

Optimal parametric control of transport across a Josephson junction

Hannah Kleine-Pollmann^{1,*}, Guido Homann¹, and Ludwig Mathey^{1,2}¹Zentrum für Optische Quantentechnologien and Institut für Quantenphysik, Universität Hamburg, 22761 Hamburg, Germany²The Hamburg Centre for Ultrafast Imaging, Luruper Chaussee 149, 22761 Hamburg, Germany

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We present optimal control strategies for the dc transport across a Josephson junction. Specifically, we consider a junction in which the Josephson coupling is driven parametrically, with either a bichromatic or a trichromatic driving protocol, and optimize the prefactor of the $1/\omega$ divergence of the imaginary part of the conductivity, which we refer to as the effective Josephson coupling J_{eff} . We demonstrate that for an optimal bichromatic protocol an enhancement of J_{eff} by a factor of 70 can be reached, and for an optimal trichromatic protocol an enhancement by a factor of 135. This is motivated by pump-probe experiments that have demonstrated light-enhanced superconductivity along the c axis of underdoped $\text{YBa}_2\text{Cu}_3\text{O}_{6+\delta}$ (YBCO), where the junction serves as a minimal model for the c -axis coupling of superconducting layers. Therefore, the significant enhancement of superconductivity that we show for multifrequency protocols demonstrates that the advancement of pump-probe technology toward these strategies is highly desirable.

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I. INTRODUCTION

Pump-probe experiments on materials have established remarkable phenomena, such as light-enhanced and light-induced superconductivity in high-temperature superconductors such as $\text{YBa}_2\text{Cu}_3\text{O}_{6+\delta}$ (YBCO) and other materials [1,2]. These phenomena consist of either enhancing the superconducting response of a material that is a superconductor initially or achieving a superconducting response of a material that is initially above the critical temperature [3–9]. The observation of these phenomena suggests that dynamical control of materials via light is possible. This poses the question of how and how much a material property can be controlled in a desirable manner. A proposed mechanism for light-enhanced superconductivity in cuprates is parametric enhancement, exemplified for a Josephson junction in Ref. [10]. Here, the Josephson coupling is modulated parametrically, resulting in an enhancement of the dc transport across the junction, which can also be expressed as an enhancement of the effective Josephson coupling. Specifically, this mechanism shows a maximal increase of the effective Josephson interlayer coupling of a factor of 2–3 in a steady state, when the driving frequency is blue detuned, compared to the plasma frequency of the Josephson junction. In the transient regime, rather than the steady state, the enhancement of the effective Josephson coupling exceeds the bare coupling further [11]. This mechanism also has an influence on Meissner screening [12] and can be demonstrated in Josephson-coupled condensates [13]. Furthermore, related phenomena in the context of nonlinear

coupling between the electric field and the Higgs mode have been discussed [14–16], as well as photoinduced eta pairing [17]. In this paper, we present optimal parametric driving protocols of a Josephson junction [18]. With this, we address the question of how much the transport across a Josephson junction can be enhanced via optimal control. We optimize the dc transport across the junction, based on the low-frequency limit of the imaginary part of the conductivity. The Josephson junction serves as a minimal model for the Josephson plasma mode dynamics of a cuprate superconductor. Therefore, the transport across the junction is a minimal model for c -axis transport in a cuprate. Thus, we motivate optimal driving protocols of light-driven cuprates to maximally enhance the c -axis transport.

The magnitude of the low-frequency limit of the imaginary part of the conductivity is quantified by the effective Josephson coupling J_{eff} , which is proportional to the prefactor of the $1/\omega_{\text{pr}}$ divergence of the imaginary part of the conductivity. Here, ω_{pr} is the probing frequency. We drive the Junction parametrically and probe the junction to read out the low-frequency conductivity response in order to derive an effective Josephson coupling. We utilize a Metropolis algorithm [19,20] with the objective to enhance the effective Josephson coupling, in the steady state. We optimize two classes of driving protocols. First, we consider a bichromatic driving protocol and show that driving at the second harmonic leads to an enhancement of the effective Josephson coupling of around 70, compared to the undriven case. Second, we consider a trichromatic driving protocol and find an enhancement of the effective Josephson coupling of around 135, compared to the undriven case. Interestingly, this optimal enhancement is achieved for a set of frequencies, in which the third frequency is not at a simple integer ratio to one of the other frequencies, while the first and second frequencies are in a 1:2 ratio. Furthermore, we find an enhancement of the effective Josephson coupling of around 50 for a wide range of

*Contact author: hannah.kleinepollmann@uni-hamburg.de

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frequencies. These results suggest that multifrequency driving protocols are a desirable method to optimally enhance the optical conductivity in high- T_c superconductors, in the steady state.

This paper is organized as follows. In Sec. II, we introduce our method to estimate the effective Josephson coupling. In Sec. III, we introduce the numerical method to optimize the driving protocol. In Sec. IV, we show the enhancement of the effective Josephson coupling for optimal bichromatic and trichromatic driving protocols. In Sec. V, we show the conductivity for optimal bichromatic and trichromatic driving protocols, in comparison to the undriven case. In Sec. VI, we conclude our findings.

II. CONDUCTIVITY OF A JOSEPHSON JUNCTION

We consider a single Josephson junction [21] with thickness d , Josephson coupling J_0 , and dielectric constant ϵ . In the absence of driving, the equation of motion for the phase φ of the junction is given by the resistively and capacitively shunted junction (RCSJ) model [22]:

$$\ddot{\varphi} + \gamma \dot{\varphi} + \omega_{\text{pl}}^2 \sin \varphi = \frac{\omega_{\text{pl}}^2 J}{J_0}, \quad (1)$$

where $\omega_{\text{pl}} = \sqrt{2edJ_0/\epsilon\epsilon_0\hbar}$ is the plasma frequency of the junction and γ is a damping constant. We probe the junction by applying a monochromatic current

$$J = J_{\text{pr}} \cos(\omega_{\text{pr}} t), \quad (2)$$

where J_{pr} is a probing current that is small compared to the Josephson coupling $J_{\text{pr}}/J_0 \ll 1$ and ω_{pr} is the probing frequency. Note that we focus on small probing frequencies $\omega_{\text{pr}} < \omega_{\text{pl}}$ in the following. In frequency space, we compute the linear conductivity

$$\sigma(\omega_{\text{pr}}) = \frac{J(\omega_{\text{pr}})d}{V(\omega_{\text{pr}})}, \quad (3)$$

where $V(t) = (\hbar/2e)\dot{\varphi}(t)$ is the junction voltage in the time domain and $J = J(\omega_{\text{pr}})$. As we show in Appendix A, the numerical result is in excellent agreement with the analytical solution for the equilibrium case,

$$\sigma(\omega_{\text{pr}}) = \epsilon\epsilon_0 \left[\gamma + i \left(\frac{\omega_{\text{pl}}^2}{\omega_{\text{pr}}} - \omega \right) \right]. \quad (4)$$

Next, we add parametric driving to the dynamics,

$$\ddot{\varphi} + \gamma \dot{\varphi} + \omega_{\text{pl}}^2 [1 + F(t)] \sin \varphi = \frac{\omega_{\text{pl}}^2 J}{J_0}, \quad (5)$$

where $F(t)$ is a driving protocol. A schematic representation of the driven system is shown in Fig. 1(b). We define the effective Josephson coupling as

$$J_{\text{eff}} = \frac{\hbar}{2ed} [\sigma_2(\omega_{\text{pr}})\omega_{\text{pr}}]_{\omega_{\text{pr}} \rightarrow 0}. \quad (6)$$

Note that $J_{\text{eff}} = J_0$ in equilibrium. Numerically, we proceed as follows. Once the system has reached a steady state, we estimate $\sigma_2(\omega_{\text{pr}})$ as the imaginary part of $\sigma(\omega_{\text{pr}})$ according to Eq. (3), where we record the dynamics of the phase $\varphi(t)$ over a time interval of $25\,000 \times T$, with $T = 2\pi/\omega_{\text{pl}}$. Next, we evaluate the effective Josephson coupling J_{eff} from a linear fit of

