

Phase-driven precision boost in quantum compression for postselected metrology

Aiham M. Rostom^{1,2,3,*}, Saeed Haddadi^{4,†} and Vladimir A. Tomilin²

¹*Novosibirsk State University, 630090 Novosibirsk, Russia*

²*Institute of Automation and Electrometry SBRAS, 630090 Novosibirsk, Russia*

³*Department of Physics, Lattakia University, Lattakia, Syria*

⁴*School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5531, Tehran, Iran*



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We reveal the noncyclic Pancharatnam phase—arising from the coherent system-meter interaction—as a fundamental criterion that governs the optimal performance of quantum compression channels in postselected metrology. This phase embodies a geometric connection that enables precise control over the parallel evolution of the meter state, thereby maximizing the quantum Fisher information per trial and achieving lossless compression channels. Remarkably, fine-tuning the postselection parameter just below this optimal phase incurs substantial information loss, whereas tuning it just above fully suppresses undesired parallel evolution, enhancing information retention beyond that achievable in postselected protocols lacking Pancharatnam phase effects. We further reveal that leveraging qudit-meter states can unlock a substantial additional enhancement. These findings establish the Pancharatnam phase as a geometric benchmark, guiding the design of high-precision quantum parameter estimation protocols.

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I. INTRODUCTION

Harnessing quantum correlations, quantum metrology enables parameter estimation at sensitivities unattainable by classical methods, with broad implications for foundational physics and advanced measurement applications [1–6]. Compression channels in postselected quantum metrology, wherein measurement outcomes are conditioned on specific postselection events, have emerged as a powerful paradigm for amplifying weak signals and enhancing sensitivity, offering genuine quantum and technical advantages [7–9].

Quantum compression channels enable more efficient and robust quantum metrology by allowing information to be concentrated into fewer, higher-quality measurement outcomes that are particularly valuable when measurements are expensive or noisy [7,10,11].

By employing postselection protocols, compression channels offer a range of significant technical advantages for enhancing precision of quantum metrology. Among these is the effective amplification of the input quanta flux in systems characterized by low generation rates, achieved through the recycling of unpostselected quantum resources [12]. Postselection inherently circumvents detector saturation by selectively heralding rare events [13], thereby mitigat-

ing nonlinear detector response and dead-time limitations. Moreover, the exploitation of postselection frameworks enhances the discrimination fidelity between genuine single photons, substantially suppressing detector noise contributions and elevating measurement sensitivity [10,14]. Beyond these operational benefits, postselection engenders profound foundational phenomena, notably, the emergence of closed timelike curves within certain quantum circuit architectures [15,16]. Recent theoretical and experimental investigations have illuminated the capacity of such nonclassical causal structures to confer metrological advantages that surpass classical bounds [17], opening avenues for exploiting temporal quantum correlations in parameter estimation.

Nevertheless, the intrinsic probabilistic nature of postselection and the associated backaction on the meter state introduce challenges in efficiently compressing and transferring information from the quantum system to the meter [7–9,18–21]. Addressing this challenge requires identifying the optimal operational parameters that maximize the retention of quantum Fisher information (QFI) per trial, a central figure of merit quantifying achievable precision [7].

In this work, we unveil a fundamentally geometric solution to this problem by demonstrating that the operation on the noncyclic Pancharatnam phase uniquely optimizes the quality of quantum compression channels. We concentrate on the Pancharatnam phase that emerges naturally from the coherent system-meter interaction, where the parameter-dependent coupling imprints a structural phase relationship on the composite evolution.

The Pancharatnam phase, originally introduced in the context of geometric phases for polarized light [22] and later extended to general quantum evolutions [23], prescribes a phase relation that captures the intrinsic geometry of quantum

*Contact author: a.rostom@g.nsu.ru

†Contact author: haddadi@ipm.ir

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state transformations. Its robustness and operational significance have been verified theoretically and experimentally using interferometric and polarimetric methods [24–27]. For both pure and mixed states, the Pancharatnam phase manifests as a measurable shift in the interference pattern [23,28]. Our primary focus is on the Pancharatnam phase as a relative phase difference that can be experimentally accessed and quantified. Although the Pancharatnam phase has been extensively studied, its role in optimizing quantum compression channels remains unexplored.

In the following section, we derive the QFI associated with the system-meter interaction parameter as an explicit function of the channel operator. We then specify the channel and meter operators, expressing the QFI accordingly. Building on this foundation, we derive the Pancharatnam phase that characterizes the quantum channel and establishes the parallel transport condition necessary to control the unwanted parallel evolution of the meter state. In Sec. IV, the optimality of the Pancharatnam phase is examined as a natural and physically meaningful parametrization for quantum compression channels. This analysis includes a treatment of general qudit-meter states and provides a clear comparison between scenarios where the Pancharatnam phase is incorporated and those where it is not, emphasizing its crucial role in enhancing compression performance and metrological information encoding.

II. PRELIMINARIES

A. System-meter coupling and channel operator

Consider a quantum system described by the Hilbert space $\mathcal{H}_S = \text{span}\{|S_1\rangle, |S_2\rangle, \dots, |S_i\rangle, \dots\}$, initially prepared in state $|S_i\rangle$. Suppose an ancillary system—the meter—initially in the state $|\mathcal{M}_i\rangle$. Assuming that the system and meter are initially unentangled, their joint initial state is described by the tensor product

$$|\Upsilon\rangle = |S_i\rangle \otimes |\mathcal{M}_i\rangle. \quad (1)$$

Suppose an evolution operator $\hat{J}_{(\lambda,\Theta)}$ acting on the combined system-meter Hilbert space, where Θ denotes an experimentally tunable postselection parameter controlling the quantum system, and λ is the coupling strength between the system and the meter. The joint state is given by

$$|\Upsilon_{(\lambda,\Theta)}\rangle = \hat{J}_{(\lambda,\Theta)}|\Upsilon\rangle = \hat{J}_{(\lambda,\Theta)}|S_i\rangle \otimes |\mathcal{M}_i\rangle. \quad (2)$$

By postselecting the system in the final state $|S_f\rangle$, the output conditional state becomes

$$\begin{aligned} |\Psi_{(\lambda,\Theta)}\rangle &\propto (|S_f\rangle\langle S_f| \otimes \hat{\mathbb{I}})|\Upsilon_{(\lambda,\Theta)}\rangle \\ &= (|S_f\rangle\langle S_f| \otimes \hat{\mathbb{I}})\hat{J}_{(\lambda,\Theta)}|S_i\rangle \otimes |\mathcal{M}_i\rangle \\ &= |S_f\rangle \otimes \hat{\mathbb{K}}_{(\lambda,\Theta)}|\mathcal{M}_i\rangle, \end{aligned} \quad (3)$$

where $\hat{\mathbb{K}}_{(\lambda,\Theta)} = \langle S_f|\hat{J}_{(\lambda,\Theta)}|S_i\rangle$ is the channel operator, representing an effective transformation acting solely on the meter state. Under the action of $\hat{\mathbb{K}}_{(\lambda,\Theta)}$, the output normalized state

$$|\Psi_{(\lambda,\Theta)}\rangle = \mathcal{P}^{-1/2}_{(\lambda,\Theta)}|S_f\rangle \otimes \hat{\mathbb{K}}_{(\lambda,\Theta)}|\mathcal{M}_i\rangle \quad (4)$$

exhibits an amplified sensitivity to weak coupling strength λ . The normalization function $\mathcal{P}_{(\lambda,\Theta)}$ determines the postselection probability:

tion probability:

$$\mathcal{P}_{(\lambda,\Theta)} = \langle \mathcal{M}_i | \hat{\mathbb{K}}^\dagger_{(\lambda,\Theta)} \hat{\mathbb{K}}_{(\lambda,\Theta)} | \mathcal{M}_i \rangle = \langle \hat{\mathbb{K}}^\dagger_{(\lambda,\Theta)} \hat{\mathbb{K}}_{(\lambda,\Theta)} \rangle. \quad (5)$$

B. QFI: Total and parallel evolutions characterized by $\hat{\mathbb{K}}_{(\lambda,\Theta)}$

In this subsection, we analyze the total and parallel contributions to the QFI as expressed through the operator $\hat{\mathbb{K}}_{(\lambda,\Theta)}$.

QFI quantifies the sensitivity of the quantum state $|\Psi_{(\lambda,\Theta)}\rangle$ to changes in the parameter λ and sets the theoretical limits for estimation precision via the quantum Cramér-Rao bound [7]

$$\text{Var}(\lambda) \geq \frac{1}{M\mathcal{I}^\perp_{(\lambda,\Theta)}}, \quad (6)$$

where M is the number of measurements and $\mathcal{I}^\perp_{(\lambda,\Theta)}$ is the observed QFI of a single measurement. QFI is a gauge-invariant quantity (Appendix C), as it depends only on the orthogonal component $|\partial_\lambda \Psi_{(\lambda,\Theta)}\rangle^\perp$ (Appendix B):

$$\begin{aligned} \mathcal{I}^\perp_{(\lambda,\Theta)} &= 4\|\partial_\lambda \Psi_{(\lambda,\Theta)}\|^\perp{}^2 \\ &= 4(\langle \partial_\lambda \Psi_{(\lambda,\Theta)} | \partial_\lambda \Psi_{(\lambda,\Theta)} \rangle - |\langle \Psi_{(\lambda,\Theta)} | \partial_\lambda \Psi_{(\lambda,\Theta)} \rangle|^2) \\ &= 4(\|\partial_\lambda \Psi_{(\lambda,\Theta)}\|^2 - \|\partial_\lambda \Psi_{(\lambda,\Theta)}\|^\parallel{}^2). \end{aligned} \quad (7)$$

The term $\|\partial_\lambda \Psi_{(\lambda,\Theta)}\|^2$ quantifies the total rate of change of the quantum state $|\Psi_{(\lambda,\Theta)}\rangle$ in Hilbert space with respect to parameter λ . The second term corresponds to the parallel component $\|\partial_\lambda \Psi_{(\lambda,\Theta)}\|^\parallel{}^2$ —a gauge-dependent contribution and thus does not contribute to physically accessible information.

By substituting the postselected state $|\Psi_{(\lambda,\Theta)}\rangle$, the QFI under postselection becomes (Appendix D)

$$\begin{aligned} \mathcal{I}^\perp_{(\lambda,\Theta)} &= 4\mathcal{P}^{-2}_{(\lambda,\Theta)}[\mathcal{Q}^T_{(\lambda,\Theta)}\mathcal{P}_{(\lambda,\Theta)} - \mathcal{Q}^\parallel_{(\lambda,\Theta)}] \\ &= \mathcal{I}^T_{(\lambda,\Theta)} - \mathcal{I}^\parallel_{(\lambda,\Theta)}, \end{aligned} \quad (8)$$

where $\mathcal{Q}^T_{(\lambda,\Theta)} = \langle \partial_\lambda \hat{\mathbb{K}}^\dagger_{(\lambda,\Theta)} \partial_\lambda \hat{\mathbb{K}}_{(\lambda,\Theta)} \rangle$ is the average total change in the channel operator with respect to the parameter λ , and $\mathcal{Q}^\parallel_{(\lambda,\Theta)} = |\langle \hat{\mathbb{K}}^\dagger_{(\lambda,\Theta)} \partial_\lambda \hat{\mathbb{K}}_{(\lambda,\Theta)} \rangle|^2$ is its parallel evolution. The parallel component here is not simply subtracted as a gauge artifact but interacts nontrivially with $\mathcal{P}_{(\lambda,\Theta)}$.

The recent theory of compression quantum channels [9] establishes necessary and sufficient conditions for a postselection channel to be lossless, meaning the average QFI of the retained outcomes equals that of the standard QFI (i.e., without postselection), thereby preserving metrological sensitivity despite compression. This condition, detailed in Table I, Theorem 1, and supplementary materials of Ref. [9], is mathematically equivalent to the condition $\mathcal{Q}^\parallel_{(\lambda,\Theta)} = 0$ in this work.

We demonstrate in Secs. III C and IV C that when the interaction induces Pancharatnam phase shifts in the postselected system, the parallel component $\mathcal{Q}^\parallel_{(\lambda,\Theta)}$ exhibits phase-dependent interference controlled by the parameter Θ . This interference allows precise tuning and complete suppression of $\mathcal{Q}^\parallel_{(\lambda,\Theta)}$. Conversely, in the absence of Pancharatnam phase shifts, $\mathcal{Q}^\parallel_{(\lambda,\Theta)}$ reduces to a nonvanishing, Θ -independent squared magnitude of the expectation derivative. This critical difference in Θ -sensitive coherence effects in the former case is essential for enhanced quantum parameter estimation control.

C. QFI yield per trial

In postselected quantum metrology, the number of successful postselections is given by $M = \mathcal{N}\mathcal{P}_{(\lambda,\Theta)}$, where $\mathcal{N}$ denotes the total number of experimental trials and accounts for both successful and unsuccessful postselection attempts. The achievable precision reads

$$\text{Var}(\lambda) \geq \frac{1}{M\mathcal{I}^\perp_{(\lambda,\Theta)}} = \frac{1}{\mathcal{N}\mathcal{P}_{(\lambda,\Theta)}\mathcal{I}^\perp_{(\lambda,\Theta)}} = \frac{1}{\mathcal{N}\mathcal{T}_{(\lambda,\Theta)}}, \quad (9)$$

and the quantum advantage gained via postselection is quantified by the total QFI per trial [2,7]

$$\mathcal{T}_{(\lambda,\Theta)} = \mathcal{P}_{(\lambda,\Theta)}\mathcal{I}^\perp_{(\lambda,\Theta)}, \quad (10)$$

which serves as the central metric for evaluating the efficiency of the compression channel. This metric balances high precision—potentially large $\mathcal{I}^\perp_{(\lambda,\Theta)}$ —against the typically reduced success probability $\mathcal{P}_{(\lambda,\Theta)}$ arising from postselection on rare events [29]. Specifically, when $\mathcal{T}_{(\lambda,\Theta)}$ scales proportionally with the square of the total number of probes, the metrological protocol attains the Heisenberg limit for precision [30].

Our main objective is to identify the optimal postselection parameter $\Theta^\parallel$ that satisfies $\mathcal{Q}^\parallel = 0$, where all parameter-induced changes are encoded fully in physically distinguishable directions in Hilbert space. This is a necessary condition for maximizing $\mathcal{T}_{(\lambda,\Theta)}$ as we will demonstrate below.

