

Thermodynamic approach to quantum cooling limit of continuous Gaussian feedback

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Feedback cooling plays a critical role in stabilizing quantum systems and achieving low temperatures, where a key question is to determine the fundamental thermodynamic limits on cooling performance. We establish a fundamental bound on quantum feedback cooling in Gaussian systems, by deriving a generalized second law of thermodynamics involving the kinetic temperatures of the system and a measure of quantum information flow obtained by continuous measurement. In contrast to previously known bounds, the obtained bound can be saturated by experimentally feasible situations using the quantum Kalman filter with a large feedback gain, where the cooling efficiency approaches its maximum. Our theoretical result is numerically demonstrated using parameters from an experiment of levitated nanoparticles. Our theory provides a general framework for understanding the thermodynamic constraints on quantum feedback cooling.

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I. INTRODUCTION

Measurement and feedback are fundamental tools that enable noise suppression, stabilization, optimal control, and adaptive control in diverse applications. In quantum systems, continuous measurement and feedback based on real-time data have been theoretically studied [1–6] and experimentally realized for qubit stabilization [7,8], error correction [9], and cooling of oscillatory systems [10–13]. In nanomechanical resonators [10] and optically trapped nanoparticles [11–14], state-based feedback control reduces vibrational energy, enabling cooling close to the ground state. In Gaussian systems, the quantum Kalman filter [15] provides optimal filtering that enhances the precision of feedback and has been demonstrated experimentally [12].

In feedback cooling, the energy of the system is reduced by extracting work through measurement-based control, which can be interpreted as a “Maxwell’s demon” setup in the context of thermodynamics of information [16]. By incorporating information terms, fundamental thermodynamic relations, such as the second law and the fluctuation theorem, have been generalized to systems under measurement and feedback in both classical and quantum regimes [17–22], including the case of continuous measurement and feedback [23–29]. Experimental studies have further validated those information-thermodynamic relations [30–40].

A fundamental challenge in feedback cooling is to determine the ultimate limits imposed by thermodynamics. While

the standard formulation of the second law relates entropy production to the temperature of the heat bath, the primary objective in feedback cooling is to reduce the temperature of the system below the bath temperature. Thermodynamics of information for classical systems have revealed second-law-like inequalities for Langevin systems, which relate the kinetic temperatures of the system itself to the rate of information acquisition through measurement, quantified by transfer entropy [41–43]. However, a quantum counterpart to these results remains elusive, and a general characterization of cooling limits based on thermodynamic laws is still lacking for quantum systems.

In this study, we consider a setup in which cooling is achieved by a continuous position measurement [2] together with feedback based on its measurement outcomes [1,5] (see Fig. 1). In this framework, we establish a fundamental bound of feedback cooling in Gaussian quantum systems. Specifically, we derive a second-law-like inequality that imposes a constraint on work extraction by a measure of quantum information flow called quantum-classical-transfer (QC-transfer) entropy [29]. Since the obtained bound is based on the kinetic temperatures of the system, it enables characterization of feedback cooling (i.e., reduction of the temperature of the system) and is achievable with a sufficiently large feedback gain along with using the quantum Kalman filter, in contrast to previously known bounds [29]. We numerically demonstrate these results with experimentally relevant parameters adopted from a recent nanoparticle experiment [12]. Our results provide a theoretical foundation for the limits of feedback cooling from the perspective of thermodynamics of information in the quantum regime.

This paper is organized as follows. In Sec. II, we introduce the setup for Gaussian feedback cooling and define key quantities including the transfer entropy and kinetic temperatures. In Sec. III, we present the main result, i.e., the information-thermodynamic limit of quantum feedback

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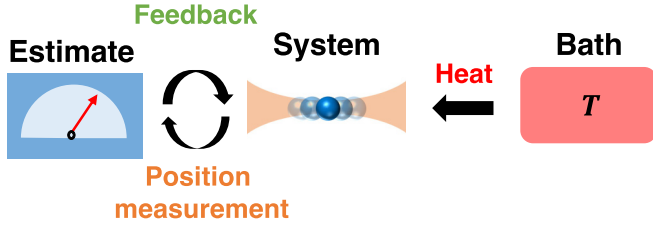


FIG. 1. Schematic of the setup. The system undergoes feedback control based on estimated values obtained from the outcomes of position measurements. The system is also coupled to a heat bath at temperature T .

cooling. In Sec. IV, we discuss the asymptotic achievement of the equality condition of the obtained bound. Finally, in Sec. V, we summarize our results and outline possible future directions.