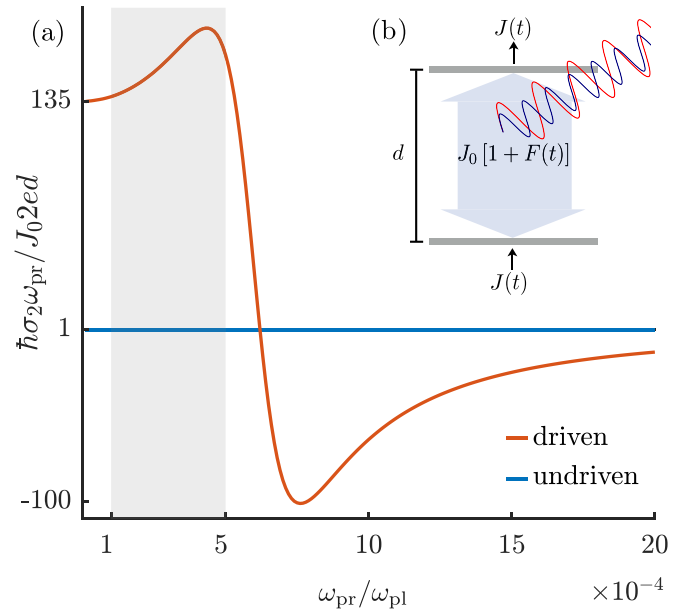


FIG. 1. (a) Imaginary part of the conductivity $\sigma_2(\omega_{\text{pr}})$ as a function of the probing frequency ω_{pr} in the steady state. The undriven case (blue) is compared to the case with an optimal trichromatic driving protocol (red). The gray area depicts the interval in which the conductivity is evaluated numerically to estimate the effective Josephson coupling J_{eff}/J_0 . For this optimal protocol, the dc transport across the junction is enhanced by a factor of 135. (b) Schematic representation of a parametrically driven Josephson junction with the thickness d and the Josephson coupling J_0 . The driving protocol $F(t)$ contains several frequency components as indicated by the wavy lines. To measure the conductivity, a probe current $J(t)$ is applied.

$\sigma_2(\omega_{\text{pr}})\omega_{\text{pr}}$ in the range of $\omega_{\text{pr}} \in [0.001 \times \omega_{\text{pl}}, 0.005 \times \omega_{\text{pl}}]$ as shown as the gray shaded area in Fig. 1(a). Because we estimate the effective Josephson coupling based on a finite number of samples, we obtain an error ϵ_J as the standard deviation in units of J_{eff} . This error is used below to exclude protocols for which J_{eff} cannot be determined with sufficient clarity.

III. MONTE CARLO OPTIMIZATION

We consider a parametrized driving protocol $F(t) = F_q(t)$, where $q = \{q_k\}$ is the set of driving parameters q_k that parametrize the driving protocol. We utilize the Metropolis algorithm [19] to optimize the driving parameters q with the objective to enhance the effective Josephson coupling J_{eff}/J_0 according to Eq. (6).

We initialize the driving parameters q and calculate $J_{\text{eff}}(q)$. For perfect convergence of the algorithm, the choice of the initial driving parameters is arbitrary. However, we find that specific initialization strategies, depending on the specific driving protocol $F_q(t)$, have a strong impact on the optimization process, as discussed in Appendix C. Starting from an initial choice of the parameters q_k , the Metropolis algorithm generates a Markov chain of parameters, to search for an optimal protocol. The update step of the algorithm generates a set of parameters p_k out of the parameters q_k via

$$p_k = q_k + x_k, \quad (7)$$

where $x_k \sim U[-r_k, r_k]$ is uniformly sampled and r_k is a hyperparameter that determines the maximum change of the k th parameter. We define the set of parameters of maximum change as $r = \{r_k\}$. We exclude a parameter set p if $F_p(t) < -1$ for any time t . This ensures positive Josephson coupling, as is implied by the form of the driving term in Eq. (5), and $\omega_{\text{pl}}^2 \sim J_0$. Depending on the specific physical implementation of the Josephson junction, this condition could be modified to capture the physically realizable range of J_0 or ω_{pl} . We also exclude regions where the parameter choices in the driving protocol leads to Floquet parametric instabilities [23,24], which are associated with heating of the system, as further discussed in Appendix B. Next, we calculate the effective Josephson coupling $J_{\text{eff}}(p)$, corresponding to the trial driving parameters p , based on Eq. (6). If we obtain an error of $\epsilon_J > 0.1$, we sample a new set of trial driving parameters p according to Eq. (7). Otherwise, we compute the probability $P(q, p)$ that the trial driving parameters p are accepted over the current driving parameters q as

$$P(q, p) = \min \left(1, \exp \left(-\frac{J_{\text{eff}}(q) - J_{\text{eff}}(p)}{J_0 \alpha} \right) \right), \quad (8)$$

where α is a constant hyperparameter. Throughout this study, we use a value of $\alpha = 0.3$. We note that in the context of sampling a thermal ensemble, the hyperparameter α is related to the temperature of the ensemble. Next, we uniformly sample a number $z \sim U[0, 1]$ and update the current driving parameters as $q \rightarrow p$ if $P(q, p) \geq z$. We repeat this procedure for 1500 iterations. We note that the possibility to accept p as a new set of driving parameters even if $J(p) < J(q)$ reduces the probability to converge into local minima.

We consider two driving protocols $F(t)$. First, we consider a bichromatic driving protocol

$$F_B(t) = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t + \theta), \quad (9)$$

which contains cosine modes of frequencies ω_1 and ω_2 with the amplitudes A_1 and A_2 and a phase shift θ . The set of parameters of maximum change is $r = \{0.05, 0.05, 0.05, 0.05, 2\pi \times 0.05\}$. Second, we consider a trichromatic driving protocol

$$F_T(t) = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t + \theta_1) + A_3 \cos(\omega_3 t + \theta_2), \quad (10)$$

which contains cosine modes with the frequencies ω_1 , ω_2 , and ω_3 and the amplitudes A_1 , A_2 , and A_3 and the phase shifts θ_1 and θ_2 . Here, the set of driving parameters is $q = \{\omega_1, \omega_3, A_1, A_2, A_3, \theta_1, \theta_2\}$. We fix the frequency ω_2 to $\omega_2 = 2 \times \omega_1$, i.e., the second harmonic, as we discuss below. The set of parameters of maximum change is $r = \{0.05, 0.05, 0.05, 0.05, 0.05, 2\pi \times 0.05, 2\pi \times 0.05\}$.

IV. RESULTS

A. Bichromatic driving protocol

To determine the optimal bichromatic driving protocol $F_B(t)$, we generate 600 optimization trajectories for various initial driving parameters q_0 (Table I). In Fig. 2, we show an example for an optimization trajectory. We find that the effective Josephson coupling J_{eff} converges to significantly

TABLE I. Initial driving parameter q_0 used for the optimization of a bichromatic driving protocol $F_B(t)$.

A_1	A_2	$\omega_1/\omega_{\text{pl}}$	$\omega_2/\omega_{\text{pl}}$	θ
0.1	0.1	0.7	1.7	0
0.6	0.1	0.7	1.7	0
0.1	0.1	0.7	2.5	0
0.6	0.1	0.7	2.5	0
0.1	0.1	1.15	1.5	0
0.6	0.1	1.15	1.5	0
0.1	0.1	1.15	1.7	0
0.6	0.1	1.15	1.7	0
0.1	0.1	1.15	2.5	0
0.6	0.1	1.15	2.5	0
0.71	0.25	1.05	2.06	-3.26

increased values up to $J_{\text{eff}}/J_0 \approx 70$, compared to the Josephson coupling J_0 . At the beginning of the optimization, the driving parameters q change frequently. As soon as a significant increase of the effective Josephson coupling J_{eff}/J_0 is achieved, the driving parameters q converge. In almost all