III. DETERMINATION OF THE CHANNEL OPERATOR

$$\hat{\mathbb{K}}_{(\lambda,\Theta)}$$

Consider a quantum system described by a two-mode Hilbert space, whose structure can be elegantly and compactly captured through the Jordan-Schwinger map [31]

$$\begin{aligned} \hat{J}_x &= \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1), & \hat{J}_y &= \frac{i}{2}(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2), \\ \hat{J}_z &= \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2), & \hat{J}_t &= \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2), \end{aligned} \quad (11)$$

where $(\hat{a}_1^\dagger, \hat{a}_2^\dagger)$ and $(\hat{a}_1, \hat{a}_2)$ are boson creation and annihilation operators for modes 1 and 2, respectively. These operators form a closed algebra under the commutator bracket, satisfying $[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k$, where ϵ_{ijk} is the Levi-Civita symbol and $\hat{J}_t$ commutes with all other $\hat{J}_i$.

The system-meter interaction is commonly modeled by a Hamiltonian that couples the meter to one or more system observables. The total Hilbert space for the combined system is given by the tensor product $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$, where $\mathcal{H}_M$ denote the Hilbert space of the meter. The interaction dynamics can be described by the unitary operator

$$\hat{\Lambda}_{(\lambda)} = e^{i\lambda(\hat{J}_t^{(S)} + \hat{J}_z^{(S)}) \otimes \hat{\mathcal{M}}}, \quad (12)$$

where $\hat{\mathcal{M}}$ is a Hermitian operator acting on the meter's state. The operator $\hat{\Lambda}_{(\lambda)}$ encapsulates the interaction between the system's mode 1 and the meter [32]. Incorporating a controlled phase Θ applied to mode 2, the overall operator governing the evolution is then defined as

$$\begin{aligned} \hat{\mathbb{J}}_{(\lambda,\Theta)} &= [e^{-i\frac{\pi}{2}\hat{J}_x^{(S)}} \otimes \hat{\mathbb{I}}] [e^{i\Theta(\hat{J}_t^{(S)} - \hat{J}_z^{(S)})} \otimes \hat{\mathbb{I}}] \\ &\times \hat{\Lambda}_{(\lambda)} [e^{+i\frac{\pi}{2}\hat{J}_x^{(S)}} \otimes \hat{\mathbb{I}}]. \end{aligned} \quad (13)$$

The operator $e^{\pm i\frac{\pi}{2}\hat{J}_x^{(S)}}$ generates a rotation about the x axis, effectively implementing the 50:50 splitting and recombination of the two quantum modes. Using the identity $\hat{U}e^{\hat{A}}\hat{U}^\dagger = e^{\hat{U}\hat{A}\hat{U}^\dagger}$, and Baker-Campbell-Hausdorff relations $e^{-i\frac{\pi}{2}\hat{J}_x^{(S)}}\hat{J}_z^{(S)}e^{+i\frac{\pi}{2}\hat{J}_x^{(S)}} = -\hat{J}_y^{(S)}$, $e^{-i\frac{\pi}{2}\hat{J}_x^{(S)}}\hat{J}_t^{(S)}e^{+i\frac{\pi}{2}\hat{J}_x^{(S)}} = \hat{J}_t^{(S)}$, the operator $\hat{\mathbb{J}}_{(\lambda,\Theta)}$ can be expressed as

$$\hat{\mathbb{J}}_{(\lambda,\Theta)} = [e^{i\Theta(\hat{J}_t^{(S)} + \hat{J}_y^{(S)})} \otimes \hat{\mathbb{I}}] e^{i\lambda(\hat{J}_t^{(S)} - \hat{J}_y^{(S)}) \otimes \hat{\mathcal{M}}}. \quad (14)$$

The pre- and postselected states can be chosen as $(|S_i\rangle = |j, m_i\rangle$ and $|S_f\rangle = |j, m_f\rangle)$, where [3]

$$|j, m\rangle \equiv |n_1, n_2\rangle = \frac{(a_1^\dagger)^{j+m}(a_2^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0, 0\rangle,$$

here $j = \frac{n_1+n_2}{2}$ is the total angular-momentum quantum number, $m = \frac{n_1-n_2}{2}$ is the eigenvalue of $\hat{J}_z$ (the magnetic quantum number), and $|0, 0\rangle$ is the vacuum state. Consequently, the operator $\hat{\mathbb{K}}_{(\lambda,\Theta)}$ can be obtained by projecting $\hat{\mathbb{J}}_{(\lambda,\Theta)}$ onto the pre- and postselected states, yielding

$$\begin{aligned} \hat{\mathbb{K}}_{m_f, m_i}^{(j)}(\lambda, \Theta) &= \langle S_f | \hat{\mathbb{J}}_{(\lambda,\Theta)} | S_i \rangle \\ &= \langle j, m_f | \hat{\mathbb{J}}_{(\lambda,\Theta)} | j, m_i \rangle \\ &= e^{i\Theta} e^{i\lambda\hat{\mathcal{M}}} \hat{\mathcal{F}}_{m_f, m_i}^{(j)}(\lambda, \Theta), \end{aligned} \quad (15)$$

where $\hat{J}_t^{(S)}|j, m_i\rangle = j|j, m_i\rangle$, $\hat{J}_z^{(S)}|j, m\rangle = m|j, m\rangle$, and

$$\hat{\mathcal{F}}_{m_f, m_i}^{(j)}(\lambda, \Theta) = \langle j, m_f | e^{i\hat{J}_y^{(S)} \otimes (\Theta\hat{\mathbb{I}} - \lambda\hat{\mathcal{M}})} | j, m_i \rangle. \quad (16)$$

Applying $\hat{\mathcal{F}}_{m_f, m_i}^{(j)}(\lambda, \Theta)$ on the meter eigenstate $|b_k\rangle$ of the operator $\hat{\mathcal{M}}$ gives

$$\hat{\mathcal{F}}_{m_f, m_i}^{(j)}(\lambda, \Theta)|b_k\rangle = d_{m_f, m_i}^{(j)}(\Theta - b_k\lambda)|b_k\rangle,$$

where $d_{m_f, m_i}^{(j)}(\Theta - b_k\lambda)$ is the Wigner d -matrix element [33] that corresponds to the eigenvalues b_k of the meter operator $\hat{\mathcal{M}}$.

Note that the operator in Eq. (14) can be equivalently expressed as

$$\hat{\mathbb{J}}_{(\lambda,\Theta)} = e^{i\hat{J}_t^{(S)} \otimes (\Theta\hat{\mathbb{I}} + \lambda\hat{\mathcal{M}})} e^{i\hat{J}_y^{(S)} \otimes (\Theta\hat{\mathbb{I}} - \lambda\hat{\mathcal{M}})}. \quad (17)$$

Thus, $\hat{\mathcal{F}}_{m_f, m_i}^{(j)}(\lambda, \Theta)$ [Eqs. (15) and (16)] reveals that the postselection probability is primarily governed by the second term in Eq. (17), which incorporates contributions from both the postselection parameter Θ and interaction strength λ . The first term in Eq. (15) shows that $e^{i\Theta\hat{J}_t^{(S)}}$ from Eq. (17) induces only a trivial global phase. Conversely, the second term in Eq. (15) demonstrates that $e^{i\lambda\hat{J}_t^{(S)} \otimes \hat{\mathcal{M}}}$ from Eq. (17) imparts a nontrivial rotation to the postselected state, which can impact the QFI in general scenarios.

From now on and for simplicity, the explicit dependence on the parameters λ and Θ will be omitted in the notation of operators and functions whenever no ambiguity arises.

A. General meter operators

Here, we present a mathematical formula that facilitates the determination of the channel operator $\hat{\mathbb{K}}_{m_f, m_i}^{(j)}$ for an arbitrary meter state.

The states $|j, m_i\rangle$ and $|j, m_f\rangle$ considered above are eigenstates of $\hat{J}_z^{(S)}$, and the operator $\hat{\mathcal{M}}$ acts solely on the meter state. To evaluate the matrix element $\hat{\mathcal{F}}_{m_f, m_i}^{(j)}$, we first express the system states in the eigenbasis of $\hat{J}_y^{(S)}$ using the Wigner d -matrix

$$|j, m\rangle = \sum_{m_y=-j}^j d_{m_y, m}^{(j)}\left(\frac{\pi}{2}\right) |j, m_y\rangle, \quad (18)$$

where $d_{m_y, m}^{(j)}(\pi/2)$ is the Wigner d -matrix element for a rotation by angle $\pi/2$ about the y axis. The operator

$$e^{i\hat{J}_y^{(S)} \otimes (\Theta \hat{\mathbb{I}} - \lambda \hat{\mathcal{M}})} \quad (19)$$

acts diagonally in this basis, yielding eigenvalues m_y for $\hat{J}_y^{(S)}$, so that its action on $|j, m_y\rangle$ is simply a multiplication by $\exp[i m_y (\Theta \hat{\mathbb{I}} - \lambda \hat{\mathcal{M}})]$. Substituting the expansions for both $|j, m_i\rangle$ and $|j, m_f\rangle$, and using the orthogonality of the $\hat{J}_y$ eigenstates, the matrix element reduces to a sum over products of Wigner d -matrix elements and the exponential operator:

$$\hat{\mathcal{F}}_{m_f, m_i}^{(j)} = \sum_{m_y=-j}^j d_{m_y, m_f}^{(j)}\left(\frac{\pi}{2}\right) d_{m_y, m_i}^{(j)}\left(\frac{\pi}{2}\right) e^{i m_y (\Theta \hat{\mathbb{I}} - \lambda \hat{\mathcal{M}})}, \quad (20)$$

where the sum runs over all eigenvalues m_y of $\hat{J}_y^{(S)}$. This construction relates the matrix element $\hat{\mathcal{F}}_{m_f, m_i}^{(j)}$ to a weighted sum over all possible projections onto the $\hat{J}_y$ eigenbasis, with each term governed by the system's rotation properties and the meter operator.

B. The Pancharatnam phase

In this section, we demonstrate that, for a general meter state, the postselected system operates as an interferometric setup exhibiting characteristic Pancharatnam phase effects.

From Eq. (20), the matrix element $\hat{\mathcal{F}}_{m_f, m_i}^{(j)}$ can be written as a finite sum of exponentials in Θ and $\hat{\mathcal{M}}$ for any j . By pre- and postselecting the highest and lowest weight states denoted by $|j, m_i\rangle = |j, j\rangle$ and $|j, m_f\rangle = |j, -j\rangle$, respectively, $\Delta m = m_i - m_f = 2j$ and the channel operator (15) yields

$$\begin{aligned} \hat{\mathbb{K}}_{-j, j}^{(j)} &= e^{i j \Theta} e^{i j \lambda \hat{\mathcal{M}}} \hat{\mathcal{F}}_{-j, j}^{(j)} \\ &= (-1)^j e^{i j \Theta} e^{i j \lambda \hat{\mathcal{M}}} \sin^{2j} \left(\frac{\Theta \hat{\mathbb{I}} - \lambda \hat{\mathcal{M}}}{2} \right) \\ &= \frac{(-1)^j}{(2i)^{2j}} (e^{i \Theta} \hat{\mathbb{I}} - \hat{\mathcal{O}}_\lambda)^{2j}, \end{aligned} \quad (21)$$

where $\hat{\mathcal{O}}_\lambda = e^{i \lambda \hat{\mathcal{M}}}$.

Since $\hat{\mathcal{O}}_\lambda$ is unitary, then $\hat{\mathcal{O}}_\lambda^\dagger \hat{\mathcal{O}}_\lambda = \hat{\mathbb{I}}$, $\langle \hat{\mathcal{O}}_\lambda^\dagger \hat{\mathcal{O}}_\lambda \rangle = 1$, and its expectation value satisfies $\langle \hat{\mathcal{O}}_\lambda^\dagger \rangle = \langle \hat{\mathcal{O}}_\lambda \rangle^*$. Define $\langle \hat{\mathcal{O}}_\lambda \rangle = |\langle \hat{\mathcal{O}}_\lambda \rangle| e^{i \text{Im} \ln \langle \hat{\mathcal{O}}_\lambda \rangle}$. The postselection probability $\mathcal{P}_{m_f, m_i}^{(j)} = \mathcal{P}_{-j, j}^{(j)}$ for $j = 1/2$ can be expressed as

$$\begin{aligned} \mathcal{P}_{-\frac{1}{2}, \frac{1}{2}}^{(\frac{1}{2})} &= \frac{1}{4} (2 - e^{i \Theta} \langle \hat{\mathcal{O}}_\lambda^\dagger \rangle - e^{-i \Theta} \langle \hat{\mathcal{O}}_\lambda \rangle) \\ &= \frac{1}{2} [1 - |\langle \hat{\mathcal{O}}_\lambda \rangle| \cos(\Theta - \text{Im} \ln \langle \hat{\mathcal{O}}_\lambda \rangle)], \end{aligned} \quad (22)$$

where $|\langle \hat{\mathcal{O}}_\lambda \rangle|$ is the visibility and $\text{Im} \ln \langle \hat{\mathcal{O}}_\lambda \rangle$ is Pancharatnam phase difference between the two system's modes [34]; see

Appendix E for more details on the interpretation of this phase. It can be defined to be the phase by which one mode must be retarded or advanced to make the postselection probability in the system reach its minimum [22].

For higher values of j , the postselection probability is given by

$$\mathcal{P}_{-j, j}^{(j)} = \frac{1}{16^j} \langle (2\hat{\mathbb{I}} - e^{i \Theta} \hat{\mathcal{O}}_\lambda^\dagger - e^{-i \Theta} \hat{\mathcal{O}}_\lambda)^{2j} \rangle, \quad (23)$$

which represents a $2j$ th-order interference kernel that generates quantum fringes with phase sensitivity controlled by Θ , phase shift inherited from the eigenvalues of $\hat{\mathcal{O}}_\lambda$, and contrast scaling determined by j ; see Fig. 1(a). It is crucial to emphasize that the Pancharatnam phase $\text{Im} \ln \langle \hat{\mathcal{O}}_\lambda \rangle$ remains invariant with respect to j , m_i , and m_f ; see Fig. 1.