II. SETUP

A. Model of continuous Gaussian feedback cooling

We describe the framework for feedback cooling in Gaussian systems, where the state remains Gaussian throughout its evolution. We consider continuous position measurement, whose measurement operator commutes with $\hat{x}$, and real-time feedback control based on the measurement record, applied to a Gaussian system coupled to a single heat bath that satisfies detailed balance (see Fig. 1). Specifically, the Hamiltonian of the system is supposed to be harmonic: $\hat{H}_{\text{HO}} = \frac{\hbar\omega}{2}(\hat{p}^2 + \hat{x}^2)$. Here, $\hat{x}$ and $\hat{p}$ represent the dimensionless position and momentum operators, respectively, satisfying the commutation relation $[\hat{x}, \hat{p}] = i$, and ω is the angular frequency of the oscillator. In addition, the system is externally controlled by a feedback Hamiltonian $\hat{H}_{\text{FB}}(t)$ based on the past measurement outcomes. We assume that the expectation values of the initial position and momentum are zero. The dynamics of feedback cooling are governed by the stochastic master equation [2]:

$$d\hat{\rho}_c(t) = -\frac{i}{\hbar}[\hat{H}_{\text{HO}} + \hat{H}_{\text{FB}}(t), \hat{\rho}_c(t)]dt + \mathcal{L}_{\text{bath}}[\hat{\rho}_c(t)]dt + 2k\mathcal{D}_{\hat{x}}[\hat{\rho}_c(t)]dt + \sqrt{2k\eta}\mathcal{H}_{\hat{x}}[\hat{\rho}_c(t)]dW, \quad (1)$$

where $\mathcal{D}_{\hat{x}}[\hat{\rho}_c] := \hat{z}\hat{\rho}_c\hat{z}^\dagger - \frac{1}{2}[\hat{z}^\dagger\hat{z}, \hat{\rho}_c]$ is the dissipator and $\hat{\rho}_c(t)$ represents a state conditioned on all measurement outcomes up to time t . The superoperator $\mathcal{L}_{\text{bath}}[\hat{\rho}_c] := \gamma(\bar{n} + 1)\mathcal{D}_{\hat{a}}[\hat{\rho}_c] + \gamma\bar{n}\mathcal{D}_{\hat{a}^\dagger}[\hat{\rho}_c]$ represents the contribution of the heat bath, where $\hat{a}$ and $\hat{a}^\dagger$ represent the annihilation and creation operators, respectively, and γ is the dissipation rate. The ratio between $\bar{n}$ and $\bar{n} + 1$ satisfies the detailed balance condition $\bar{n}/(\bar{n} + 1) = e^{-\hbar\omega/T}$, where T is the temperature of the heat bath and the Boltzmann constant is set equal to unity for the sake of simplicity. The third term on the right-hand side of Eq. (1) represents momentum diffusion at a rate $2k$, resulting from the backaction of the position measurement, while the fourth term represents the information obtained from the measurement process [2]. Because the measurement operator $\sqrt{2k}\hat{x}$ commutes with $\hat{x}$, the measurement is a quantum-nondemolition position measurement if the Hamiltonian term $\hat{H}_{\text{HO}}$ is absent. The superoperator $\mathcal{H}_{\hat{x}}$ is defined as $\mathcal{H}_{\hat{x}}[\hat{\rho}_c] := \hat{x}\hat{\rho}_c + \hat{\rho}_c\hat{x} - 2\text{tr}(\hat{x}\hat{\rho}_c)\hat{\rho}_c$. The parameter

k represents the measurement rate, while the Wiener increment dW is a Gaussian random variable with zero mean and variance dt . The measurement outcome obtained via measurement of the position over an infinitesimal time interval dt is described by $dy(t) = \langle \hat{x}(t) \rangle_c dt + dW/\sqrt{8k\eta}$, where η represents the detection efficiency and $\langle \cdot(t) \rangle_c := \text{tr}(\hat{\rho}_c(t) \cdot)$ denotes the expectation value under the conditional state. The set of all measurement outcomes until time t is defined as $Y_t := \{dy(s)\}_{s=0}^t$. Note that $\hat{\rho}_c(t)$ is the state of the system for a given realization of measurement outcomes Y_t . We also denote $\hat{\rho} = \mathbb{E}_{Y_t}[\hat{\rho}_c]$ as the averaged state by taking average over all measurement outcomes, where $\mathbb{E}_X$ represents taking the expectation value over a random variable X .