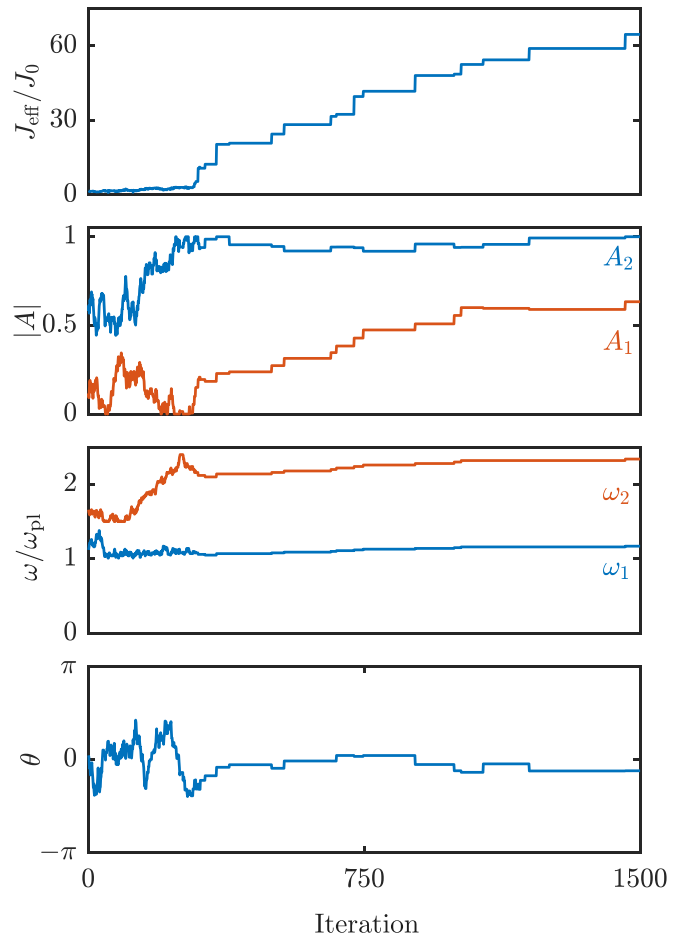


FIG. 2. Optimization of the driving parameter over 1500 optimization iterations and the corresponding effective Josephson coupling J_{eff}/J_0 for the bichromatic driving protocol $F_B(t)$. We observe a strong enhancement of the effective Josephson coupling J_{eff}/J_0 .

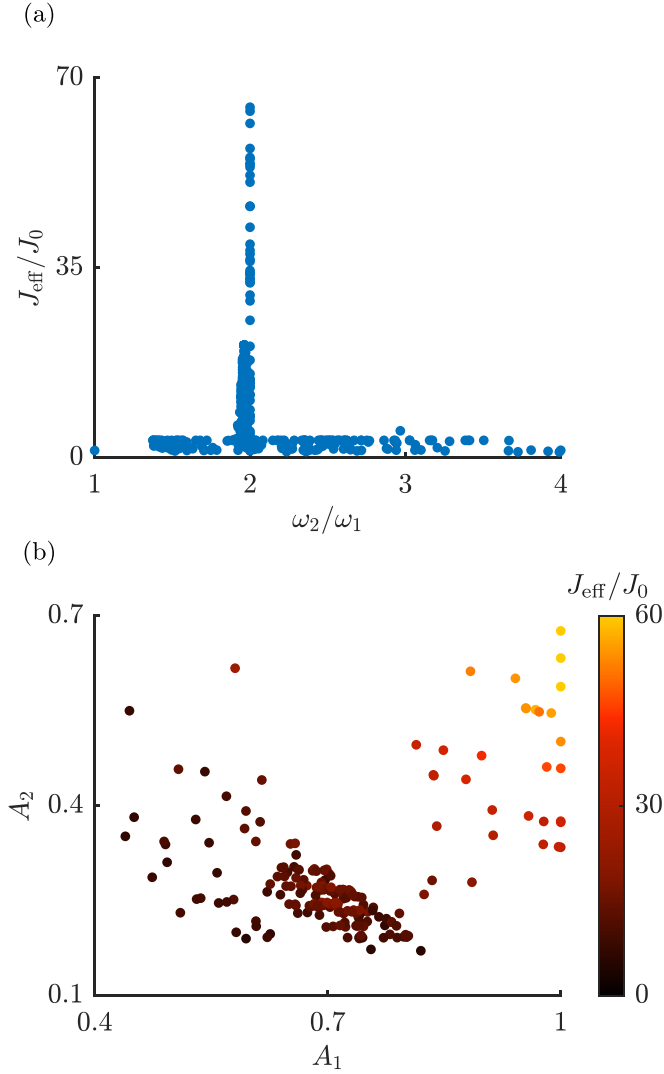


FIG. 3. Optimization of the bichromatic driving protocol $F_B(t)$. (a) Maximal effective Josephson coupling J_{eff}/J_0 as a function of the driving frequencies ω_2/ω_1 for 600 optimization trajectories. The driving frequency ω_1 is usually within the blue-detuned regime, around $\omega_1/\omega_{\text{pl}} \approx 1.1$. We obtain a strong enhancement at $\omega_2/\omega_1 \approx 2$. (b) Maximal effective Josephson coupling J_{eff}/J_0 as a function of the driving amplitudes A_1 and A_2 for 600 optimization trajectories where we only depict trajectories having an effective Josephson coupling $J_{\text{eff}}/J_0 > 10$. We obtain a strong enhancement for $A_1 = 1$.

cases, we obtain a blue-detuned driving frequency ω_1 with respect to the plasma frequency of around $\omega_1/\omega_{\text{pl}} \approx 1.1$. The second driving frequency converges close to $\omega_2/\omega_1 \approx 2$, i.e., the second harmonic, as we illustrate in Fig. 3(a).

In Fig. 3(b), we show the subspace of the driving amplitudes A_1, A_2 and the corresponding effective Josephson coupling J_{eff}/J_0 . A driving amplitude of $A_1 \approx 1$ and driving amplitude of $A_2 \approx 0.7$ lead to a significant increase of the effective Josephson coupling J_{eff}/J_0 of around 70. Preferably, the phase difference θ is close to 0 for a maximized effective Josephson coupling J_{eff}/J_0 , which has been shown in previous studies [11]. We obtain that a maximized amplitude $|F_B(t)|$ from the driving protocol $F_B(t)$ is preferable. We observe that we deviate from a phase difference of $\theta = 0$ in order to avoid

$F_B(t) < -1$ for all times, which would lead to a negative Josephson coupling, in favor of the maximized $|F_B(t)|$, (see also Fig. 10).

Overall, we find a remarkable increase in the effective Josephson coupling J_{eff}/J_0 of 71.69 with the driving parameters $A_1 = 1$, $A_2 = 0.6334$, $\omega_1/\omega_{\text{pl}} = 1.17$, $\omega_2/\omega_{\text{pl}} = 2.34$, and $\theta = -0.3812$.

In addition to the numerical results, we analyze the low-frequency behavior of the conductivity analytically. We consider a bichromatic driving protocol of the form

$$F_B(t) = A \cos(\omega_d t) + B \cos(2\omega_d t), \quad (11)$$

where the second driving frequency is fixed to the second harmonic of the fundamental driving frequency ω_d . Treating the driving amplitudes A and B as small expansion parameters, we linearize the phase dynamics and perform the perturbative expansion up to third order. Within this approach, we obtain an analytical expression for the effective Josephson coupling J_{eff} as

$$J_{\text{eff}} = J_0 \left[1 - \frac{\omega_{\text{pl}}^2 A^2}{2} J_A^{(2)} - \frac{\omega_{\text{pl}}^2 B^2}{2} J_B^{(2)} + \frac{3\omega_{\text{pl}}^4 A^2 B}{4} J_{AB}^{(3)} \right], \quad (12)$$

with

$$\begin{aligned} J_A^{(2)} &= \frac{\omega_{\text{pl}}^2 - \omega_d^2}{(\omega_{\text{pl}}^2 - \omega_d^2)^2 + \gamma^2 \omega_d^2}, \\ J_B^{(2)} &= \frac{\omega_{\text{pl}}^2 - 4\omega_d^2}{(\omega_{\text{pl}}^2 - 4\omega_d^2)^2 + 4\gamma^2 \omega_d^2}, \\ J_{AB}^{(3)} &= \frac{\omega_{\text{pl}}^4 - 6\omega_{\text{pl}}^2 \omega_d^2 + 8\omega_d^4}{((\omega_{\text{pl}}^2 - \omega_d^2)^2 + \gamma^2 \omega_d^2)((\omega_{\text{pl}}^2 - 4\omega_d^2)^2 + 4\gamma^2 \omega_d^2)}. \end{aligned} \quad (13)$$

These expressions show that both the second-order terms $J_A^{(2)}, J_B^{(2)}$ as well as the mixed third-order term $J_{AB}^{(3)}$ give a positive contribution to the dc conductivity when operating in the blue-detuned regime $\omega_{\text{pl}} < \omega_d$. As a result, we find that the effective Josephson coupling J_{eff} is enhanced relative to its equilibrium value. Comparing this analytical result with the numerical data, we observe that the conductivity $\sigma(\omega_{\text{pr}})$ generally follows the same qualitative trend of the numerical conductivity of the optimized protocol; their magnitudes do not match exactly. In particular, as $\omega_{\text{pr}} \rightarrow 0$, the analytically predicted enhancement is smaller than the numerical value. We attribute this discrepancy to the neglect of phase differences in the analytical protocol and higher-order nonlinear contributions that are not captured in the third-order perturbative expansion. For a detailed derivation of the effective Josephson coupling Eq. (12), see Appendix D.