However, due to the exponential dependence of the postselection probability on $\Delta m = 2j$, the optimal postselection—which can be viewed as a lossless compression channel—for higher values of j ($j > 1/2$) can be realized when the transition between pre- and postselected states corresponds exactly to a single quantum $\Delta m = 1$. A natural and effective choice fulfilling this condition is $m_i = j$ and $m_f = j - 1$, which corresponds to

$$\hat{\mathcal{F}}_{j-1, j}^{(j)} = \sqrt{2j} \cos^{2j-1} \left(\frac{\Theta \hat{\mathbb{I}} - \lambda \hat{\mathcal{M}}}{2} \right) \sin \left(\frac{\Theta \hat{\mathbb{I}} - \lambda \hat{\mathcal{M}}}{2} \right). \quad (24)$$

For $j = 1$, the channel operator takes the form

$$\hat{\mathbb{K}}_{0,1}^{(1)} = \frac{1}{\sqrt{2}} (e^{i 2 \Theta} \hat{\mathbb{I}} - e^{i 2 \lambda \hat{\mathcal{M}}}), \quad (25)$$

and the corresponding postselection probability is given by

$$\mathcal{P}_{0,1}^{(1)} = \frac{1}{4} [1 - |\langle \hat{\mathcal{O}}_\lambda^2 \rangle| \cos 2(\Theta - \text{Im} \ln \langle \hat{\mathcal{O}}_\lambda \rangle)], \quad (26)$$

where $\hat{\mathcal{O}}_\lambda$ is diagonal in the meter basis and hence $\text{Im} \ln \langle \hat{\mathcal{O}}_\lambda^2 \rangle = 2 \text{Im} \ln \langle \hat{\mathcal{O}}_\lambda \rangle$. The oscillatory behavior of $\mathcal{P}_{0,1}^{(1)}$ manifests at twice the frequency of $\mathcal{P}_{-\frac{1}{2}, \frac{1}{2}}^{(\frac{1}{2})}$, reflecting a harmonic doubling in the interference pattern, while both retain identical Pancharatnam phase shifts.

For higher values of j , analytical treatment becomes increasingly intricate due to the complexities involved in handling powers of operator-valued functions. Nevertheless, numerical simulations, such as those illustrated in Fig. 1, confirm the invariance of the Pancharatnam phase with respect to variations in j , m_i , and m_f .

It is important to note here that alternative pre- and postselected states m_i and m_f can be explored. For instance, with $j = 2$, choosing $m_i = 1$ and $m_f = 0$, entails a single quantum transition governed by the channel operator

$$\hat{\mathbb{K}}_{0,1}^{(2)} = \sqrt{\frac{3}{8}} (e^{i 4 \Theta} \hat{\mathbb{I}} - \hat{\mathcal{O}}_\lambda^4), \quad (27)$$

which likewise induces the identical Pancharatnam phase shift as in Eq. (22), while manifesting more rapid oscillatory behavior.

C. Controlling parallel evolution $\mathcal{Q}^{\parallel}$

If the postselected quantum system acquires a noncyclic Pancharatnam phase characterized by $\text{Im} \ln \langle \hat{O}_\lambda \rangle \neq 0$, the geometric connection $\langle \hat{O}_\lambda^\dagger \partial_\lambda \hat{O}_\lambda \rangle$ must remain nonzero (see Appendixes A and E). This nonvanishing connection enables precise control over the parallel term contribution to the QFI via the parameter Θ , thereby enabling the realization of a lossless quantum compression channel.

To illustrate this optimization mechanism, the QFI can be conveniently reformulated in terms of the meter's interaction operators $\hat{O}_\lambda$. For instance, when $j = 1/2$, the first term, which corresponds to the total rate of change, can be expressed as

$$\mathcal{Q}^T = \left\langle \partial_\lambda \left(\hat{\mathbb{K}}_{-\frac{1}{2}, \frac{1}{2}}^{(\frac{1}{2})} \right)^\dagger \partial_\lambda \hat{\mathbb{K}}_{-\frac{1}{2}, \frac{1}{2}}^{(\frac{1}{2})} \right\rangle = \frac{1}{4} \langle \partial_\lambda \hat{O}_\lambda^\dagger \partial_\lambda \hat{O}_\lambda \rangle. \quad (28)$$

At the same time, the parallel term becomes

$$\begin{aligned} \mathcal{Q}^{\parallel} &= \left| \left\langle \left(\hat{\mathbb{K}}_{-\frac{1}{2}, \frac{1}{2}}^{(\frac{1}{2})} \right)^\dagger \partial_\lambda \hat{\mathbb{K}}_{-\frac{1}{2}, \frac{1}{2}}^{(\frac{1}{2})} \right\rangle \right|^2 \\ &= \frac{1}{16} |\langle \partial_\lambda \hat{O}_\lambda \rangle - e^{i\Theta} \langle \hat{O}_\lambda^\dagger \partial_\lambda \hat{O}_\lambda \rangle|^2. \end{aligned} \quad (29)$$

When the connection $\langle \hat{O}_\lambda^\dagger \partial_\lambda \hat{O}_\lambda \rangle \neq 0$, active suppression of the parallel term $\mathcal{Q}^{\parallel}$ becomes possible: By tuning Θ to align with the operator's intrinsic phase, $\mathcal{Q}^{\parallel}$ can be driven to zero, thus surpassing the precision achievable under parallel transport $\langle \hat{O}_\lambda^\dagger \partial_\lambda \hat{O}_\lambda \rangle = 0$.

Conversely, when the connection vanishes $\langle \hat{O}_\lambda^\dagger \partial_\lambda \hat{O}_\lambda \rangle = 0$, $\mathcal{Q}^{\parallel}$ simplifies to $|\langle \partial_\lambda \hat{O}_\lambda \rangle|^2$, reflecting orthogonal fluctuations determined by the magnitude of the derivative and unaffected by Θ .

The extension to higher values of j (particularly, $\hat{\mathbb{K}}_{0,1}^{(1)}$ and $\hat{\mathbb{K}}_{0,1}^{(2)}$) proceeds analogously to the framework established in the preceding section. In Sec. IV C, we demonstrate that for any value of j , the parallel component $\mathcal{Q}^{\parallel}$ can be fully controlled and nullified through the same value of the parameter Θ , especially when the pre- and postselection processes induce transitions involving a single quantum change (lossless quantum channel).

D. Qudit-Meter State

Additionally, we consider a specific yet sufficiently general qudit-meter state $|\mathcal{M}_i\rangle = |\mathcal{M}_d^{(n)}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |b_k\rangle^{\otimes n}$, for n subsystems and d -dimensional Hilbert space spanned by the subspace $\mathcal{B} = \{|b_0\rangle^{\otimes n}, |b_1\rangle^{\otimes n}, \dots, |b_{d-1}\rangle^{\otimes n}\}$.

The meter operator can be defined as

$$e^{i n \lambda \hat{\mathcal{M}}} = \sum_{k=0}^{d-1} e^{i n b_k \lambda} (|b_k\rangle \langle b_k|)^{\otimes n}, \quad (30)$$

which imprints a mode-dependent relative phase $e^{i n b_k \lambda}$ on each component $|b_k\rangle^{\otimes n}$, yielding

$$e^{i n \lambda \hat{\mathcal{M}}} |\mathcal{M}_d^{(n)}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{i n b_k \lambda} |b_k\rangle^{\otimes n}. \quad (31)$$

This ensures a phase sensitivity of the meter state scales linearly with the number of subsystems n . The operator $e^{i n \lambda \hat{\mathcal{M}}}$ reduces to the familiar single-qubit phase gate in case $d = 2$,

$n = 1$, thus providing a natural generalization to multipartite, high-dimensional entangled meter states (see Appendix D for more details). By acting on $|\mathcal{M}_d^{(n)}\rangle$ using the channel operator, we get

$$\hat{\mathbb{K}}_{m_f, m_i}^{(j)} |\mathcal{M}_d^{(n)}\rangle = \frac{e^{i j \Theta}}{\sqrt{d}} \sum_{k=0}^{d-1} e^{i j n b_k \lambda} d_{m_f, m_i}^{(j)}(\beta_k) |b_k\rangle^{\otimes n}, \quad (32)$$

where $\beta_k = \Theta - n b_k \lambda$. Thus, the postselection probability gives

$$\mathcal{P}_{m_f, m_i}^{(j)} = \frac{1}{d} \sum_{k=0}^{d-1} [d_{m_f, m_i}^{(j)}(\beta_k)]^2, \quad (33)$$

here the squared d -matrix elements quantify the transition probabilities between the states $|j, m_i\rangle$ and $|j, m_f\rangle$ induced by a rotation through the angle β_k . Averaging these elements over d terms reflects a uniform sampling of the rotation parameter β_k , ensuring an unbiased characterization. For the Wigner d matrix $\sum_{m_f} d_{m_f, m_i}^{(j)}(\beta_k) d_{m_f, m_j}^{(j)}(\beta_k) = \delta_{m_i, m_j}$, and hence $\sum_{m_f} [d_{m_f, m_i}^{(j)}(\beta_k)]^2 = 1$, which confirms that $\sum_{m_f} \mathcal{P}_{m_f, m_i}^{(j)} = 1$.

Taking the derivative of Eq. (32) with respect to the parameter λ gives

$$\partial_\lambda \hat{\mathbb{K}}_{m_f, m_i}^{(j)} |\mathcal{M}_d^{(n)}\rangle = \frac{n e^{i j \Theta}}{\sqrt{d}} \sum_{k=0}^{d-1} b_k e^{i j n b_k \lambda} A_k |b_k\rangle^{\otimes n}, \quad (34)$$

where $A_k := i j d_{m_f, m_i}^{(j)}(\beta_k) - \frac{d}{d\beta} d_{m_f, m_i}^{(j)}(\beta)|_{\beta=\beta_k}$. The total change in the channel operator captured by $\mathcal{Q}^T$ can subsequently be expressed as

$$\begin{aligned} \mathcal{Q}^T &= \frac{n^2}{d} \sum_{k=0}^{d-1} \sum_{k'=0}^{d-1} b_k u_{k'} e^{-i j n u_{k'} \lambda} e^{i j n b_k \lambda} \\ &\quad \times A_k A_{k'}^* \langle b_{k'} |^{\otimes n} |b_k\rangle^{\otimes n} \\ &= \frac{n^2}{d} \sum_{k=0}^{d-1} b_k^2 |A_k|^2, \end{aligned} \quad (35)$$

where $\langle b_{k'} |^{\otimes n} |b_k\rangle^{\otimes n} = \delta_{k, k'}$. Equivalently, the parallel term takes the form

$$\mathcal{Q}^{\parallel} = \frac{n^2}{d^2} \left| \sum_{k=0}^{d-1} b_k d_{m_f, m_i}^{(j)}(\beta_k) A_k \right|^2. \quad (36)$$

IV. RESULTS AND DISCUSSION

A. Meter operator design

Here, we analyze two representative classes of meter states that critically shape and maximize the QFI.

The first case corresponds to the generator $\hat{\mathcal{M}} = \hat{J}_t^{(\mathcal{M})} + \hat{J}_z^{(\mathcal{M})}$, which leads to the computational basis for the n -particle system $\mathcal{B} = \{|k_i\rangle^{\otimes n}\}$, where $k_i \in \{0, 1, \dots, d-1\}$, and $|k\rangle$ denoting the Fock state with k bosons in the meter mode. The meter state represents a multimode boson state that can be defined as $|\mathcal{M}_d^{(n)}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle^{\otimes n}$. The corresponding

meter operator is given by

$$\hat{\mathcal{O}}_\lambda^P = e^{i n \lambda \hat{\mathcal{M}}} = \sum_{k=0}^{d-1} e^{i n k \lambda} (|k\rangle\langle k|)^{\otimes n}. \quad (37)$$

We denote this meter operator by $\hat{\mathcal{O}}_\lambda^P$ to emphasize its role as the generator of the Pancharatnam phase:

$$\text{Im} \ln \langle \hat{\mathcal{O}}_\lambda^P \rangle = \text{Im} \ln \sum_{k=0}^{d-1} e^{i n k \lambda} = n \frac{(d-1)\lambda}{2}. \quad (38)$$

This operator enables efficient Θ optimization of $\mathcal{Q}^\parallel$ since

$$\begin{aligned} \langle (\hat{\mathcal{O}}_\lambda^P)^\dagger \partial_\lambda \hat{\mathcal{O}}_\lambda^P \rangle &= \frac{1}{d} \sum_{k=0}^{d-1} \langle k |^{\otimes n} (\hat{\mathcal{O}}_\lambda^P)^\dagger \partial_\lambda \hat{\mathcal{O}}_\lambda^P | k \rangle^{\otimes n} \\ &= i n \frac{1}{d} \sum_{k=0}^{d-1} k = i n \frac{d-1}{2} \neq 0. \end{aligned} \quad (39)$$

The second case corresponds to the generator $\hat{\mathcal{M}} = \hat{J}_z^{(\mathcal{M})}$, which produces the angular-momentum basis, $|b_k\rangle = |\tilde{j}, m_k\rangle$, where symbol $\tilde{j}$ is used to distinguish it from the j of the postselected system. Here, $m_k = k - \tilde{j}$ runs over all integer or half-integer values from $-\tilde{j}$ to $\tilde{j}$, and $\tilde{j} = \frac{d-1}{2}$. This yields

$$|\mathcal{M}_d^{(n)}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |\tilde{j}, k - \tilde{j}\rangle^{\otimes n} = \frac{1}{\sqrt{d}} \sum_{m=-\tilde{j}}^{\tilde{j}} |\tilde{j}, m\rangle^{\otimes n}, \quad (40)$$

with the angular-momentum basis being simply a reindexed computational basis. The meter operator in this case takes the form

$$\hat{\mathcal{O}}_\lambda^{(0)} = e^{i n \lambda \hat{\mathcal{M}}} = \sum_{m=-\tilde{j}}^{\tilde{j}} e^{i n m \lambda} (|\tilde{j}, m\rangle\langle \tilde{j}, m|)^{\otimes n}. \quad (41)$$

We label this operator as $\hat{\mathcal{O}}_\lambda^{(0)}$ to indicate the absence of the Pancharatnam phase:

$$\text{Im} \ln \langle \hat{\mathcal{O}}_\lambda^{(0)} \rangle = \text{Im} \ln \sum_{m=-\tilde{j}}^{\tilde{j}} e^{i n m \lambda} = 0. \quad (42)$$

The operator $\hat{\mathcal{O}}_\lambda^{(0)}$ obstructs the Θ optimization of $\mathcal{Q}^\parallel$ because

$$\begin{aligned} \langle (\hat{\mathcal{O}}_\lambda^{(0)})^\dagger \partial_\lambda \hat{\mathcal{O}}_\lambda^{(0)} \rangle &= \frac{1}{d} \sum_{m=-\tilde{j}}^{\tilde{j}} \langle \tilde{j}, m |^{\otimes n} (\hat{\mathcal{O}}_\lambda^{(0)})^\dagger \partial_\lambda \hat{\mathcal{O}}_\lambda^{(0)} | \tilde{j}, m \rangle^{\otimes n} \\ &= \frac{i n}{d} \sum_{m=-\tilde{j}}^{\tilde{j}} m = 0. \end{aligned} \quad (43)$$