Estimates of the observable are constructed from the set of obtained measurement outcomes and used for feedback. Using measurement outcomes Y_t , estimates are defined as $e_t = h(Y_t)$, where $h(\cdot)$ compresses the past measurement data into estimates e_t . The posterior state conditioned on this estimate is denoted by $\hat{\rho}_t^{e_t}$. Using the estimates, the feedback action is determined. The linear feedback Hamiltonian $\hat{H}_{\text{FB}}$ is given by

$$\hat{H}_{\text{FB}}(t) = \hbar c_x(e_t)\hat{x} + \hbar c_p(e_t)\hat{p}, \quad (2)$$

where c_x and c_p are linear functions of the estimate e_t , defining the feedback coefficients. The special case of the estimate is the quantum Kalman filter [15], which is given by the conditional expectation values

$$e_t = (\langle \hat{x}(t) \rangle_c, \langle \hat{p}(t) \rangle_c). \quad (3)$$

Note that obtaining the measurement outcomes reduces the uncertainty in position, although $\langle \hat{x} \rangle_c$ and $\langle \hat{p} \rangle_c$ fluctuate due to random measurement outcomes. The feedback Hamiltonian is then designed to decrease the absolute values of $\langle \hat{x} \rangle_c$ and $\langle \hat{p} \rangle_c$, and thus the fluctuation of position and momentum is reduced at the ensemble average level. Therefore, the dynamics described by Eq. (1) allows reducing the energy of the system and realizes feedback cooling.

B. QC-transfer entropy

As our goal is to obtain an information-thermodynamic relation to characterize the cooling limit of Gaussian systems, we consider the QC-transfer entropy introduced in Ref. [29], which quantifies the amount of information gain through continuous measurement. It is defined as

$$\dot{I}_{\text{QCT}} := \lim_{dt \rightarrow 0} \frac{1}{dt} \mathbb{E}_{Y_t}[I_{\text{QC}}(\hat{\rho}_c : dy(t))]. \quad (4)$$

Here, $I_{\text{QC}}(\hat{\rho}_c : dy(t))$ is the quantum-classical-mutual (QC-mutual) information [44,45] given by

$$I_{\text{QC}}(\hat{\rho}_c : dy(t)) := S(\hat{\rho}_c) - \mathbb{E}_{dy(t)}[S(\hat{\rho}_{c,M})], \quad (5)$$

where $\hat{\rho}_{c,M} := \hat{\rho}_c + 2k\mathcal{D}_{\hat{x}}[\hat{\rho}_c]dt + 4k\eta\mathcal{H}_{\hat{x}}[\hat{\rho}_c](dy(t) - \langle \hat{x}(t) \rangle_c dt)$ is the conditional state after measurement at time t and $S(\hat{\rho}) := -\text{tr}[\hat{\rho} \ln \hat{\rho}]$ represents the von Neumann entropy. We note that Eq. (5) is regarded as the conditional QC-mutual information given all prior measurement outcomes Y_t , because $\hat{\rho}_c$ is the conditional state. Since the QC-transfer entropy quantifies the total information gain, it provides an upper bound for the information available in the multiple feedback processes.

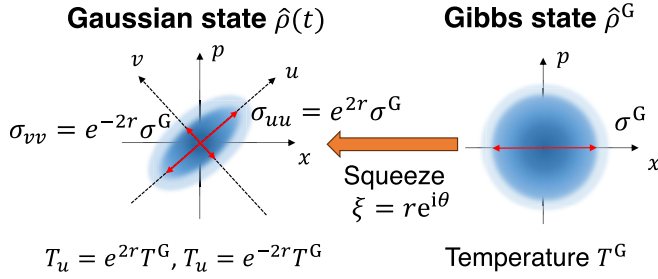


FIG. 2. Schematic of the distribution of a Gaussian state $\hat{\rho}(t)$ in the phase space (i.e., the Wigner distribution). The kinetic temperatures T_u and T_v characterize the variances in the eigendirections of the covariance matrix σ .