B. Trichromatic driving protocol

Next, we discuss the case of the trichromatic driving protocol $F_T(t)$. We set the driving frequency ω_2 to the second harmonic of ω_1 motivated by the optimal bichromatic driving protocol obtained above. We generate 950 optimization trajectories for various initial driving parameters q_0 (Table II) in

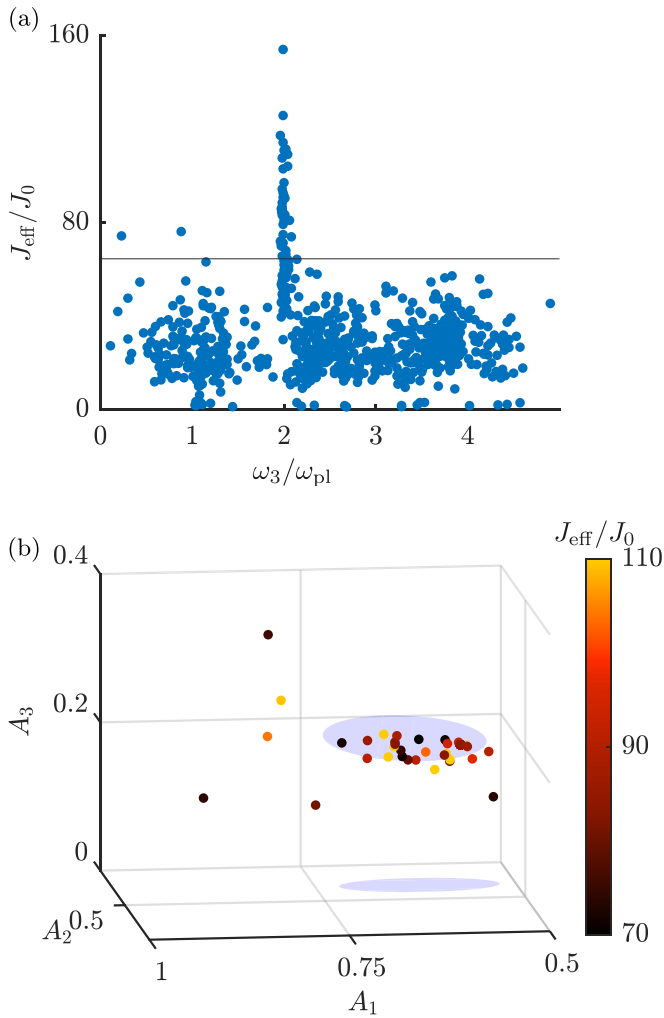


FIG. 4. Optimization of the trichromatic driving protocol $F_T(t)$. (a) Maximal effective Josephson coupling J_{eff}/J_0 as a function of the driving frequency $\omega_3/\omega_{\text{pl}}$ for 950 optimization trajectories. The driving frequency ω_1 is usually within the blue-detuned regime, i.e., $\omega_1/\omega_{\text{pl}} \approx 1.1$. We obtain a strong enhancement at $\omega_3/\omega_{\text{pl}} \approx 2$. The black line indicates the highest effective Josephson coupling J_{eff}/J_0 (≈ 70) that we find with a bichromatic driving protocol $F_B(t)$. (b) Maximal effective Josephson coupling J_{eff}/J_0 as a function of the driving amplitudes A_1, A_2 , and A_3 for 950 optimization trajectories where we only depict trajectories having an effective Josephson coupling $J_{\text{eff}}/J_0 > 70$. The shadowed elliptic areas are the projection on the A_1, A_2 and A_1, A_3 planes.

order to optimize the driving parameters q of the trichromatic driving protocol $F_T(t)$. In Fig. 4(a), we show the maximal effective Josephson coupling J_{eff}/J_0 as a function of the driving frequency ω_3 for all optimization trajectories. We find a significant increase of the effective Josephson coupling $J_{\text{eff}}/J_0 \approx 155$ for driving frequencies $\omega_1/\omega_{\text{pl}} \approx 1.1$ and $\omega_3/\omega_{\text{pl}} \approx 2$. A driving amplitude of $A_1 \sim [0.5-0.75]$ and a driving amplitude of $A_2 \sim [0.5-0.2]$ with a slightly smaller A_3 constitute the optimal regime, as shown in Fig. 4(b). Note that in contrast to the bichromatic driving protocol $F_B(t)$, the initial driving parameter q_0 , especially ω_3 , is crucial for the convergence of the algorithm, which we further analyze in Appendix C.

Overall, we obtain a remarkable increase in the effective Josephson coupling J_{eff}/J_0 of 155. The corresponding optimal driving parameters are $A_1 = 0.6514$, $A_2 = 0.2991$, $A_3 = 0.1873$, $\omega_1/\omega_{\text{pl}} = 1.06$, $\omega_3/\omega_{\text{pl}} = 1.99$, $\theta_1 = -0.9230$, and $\theta_2 = 0.6188$.

The result of the optimization algorithm is limited by the minimal probe frequency ω_{pr} . We have therefore examined the optimal protocol more closely for even smaller probe frequencies $\omega_{\text{pr}} \in [0.0001 \times \omega_{\text{pl}}, 0.0005 \times \omega_{\text{pl}}]$ and have reached a value of 134.9 for the effective Josephson coupling J_{eff}/J_0 with the optimized driving parameters as we show in Fig. 1(a). This probing regime corresponds to the left side of the shaded gray area, which indicates the probe frequencies explored in the optimization process.