As we shall see below, the QFI in the latter case is smaller than in the former. Remarkably, however, the two cases are physically equivalent with respect to the meter state, differing only by an overall phase factor. With the generators $\hat{\mathcal{M}} = \hat{J}_t^{(\mathcal{M})} + \hat{J}_z^{(\mathcal{M})}$ and $\hat{\mathcal{M}} = \hat{J}_z^{(\mathcal{M})}$ of $\hat{\mathcal{O}}_\lambda^P$ and $\hat{\mathcal{O}}_\lambda^{(0)}$, respectively, we write

$$\hat{\mathcal{O}}_\lambda^P = e^{i n \lambda (\hat{J}_t^{(\mathcal{M})} + \hat{J}_z^{(\mathcal{M})})} = e^{i n \lambda \hat{J}_t^{(\mathcal{M})}} \hat{\mathcal{O}}_\lambda^{(0)}. \quad (44)$$

For a fixed dimension of the meter subspace, $\hat{J}_t^{(\mathcal{M})}$ acts as multiplication by a scalar $\tilde{j}$, contributing a global phase factor $e^{i n \tilde{j} \lambda}$, independent of the basis states $|k\rangle$ or $|\tilde{j}, m\rangle$. For example, when $d = 3$ ($\tilde{j} = 1$)

$$\hat{\mathcal{O}}_\lambda^P = (|0\rangle\langle 0| + e^{i\lambda}|1\rangle\langle 1| + e^{2i\lambda}|2\rangle\langle 2|)^{\otimes n}, \quad (45)$$

and

$$\begin{aligned} \hat{\mathcal{O}}_\lambda^{(0)} &= (e^{-i\lambda}|1, -1\rangle\langle 1, -1| + |1, 0\rangle\langle 1, 0| + e^{i\lambda}|1, 1\rangle\langle 1, 1|)^{\otimes n} \\ &\equiv e^{-i n \lambda} (|0\rangle\langle 0| + e^{i\lambda}|1\rangle\langle 1| + e^{2i\lambda}|2\rangle\langle 2|)^{\otimes n} \\ &= e^{-i n \lambda} \hat{\mathcal{O}}_\lambda^P, \end{aligned} \quad (46)$$

where $e^{-i n \lambda}$ compensates for the index shift $k = m + 1$. Considering the structure of the interaction unitary operator in Eq. (12), the operator $\hat{\mathcal{O}}_\lambda^P$ corresponds to an experimental scenario where a single system mode interacts with a single meter mode, whereas $\hat{\mathcal{O}}_\lambda^{(0)}$ describes an interaction involving the two meter modes. Notably, both operators generate the same relative phase in the meter state.

B. The pancharatnam phase shift for arbitrary j

In this section, we verify the previously stated result that the postselected system's phase shift is characterized by the identical Pancharatnam phase, independent of the quantum number j and irrespective of the specific choices of pre- and postselection states.

Consider pre- and postselecting the highest and lowest weight states denoted by $|j, m_i\rangle = |j, j\rangle$ and $|j, m_f\rangle = |j, -j\rangle$, respectively, wherein the relevant Wigner d -matrix element takes the form $d_{-j,j}^{(j)}(\beta_k) = \sin^{2j}(\frac{\beta_k}{2})$, with $\beta_k = \Theta - n k \lambda$. The expressions given in Eqs. (33), (35), and (36) can then be, respectively, rewritten as

$$\mathcal{P}_{-j,j}^{(j)} = \frac{1}{d} \sum_{k=0}^{d-1} \sin^{4j} \left(\frac{\beta_k}{2} \right), \quad (47)$$

$$\mathcal{Q}^T = \frac{n^2 j^2}{d} \sum_{k=0}^{d-1} b_k^2 \sin^{4j-2} \left(\frac{\beta_k}{2} \right), \quad (48)$$

$$\mathcal{Q}^\parallel = \frac{n^2 j^2}{d^2} \left| \sum_{k=0}^{d-1} b_k \sin^{4j-1} \left(\frac{\beta_k}{2} \right) e^{i \frac{n \beta_k \lambda}{2}} \right|^2, \quad (49)$$

representing the postselection probability, total change, and parallel change.

When the transition between pre- and postselected states is by a single quantum with $m_i = j$ and $m_f = j - 1$, the corresponding Wigner d -matrix element takes the well-known form

$$d_{j-1,j}^{(j)}(\beta_k) = \sqrt{2j} \sin \left(\frac{\beta_k}{2} \right) \cos^{2j-1} \left(\frac{\beta_k}{2} \right). \quad (50)$$

Using this expression, the postselection probability can be written as

$$\mathcal{P}_{j-1,j}^{(j)} = \frac{2j}{d} \sum_{k=0}^{d-1} \sin^2 \left(\frac{\beta_k}{2} \right) \cos^{4j-2} \left(\frac{\beta_k}{2} \right). \quad (51)$$

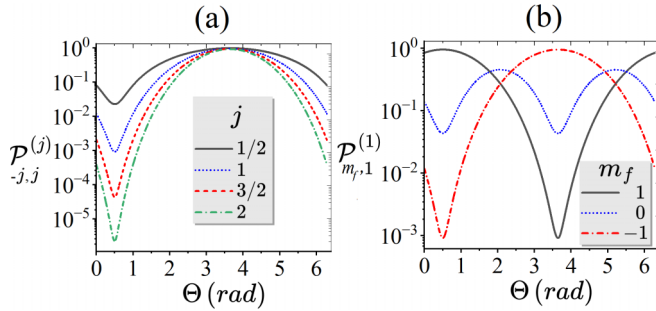


FIG. 1. Postselection probability as a function of the postselection parameter Θ , for $\lambda = 10^{-3}$ rad, $d = 21$, and $n = 50$. This produces a Pancharatnam phase shift $\text{Im} \ln \langle \hat{O}_\lambda^P \rangle = \frac{(d-1)n\lambda}{2} = 0.5$. (a) For different values of j with preselected state $m_i = j$, and post-selected state $m_f = -j$. (b) For $j = 1$, $m_i = j$, and different values of m_f .

Notably, for $j = 1/2, 1$, exact closed-form expressions for the postselection probability are obtained:

$$\mathcal{P}_{-\frac{1}{2}, \frac{1}{2}}^{(\frac{1}{2})} = \frac{1}{2} \left[1 - \frac{\sin(\frac{dn\lambda}{2})}{d \sin(\frac{n\lambda}{2})} \cos \left(\Theta - \frac{(d-1)n\lambda}{2} \right) \right], \quad (52)$$

$$\mathcal{P}_{0,1}^{(1)} = \frac{1}{4} \left[1 - \frac{\sin(dn\lambda)}{d \sin(n\lambda)} \cos 2 \left(\Theta - \frac{(d-1)n\lambda}{2} \right) \right]. \quad (53)$$

For higher j values, the postselection probabilities generalize to sums involving harmonic terms S_σ :

$$\mathcal{P}_{\frac{1}{2}, \frac{3}{2}}^{(\frac{3}{2})} = \frac{3}{16} \left(1 + \frac{1}{2} S_1 - S_2 - \frac{1}{2} S_3 \right), \quad (54)$$

$$\mathcal{P}_{1,2}^{(2)} = \frac{1}{8} \left(\frac{5}{4} + S_1 - S_2 - S_3 - \frac{1}{4} S_4 \right), \quad (55)$$

where each S_σ encodes multipath interference effects through

$$S_\sigma = \frac{\sin(\sigma \frac{dn\lambda}{2})}{d \sin(\sigma \frac{n\lambda}{2})} \cos \sigma \left(\Theta - \frac{(d-1)n\lambda}{2} \right), \quad (56)$$

where $\sigma = 1, 2, 3, \dots$

Figure 1 presents the postselection probability $\mathcal{P}_{m_f, m_i}^{(j)}$ as a function of the parameter Θ for various quantum numbers j , alongside results for fixed $j = 1$ with varying postselected states m_f . While the visibility and oscillation frequency of the interference fringes depend on these quantum numbers, the Pancharatnam phase shift $\text{Im} \ln \langle \hat{O}_\lambda^P \rangle$ emerges as the fundamental invariant quantity governing the postselection probability of the compression channels.

C. Complete nullification of $\mathcal{Q}^\parallel$ for arbitrary j

As introduced in Sec. III C, by postselecting on system characterized by the Pancharatnam phase, one gains direct control over the parallel contribution to QFI in the quantum channel, enabling its complete suppression and rendering the compression channel lossless.

To demonstrate this quantitatively for a general quantum number j , we consider the expressions derived from the

Wigner d -matrix formalism [Eqs. (35), (36) and (50)]. The total $\mathcal{Q}^T$ and parallel $\mathcal{Q}^\parallel$ components can be expressed as

$$\mathcal{Q}^T = 2 \frac{n^2 j}{d} \sum_{k=0}^{d-1} b_k^2 |A_k|^2, \quad (57)$$

$$\mathcal{Q}^\parallel = 4 \frac{n^2 j^2}{d^2} \left| \sum_{k=0}^{d-1} b_k \sin \left(\frac{\beta_k}{2} \right) \cos^{2j-1} \left(\frac{\beta_k}{2} \right) A_k \right|^2, \quad (58)$$

where the complex amplitudes A_k are given by

$$A_k = \iota j \left[\sin \left(\frac{\beta_k}{2} \right) \cos^{2j-1} \left(\frac{\beta_k}{2} \right) - \frac{1}{2} \cos^{2j-2} \left(\frac{\beta_k}{2} \right) \left[\cos^2 \left(\frac{\beta_k}{2} \right) - (2j-1) \sin^2 \left(\frac{\beta_k}{2} \right) \right] \right].$$

For the operator $\hat{O}_\lambda^P$, in the simplest nontrivial case $d = 2$, the parallel component simplifies to

$$\mathcal{Q}^\parallel = \frac{n^2 j^2}{4} \sin^2 \left(\frac{\Theta - n\lambda}{2} \right) \cos^{8j-6} \left(\frac{\Theta - n\lambda}{2} \right) \times \left[\cos^2 \left(\frac{\Theta - n\lambda}{2} \right) + (2j-1)^2 \sin^2 \left(\frac{\Theta - n\lambda}{2} \right) \right]. \quad (59)$$

By choosing $\Theta = n\lambda$, this term is identically null for all j , demonstrating that the parallel contribution to the QFI can be completely suppressed through this parameter tuning.

For the operator $\hat{O}_\lambda^{(0)}$, the parallel component of the QFI simplifies to $\mathcal{Q}^\parallel = \frac{n^2}{16} \sin^2(\frac{n\lambda}{2}) \neq 0$ for $j = \frac{1}{2}$, and $\mathcal{Q}^\parallel = \frac{n^2}{64} \sin^2(n\lambda) \neq 0$ for $j = 1$. Analytical expressions become intractable for higher j , yet in the small- λ regime, one obtains the asymptotic scaling, for instance, when $j = 3/2$, $\mathcal{Q}^\parallel \approx \frac{9n^4}{256} \lambda^2$. This behavior reveals a fundamental limitation on loss mitigation achievable through compression channels.

It is important to highlight that Eq. (59) encompasses parameter regimes where the parallel term vanishes, specifically for cases with $j \geq \frac{3}{2}$ and $\Theta = \pi + n\lambda$. This behavior aligns with a high postselection probability regime, indicating no compression of metrological information, as evident from the postselection probability profile [Eq. (51)]. In such scenarios, contingent on the interaction strength λ and meter dimensionality d , we emphasize the possibility of exploring alternative postselection schemes on single quantum transitions—such as $d_{0,1}^j$ for integer j and $d_{-\frac{1}{2}, \frac{1}{2}}^j$ for half-integer j —to realize a lossless compression channel. The forthcoming analysis illustrates the core concept using Eq. (59), but for larger values of j , the aforementioned alternatives remain valid, adhering to the same methodological framework and logic presented here.

D. QFI yield: $d_{j-1,j}^{(j)}$ versus $d_{-j,j}^{(j)}$

It is well established that compressing metrological information into a small subset of postselection outcomes requires the success events to be intrinsically rare. However, this condition is necessary but not sufficient. Crucially, the postselection must be performed on a single quantum transition to realize a lossless quantum compression channel.

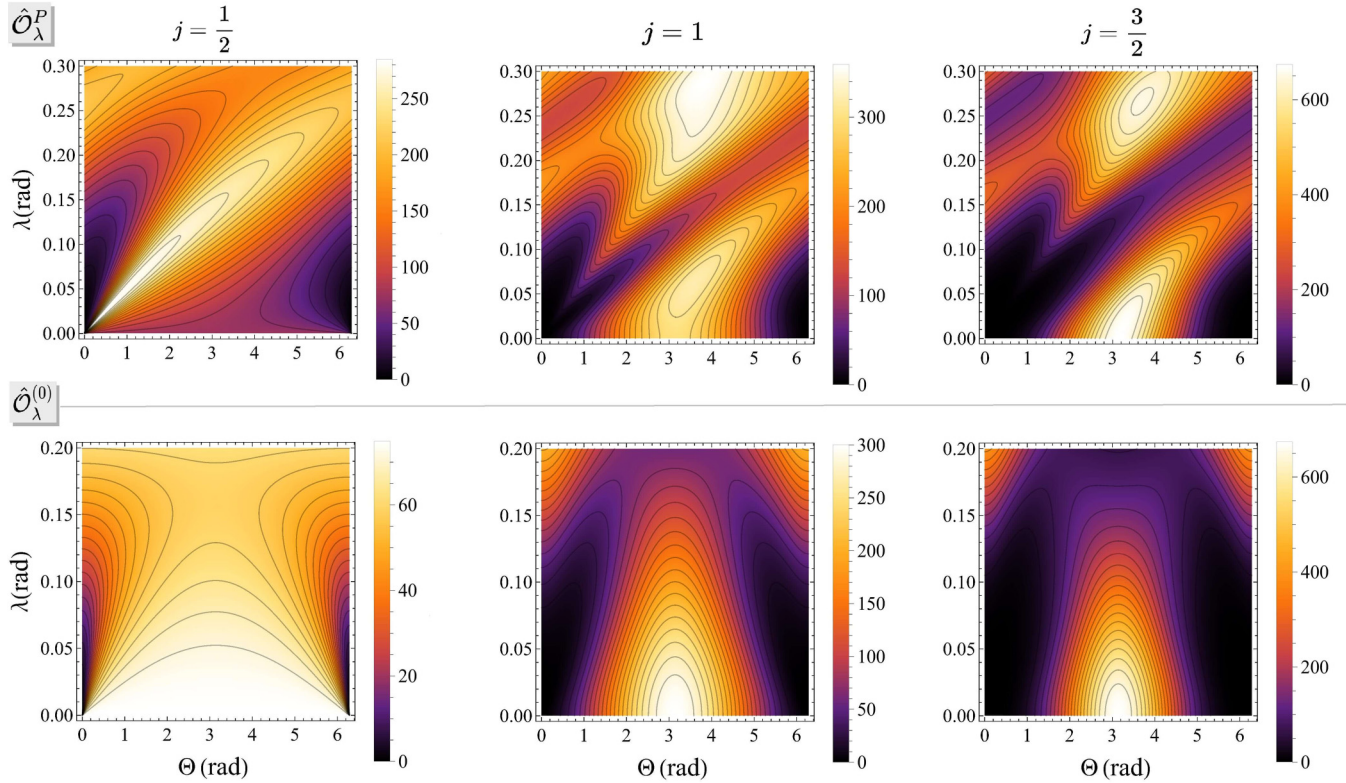


FIG. 2. Contour plots of the QFI per trial $\mathcal{T}_{(\lambda, \Theta)}$ depicted as functions of the coupling and postselection parameters λ and Θ , for quantum numbers $j = 1/2, 1$, and $3/2$, $n = 1$, and $d = 30$. The top row corresponds to calculations using the meter operator $\hat{O}_\lambda^P$, while the bottom row displays results obtained with $\hat{O}_\lambda^{(0)}$. In each panel, the color scale encodes the $\mathcal{T}_{(\lambda, \Theta)}$ magnitude, with lighter shades indicating higher sensitivity. Further discussion and detailed physical interpretation are provided in the main text.

