large energy gap. Candidate materials include the twisted bilayer graphene [32], metal-organic frameworks [39–41], and several chain-type lattice materials [26].

Summary. We have investigated the Hall effect in isolated flat-band systems in the weak-field limit. The derived gauge-invariant formula for the Hall conductivity [Eqs. (4) and (5)], and the exact relation between the orbital magnetic susceptibility and the thermodynamic contribution of the Hall conductivity [Eq. (6)] can be applied to any lattice models. The Hall conductivity in isolated flat-band systems [Eq. (7)] shows the universal form consisting of the non-Abelian quantum geometric tensor (the geometric part) and energy difference between the dispersive and isolated flat bands (the band-structure part), the former of which arises from the interband effect. This indicates the breakdown of the semiclassical theory based on the Boltzmann equation, and a quantum-mechanical treatment is essential to examine the interband effect in systems with isolated flat bands. Future problems include addressing the effects of interactions, disorders, and strong magnetic fields that exceed the linear response regime, as well as exploring other physical quantities that display the effect of the non-Abelian quantum geometric tensor in isolated flat-band systems.

Acknowledgments. We appreciate the useful comments from Jörg Schmalian, Alexander Mirlin, Kazuaki Takasan, Shohei Imai, Taiga Nakamoto, and Shuta Matsuura. This work was supported by JST FOREST (Grant No. JPMJFR2131) and JSPS KAKENHI (Grants No. JP24H00191, No. JP25H01246, and No. JP25H01251).

Data availability. The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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- [1] E. H. Hall, On a new action of the magnet on electric currents, *Am. J. Math.* **2**, 287 (1879).
 - [2] C. Kittel, *Introduction to Solid State Physics* (Wiley, New York, 2004).
 - [3] G. Grosso and G. P. Parravicini, *Solid State Physics* (Elsevier, Amsterdam, 2013).
 - [4] L. Chen, T. Mazaheri, A. Seidel, and X. Tang, The impossibility of exactly flat non-trivial Chern bands in strictly local periodic tight binding models, *J. Phys. A: Math. Theor.* **47**, 152001 (2014).
 - [5] E. H. Lieb, Two theorems on the Hubbard model, *Phys. Rev. Lett.* **62**, 1201 (1989).
 - [6] A. Mielke, Ferromagnetism in the Hubbard model on line graphs and further considerations, *J. Phys. A: Math. Gen.* **24**, 3311 (1991).
 - [7] H. Tasaki, Ferromagnetism in the Hubbard models with degenerate single-electron ground states, *Phys. Rev. Lett.* **69**, 1608 (1992).
 - [8] A. Mielke and H. Tasaki, Ferromagnetism in the Hubbard model, *Commun. Math. Phys.* **158**, 341 (1993).
 - [9] H. Tasaki, Ferromagnetism in Hubbard models, *Phys. Rev. Lett.* **75**, 4678 (1995).
 - [10] E. Tang, J.-W. Mei, and X.-G. Wen, High-temperature fractional quantum Hall states, *Phys. Rev. Lett.* **106**, 236802 (2011).
 - [11] K. Sun, Z. Gu, H. Katsura, and S. Das Sarma, Nearly flatbands with nontrivial topology, *Phys. Rev. Lett.* **106**, 236803 (2011).
 - [12] T. Neupert, L. Santos, C. Chamon, and C. Mudry, Fractional quantum Hall states at zero magnetic field, *Phys. Rev. Lett.* **106**, 236804 (2011).
 - [13] N. Regnault and B. A. Bernevig, Fractional Chern insulator, *Phys. Rev. X* **1**, 021014 (2011).
 - [14] E. J. Bergholtz and Z. Liu, Topological flat band models and fractional Chern insulators, *Int. J. Mod. Phys. B* **27**, 1330017 (2013).
 - [15] A. Zhao and S.-Q. Shen, Quantum anomalous Hall effect in a flat band ferromagnet, *Phys. Rev. B* **85**, 085209 (2012).

- [16] Y. Zhou, G. Sethi, H. Liu, Z. Wang, and F. Liu, Excited quantum anomalous and spin Hall effect: Dissociation of flat-bands-enabled excitonic insulator state, *Nanotechnology* **33**, 415001 (2022).