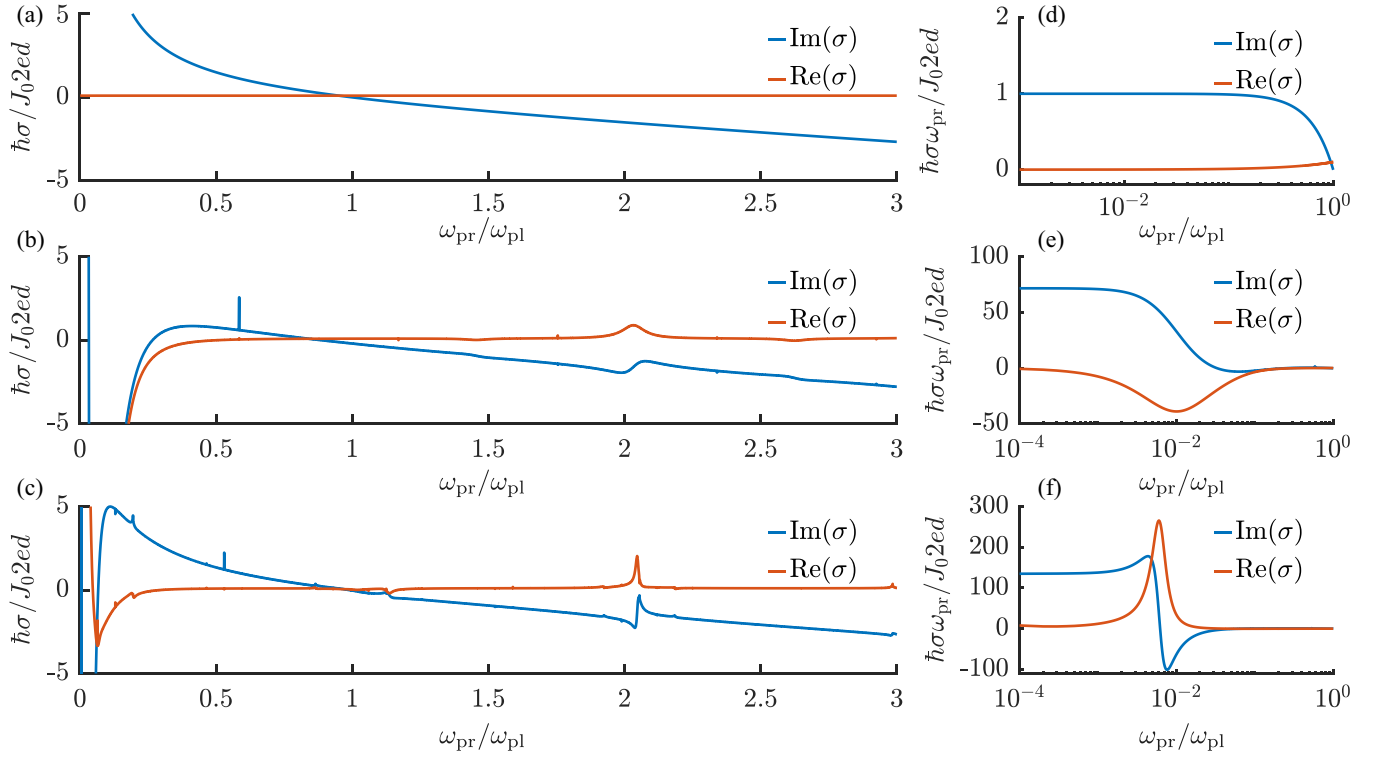
V. CONDUCTIVITY OF OPTIMALLY DRIVEN JUNCTIONS

In this section, we investigate the frequency spectrum from 0 to $3 \times \omega_{\text{pl}}$ and calculate numerically the real and imaginary parts of the conductivity $\sigma(\omega_{\text{pr}})$ according to Eq. (3). We show the results in Fig. 5 for the undriven case [panels (a) and (d)], the optimal bichromatic driving protocol [panels (b) and (e)], and the optimal trichromatic driving protocol [panels (c) and (f)]. In the undriven case, [see Fig. 5(a)], we obtain the expected results with a constant real part $\text{Re}(\sigma) \equiv \sigma_1 = \gamma$ and a $1/\omega_{\text{pr}}$ divergence for the imaginary part $\text{Im}(\sigma) \equiv \sigma_2$ of the conductivity σ . In Fig. 5(d), we display the product $\sigma(\omega_{\text{pr}})\omega_{\text{pr}}$ in the low-frequency limit for the undriven case on a logarithmic x axis. We obtain a constant value for the imaginary (blue) part and a linear behavior for the real (red) part.

As shown in Fig. 5(b), the conductivity of the bichromatically driven junction displays several additional features, compared to the undriven junction. We observe a peak around the probe frequency $\omega_{\text{pr}}/\omega_{\text{pl}} \approx 2$ for both the real (red) and the imaginary (blue) part of the conductivity σ . This is near the second driving frequency ω_2 . Furthermore, while the imaginary part σ_2 has a zero crossing near $\omega_{\text{pr}}/\omega_{\text{pl}} = 1$, similar to the undriven case, both parts σ_1 and σ_2 are negative for probe frequencies smaller than $\omega_{\text{pr}}/\omega_{\text{pl}} = 0.2$. However, in contrast to the real part σ_1 , the imaginary part σ_2 becomes positive again in the limit $\omega_{\text{pr}}/\omega_{\text{pl}} \rightarrow 0$, resulting in the large positive value of the effective Josephson coupling J_{eff}/J_0 discussed before.

In Fig. 5(e), we resolve the low-frequency limit of the product $\sigma(\omega_{\text{pr}})\omega_{\text{pr}}$ for an optimal bichromatic driving protocol with a logarithmic x axis. We obtain a negative real part of the conductivity σ_1 , which is related to optical gain with a maximal value of approximately -30 at $\omega_{\text{pr}}/\omega_{\text{pl}} \approx 0.02$. In the low-frequency limit of $\omega_{\text{pr}}/\omega_{\text{pl}} \rightarrow 0$, σ_1 converges to 0, but it stays negative. The imaginary part σ_2 converges to a value of approximately 70 in the limit, which we defined as the effective Josephson coupling J_{eff}/J_0 according to Eq. (6).

In Fig. 5(c), we show the conductivity of a junction driven by the optimal trichromatic protocol. Similar to the bichromatic protocol, the conductivity displays a peak near $\omega_{\text{pr}}/\omega_{\text{pl}} \approx 2$. However, for the trichromatic protocol, the peak is narrower. Furthermore, the qualitative behavior of the imaginary part σ_2 is similar, with a negative regime below $\omega_{\text{pr}}/\omega_{\text{pl}} \approx 0.1$, down to $\omega_{\text{pr}}/\omega_{\text{pl}} \approx 0.01$. Below this



frequency, σ_2 displays the $1/\omega_{\text{pr}}$ dependence, with a large positive prefactor of around 135, discussed above. The real part of the conductivity σ_1 also changes sign, near $\omega_{\text{pr}}/\omega_{\text{pl}} \approx 0.2$. However, in contrast to the bichromatic protocol, its sign changes again to a positive value near $\omega_{\text{pr}}/\omega_{\text{pl}} \approx 0.01$, therefore not giving rise to optical gain.

VI. CONCLUSION

In conclusion, we have developed optimal control protocols to maximally enhance the dc transport across a Josephson junction. Specifically, we have identified bichromatic and trichromatic protocols, characterized by two or three frequencies, amplitudes, and phases, which modulate the Josephson coupling of the junction parametrically. To develop these protocols, we have utilized a Monte Carlo search, which optimizes the prefactor of the $1/\omega_{\text{pr}}$ divergence of the imaginary part of the conductivity at low frequencies, i.e., the effective Josephson coupling. The bichromatic protocols utilize a driving frequency that is blue detuned relative to the plasma frequency of the junction, and a second frequency at twice the value of the base frequency. The trichromatic protocol also uses approximately these two frequencies, with a third frequency that is not at an integer ratio to the base frequency, but rather near twice the plasma frequency. This trichromatic protocol shows an enhancement of the effective Josephson coupling by a factor of 135, while the bichromatic protocol shows an enhancement by a factor of 70. This

remarkable enhancement suggests that the phenomenon of light-enhanced superconductivity can be strongly advanced by utilizing multifrequency protocols, as a next generation of dynamical control of materials via light.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: ANALYTICAL SOLUTION

In this Appendix, we show the agreement of the numerical solution with the analytical solution in the equilibrium case. First, we linearize Eq. (1):

$$\ddot{\varphi} + \gamma \dot{\varphi} + \omega_{\text{pl}}^2 \varphi = \omega_{\text{pl}}^2 J(t), \quad (\text{A1})$$

with $J(t) = J_0 e^{-i\omega_{\text{pr}} t}$. We choose as our ansatz

$$\varphi(t) = \varphi_0 e^{-i\omega_{\text{pr}} t}. \quad (\text{A2})$$

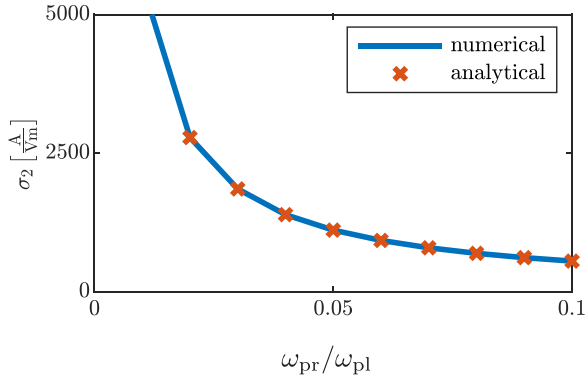


FIG. 6. The imaginary conductivity of a single Josephson junction without driving. The numerical result matches the analytical solution [Eq. (A4)].

Note that $\varphi_0 \propto \frac{E_0}{-i\omega_{pr}}$, which implies that J_0 is proportional to E_0 in form of

$$E_0 \left(\frac{-\omega_{pr}^2 - i\gamma\omega_{pr} + \omega_{pl}^2}{-i\omega_{pr}} \right) \propto J_0, \quad (\text{A3})$$

with the conductivity $\sigma = \sigma_1 + i\sigma_2$ as the proportionality constant. We obtain the real part as $\sigma_1 = \gamma$ and the imaginary part as

$$\sigma_2 = \left(\frac{\omega_{pl}^2}{\omega_{pr}} - \omega_{pr} \right). \quad (\text{A4})$$

In Fig. 6, we show the imaginary part of the conductivity for both numerical and analytical solutions, which are in excellent agreement with each other.