Figure 2 presents contour plots of the total QFI per trial $\mathcal{T}_{(\lambda, \Theta)} = \mathcal{P}_{-j, j}^{(j)} \mathcal{I}^\perp$ as a function of the coupling parameter λ and Θ , for various quantum numbers j . The results correspond to $d = 30$, obtained directly from Eqs. (47)–(49).

For $j = 1/2$, the operator $\hat{O}_\lambda^P$ displays a pronounced diagonal ridge even at small values of λ , indicating a significant enhancement in quantum sensitivity, $\mathcal{T}_{(\lambda, \Theta)} \gg 1$, along correlated values of λ and Θ . In contrast, the operator $\hat{O}_\lambda^{(0)}$ exhibits a broad, symmetric maximum centered at small λ and intermediate Θ , decreasing rapidly outside this region. Overall, the quantum advantage in sensitivity achievable with $\hat{O}_\lambda^{(0)}$ is notably limited compared to that of $\hat{O}_\lambda^P$.

For higher values $j = 1, 3/2$, the landscape exhibits multiple local maxima and complex contour structures, with prominent high- $\mathcal{T}_{(\lambda, \Theta)}$ regions occurring at intermediate parameter values. This behavior highlights the suboptimality of the small- (Θ, λ) regime as j increases, where $\mathcal{T}_{(\lambda, \Theta)} \ll 1$. For smaller λ , $\mathcal{T}_{(\lambda, \Theta)}$ reaches its highest amplitudes near $\Theta \approx \pi$, corresponding to a high postselection probability (no compression). As the quantum number j increases, these peaks become increasingly localized around $\Theta \approx \pi$ and remain constrained to the small- λ regime. This behavior arises because the postselection probability decreases exponentially with increasing quantum number j , as illustrated in Fig. 1(a). Such decay can be well approximated by the expression

$$\mathcal{P}_{-j, j}^{(j)} \approx [\mathcal{P}_{-\frac{1}{2}, \frac{1}{2}}^{(\frac{1}{2})}]^{2j}, \quad (60)$$

whereas, the QFI grows linearly with j , as shown in Fig. 3(a). Consequently, the total QFI per trial $\mathcal{T}_{(\lambda, \Theta)}$ decreases rapidly with increasing j in the weak-coupling regime.

To achieve optimal quantum compression for higher j values, we postselect on quantum states that induce a single quantum transition given by Eq. (50). The postselection probability for this case is given by Eq. (51) where $\mathcal{P}_{j-1, j}^{(j)} \sim j \mathcal{P}_{-\frac{1}{2}, \frac{1}{2}}^{(\frac{1}{2})}$. This procedure restores the metrological information

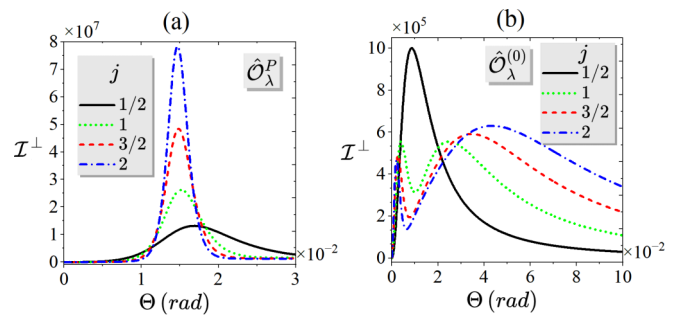


FIG. 3. The postselected QFI $\mathcal{I}^\perp$ as a function of Θ for $n\lambda = 10^{-3}$ and $d = 30$. (a) QFI exhibits a linear growth as a function of j . Notably, the QFI peak shifts toward the Pancharatnam phase as j increases. (b) Reference protocol $\hat{O}_\lambda^{(0)}$ shows qualitative differences in scaling and suppression of QFI as j increases.

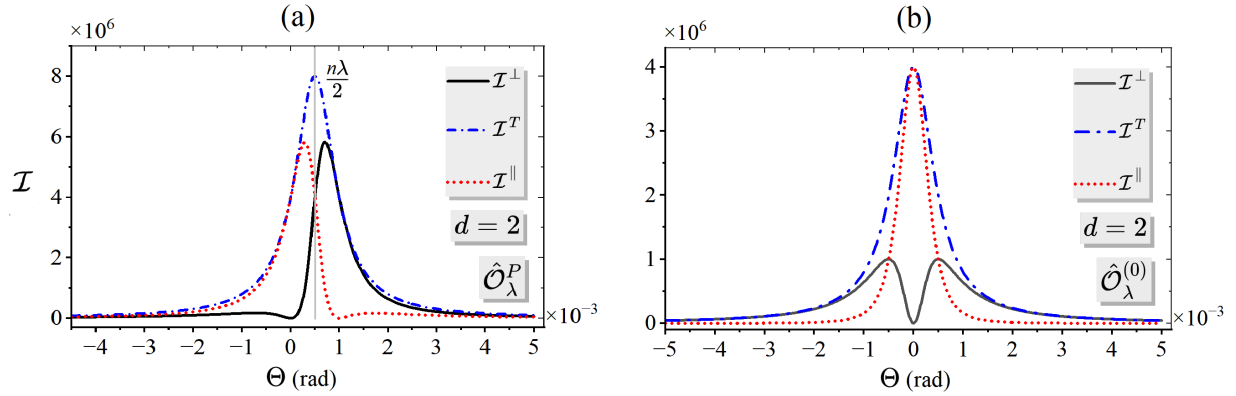


FIG. 4. Postselected QFI $\mathcal{I}^\perp$, $\mathcal{I}^T$, and $\mathcal{I}^\parallel$ as a function of Θ for $n\lambda = 10^{-3}$, $d = 2$, and arbitrary j . (a) For the meter operator $\hat{\mathcal{O}}_\lambda^P$, the characteristic trade-off between $\mathcal{I}^\perp$ and $\mathcal{I}^\parallel$ occurs in the vicinity of the Pancharatnam phase $\frac{n\lambda}{2}$. (b) For the meter operator $\hat{\mathcal{O}}_\lambda^{(0)}$, both $\mathcal{I}^\perp$ and $\mathcal{I}^T$ exhibit cooperative behavior, whereas the postselected QFI $\mathcal{I}^\perp$ suffers an approximately 83% reduction at $\Theta^\perp = \pm \frac{n\lambda}{2}$, indicating suboptimal performance of operator $\hat{\mathcal{O}}_\lambda^{(0)}$ in comparison with $\hat{\mathcal{O}}_\lambda^P$.

in $\mathcal{T}_{(\lambda, \Theta)}$ to j times that of the $j = \frac{1}{2}$ case [Eq. (57)], thereby generalizing the optimal compression to arbitrary j .

It is straightforward to verify whether the compression channel associated with the operator $\hat{\mathcal{O}}_\lambda^P$ at $\Theta = \Theta^\parallel$ is lossless. The QFI per trial is given by

$$\mathcal{T}_{(\lambda, \Theta^\parallel)} = \mathcal{P}_{j-1, j}^{(j)} \mathcal{I}^\perp = 4Q^T,$$

which exactly matches the standard QFI for pure states without postselection

$$\mathcal{I}_S = \sum_{m_f} \frac{1}{\mathcal{P}_{m_f, j}^{(j)}} \left(\frac{\partial \mathcal{P}_{m_f, j}^{(j)}}{\partial \lambda} \right)^2,$$

assuming that all quantum systems are initially prepared in the state $m_i = j$. For example, when $d = 2$, $n = 1$, it can be shown that the QFI per trial reduces to $\mathcal{T}_{(\lambda, \Theta^\parallel)} = \mathcal{I}_S = j$, taking into careful consideration the final remark presented in Sec. IV C.

E. Postselected QFI $\mathcal{I}^\perp$ analysis for a qubit meter

For $d = 2$, the operator $\hat{\mathcal{O}}_\lambda^P$ can be defined as

$$\hat{\mathcal{O}}_\lambda^P = |0\rangle\langle 0|^{\otimes n} + e^{in\lambda}|1\rangle\langle 1|^{\otimes n}, \quad (61)$$

which imparts relative phase $n\lambda$ to the meter states. Given the initial state of the meter $|\mathcal{M}_2^{(n)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$, the Pancharatnam phase is expressed as $\text{Im} \ln \langle \hat{\mathcal{O}}_\lambda^P \rangle = n\frac{\lambda}{2}$.

From Eq. (57), the total contribution for QFI Q^T for different j values is given by

$$\begin{aligned} j = \frac{1}{2} &: \frac{n^2}{8}, \\ j = 1 &: \frac{n^2}{4}, \\ j = \frac{3}{2} &: \frac{3n^2}{2} \cos^2 \left(\frac{\Theta - n\lambda}{2} \right) \left[1 - \frac{3}{4} \cos^2 \left(\frac{\Theta - n\lambda}{2} \right) \right]. \end{aligned} \quad (62)$$

The same holds for the term corresponding to parallel evolution $Q^\parallel$. From Eq. (59), we find $Q^\parallel$:

$$\begin{aligned} j = \frac{1}{2} &: \frac{n^2}{16} \sin^2 \left(\frac{\Theta - n\lambda}{2} \right), \\ j = 1 &: \frac{n^2}{4} \sin^2 \left(\frac{\Theta - n\lambda}{2} \right) \cos^2 \left(\frac{\Theta - n\lambda}{2} \right), \\ j = \frac{3}{2} &: \frac{9n^2}{16} \sin^2 \left(\frac{\Theta - n\lambda}{2} \right) \cos^6 \left(\frac{\Theta - n\lambda}{2} \right) \\ &\times \left[\cos^2 \left(\frac{\Theta - n\lambda}{2} \right) + 4 \sin^2 \left(\frac{\Theta - n\lambda}{2} \right) \right]. \end{aligned} \quad (63)$$

Accordingly, taking into account Eqs. (57), (59), and the postselection probability $\mathcal{P}_{j-1, j}^{(j)}$ in Eq. (51), we can find the postselected QFI $\mathcal{I}^\perp = \mathcal{I}^T - \mathcal{I}^\parallel$, given by Eq. (8), with

$$\mathcal{I}^T = 4Q^T (\mathcal{P}_{j-1, j}^{(j)})^{-1}, \quad \mathcal{I}^\parallel = 4Q^\parallel (\mathcal{P}_{j-1, j}^{(j)})^{-2}. \quad (64)$$

Figure 4 shows $\mathcal{I}^\perp$, $\mathcal{I}^T$, $\mathcal{I}^\parallel$, as a function of Θ for arbitrary j . $\mathcal{I}^T$ attains its maximum for

$$\Theta^T = \arg \max_{\Theta} \mathcal{I}^T_{(\lambda, \Theta)} = n\frac{\lambda}{2}$$

(Pancharatnam phase), whereas the parallel component $\mathcal{I}^\parallel$ vanishes under the condition

$$\Theta^\parallel = \arg \min_{\Theta} \mathcal{I}^\parallel_{(\lambda, \Theta)} = n\lambda.$$

Under the small-angle approximation for λ ($n\lambda < 1$ rad), and regardless of the value of j , the postselection parameter $\Theta^\perp$ that maximizes $\mathcal{I}^\perp$ is

$$\Theta^\perp = \arg \max_{\Theta} \mathcal{I}^\perp_{(\lambda, \Theta)} = \arccos \left[\cos^2 \left(n\frac{\lambda}{2} \right) \right], \quad (65)$$

a direct function of the Pancharatnam phase. Figure 4(a) illustrates the trade-off around Pancharatnam phase between $\mathcal{I}^\perp$ and $\mathcal{I}^\parallel$ as Θ varies.

Figure 4(b) shows that the operator $\hat{\mathcal{O}}_\lambda^{(0)} = e^{-in\lambda/2}|0\rangle\langle 0|^{\otimes n} + e^{in\lambda/2}|1\rangle\langle 1|^{\otimes n}$, which introduces the same relative phase $n\lambda$ to the meter state, leads to a cooperative

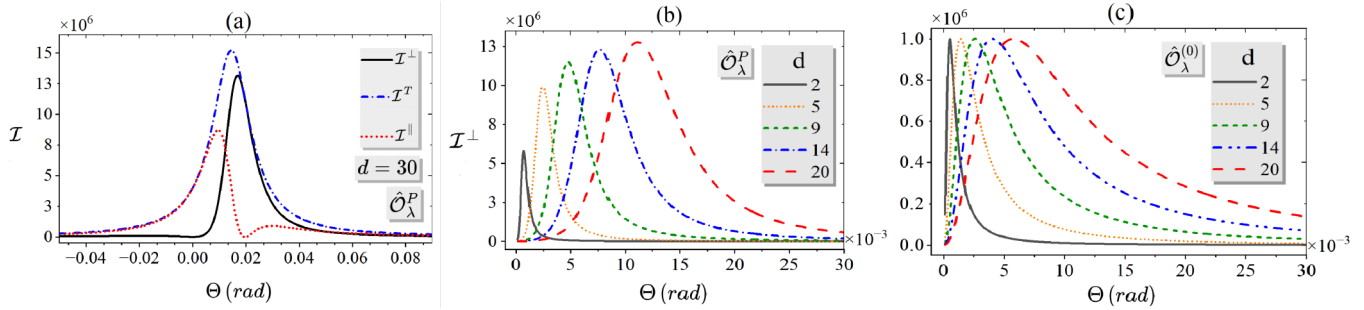


FIG. 5. (a) Variation of the postselected QFI $\mathcal{I}^{\perp}$, $\mathcal{I}^T$, and $\mathcal{I}^{\parallel}$ as functions of Θ for $d = 30$, $n\lambda = 10^{-3}$, and arbitrary j , illustrating how the operator $\hat{\mathcal{O}}_{\lambda}^P$ enables precise tuning of the trade-off between orthogonal and parallel sensitivities. (b) The QFI $\mathcal{I}^{\perp}$ plotted against Θ for various meter dimensions d , demonstrating a pronounced increase in QFI and the formation of sharply localized peaks with rising d when using $\hat{\mathcal{O}}_{\lambda}^P$. These peaks signify optimal Θ at which maximal quantum sensitivity is achieved as meter complexity grows. (c) In contrast, the QFI $\mathcal{I}^{\perp}$ corresponding to $\hat{\mathcal{O}}_{\lambda}^{(0)}$ remains essentially unchanged in its maximal numerical value as d increases, reflecting a diminished sensitivity to meter dimension d .

behavior between $\mathcal{I}^{\parallel}$ and $\mathcal{I}^T$, which reduces the postselected QFI by 83.33%, rendering $\hat{\mathcal{O}}_{\lambda}^{(0)}$ suboptimal. QFI in this case is maximized for $\Theta^{\perp} = \arg \max_{\Theta} \mathcal{I}^{\perp}(\lambda, \Theta) = \pm n\frac{\lambda}{2}$.