- [17] J.-W. Rhim, K. Kim, and B.-J. Yang, Quantum distance and anomalous Landau levels of flat bands, *Nature (London)* **584**, 59 (2020).
- [18] T. Misumi and H. Aoki, New class of flat-band models on tetragonal and hexagonal lattices: Gapped versus crossing flat bands, *Phys. Rev. B* **96**, 155137 (2017).
- [19] B. Roy and V. Juričić, Unconventional superconductivity in nearly flat bands in twisted bilayer graphene, *Phys. Rev. B* **99**, 121407 (2019).
- [20] H. Aoki, Theoretical possibilities for flat band superconductivity, *J. Supercond. Nov. Magn.* **33**, 2341 (2020).
- [21] L. H. C. M. Nunes and C. M. Smith, Flat-band superconductivity for tight-binding electrons on a square-octagon lattice, *Phys. Rev. B* **101**, 224514 (2020).
- [22] V. Peri, Z.-D. Song, B. A. Bernevig, and S. D. Huber, Fragile topology and flat-band superconductivity in the strong-coupling regime, *Phys. Rev. Lett.* **126**, 027002 (2021).
- [23] H. Tian, X. Gao, Y. Zhang, S. Che, T. Xu, P. Cheung, K. Watanabe, T. Taniguchi, M. Randeria, F. Zhang, C. N. Lau, and M. W. Bockrath, Evidence for Dirac flat band superconductivity enabled by quantum geometry, *Nature (London)* **614**, 440 (2023).
- [24] T. Mizoguchi and M. Udagawa, Flat-band engineering in tight-binding models: Beyond the nearest-neighbor hopping, *Phys. Rev. B* **99**, 235118 (2019).
- [25] N. Regnault, Y. Xu, M.-R. Li, D.-S. Ma, M. Jovanovic, A. Yazdani, S. S. P. Parkin, C. Felser, L. M. Schoop, N. P. Ong, R. J. Cava, L. Elcoro, Z.-D. Song, and B. A. Bernevig, Catalogue of flat-band stoichiometric materials, *Nature (London)* **603**, 824 (2022).
- [26] P. M. Neves, J. P. Wakefield, S. Fang, H. Nguyen, L. Ye, and J. G. Checkelsky, Crystal net catalog of model flat band materials, *npj Comput. Mater.* **10**, 39 (2024).
- [27] M. Kang, L. Ye, S. Fang, J.-S. You, M. Levitan, Abe Han, J. I. Facio, C. Jozwiak, A. Bostwick, E. Rotenberg, *et al.*, Dirac fermions and flat bands in the ideal kagome metal FeSn, *Nat. Mater.* **19**, 163 (2020).
- [28] J. P. Wakefield, M. Kang, P. M. Neves, D. Oh, S. Fang, R. McTigue, S. Y. F. Zhao, T. N. Lamichhane, A. Chen, S. Lee, *et al.*, Three-dimensional flat bands in pyrochlore metal CaNi_2 , *Nature (London)* **623**, 301 (2023).
- [29] E. Suárez Morell, J. D. Correa, P. Vargas, M. Pacheco, and Z. Barticevic, Flat bands in slightly twisted bilayer graphene: Tight-binding calculations, *Phys. Rev. B* **82**, 121407 (2010).
- [30] R. Bistritzer and A. H. MacDonald, Moiré bands in twisted double-layer graphene, *Proc. Natl. Acad. Sci. USA* **108**, 12233 (2011).
- [31] N. R. Chebrolu, B. L. Chittari, and J. Jung, Flat bands in twisted double bilayer graphene, *Phys. Rev. B* **99**, 235417 (2019).
- [32] S. Lisi, X. Lu, T. Benschop, T. A. de Jong, P. Stepanov, J. R. Duran, F. Margot, I. Cucchi, E. Cappelli, A. Hunter, *et al.*, Observation of flat bands in twisted bilayer graphene, *Nat. Phys.* **17**, 189 (2021).
- [33] J. Schulenburg, A. Honecker, J. Schnack, J. Richter, and H.-J. Schmidt, Macroscopic magnetization jumps due to independent magnons in frustrated quantum spin lattices, *Phys. Rev. Lett.* **88**, 167207 (2002).