APPENDIX B: FLOQUET ANALYSIS

We briefly explain the Floquet analysis [23,24] that we use to identify protocols that generate strong heating. The Floquet theorem enables statements about homogeneous, ordinary differential equation systems of the form

$$\ddot{\vec{x}}(t) = \mathbf{A}(t)\vec{x}(t), \quad (\text{B1})$$

with a periodic coefficient matrix $\mathbf{A}(t) = \mathbf{A}(t + T)$. The fundamental matrix takes the following form:

$$\varphi(t) = P(t)e^{tR}, \quad (\text{B2})$$

with $P(t) = P(t + T)$ a time periodic function. The eigenvalues λ_i of R are the Floquet exponents. If the absolute value of $|\lambda_i|$ is larger than 1, this indicates dynamical instability and diverging of Eq. (B2). In our model, we linearize Eq. (1) and set $J = 0$, and add parametric driving to the dynamics. We numerically solve the equation of motion

$$\begin{pmatrix} \dot{\varphi}_1 & \dot{\varphi}_2 \\ \ddot{\varphi}_1 & \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_{pl}^2[1 + A \cos(\omega t)] & -\gamma \end{pmatrix} \begin{pmatrix} \varphi_1 & \varphi_2 \\ \dot{\varphi}_1 & \dot{\varphi}_2 \end{pmatrix} \quad (\text{B3})$$

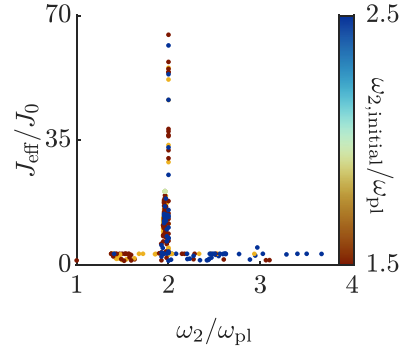


FIG. 7. $\omega_{2,\text{initial}}/\omega_{pl}$ as a function of the final ω_2/ω_{pl} and the corresponding effective Josephson coupling J_{eff}/J_0 for 600 optimization trajectories with a bichromatic driving protocol $F_B(t)$.

over a period from $t = 0$ to $t = 2\pi/\omega$ with the initial conditions

$$(\varphi_1(0), \dot{\varphi}_1(0)) = (1, 0), \quad (\text{B4})$$

$$(\varphi_2(0), \dot{\varphi}_2(0)) = (0, 1). \quad (\text{B5})$$

We obtain the eigenvalues of the resulting matrix

$$\lambda_{1,2} = \pm \sqrt{\frac{1}{4}(\varphi_1 - \dot{\varphi}_2)^2 + \dot{\varphi}_1\varphi_2} + \frac{1}{2}\varphi_1 + \dot{\varphi}_2, \quad (\text{B6})$$

which are the Floquet exponents. These instabilities, which we are identifying with the Floquet exponents, correspond to heating, where energy is absorbed uncontrollably, leading to unbounded energy growth in the system. Dynamical instability often arises when the driving frequency matches or is harmonically related to the natural frequency of the system, in our case, e.g., the plasma frequency ω_{pl} . Performing a Floquet analysis and determining the Floquet exponents leads us to avoid such critical regimes in our parameter choice.

APPENDIX C: IMPACT OF THE INITIAL STATE

In this Appendix, we discuss the impact of the initial state. For a bichromatic driving protocol, the choice of the initial state does not affect the ability of the algorithm to find an optimal protocol.

Our choice of the initial driving parameters q_0 is based on previous results [10,11]. We examine 11 different initial parameter sets q_0 . We choose as combinations for $(\omega_1/\omega_{pl}, \omega_2/\omega_{pl}) = \{(0.7, 1.7), (0.7, 2.5), (1.15, 1.5), (1.15, 1.7), \text{ and } (1.15, 2.5)\}$ for two driving amplitudes $A_1 = 0.1$ and $A_1 = 0.6$. In every initial set, A_2 was set to 0.1 and we tested two different values for $A_1 = (0.1, 0.6)$. The phase difference θ was chosen to be 0.

In Fig. 7, we present Fig. 3(a) once again, as previously presented in the main text. We show $\omega_{2,\text{initial}}/\omega_{pl}$ as a function of the final ω_2/ω_{pl} and the corresponding effective Josephson coupling J_{eff}/J_0 in the optimization. Regardless of the choice of the initial ω_2 , the parameter space was screened and a significant increase in coupling was found around the second harmonic.

TABLE II. Initial driving parameter q_0 used for the optimization of a trichromatic driving protocol $F_T(t)$.

A_1	A_2	A_3	ω_1/ω_{pl}	ω_2/ω_{pl}	ω_3/ω_{pl}	θ_1	θ_2
0.4	0.2	0.1	1.15	2.3	0.7	0	0
0.4	0.2	0.1	1.15	2.3	1.15	0	0
0.4	0.2	0.1	1.15	2.3	1.3	0	0
0.4	0.2	0.1	1.15	2.3	1.7	0	0
0.4	0.2	0.1	1.15	2.3	2.3	0	0
0.4	0.2	0.1	1.15	2.3	2.7	0	0
0.4	0.2	0.1	1.15	2.3	3.3	0	0
0.4	0.2	0.1	1.15	2.3	3.7	0	0
0.4	0.2	0.1	1.15	2.3	4.3	0	0
0.67	0.46	0.36	1.12	2.24	3.8	-0.11	1.95
0.57	0.28	0.19	1.05	2.1	1.98	-0.89	0.95
0.61	0.27	0.16	1.06	2.12	2.01	-0.95	0.64

In contrast, the initial state, more precisely the value of ω_3/ω_{pl} , in the trichromatic driving protocol has a significant influence on the success of the optimization algorithm. We examine the following 13 different initial parameter sets, displayed in Table II.

We illustrate in Fig. 8 the initial third driving frequency $\omega_{3,\text{initial}}/\omega_{pl}$ as a function of the final frequency ω_3/ω_{pl} and the corresponding effective Josephson coupling J_{eff}/J_0 in the optimization. We observe that the choice of the initial ω_3/ω_{pl} has an influence on the range in which the parameter space of ω_3/ω_{pl} is screened. The operating distance is significantly lower. We suspect that this could be due to the restrictive conditions on the parameters q such as the limited driving protocol $F(t) > -1$ or our Floquet analysis to avoid dynamical instability that makes the hopping over forbidden areas more difficult or impossible.

APPENDIX D: PERTURBATION ANALYSIS

In this Appendix, we consider a perturbative ansatz to analytical estimate $\sigma(\omega_{pr})$ for weak probe strengths. We recall Eq. (5). We linearize $\sin(\varphi) \approx \varphi$ by assuming small phase oscillation amplitudes, yielding

$$\ddot{\varphi} + \gamma \dot{\varphi} + \omega_{pl}^2 [1 + F_B(t)] \sin \varphi = \frac{\omega_{pl}^2 J}{J_0}. \quad (\text{D1})$$

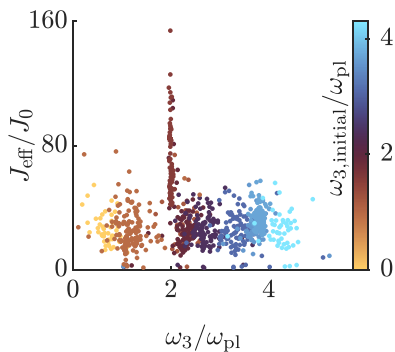


FIG. 8. $\omega_{3,\text{initial}}/\omega_{pl}$ as a function of the final ω_3/ω_{pl} and the corresponding effective Josephson coupling J_{eff}/J_0 for 950 optimization trajectories with a trichromatic driving protocol $F_T(t)$.

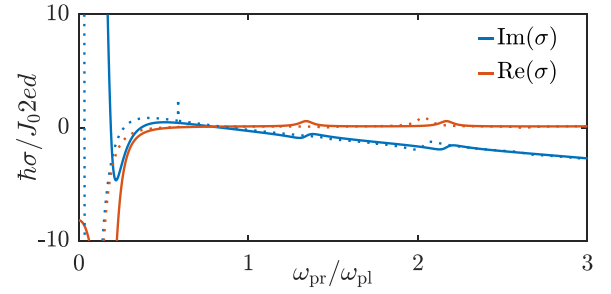


FIG. 9. Real (red) and imaginary (blue) parts of the conductivity $\sigma(\omega_{pr})$ for probe frequencies up to $3 \times \omega_{pr}/\omega_{pl}$ for an optimal bichromatic driving protocol. We compare the analytical result (solid line) of the perturbation analysis with our numerical findings (dotted lines).