The same information compression, characterized by identical postselection probabilities, occurs when $\Theta = \Theta^{\parallel} = n\lambda$ and $\Theta = \Theta^{\perp} = n\frac{\lambda}{2}$ for the operators $\hat{\mathcal{O}}_{\lambda}^P$ and $\hat{\mathcal{O}}_{\lambda}^{(0)}$, respectively. The metrological gain in sensitivity to λ of operator $\hat{\mathcal{O}}_{\lambda}^P$ relative to $\hat{\mathcal{O}}_{\lambda}^{(0)}$ can be quantified as

$$g = 10 \log_{10} \frac{\text{Var}(\lambda)|_{\hat{\mathcal{O}}_{\lambda}^{(0)}}}{\text{Var}(\lambda)|_{\hat{\mathcal{O}}_{\lambda}^P}} = 10 \log_{10} \frac{\mathcal{I}^{\perp}(\Theta^{\parallel})|_{\hat{\mathcal{O}}_{\lambda}^P}}{\mathcal{I}^{\perp}(\Theta^{\perp})|_{\hat{\mathcal{O}}_{\lambda}^{(0)}}} = 6 \text{ dB}, \quad (66)$$

which is comparable to the metrological gain recently demonstrated in Laser Interferometer Gravitational-Wave Observatory (LIGO) using squeezed light [35].

F. Qudit-enhanced sensitivity: Boosting $\mathcal{I}^{\perp}$, $\mathcal{T}$, $\Delta\lambda$, and signal-to-noise ratio

For a qudit meter with $d > 2$, the postselection probability is determined by Eq. (51). According to Eq. (57), $\mathcal{Q}^T$ for different j values is expressed as

$$\begin{aligned} j = \frac{1}{2} &: \frac{n^2}{24}(d-1)(2d-1), \\ j = 1 &: \frac{n^2}{12}(d-1)(2d-1), \\ j = \frac{3}{2} &: \frac{3n^2}{d} \sum_{k=0}^{d-1} k^2 \left[\cos^2\left(\frac{\beta_k}{2}\right) - \frac{3}{4} \cos^4\left(\frac{\beta_k}{2}\right) \right] \\ &\approx \frac{3n^2}{24}(d-1)(2d-1) \quad \text{for small } \beta_k. \end{aligned} \quad (67)$$

The analytical solution for the parallel component $\mathcal{Q}^{\parallel}$ remains challenging to obtain for arbitrary values of j . However, as discussed above, the explicit expression for $\Theta^{\parallel}$ can be derived for the specific case of $j = \frac{1}{2}$, and the result is expected to hold for higher j values. The accuracy and applicability of the derived $\Theta^{\parallel}$ for larger j can subsequently be validated through numerical calculations.

Thus, for $j = 1/2$, the parallel term $\mathcal{Q}^{\parallel}$ [Eq. (58)] gives

$$\mathcal{Q}^{\parallel} = \frac{n^2}{16d^2} |\mathcal{Z}(\lambda) - e^{i\Theta} \mathcal{Z}_0|^2, \quad (68)$$

where $\mathcal{Z}_0 = d(d-1)/2$ and

$$\mathcal{Z}(\lambda) = \sum_{k=0}^{d-1} k e^{in k \lambda} = \frac{e^{in \lambda} - d e^{ind \lambda} + (d-1) e^{in(d+1) \lambda}}{(1 - e^{in \lambda})^2}.$$

From Eq. (68), the optimal postselection parameter Θ that satisfies the condition $\mathcal{Q}^{\parallel} = 0$, can be expressed analytically as

$$\Theta^{\parallel} = -n\lambda + \text{Im} \ln [d e^{ind \lambda} - e^{in \lambda} + (1-d) e^{in(d+1) \lambda}], \quad (69)$$

which defines the precise tuning condition for maximizing $\mathcal{T}$ for different values of j , as illustrated in Fig. 6(a). While $\mathcal{I}^{\parallel}$ is not strictly zero for $\Theta = \Theta^{\parallel}$ as in the case $d = 2$ [Eq. (59)], its magnitude remains much smaller than the total term $\mathcal{I}^T$, such that $\frac{\mathcal{I}^{\parallel}}{\mathcal{I}^T} \ll 1$. This indicates that the parallel component contributes negligibly and does not affect the overall scaling of $\mathcal{I}^T$ for $\Theta = \Theta^{\parallel}$, taking into account the final note in Sec. IV C.

Figure 5(a) shows the postselected QFI as functions of Θ for $d = 30$, illustrating how $\hat{\mathcal{O}}_{\lambda}^P$ enables precise tuning of the trade-off between orthogonal and parallel sensitivities. The results in Fig. 5(b) reveal that the QFI associated with the operator $\hat{\mathcal{O}}_{\lambda}^P$ exhibits a pronounced upward scaling as the meter dimension d increases. Conversely, Fig. 5(c) shows that the QFI computed with the operator $\hat{\mathcal{O}}_{\lambda}^{(0)}$ yields a QFI that is comparatively insensitive to d .

The performance comparison between $\hat{\mathcal{O}}_{\lambda}^P$ and $\hat{\mathcal{O}}_{\lambda}^{(0)}$ [Figs. 6(a) and 6(b)] demonstrates that the presence of Pancharatnam phase in the postselected system is crucial for unlocking the full advantage offered by increasing the meter quantum complexity in the compression of metrological information.

For $n\lambda = 10^{-3}$ and $d = 30$, under minimal loss conditions with $\Theta = \Theta^{\parallel}$, the operator $\hat{\mathcal{O}}_{\lambda}^P$ yields a postselection probability of 25×10^{-6} . Achieving the same compression (i.e., equal postselection probability) with the operator $\hat{\mathcal{O}}_{\lambda}^{(0)}$ requires $\Theta \simeq 5.17n\lambda$. Consequently, for equivalent compression rates, the

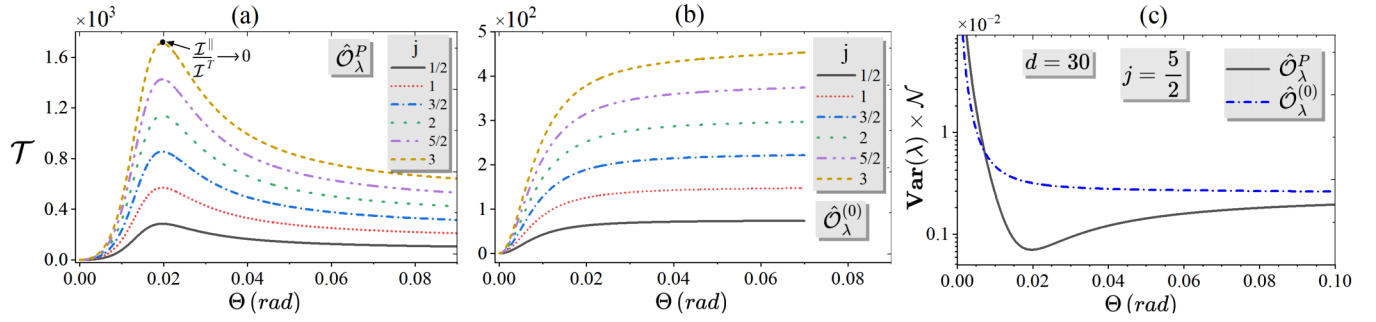


FIG. 6. Performance of $\hat{O}_\lambda^P$ and $\hat{O}_\lambda^{(0)}$ as a function of the postselection parameter Θ , shown for various values j , $d = 30$, $\lambda = 10^{-3}$, and $n = 1$. (a) The operator $\hat{O}_\lambda^P$ realizes a pronounced enhancement in the QFI per trial, $\mathcal{T}$, peaking at the optimal value $\Theta^\parallel$, where the parallel contribution $\mathcal{I}^\parallel$ vanishes. (b) The quantum Fisher information per trial for $\hat{O}_\lambda^{(0)}$ is consistently lower across all Θ . (c) Scaled variance $\text{Var}(\lambda) \times \mathcal{N}$ as a function of Θ for $j = 5/2$. Both $\hat{O}_\lambda^P$ and $\hat{O}_\lambda^{(0)}$ exhibit increased variance near $\Theta = 0$, highlighting suboptimal performance in this regime, while optimization using $\hat{O}_\lambda^P$ yields a lower variance at $\Theta^\parallel$.

metrological gain of $\hat{O}_\lambda^P$ relative to $\hat{O}_\lambda^{(0)}$ is

$$g = 10 \log_{10} \frac{\mathcal{I}^\perp(\Theta^\parallel) |\hat{O}_\lambda^P}{\mathcal{I}^\perp(\Theta \simeq 5.17n\lambda) |\hat{O}_\lambda^{(0)}} = 11.5 \text{ dB} \quad (70)$$

for the same number of trials $\mathcal{N}$.

Figure 6(c) shows that the variance $\text{Var}(\lambda)$ peaks near $\Theta = 0$, rendering this regime unsuitable for estimation. In contrast, optimizing at $\Theta^\parallel$ enables saturation of the Cramér-Rao bound (6), thus achieving optimal measurement precision.

For $\Theta = \Theta^\parallel$, $j = 1/2$, and in the large- d limit, the QFI per trial gives $\mathcal{T}_{(\lambda, \Theta^\parallel)} = 4Q^T \simeq \frac{n^2 d^2}{3}$. From Eq. (9), the uncertainty becomes

$$\Delta\lambda \geq \frac{\sqrt{3}}{nd} \frac{1}{\sqrt{\mathcal{N}}}. \quad (71)$$

The protocol achieves Heisenberg-limited scaling with respect to both the number of probes and the dimensionality of the meter quantum state, demonstrating a sensitivity gain beyond classical and standard quantum strategies [30]. Specifically, the metrological gain obtained by employing a qudit GHZ state relative to a qubit GHZ state is quantified as $g = 10 \log_{10} \frac{\text{Var}(\lambda)_{\text{qudit GHZ}}}{\text{Var}(\lambda)_{\text{qubit GHZ}}} \approx 10 \log_{10}(d)$. For $d = 30$, this corresponds to a gain of approximately 15 dB.

The signal-to-noise ratio (SNR) quantifies the precision of the estimated parameter λ and satisfies the inequality:

$$\text{SNR} = \frac{\lambda}{\sqrt{\text{Var}(\lambda)}} \leq \lambda \sqrt{M \mathcal{I}^\perp(\Theta)} = \lambda \sqrt{\mathcal{N} \mathcal{T}(\Theta)}. \quad (72)$$

For the operator $\hat{O}_\lambda^P$, and when $\Theta = \Theta^\parallel$, the SNR gives

$$\text{SNR} \leq \frac{nd\lambda}{\sqrt{3}} \sqrt{\mathcal{N}} = \frac{nd\lambda}{\sqrt{3}} \sqrt{\Gamma \tau}, \quad (73)$$

where Γ is the generation rate of the quantum states and τ is the data acquisition time. This relationship captures the fundamental dependence of measurement precision on both the quantum resources and experimental parameters. Therefore, optimizing the parameters n , d , and Γ relative to λ is vital to approach or saturate the uncertainty limit $\Delta\lambda$.

These estimates delineate the fundamental minimal experimental thresholds required to attain the theoretical lower bound of precision. We emphasize that a rigorous analysis

incorporating realistic noise models is essential in future work to accurately characterize the ultimate experimental limitations. Nonetheless, the geometric framework we establish remains fundamentally valid for optimizing protocol performance despite decoherence and operational imperfections that inevitably degrade achievable precision.

G. Challenges and key considerations for future studies

As shown in Fig. 5, the postselected QFI increases with increasing d , with its peak shifting closer toward the Pancharatnam phase. This increase, however, saturates around $d = 30$. The question remains: Can we achieve further enhancement in the postselected QFI beyond this point as d continues to grow?

We delineate three key characteristic values of the postselection parameter Θ :

$$\Theta^T = \arg \max_{\Theta} \mathcal{I}^T(\lambda, \Theta) = \text{Im} \ln \langle \hat{O}_\lambda^P \rangle, \quad (74)$$

$$\Theta^\perp = \arg \max_{\Theta} \mathcal{I}^\perp(\lambda, \Theta), \quad (75)$$

$$\Theta^\parallel = \arg \min_{\Theta} \mathcal{I}^\parallel(\lambda, \Theta), \quad (76)$$

each marking distinct operational regimes, with $\Theta^T < \Theta^\perp < \Theta^\parallel$. Achieving the most accessible enhancement in the quality of quantum compression channels requires the suppression of parallel dynamics precisely at the Pancharatnam phase. This condition leads to the equality

$$\Theta^\parallel = \Theta^\perp = \text{Im} \ln \langle \hat{O}_\lambda^P \rangle. \quad (77)$$

Future research should focus on manipulating the Pancharatnam phase and exploring meter operators capable of achieving this critical equality. Detailed investigation into the structural and operational properties of such meter operators is essential to optimize many features of postselected quantum metrology protocols. Table I summarizes the broad spectrum of quantum states considered in this study. Further investigations may directly incorporate other meter states, such as squeezed states [36] and entangled coherent states [37].