- [34] M. E. Zhitomirsky and H. Tsunetsugu, Exact low-temperature behavior of a kagomé antiferromagnet at high fields, *Phys. Rev. B* **70**, 100403 (2004).
- [35] S. Tacchi, J. Flores-Farías, D. Petti, F. Brevis, A. Cattoni, G. Scaramuzzi, D. Girardi, D. Cortés-Ortuño, R. A. Gallardo, E. Albisetti, G. Carlotti, and P. Landeros, Experimental observation of flat bands in one-dimensional chiral magnonic crystals, *Nano Lett.* **23**, 6776 (2023).
- [36] F. Baboux, L. Ge, T. Jacqmin, M. Biondi, E. Galopin, A. Lemaître, L. Le Gratiet, I. Sagnes, S. Schmidt, H. E. Türeci, A. Amo, and J. Bloch, Bosonic condensation and disorder-induced localization in a flat band, *Phys. Rev. Lett.* **116**, 066402 (2016).
- [37] A. A. Daniel Leykam and S. Flach, Artificial flat band systems: From lattice models to experiments, *Adv. Phys.: X* **3**, 1473052 (2018).
- [38] T.-X. Ma, Q.-S. Fan, C. Zhang, and Y.-S. Wang, Acoustic flat-bands in phononic crystal defect lattices, *J. Appl. Phys.* **129**, (2021).
- [39] M. Pan, X. Zhang, Y. Zhou, P. Wang, Q. Bian, H. Liu, X. Wang, X. Li, A. Chen, X. Lei, S. Li, Z. Cheng, Z. Shao, H. Ding, J. Gao, F. Li, and F. Liu, Growth of mesoscale ordered two-dimensional hydrogen-bond organic framework with the observation of flat band, *Phys. Rev. Lett.* **130**, 036203 (2023).
- [40] Z. Liu, Z.-F. Wang, J.-W. Mei, Y.-S. Wu, and F. Liu, Flat Chern band in a two-dimensional organometallic framework, *Phys. Rev. Lett.* **110**, 106804 (2013).
- [41] M. G. Yamada, T. Soejima, N. Tsuji, D. Hirai, M. Dincă, and H. Aoki, First-principles design of a half-filled flat band of the kagome lattice in two-dimensional metal-organic frameworks, *Phys. Rev. B* **94**, 081102 (2016).
- [42] T. Bilitewski and R. Moessner, Disordered flat bands on the kagome lattice, *Phys. Rev. B* **98**, 235109 (2018).
- [43] D. L. Bergman, C. Wu, and L. Balents, Band touching from real-space topology in frustrated hopping models, *Phys. Rev. B* **78**, 125104 (2008).
- [44] One way to understand this is that as one changes the strength of the magnetic field the magnetic Brillouin zone changes discontinuously, and in the weak-field limit the size of the magnetic Brillouin zone diverges. For example, electrons on a square lattice in a uniform magnetic field show a fractal-like band structure known as the Hofstadter butterfly. Such a fine structure is smeared out if we introduce dissipation on a lattice system.
- [45] H. Fukuyama, Theory of Hall effect. II: Bloch electrons, *Prog. Theor. Phys.* **42**, 1284 (1969).
- [46] Y. Fuseya, M. Ogata, and H. Fukuyama, Transport properties and diamagnetism of Dirac electrons in bismuth, *J. Phys. Soc. Jpn.* **84**, 012001 (2015).
- [47] V. Könye and M. Ogata, Microscopic theory of magnetoelectricity at low magnetic fields in terms of Berry curvature and orbital magnetic moment, *Phys. Rev. Res.* **3**, 033076 (2021).
- [48] R. Peierls, Zur theorie des diamagnetismus von leitungselektronen, *Z. Phys.* **80**, 763 (1933).
- [49] H. Fukuyama, Theory of orbital magnetism of Bloch electrons: Coulomb interactions, *Prog. Theor. Phys.* **45**, 704 (1971).

- [50] M. Ogata and H. Fukuyama, Orbital magnetism of Bloch electrons I. General formula, *J. Phys. Soc. Jpn.* **84**, 124708 (2015).