We consider a bichromatic driving protocol $F_B(t)$ of the following form:

$$F_B(t) = A \cos(\omega_d t) + B \cos(2\omega_d t), \quad (\text{D2})$$

where A and B denote the amplitudes of the cosines and ω_d denotes the driving frequency. We chose a ratio of 1:2 between the two driving terms based on our numerical findings. For the perturbative ansatz, we treat the driving amplitudes A and B as expansion parameters and we expand φ to the third order in A and B to

$$\begin{aligned} \varphi = & \varphi^{(0)} + A\varphi^{(1)} + B\varphi^{(2)} + AB\varphi^{(3)} \\ & + A^2\varphi^{(4)} + B^2\varphi^{(5)} + A^3\varphi^{(6)} + B^3\varphi^{(7)} \\ & + A^2B\varphi^{(8)} + AB^2\varphi^{(9)} + \mathcal{O}(A^4, B^4, \dots). \end{aligned} \quad (\text{D3})$$

Inserting this ansatz into Eq. (D1) leads to a system of coupled differential equations. We solve the resulting coupled equation and neglect fast-oscillating terms under the assumption $\omega_p \ll \omega_d$. We obtain the nonvanishing terms as

$$\varphi = (\varphi_{0,0} + A^2\varphi_{4,0} + B^2\varphi_{5,0} + A^2B\varphi_{8,0})e^{-i\omega_p t}, \quad (\text{D4})$$

where the coefficients $\varphi_{n,m}$ are the respective expansion coefficients from the ansatz for the solution of the differential equation, corresponding to terms oscillating at the fundamental frequency $e^{-i\omega_p t}$. The first index n refers to the corresponding equation of motion in the perturbative expansion in A and B . The second index m specifies the frequency component, meaning that the term oscillates as $e^{i(m\omega_d - \omega_p)t}$. For example, $\varphi_{4,2}$ is the coefficient of the A^2 correction oscillating at frequency $2\omega_d - \omega_p$. We obtain

$$\varphi_{0,0} = \frac{\omega_{jp}^2}{\omega_{jp}^2 - \omega_p^2 - i\gamma\omega_p}, \quad (\text{D5})$$

$$\varphi_{4,0} = -\frac{\varphi_{0,0}}{2}(\varphi_{1,-1} + \varphi_{1,1}), \quad (\text{D6})$$

$$\varphi_{5,0} = -\frac{\varphi_{0,0}}{2}(\varphi_{2,-1} + \varphi_{2,1}), \quad (\text{D7})$$

$$\varphi_{8,0} = -\frac{\varphi_{0,0}}{2}(\varphi_{4,2} + \varphi_{4,-2} + \varphi_{3,1} + \varphi_{3,-1}), \quad (\text{D8})$$

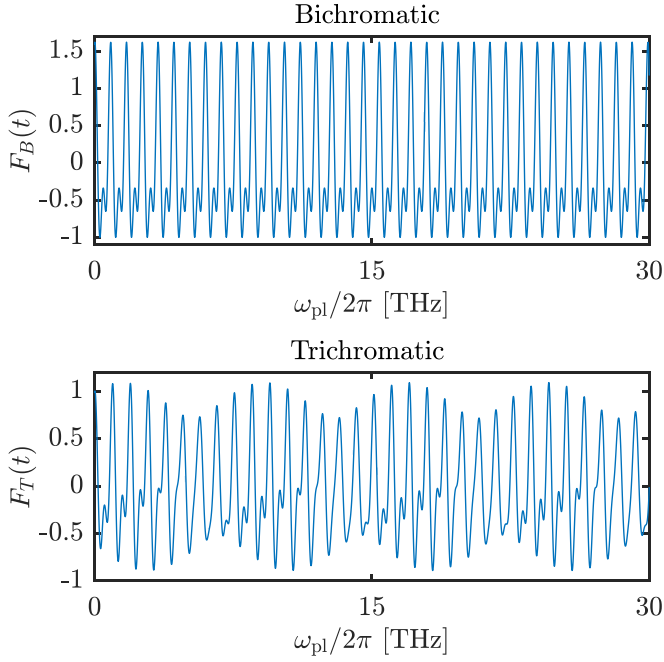


FIG. 10. The optimized driving protocols $F_B(t)$ (top) and $F_T(t)$ (bottom).

with

$$\varphi_{1,1} = -\frac{\omega_{\text{jp}}^2 \varphi_0^{(0)}}{2} \frac{1}{\omega_{\text{jp}}^2 - (\omega_d - \omega_p)^2 + i\gamma(\omega_d - \omega_p)}, \quad (\text{D9})$$

$$\varphi_{1,-1} = -\frac{\omega_{\text{jp}}^2 \varphi_0^{(0)}}{2} \frac{1}{\omega_{\text{jp}}^2 - (\omega_d + \omega_p)^2 - i\gamma(\omega_d + \omega_p)}, \quad (\text{D10})$$

$$\varphi_{2,1} = -\frac{\omega_{\text{jp}}^2 \varphi_0^{(0)}}{2} \frac{1}{\omega_{\text{jp}}^2 - (2\omega_d - \omega_p)^2 + i\gamma(2\omega_d - \omega_p)}, \quad (\text{D11})$$

$$\varphi_{2,-1} = -\frac{\omega_{\text{jp}}^2 \varphi_0^{(0)}}{2} \frac{1}{\omega_{\text{jp}}^2 - (2\omega_d + \omega_p)^2 - i\gamma(2\omega_d + \omega_p)}, \quad (\text{D12})$$

$$\varphi_{3,1} = -\frac{\omega_{\text{jp}}^2}{2} \frac{\varphi_{1,-1} + \varphi_{2,1}}{\omega_{\text{jp}}^2 - (\omega_d - \omega_p)^2 + i\gamma(\omega_d - \omega_p)}, \quad (\text{D13})$$

$$\varphi_{3,-1} = -\frac{\omega_{\text{jp}}^2}{2} \frac{\varphi_{1,1} + \varphi_{2,-1}}{\omega_{\text{jp}}^2 - (\omega_d + \omega_p)^2 - i\gamma(\omega_d + \omega_p)}, \quad (\text{D14})$$

$$\varphi_{4,2} = -\frac{\omega_{\text{jp}}^2}{2} \frac{\varphi_{1,1}}{\omega_{\text{jp}}^2 - (2\omega_d - \omega_p)^2 + i\gamma(2\omega_d - \omega_p)}, \quad (\text{D15})$$

$$\varphi_{4,-2} = -\frac{\omega_{\text{jp}}^2}{2} \frac{\varphi_{1,-1}}{\omega_{\text{jp}}^2 - (2\omega_d + \omega_p)^2 - i\gamma(2\omega_d + \omega_p)}. \quad (\text{D16})$$

We recall Eq. (3) together with the junction voltage in the time domain $V(t) = (\hbar/2e)\dot{\varphi}(t)$ from the main text as

$$\sigma(\omega_{\text{pr}}) = \frac{2ed}{\hbar} \frac{J(\omega_{\text{pr}})}{\dot{\varphi}(t)}. \quad (\text{D17})$$

Inserting Eq. (D4) into Eq. (D17), we find

$$\sigma(\omega_{\text{pr}}) = -\frac{2eJ_0d}{i\omega_{\text{pr}}\hbar\varphi_{0,0}} \left[1 + \frac{A^2}{2}(\varphi_{1,-1} + \varphi_{1,1}) \right] \quad (\text{D18})$$

$$+ \frac{B^2}{2}(\varphi_{2,-2} + \varphi_{2,2}) \quad (\text{D19})$$

$$+ \frac{A^2B}{2}(\varphi_{4,2} + \varphi_{4,-2} + \varphi_{3,1} + \varphi_{3,-1}) \right]. \quad (\text{D20})$$

We recall the effective Josephson coupling from the main text as

$$J_{\text{eff}} = \frac{\hbar}{2ed} [\sigma_2(\omega_{\text{pr}})\omega_{\text{pr}}]_{\omega_{\text{pr}} \rightarrow 0}. \quad (\text{D21})$$