To illustrate this point, consider the meter operator

$$\hat{O}_\lambda^P = \sum_{k=0}^{d-1} e^{ik^\varepsilon \lambda} |k\rangle \langle k|, \quad (78)$$

with the fractional exponent approaching zero, $\varepsilon \rightarrow 0^+$, representing a slowly varying phase. This form leads to the expectation value

$$\langle \hat{O}_\lambda^P \rangle = \frac{1}{d} \sum_{k=0}^{d-1} e^{i\lambda k^\varepsilon}. \quad (79)$$

This sum is approximated as $\sum_{k=0}^{d-1} e^{i\lambda k^\varepsilon} \approx 1 + e^{i\lambda} \sum_{k=1}^{d-1} k^{i\lambda\varepsilon}$, where $k^{i\lambda\varepsilon} = e^{i\lambda\varepsilon \ln(k)}$. Using the integral approximation for large d , we get [38]

$$\sum_{k=1}^{d-1} k^{i\lambda\varepsilon} \approx \int_1^d x^{i\lambda\varepsilon} dx = \frac{d^{1+i\lambda\varepsilon} - 1}{1+i\lambda\varepsilon}, \quad (80)$$

thus $\sum_{k=0}^{d-1} e^{i\lambda k^\varepsilon} \approx 1 + e^{i\lambda} \frac{d^{1+i\lambda\varepsilon} - 1}{1+i\lambda\varepsilon}$, and the expectation value can be rewritten as

$$\langle \hat{O}_\lambda^P \rangle \approx \frac{1}{d} \left[1 + e^{i\lambda} \frac{d^{1+i\lambda\varepsilon} - 1}{1+i\lambda\varepsilon} \right] = \frac{1}{d} + e^{i\lambda} \frac{d^{i\lambda\varepsilon} - d^{-1}}{1+i\lambda\varepsilon}, \quad (81)$$

where $d^{i\lambda\varepsilon} = e^{i\lambda\varepsilon \ln(d)} \approx 1 + i\lambda\varepsilon \ln(d) + \dots$, using the condition $|\lambda\varepsilon \ln(d)| \ll 1$, and $\frac{1}{1+i\lambda\varepsilon} \approx 1 - i\lambda\varepsilon - \lambda^2\varepsilon^2 + \dots$. This gives

$$\langle \hat{O}_\lambda^P \rangle \approx e^{i\lambda} \{1 + i\lambda\varepsilon [\ln(d) - 1]\} + \frac{1 - e^{i\lambda}}{d} + \dots \quad (82)$$

Noting that $\ln(1 + \delta) \approx \delta$ for small δ , where $\delta = i\lambda\varepsilon [\ln(d) - 1] + \frac{e^{-i\lambda} - 1}{d} + \dots$. The Pancharatnam phase gives

$$\begin{aligned} \text{Im} \ln \langle \hat{O}_\lambda^P \rangle &= \lambda + \lambda\varepsilon [\ln(d) - 1] - \frac{\sin \lambda}{d} + \dots \\ &\approx \lambda [1 - \varepsilon + \varepsilon \ln(d)] \approx \lambda. \end{aligned} \quad (83)$$

As illustrated in Fig. 7, in both the small- ε and large- d limits, equality (77) holds. At this critical phase, the unobservable parallel evolution is completely eliminated, i.e., $\mathcal{I}^\parallel \approx 0$, while the compression quality achieves its maximum by fully concentrating all accessible information contained in $\mathcal{I}^T$ into the experimentally measurable orthogonal component $\mathcal{I}^\perp$. This demonstrates that the Pancharatnam phase acts as a natural, robust benchmark for optimizing compression quality of quantum channels.

V. CONCLUSION

We identify the noncyclic Pancharatnam phase, arising from coherent system-meter coupling, as the central geometric criterion for optimal quantum compression across a broad class of postselected metrology protocols.

The realization of a truly lossless compression channel is shown to be contingent upon two principles: The system's evolution must be constrained to a Pancharatnam phase (non-vanishing connection), and the transition from the pre- to postselected state must constitute a single quantum.

This study reveals that slight deviations from the Pancharatnam phase cause pronounced reductions in both com-

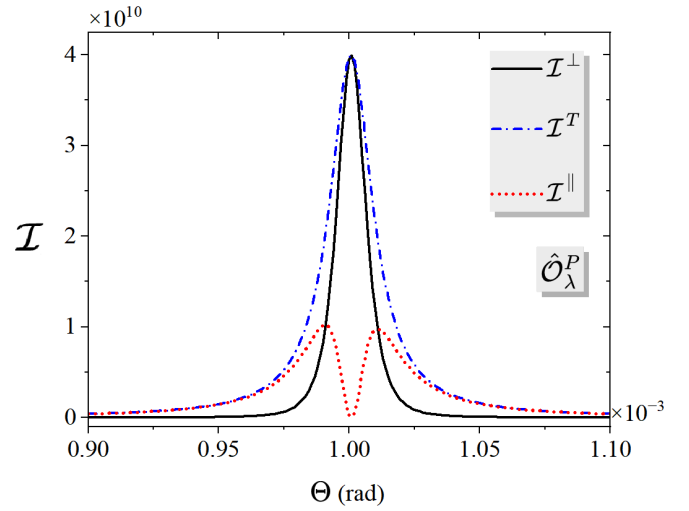


FIG. 7. QFI $\mathcal{I}^\perp$, $\mathcal{I}^T$, and $\mathcal{I}^\parallel$ are plotted as functions of the postselection parameter Θ for $\lambda = 10^{-3}$, $d = 10^4$, $\varepsilon = 10^{-4}$, and arbitrary j . The results demonstrate a pronounced central peak in the total and orthogonal components, signifying optimal quantum sensitivity at the corresponding postselection parameter Θ . The parallel component, in contrast, exhibits a split-peak structure and vanishes at the central phase point, illustrating the complete suppression of unobservable parallel evolution at the Pancharatnam phase given in Eq. (83).

pression efficiency and metrological precision, highlighting its practical role in experimental realizations.

Significantly, our framework further reveals that exploiting the extended complexity of quantum meters—specifically through high-dimensional qudit systems—offers a pathway to unlock substantial additional gains in quantum compression and measurement sensitivity.

The Pancharatnam phase thereby constitutes a foundational benchmark, advancing the theoretical foundations of quantum parameter estimation in postselected metrology.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: QUANTUM PARALLEL TRANSPORT CONDITION

A canonical example of parallel transport in classical physics is the transport of a tangent vector along a closed loop on the surface of a sphere. Consider a vector initially defined at point A and transported along the geodesic triangle $A \rightarrow B \rightarrow C \rightarrow A$. Throughout this process, the vector remains confined to the tangent plane of the sphere, and no local rotation (phase accumulation) occurs along the geodesic segments AB , BC , and CA . Upon returning to the initial point

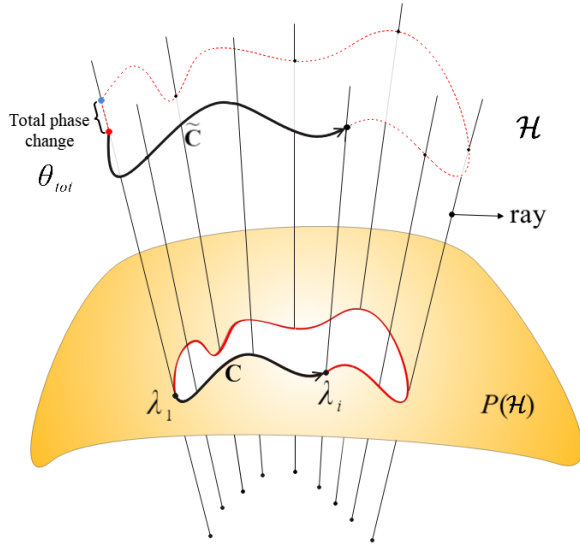


FIG. 8. The set of all equivalent state vectors (a ray) represents a point in the projective Hilbert space $P(\mathcal{H})$.

A, the vector generally undergoes a net rotation relative to its original orientation, a manifestation of the sphere's intrinsic curvature encoded by the holonomy associated with the loop.

By analogy, the concept of parallel transport extends naturally to the evolution of quantum states in a Hilbert space $\mathcal{H}$. Quantum states differing only by a global phase factor form equivalence classes known as rays, which correspond to one-dimensional subspaces spanned by normalized vectors. The collection of all such rays constitutes the projective Hilbert space $P(\mathcal{H})$, a manifold that effectively removes the physically irrelevant global phase from the state description; see Fig. 8. This space can be identified with the set of pure-state density matrices modulo global phase, serving as the natural arena for geometric phases and holonomies in quantum mechanics.

The quantum parallel transport condition can be rigorously formulated to ensure the absence of any geometric phase accumulation as the parameter λ varies infinitesimally [39]. Concretely, this condition requires that no relative phase is introduced between the states $|\Psi(\lambda)\rangle$ and $|\Psi(\lambda+d\lambda)\rangle$, which is mathematically equivalent to demanding that their overlap $\langle\Psi(\lambda)|\Psi(\lambda+d\lambda)\rangle$ remains strictly real and positive. The argument of this overlap, defined as $\arg\langle\Psi(\lambda)|\Psi(\lambda+d\lambda)\rangle = \text{Im} \ln\langle\Psi(\lambda)|\Psi(\lambda+d\lambda)\rangle$ vanishes identically. Accordingly,

$$\begin{aligned} \text{Im} \ln\langle\Psi(\lambda)|\Psi(\lambda+d\lambda)\rangle &= \text{Im} \ln(\langle\Psi(\lambda)|\Psi(\lambda)\rangle \\ &\quad + \langle\Psi(\lambda)|\partial_\lambda\Psi(\lambda)\rangle d\lambda + O(d\lambda^2)) \\ &\simeq \text{Im} \ln(1 + \langle\Psi(\lambda)|\partial_\lambda\Psi(\lambda)\rangle d\lambda) \\ &\simeq \langle\Psi(\lambda)|\partial_\lambda\Psi(\lambda)\rangle d\lambda = 0, \end{aligned}$$

and the quantum parallel transport condition can be expressed by $\langle\Psi(\lambda)|\partial_\lambda\Psi(\lambda)\rangle = 0$, which imposes that the tangent vector $|\partial_\lambda\Psi(\lambda)\rangle$ remains orthogonal to the state vector $|\Psi(\lambda)\rangle$ in Hilbert space, i.e., $|\partial_\lambda\Psi(\lambda)\rangle \perp |\Psi(\lambda)\rangle$. This orthogonality condition defines a horizontal lift of the parametrized curve from the projective Hilbert space to the full Hilbert space, eliminating gauge-dependent dynamical phase contributions.

APPENDIX B: GEOMETRICAL DESCRIPTION OF QFI

When a quantum state $|\Psi(\lambda)\rangle$ exhibits parametric dependence on λ , its derivative $|\partial_\lambda\Psi(\lambda)\rangle$ describes the state infinitesimal variations. These variations encompass two distinct contributions: (1) a physical component reflecting the state's evolution to a different ray within the projective Hilbert space and (2) a gauge-dependent component corresponding to an infinitesimal phase rotation, which is parallel to $|\Psi(\lambda)\rangle$. Thus, the derivative with respect to the parameter λ decomposes as

$$|\partial_\lambda\Psi(\lambda)\rangle = |\partial_\lambda\Psi(\lambda)\rangle^\parallel + |\partial_\lambda\Psi(\lambda)\rangle^\perp. \quad (\text{B1})$$

The component of $|\partial_\lambda\Psi(\lambda)\rangle$ parallel to $|\Psi(\lambda)\rangle$ is given by [40]

$$|\partial_\lambda\Psi(\lambda)\rangle^\parallel = \mathbb{P}_\psi |\partial_\lambda\Psi(\lambda)\rangle = |\Psi(\lambda)\rangle \langle\Psi(\lambda)|\partial_\lambda\Psi(\lambda)\rangle,$$

where $\mathbb{P}_\psi = |\Psi(\lambda)\rangle \langle\Psi(\lambda)|$ is the projector onto the state $|\Psi(\lambda)\rangle$. The overlap $\langle\Psi(\lambda)|\partial_\lambda\Psi(\lambda)\rangle$ causes a global phase change that does not affect measurement outcomes. The orthogonal component is derived simply by subtraction of the parallel component from $|\partial_\lambda\Psi(\lambda)\rangle$ and can be written as [40]

$$\begin{aligned} |\partial_\lambda\Psi(\lambda)\rangle^\perp &= |\partial_\lambda\Psi(\lambda)\rangle - |\partial_\lambda\Psi(\lambda)\rangle^\parallel \\ &= (\mathbb{I} - |\Psi(\lambda)\rangle \langle\Psi(\lambda)|) |\partial_\lambda\Psi(\lambda)\rangle, \end{aligned} \quad (\text{B2})$$

which leads to observable differences in the quantum state. The decomposition (B1) of the state derivative becomes

$$|\partial_\lambda\Psi(\lambda)\rangle = \underbrace{\langle\Psi(\lambda)|\partial_\lambda\Psi(\lambda)\rangle |\Psi(\lambda)\rangle}_{\text{Parallel(global phase)}} + \underbrace{(\mathbb{I} - |\Psi(\lambda)\rangle \langle\Psi(\lambda)|) |\partial_\lambda\Psi(\lambda)\rangle}_{\text{Orthogonal(physical)}}, \quad (\text{B3})$$

which clearly isolates the physically relevant sensitivity of the state to λ . The QFI can then be written as [7,9]

$$\begin{aligned} \mathcal{I}^\perp &= 4 \| |\partial_\lambda\Psi(\lambda)\rangle^\perp \|^2 \\ &= 4 (\langle\partial_\lambda\Psi(\lambda)|\partial_\lambda\Psi(\lambda)\rangle - |\langle\Psi(\lambda)|\partial_\lambda\Psi(\lambda)\rangle|^2) \\ &= 4 (\| |\partial_\lambda\Psi(\lambda)\rangle \|^2 - \| |\partial_\lambda\Psi(\lambda)\rangle^\parallel \|^2), \end{aligned} \quad (\text{B4})$$

which rigorously quantifies the parametric sensitivity of a quantum state by evaluating the squared norm of the component of the state's derivative that is orthogonal to the state vector itself in Hilbert space. This orthogonal component encapsulates the physically meaningful variation of the state within the projective Hilbert space, effectively filtering out gauge-dependent phase fluctuations parallel to the state vector.

APPENDIX C: GAUGE INVARIANCE OF QFI

By performing a phase redefinition (gauge transformation) for the quantum state

$$|\Psi(\lambda)\rangle \rightarrow e^{i\varphi(\lambda)} |\Psi(\lambda)\rangle,$$

where $\varphi(\lambda)$ is an arbitrary real function, the total derivative reads

$$|\partial_\lambda\Psi(\lambda)\rangle \rightarrow e^{i\varphi(\lambda)} (i|\Psi(\lambda)\rangle \partial_\lambda\varphi(\lambda) + |\partial_\lambda\Psi(\lambda)\rangle),$$

and its norm gives

$$\begin{aligned} \| |\partial_\lambda\Psi(\lambda)\rangle \|^2 &\rightarrow \langle\partial_\lambda\Psi(\lambda)|\partial_\lambda\Psi(\lambda)\rangle + (\partial_\lambda\varphi(\lambda))^2 \\ &\quad + 2i\partial_\lambda\varphi(\lambda) \langle\Psi(\lambda)|\partial_\lambda\Psi(\lambda)\rangle, \end{aligned} \quad (\text{C1})$$

TABLE I. Meter state $|\mathcal{M}_d^{(n)}\rangle$ classification for different values of d and n .