- [51] M. Ogata, Theory of magnetization in Bloch electron systems, *J. Phys. Soc. Jpn.* **86**, 044713 (2017).
- [52] M. Ogata, Theory of Hall conductivity and its exact relationship to orbital magnetic susceptibility, *J. Phys. Soc. Jpn.* **93**, 013703 (2024).
- [53] J. Mitscherling and T. Holder, Bound on resistivity in flat-band materials due to the quantum metric, *Phys. Rev. B* **105**, 085154 (2022).
- [54] B. Mera and J. Mitscherling, Nontrivial quantum geometry of degenerate flat bands, *Phys. Rev. B* **106**, 165133 (2022).
- [55] K.-E. Huhtinen and P. Törmä, Conductivity in flat bands from the Kubo-Greenwood formula, *Phys. Rev. B* **108**, 155108 (2023).
- [56] Y.-Q. Ma, S. Chen, H. Fan, and W.-M. Liu, Abelian and non-Abelian quantum geometric tensor, *Phys. Rev. B* **81**, 245129 (2010).
- [57] H.-T. Ding, C.-X. Zhang, J.-X. Liu, J.-T. Wang, D.-W. Zhang, and S.-L. Zhu, Non-Abelian quantum geometric tensor in degenerate topological semimetals, *Phys. Rev. A* **109**, 043305 (2024).
- [58] In principle, our approach can include the spin-orbit coupling. However, our formulation for including the effect of an external magnetic field relies on the Peierls substitution, which introduces a phase arising from the magnetic field when an electron hops from one site to another. When we introduce spin-orbit coupling, the term arising from an electron hopping intertwined with spin degrees of freedom is allowed. Since this hopping term acquires an additional Peierls phase, the weak-field limit will be much more complicated. Including the Zeeman effect should be straightforward.
- [59] A. Raoux, F. Piéchon, J.-N. Fuchs, and G. Montambaux, Orbital magnetism in coupled-bands models, *Phys. Rev. B* **91**, 085120 (2015).
- [60] K.-T. Chen and P. A. Lee, Unified formalism for calculating polarization, magnetization, and more in a periodic insulator, *Phys. Rev. B* **84**, 205137 (2011).
- [61] B. Savoie, On the zero-field orbital magnetic susceptibility of Bloch electrons in graphene-like solids: Some rigorous results, *J. Math. Phys.* **53**, 073302 (2012).
- [62] R. Nourafkan and A.-M. S. Tremblay, Hall and Faraday effects in interacting multiband systems with arbitrary band topology and spin-orbit coupling, *Phys. Rev. B* **98**, 165130 (2018).
- [63] G. Gómez-Santos and T. Stauber, Measurable lattice effects on the charge and magnetic response in graphene, *Phys. Rev. Lett.* **106**, 045504 (2011).
- [64] P. Streda, Theory of quantised Hall conductivity in two dimensions, *J. Phys. C* **15**, L717 (1982).
- [65] A. Widom, Thermodynamic derivation of the Hall effect current, *Phys. Lett. A* **90**, 474 (1982).
- [66] P. Streda and L. Smrcka, Thermodynamic derivation of the Hall current and the thermopower in quantising magnetic field, *J. Phys. C* **16**, L895 (1983).
- [67] We note that we consider the strong topology (independent of the spatial symmetry and characterized by the dimension of the whole system) to define the topologically trivial isolated flat bands. The trace of the non-Abelian Berry curvature vanishes in our setting, since the system preserves time-reversal symmetry (when the external magnetic field is absent) and lacks spin-orbit coupling; hence, we call the system topologically trivial.
- [68] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/5s6l-vc71> for details of the derivation, model descriptions, plots of the geometric part of the formula, the parameter dependence of the two components that comprise the formula, and the modification of the formula by a small finite dispersion of the flat band for the two-band model.
- [69] A. Mielke, Ferromagnetism in the Hubbard model and Hund's rule, *Phys. Lett. A* **174**, 443 (1993).
- [70] A. Mielke, Stability of ferromagnetism in Hubbard models with degenerate single-particle ground states, *J. Phys. A: Math. Gen.* **32**, 8411 (1999).