In the limit of $\omega_{\text{pr}} \rightarrow 0$, this leads to the analytical prediction

$$J_{\text{eff}} = J_0 \left[1 - \frac{\omega_{\text{pl}}^2 A^2}{2} \frac{\omega_{\text{pl}}^2 - \omega_d^2}{(\omega_{\text{pl}}^2 - \omega_d^2)^2 + \gamma^2 \omega_d^2} - \frac{\omega_{\text{pl}}^2 B^2}{2} \frac{\omega_{\text{pl}}^2 - 4\omega_d^2}{(\omega_{\text{pl}}^2 - 4\omega_d^2)^2 + 4\gamma^2 \omega_d^2} + \frac{3\omega_{\text{pl}}^4 A^2 B}{4} \frac{\omega_{\text{pl}}^4 - 6\omega_{\text{pl}}^2 \omega_d^2 + 8\omega_d^4}{(\omega_{\text{pl}}^2 - 4\omega_d^2)^2 + 4\gamma^2 \omega_d^2} \times \frac{1}{(\omega_{\text{pl}}^2 - \omega_d^2)^2 + \gamma^2 \omega_d^2} \right] \quad (\text{D22})$$

of the effective Josephson coupling. The A^2 and B^2 terms correspond to the first and second harmonic driving, respectively. Both enhance the dc conductivity in the blue-detuned regime ($\omega_d > \omega_{\text{jp}}$). The third-order A^2B term also gives a positive contribution in this regime, as its numerator remains positive for $\omega_d > \omega_{\text{jp}}$. Thus, all terms act to increase the effective Josephson coupling under optimal driving conditions. In Fig. 9, we compare the analytical estimation (solid line) with our numerical findings (dashed line) for the optimal bichromatic driving protocol $F_B(t)$. Our analytical estimate is in qualitative agreement with the numerical results.

APPENDIX E: PULSE SHAPE

The two optimal protocols are depicted in Fig. 10, as a visualization over 30 periods of the plasma frequency ω_{pl} . In the bichromatic driving protocol $F_B(t)$ (top), the amplitude of the drive remains constant. In the trichromatic driving protocol $F_T(t)$ (bottom), the amplitude of the drive is not constant but periodically modeled.

- [1] W. Hu, S. Kaiser, D. Nicoletti, C. R. Hunt, I. Gierz, M. C. Hoffmann, M. Le Tacon, T. Loew, B. Keimer, and A. Cavalleri, Optically enhanced coherent transport in $\text{YBa}_2\text{Cu}_3\text{O}_{6.5}$ by ultrafast redistribution of interlayer coupling, *Nat. Mater.* **13**, 705 (2014).
- [2] D. Fausti, R. I. Tobey, N. Dean, S. Kaiser, A. Dienst, M. C. Hoffmann, S. Pyon, T. Takayama, H. Takagi, and A. Cavalleri, Light-induced superconductivity in a stripe-ordered cuprate, *Science* **331**, 189 (2011).
- [3] M. Först, A. Frano, S. Kaiser, R. Mankowsky, C. R. Hunt, J. J. Turner, G. L. Dakovski, M. P. Minitti, J. Robinson, T. Loew, M. Le Tacon, B. Keimer, J. P. Hill, A. Cavalleri, and S. S. Dhesi, Femtosecond x rays link melting of charge-density wave correlations and light-enhanced coherent transport in $\text{YBa}_2\text{Cu}_3\text{O}_{6.6}$, *Phys. Rev. B* **90**, 184514 (2014).
- [4] M. Först, R. Mankowsky, and A. Cavalleri, Mode-selective control of the crystal lattice, *Acc. Chem. Res.* **48**, 380 (2015).
- [5] S. Kaiser, C. R. Hunt, D. Nicoletti, W. Hu, I. Gierz, H. Y. Liu, M. Le Tacon, T. Loew, D. Haug, B. Keimer, and A. Cavalleri, Optically induced coherent transport far above T_c in underdoped $\text{YBa}_2\text{Cu}_3\text{O}_{6+\delta}$, *Phys. Rev. B* **89**, 184516 (2014).
- [6] D. Nicoletti and A. Cavalleri, Nonlinear light-matter interaction at terahertz frequencies, *Adv. Opt. Photon.* **8**, 401 (2016).
- [7] K. A. Cremin, J. Zhang, C. C. Homes, G. D. Gu, Z. Sun, M. M. Fogler, A. J. Millis, D. N. Basov, and R. D. Averitt, Photoenhanced metastable c -axis electrodynamics in stripe-ordered cuprate $\text{La}_{1.885}\text{Ba}_{0.115}\text{CuO}_4$, *Proc. Natl. Acad. Sci. USA* **116**, 19875 (2019).
- [8] K. Katsumi, M. Nishida, S. Kaiser, S. Miyasaka, S. Tajima, and R. Shimano, Near-infrared light-induced superconducting-like state in underdoped $\text{YBa}_2\text{Cu}_3\text{O}_y$ studied by c -axis terahertz third-harmonic generation, *Phys. Rev. B* **107**, 214506 (2023).
- [9] M. Nishida, K. Katsumi, D. Song, H. Eisaki, and R. Shimano, Light-induced coherent interlayer transport in stripe-ordered $\text{La}_{1.6-x}\text{Nd}_{0.4}\text{Sr}_x\text{CuO}_4$, *Phys. Rev. B* **107**, 174523 (2023).
- [10] J.-I. Okamoto, A. Cavalleri, and L. Mathey, Theory of enhanced interlayer tunneling in optically driven high- T_c superconductors, *Phys. Rev. Lett.* **117**, 227001 (2016).
- [11] J.-I. Okamoto, W. Hu, A. Cavalleri, and L. Mathey, Transiently enhanced interlayer tunneling in optically driven high- T_c superconductors, *Phys. Rev. B* **96**, 144505 (2017).
- [12] G. Homann, J. G. Cosme, and L. Mathey, Parametric control of Meissner screening in light-driven superconductors, *New J. Phys.* **24**, 113007 (2022).
- [13] B. Zhu, V. P. Singh, J. Okamoto, and L. Mathey, Dynamical control of the conductivity of an atomic Josephson junction, *Phys. Rev. Res.* **3**, 013111 (2021).
- [14] G. Homann, J. G. Cosme, J. Okamoto, and L. Mathey, Higgs mode mediated enhancement of interlayer transport in high- T_c cuprate superconductors, *Phys. Rev. B* **103**, 224503 (2021).
- [15] G. Homann, J. G. Cosme, and L. Mathey, Higgs time crystal in a high- T_c superconductor, *Phys. Rev. Res.* **2**, 043214 (2020).
- [16] T. E. Glier, M. Rerrer, L. Westphal, G. Lüllau, L. Feng, S. Tian, R. Haenel, M. Zonno, H. Eisaki, M. Greven, A. Damascelli, S. Kaiser, D. Manske, and M. Rübhausen, Direct observation of the Higgs mode in a superconductor by non-equilibrium Raman scattering, *Nat. Commun.* **16**, 7027 (2023).
- [17] N. Bittner, T. Tohyama, S. Kaiser, and D. Manske, Possible light-induced superconductivity in a strongly correlated electron system, *J. Phys. Soc. Jpn.* **88**, 044704 (2019).
- [18] B. Josephson, Possible new effects in superconductive tunnelling, *Phys. Lett.* **1**, 251 (1962).
- [19] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, Equation of state calculations by fast computing machines, *J. Chem. Phys.* **21**, 1087 (1953).
- [20] G. Ashton, N. Bernstein, J. Buchner, X. Chen, G. Csányi, A. Fowle, F. Feroz, M. Griffiths, W. Handley, M. Habeck, *et al.*, Nested sampling for physical scientists, *Nat. Rev. Methods Primers* **2**, 39 (2022).
- [21] B. Zhu, T. Rexin, and L. Mathey, Magnus expansion approach to parametric oscillator systems in a thermal bath, *Z. Naturforsch. A* **71**, 921 (2016).
- [22] D. E. McCumber, Effect of ac impedance on dc voltage-current characteristics of superconductor weak-link junctions, *J. Appl. Phys.* **39**, 3113 (1968).
- [23] A. Coddington and R. Carlson, *Linear Ordinary Differential Equations* (Society for Industrial and Applied Mathematics, Philadelphia, PA, 1997).
- [24] L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations* (Springer, Berlin, 2013).