C. Kinetic temperatures

We here introduce kinetic temperatures and work rate. We begin by defining the kinetic temperatures of a given Gaussian state. The one-mode Gaussian state is fully characterized by the mean values of position and momentum and the variance-covariance matrix. Let $\sigma := [\sigma_{ij}]_{i,j=x,p}$ denote the covariance matrix of the averaged state $\hat{\rho}(t)$. The position and momentum variances are given by $\sigma_{xx} := \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$ and $\sigma_{pp} := \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$, respectively, while the covariance is defined as $\sigma_{xp} := \frac{1}{2} \langle \{\hat{x}, \hat{p}\} \rangle - \langle \hat{x} \rangle \langle \hat{p} \rangle$, where $\langle \cdot \rangle$ denotes the expectation value with respect to $\hat{\rho}(t)$. Let the eigenvalues of σ be σ_{uu} and σ_{vv} . Noting that a single-mode Gaussian state can always be expressed as a squeezed thermal state [46], we define the kinetic temperatures along eigendirections. The single-mode squeezed thermal state is generated by applying a squeezing operation $\hat{S}(\xi) := \exp(\frac{1}{2}\xi(\hat{a}^\dagger)^2 - \frac{1}{2}\xi^*\hat{a}^2)$, with a squeezing parameter $\xi = re^{i\theta}$ to the Gibbs state $\hat{\rho}^G := \exp(-\hat{H}_{\text{HO}}/T^G)/\text{tr}[\exp(-\hat{H}_{\text{HO}}/T^G)]$ [47], where T^G is the temperature of the Gibbs state, as $\hat{\rho}_t = \hat{S}(\xi)\hat{\rho}^G\hat{S}(\xi)^\dagger$. The eigenvalues of σ satisfy $\sigma_{uu} = e^{2r}\sigma^G$ and $\sigma_{vv} = e^{-2r}\sigma^G$, where

$$\sigma^G := \text{tr}(\hat{x}^2\hat{\rho}^G) = \text{tr}(\hat{p}^2\hat{\rho}^G) = \frac{1}{2} \coth \frac{\hbar\omega\beta^G}{2} \quad (6)$$

is the common position and momentum variance of the state $\hat{\rho}^G$. Accordingly, the kinetic temperatures T_u and T_v , associated with these eigenvalues, are defined as

$$T_u := e^{2r}T^G, \quad T_v := e^{-2r}T^G. \quad (7)$$

A schematic of kinetic temperatures is shown in Fig. 2. The kinetic temperatures T_u and T_v correspond to scaling the Gibbs state temperature

$$T^G = \hbar\omega \left[\ln \left(\frac{2\sqrt{\det \sigma} + 1}{2\sqrt{\det \sigma} - 1} \right) \right]^{-1}, \quad (8)$$

according to the widths in eigendirections of the distribution. We further note that

$$\begin{aligned} \ln \hat{\rho}_t &= \hat{S}(\xi)(\ln \hat{\rho}^G)\hat{S}(\xi)^\dagger \\ &= -\beta^G \hat{S}(\xi)\hat{H}_{\text{HO}}\hat{S}(\xi)^\dagger - \ln Z(\beta^G) \\ &= -\frac{\hbar\omega\hat{u}^2}{2T_u} - \frac{\hbar\omega\hat{v}^2}{2T_v} - \ln Z(\beta^G), \end{aligned} \quad (9)$$

where

$$\hat{u} := e^r \hat{S}(\xi) \hat{x} \hat{S}(\xi)^\dagger, \quad (10)$$

$$\hat{v} := e^{-r} \hat{S}(\xi) \hat{p} \hat{S}(\xi)^\dagger \quad (11)$$

are the quadratures that correspond to the eigenvalues σ_{uu} and σ_{vv} , i.e., $\sigma_{uu} = \text{tr}(\hat{u}^2\hat{\rho}_t) - \text{tr}(\hat{u}\hat{\rho}_t)^2$ and $\sigma_{vv} = \text{tr}(\hat{v}^2\hat{\rho}_t) - \text{tr}(\hat{v}\hat{\rho}_t)^2$, respectively.

We next define the work rate as follows. First, the internal energy of the system is given by $E(t) = \text{tr}[\hat{H}_{\text{HO}}\hat{\rho}(t)]$. Then, based on the first law of thermodynamics, the work rate corresponding to the energy extracted from the system is defined as $\dot{W}_{\text{ext}} := \dot{Q} - \frac{d}{dt}E(t)$, where

$$\dot{Q} := \text{tr}(\hat{H}_{\text{HO}}\mathcal{L}_{\text{bath}}[\rho]) \quad (12)$$

is the heat [22]. The time variation of the energy can be expressed as contributions along the directions of σ_{uu} and σ_{vv} , using the quadrature operators $\hat{u}$ and $\hat{v}$ associated with these directions, as $\frac{d}{dt}E(t) = \frac{\hbar\omega}{2} \frac{d}{dt} \text{tr}[\hat{\rho}\hat{u}^2] + \frac{\hbar\omega}{2} \frac{d}{dt} \text{tr}[\hat{\rho}\hat{v}^2]$. Similarly, the work rate can be further decomposed as

$$\dot{W}_{\text{ext}} = \dot{W}_{\text{ext},u} + \dot{W}_{\text{ext},v}, \quad (13)$$

where the terms $\dot{W}_{\text{ext},u}$ and $\dot{W}_{\text{ext},v}$ represent the contributions of the work rate along the directions of σ_{uu} and σ_{vv} , defined by Eqs. (B7) and (B8).