Number of parties (n)	Dimension (d)	Meter state $ \mathcal{M}_d^{(n)}\rangle$	State classification
1	1	$ b_0\rangle$	Basis
	2	$\frac{1}{\sqrt{2}}(b_0\rangle + b_1\rangle)$	Qubit
	3	$\frac{1}{\sqrt{3}}(b_0\rangle + b_1\rangle + b_2\rangle)$	Qutrit
	d	$\frac{1}{\sqrt{d}}(b_0\rangle + b_1\rangle + \dots + b_{d-1}\rangle)$	Qudit
2	1	$ b_0\rangle b_0\rangle$	Bipartite product (twin)
	2	$\frac{1}{\sqrt{2}}(b_0\rangle b_0\rangle + b_1\rangle b_1\rangle)$	Maximally entangled two-qubit
	3	$\frac{1}{\sqrt{3}}(b_0\rangle b_0\rangle + b_1\rangle b_1\rangle + b_2\rangle b_2\rangle)$	Maximally entangled two-qutrit
	d	$\frac{1}{\sqrt{d}}(b_0\rangle b_0\rangle + b_1\rangle b_1\rangle + \dots + b_{d-1}\rangle b_{d-1}\rangle)$	Maximally entangled two-qudit
3	1	$ b_0\rangle b_0\rangle b_0\rangle$	Tripartite product
	2	$\frac{1}{\sqrt{2}}(b_0\rangle b_0\rangle b_0\rangle + b_1\rangle b_1\rangle b_1\rangle)$	Greenberger-Horne-Zeilinger (GHZ)
	3	$\frac{1}{\sqrt{3}}(b_0\rangle b_0\rangle b_0\rangle + b_1\rangle b_1\rangle b_1\rangle + b_2\rangle b_2\rangle b_2\rangle)$	Qutrit (generalized) GHZ
	d	$\frac{1}{\sqrt{d}}(b_0\rangle b_0\rangle b_0\rangle + b_1\rangle b_1\rangle b_1\rangle + \dots + b_{d-1}\rangle b_{d-1}\rangle b_{d-1}\rangle)$	Qudit (generalized) GHZ
n	d	$\frac{1}{\sqrt{d}}(b_0\rangle^{\otimes n} + b_1\rangle^{\otimes n} + \dots + b_{d-1}\rangle^{\otimes n})$	n -partite qudit (generalized) GHZ

where $\langle\Psi(\lambda)|\Psi(\lambda)\rangle = 1$. The parallel term transforms as

$$\begin{aligned} \|\partial_\lambda \Psi(\lambda)\|^2 &\rightarrow |\langle\Psi(\lambda)|\partial_\lambda \Psi(\lambda)\rangle + i\partial_\lambda \varphi(\lambda)|^2 \\ &= |\langle\Psi(\lambda)|\partial_\lambda \Psi(\lambda)\rangle|^2 + (\partial_\lambda \varphi(\lambda))^2 \\ &\quad + 2i\partial_\lambda \varphi(\lambda)\langle\Psi(\lambda)|\partial_\lambda \Psi(\lambda)\rangle. \end{aligned} \quad (\text{C2})$$

Substituting the last two expressions into the QFI formula (7), we find that the quantities involving $\varphi(\lambda)$ cancel exactly:

$$\mathcal{I}^\perp = 4\langle\partial_\lambda \Psi(\lambda)|\partial_\lambda \Psi(\lambda)\rangle - 4|\langle\Psi(\lambda)|\partial_\lambda \Psi(\lambda)\rangle|^2. \quad (\text{C3})$$

The QFI formula is gauge invariant.

APPENDIX D: DERIVATION OF QFI ASSOCIATED WITH $|\Psi(\lambda)\rangle$

To derive the QFI formula, we start from the compressed state [9]

$$|\Psi(\lambda)\rangle = \mathcal{P}^{-1/2}(\lambda)|S_f\rangle \otimes \hat{\mathbb{K}}(\lambda)|\mathcal{M}_i\rangle = |S_f\rangle \otimes |\Phi(\lambda)\rangle, \quad (\text{D1})$$

and find the derivative

$$|\partial_\lambda \Phi(\lambda)\rangle = \mathcal{P}^{-1/2}(\lambda)[\partial_\lambda \hat{\mathbb{K}}(\lambda) - \frac{1}{2}\hat{\mathbb{K}}(\lambda)\partial_\lambda \ln \mathcal{P}(\lambda)]|\mathcal{M}_i\rangle. \quad (\text{D2})$$

Thus, the magnitude of the total derivative is given by

$$\begin{aligned} \|\partial_\lambda \Psi(\lambda)\|^2 &= \mathcal{P}^{-1}(\lambda) \left[\langle\partial_\lambda \hat{\mathbb{K}}^\dagger(\lambda)\partial_\lambda \hat{\mathbb{K}}(\lambda)\rangle \right. \\ &\quad \left. - \frac{\mathcal{P}^{-1}(\lambda)}{2} \langle\partial_\lambda (\hat{\mathbb{K}}^\dagger(\lambda)\hat{\mathbb{K}}(\lambda))\partial_\lambda \ln \mathcal{P}(\lambda)\rangle \right] \\ &\quad + \frac{1}{4}(\partial_\lambda \ln \mathcal{P}(\lambda))^2. \end{aligned} \quad (\text{D3})$$

Using expressions (D1) and (D2), one can find

$$\langle\Phi(\lambda)|\partial_\lambda \Phi(\lambda)\rangle = \mathcal{P}^{-1}(\lambda)\langle\hat{\mathbb{K}}^\dagger(\lambda)\partial_\lambda \hat{\mathbb{K}}(\lambda)\rangle - \frac{1}{2}\partial_\lambda \ln \mathcal{P}(\lambda), \quad (\text{D4})$$

$$\langle\partial_\lambda \Phi(\lambda)|\Phi(\lambda)\rangle = \mathcal{P}^{-1}(\lambda)\langle\partial_\lambda \hat{\mathbb{K}}^\dagger(\lambda)\hat{\mathbb{K}}(\lambda)\rangle - \frac{1}{2}\partial_\lambda \ln \mathcal{P}(\lambda), \quad (\text{D5})$$

and the parallel term in the QFI gives

$$\begin{aligned} \|\partial_\lambda \Psi(\lambda)\|^2 &= \mathcal{P}^{-2}(\lambda)\langle\hat{\mathbb{K}}^\dagger(\lambda)\partial_\lambda \hat{\mathbb{K}}(\lambda)\rangle^2 \\ &\quad - \frac{\mathcal{P}^{-1}(\lambda)}{2} \langle\partial_\lambda (\hat{\mathbb{K}}^\dagger(\lambda)\hat{\mathbb{K}}(\lambda))\partial_\lambda \ln \mathcal{P}(\lambda)\rangle \\ &\quad + \frac{1}{4}(\partial_\lambda \ln \mathcal{P}(\lambda))^2. \end{aligned} \quad (\text{D6})$$

The final form of the QFI becomes

$$\begin{aligned} \mathcal{I}^\perp &= 4[\|\partial_\lambda \Psi(\lambda)\|^2 - \|\partial_\lambda \Psi(\lambda)\|^2] \\ &= 4\mathcal{P}^{-2}(\lambda)[\langle\partial_\lambda \hat{\mathbb{K}}^\dagger(\lambda)\partial_\lambda \hat{\mathbb{K}}(\lambda)\rangle\mathcal{P}(\lambda) - \langle\hat{\mathbb{K}}^\dagger(\lambda)\partial_\lambda \hat{\mathbb{K}}(\lambda)\rangle^2], \end{aligned} \quad (\text{D7})$$

which is the QFI written as a function of the channel operator $\hat{\mathbb{K}}(\lambda)$.

We note that the averages in all previous calculations are evaluated with respect to the meter state, which is defined as the qudit state $|\mathcal{M}_i\rangle = |\mathcal{M}_d^{(n)}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |b_k\rangle^{\otimes n}$, as introduced in Sec. III D. The corresponding states for different values of n and d are listed in Table I.

APPENDIX E: NONCYCLIC PANCHARATNAM PHASE

To elucidate the nature of the Pancharatnam phase in Eq. (22), we exploit its characteristic nontransitivity [23], which directly reflects the geometric origin of the phase difference. Specifically, we consider the evolution generated by the operator $\hat{\mathcal{O}}_\lambda$ as a sequence of i discrete transitions between neighboring states, defined by parameters λ with $\lambda_1 = d\lambda$, $\lambda_2 = \lambda_1 + d\lambda$, $\dots$, $\lambda_i = \lambda_{i-1} + d\lambda$. The phase difference between the initial and final states is then constructed via successive projections: The initial state $|\phi_1\rangle$ is projected onto the second state $|\phi_2\rangle\langle\phi_2|\phi_1\rangle$, the second onto the third, and so forth, until the final projection onto the original state $|\phi_1\rangle\langle\phi_1|\phi_i\rangle$. Consequently, the final state is expressed

as [41]

$$|\tilde{\phi}_1\rangle = |\phi_1\rangle\langle\phi_1|\phi_i\rangle\langle\phi_i|\cdots\langle\phi_4|\phi_3\rangle\langle\phi_3|\phi_2\rangle\langle\phi_2|\phi_1\rangle, \quad (\text{E1})$$

which differs from the initial state $|\phi_1\rangle$, by the phase factor

$$\begin{aligned} \theta_{\text{geo}} &= \text{Im} \ln[\langle\phi_1|\phi_i\rangle\langle\phi_i|\cdots\langle\phi_4|\phi_3\rangle\langle\phi_3|\phi_2\rangle\langle\phi_2|\phi_1\rangle] \\ &= \text{Im} \ln[\text{Tr}(|\phi_i\rangle\langle\phi_i|\cdots\langle\phi_3|\phi_2\rangle\langle\phi_2|\phi_1\rangle\langle\phi_1|)] \\ &= \text{Im} \ln\left[\text{Tr} \prod_{k=1}^i \varrho_k\right] \end{aligned} \quad (\text{E2})$$

corresponds to the geometric phase, which is a gauge-invariant quantity for cyclic evolution, where the initial and final states coincide. In contrast, for noncyclic evolution [41], the final state $|\phi_i\rangle$, does not project back onto the initial state $|\phi_1\rangle$, and the geometric phase can be generalized accordingly as

$$\begin{aligned} \theta_{\text{geo}} &= \text{Im} \ln[\langle\phi_1|\phi_i\rangle\langle\phi_i|\cdots\langle\phi_4|\phi_3\rangle\langle\phi_3|\phi_2\rangle\langle\phi_2|\phi_1\rangle] \\ &= \text{Im} \ln\langle\phi_1|\phi_i\rangle - \text{Im} \ln \prod_{k=1}^{i-1} \langle\phi_k|\phi_{k+1}\rangle \\ &\approx \text{Im} \ln\langle\phi_1|\phi_i\rangle - \text{Im} \sum_{k=1}^{i-1} \ln[1 + \langle\phi_k|\partial_\lambda\phi_k\rangle d\lambda + \cdots] \\ &\approx \text{Im} \ln\langle\phi_1|\hat{\mathcal{O}}_\lambda|\phi_i\rangle - \text{Im} \sum_{k=1}^{i-1} \langle\phi_k|\partial_\lambda\phi_k\rangle d\lambda, \end{aligned} \quad (\text{E3})$$

where $\text{Im} \ln\langle\phi_1|\phi_i\rangle = \text{Im} \ln\langle\hat{\mathcal{O}}_\lambda\rangle$ is the total phase difference between initial and final states. The second term involving $\text{Im} \sum_{k=1}^{i-1} \langle\phi_k|\partial_\lambda\phi_k\rangle d\lambda$ is the total dynamical phase. In the limit

$i \rightarrow \infty$, the geometric phase becomes [41]

$$\begin{aligned} \theta_{\text{geo}} &= \text{Im} \ln\langle\hat{\mathcal{O}}_\lambda\rangle - \lim_{i \rightarrow \infty} \left\{ \text{Im} \sum_{k=1}^{i-1} \langle\phi_k|\partial_\lambda\phi_k\rangle d\lambda \right\} \\ &= \text{Im} \ln\langle\hat{\mathcal{O}}_\lambda\rangle - \text{Im} \int_{\lambda_1}^{\lambda_i} \langle\hat{\mathcal{O}}_\lambda^\dagger \partial_\lambda \hat{\mathcal{O}}_\lambda\rangle d\lambda \\ &= \theta_{\text{tot}} - \theta_{\text{dyn}}, \end{aligned} \quad (\text{E4})$$

where the integral is taken along the evolution path $\mathcal{C}$ in projective Hilbert space $P(\mathcal{H})$, the space of density matrices ϱ . This generalization extends the notion of the Pancharatnam phase beyond cyclic trajectories. When the parallel transport condition $\langle\hat{\mathcal{O}}_\lambda^\dagger \partial_\lambda \hat{\mathcal{O}}_\lambda\rangle \rightarrow 0$ is achieved (see Appendix A), the noncyclic Pancharatnam phase θ_{tot} becomes purely geometric. Figure 1 shows the postselection probability $\mathcal{P}^{(j)}$ as a function of Θ for the case of $\hat{\mathcal{O}}_\lambda^P$ for different j values. All curves are uniformly phase shifted (by the Pancharatnam phase $\text{Im} \ln\langle\hat{\mathcal{O}}_\lambda\rangle$), resulting in coincident fringe positions that are independent of j .

As a final remark, the noncyclic Pancharatnam phase represents the most general form in which geometric phases can be defined. Consequently, our formalism is directly extendable to other scenarios by suitably modifying the channel operator to incorporate the parameter-dependent evolution. For example, our framework naturally reduces to the Berry phase in the special case of adiabatic cyclic evolution [23]. Specifically, defining the Berry phase requires specifying a parameter space $\lambda = \lambda(\mathbf{R})$ associated with the system, followed by a cyclic evolution of λ within this space [42]. In general, we expect that optimization remains effective provided the Pancharatnam phase shift in the interference pattern is nonzero. However, if the geometric phase contribution exactly cancels the dynamical phase, resulting in a net zero phase shift, optimization is expected to fail.

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