III. MAIN RESULTS

A. Information-thermodynamic cooling limit

We now present the main result of this study, a fundamental bound of the work rate in terms of kinetic temperatures. Suppose that the system is in the steady state; the averaged state $\hat{\rho}$ is given by the steady solution of Eq. (1). Then, we have a generalized second law of thermodynamics in the form

$$\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \leq \dot{I}_{\text{QCT}}. \quad (14)$$

This inequality indicates how much QC-transfer entropy can be utilized to enhance work extraction and lower the kinetic temperatures, and serves as a quantum generalization of the second-law-like bound for classical Langevin systems [41]. Since cooling is achieved by extracting work from the system and lowering its kinetic temperatures, inequality (14) represents the fundamental cooling limit.

We compare our result (14) with the generalized second law using the temperature T of the heat bath, given in the form [29]

$$\frac{\dot{W}_{\text{ext}}}{T} \leq \dot{I}_{\text{QCT}}. \quad (15)$$

In Appendix B 3, we show that the following relation holds in the steady state:

$$\frac{\dot{W}_{\text{ext}}}{T} \leq \frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v}. \quad (16)$$

This relation shows that inequality (14) provides a tighter bound compared to Eq. (15) in the steady-state regime, since the former explicitly involves the kinetic temperatures of the system itself.

We next consider the conditions for achieving the equality in Eq. (14). A key ingredient is the quantum Kalman filter, which provides the conditional expectation values as the estimates, as presented in Eq. (3). Note that $\hat{\rho}_t^{e_i} = \hat{\rho}_c$ holds in this case. By using this estimate based on the quantum Kalman filter, one can implement the so-called state-based feedback [5,12], where the feedback Hamiltonian (2) is given by the coefficients $c_x = b_x \langle \hat{x} \rangle_c + b_p \langle \hat{p} \rangle_c$ and $c_p = -a_x \langle \hat{x} \rangle_c - a_p \langle \hat{p} \rangle_c$. This feedback Hamiltonian aims to reduce the mean position and momentum to the origin in the phase space. Another requirement for achieving the equality in Eq. (14) is a large feedback gain. In fact, the equality in Eq. (14) is asymptotically achieved as the feedback gain goes to infinity if the estimate is implemented by the quantum Kalman filter (see Sec. IV for the proof). For simplicity, let the parameters of the feedback Hamiltonian be $a_x = b_p = g$ and $a_p = b_x = 0$. In this case, the equality in Eq. (14) is asymptotically achieved in the limit $g \rightarrow \infty$. This implies that, under these conditions, the feedback system perfectly converts the acquired information into work.

Under the same conditions, the occupancy number of the harmonic Hamiltonian $\hat{H}_{\text{HO}}$, $\langle a^\dagger a \rangle$ also reaches its minimal value determined by the bath temperature and the measurement rate (see Sec. IV), as the optimal feedback minimizes the sum of the variances of position and momentum, which is equivalent to the occupation number in our setup. This highlights the connection between the information-theoretic bound on thermodynamic energy conversion and the cooling limit of Gaussian systems.

B. Numerical demonstration

We present a numerical demonstration of our main inequality (14), by numerically implementing the quantum Kalman filter. Figure 3 shows the behavior of $\dot{W}_{\text{ext},u}/T_u + \dot{W}_{\text{ext},v}/T_v$ and $\dot{W}_{\text{ext}}/T$ as functions of the feedback gain in the steady state of Eq. (1). The parameters are taken from Ref. [12], an experiment of feedback cooling of levitated nanoparticles.

Figure 3 compares the two feedback Hamiltonians and shows that only $H_{\text{FB}}^{\hat{x},\hat{p}} = \hbar g(-\langle x \rangle_c \hat{p} + \langle p \rangle_c \hat{x})$, plotted in the orange curve, ensures that $\dot{W}_{\text{ext},u}/T_u + \dot{W}_{\text{ext},v}/T_v$ approaches the upper bound $\dot{I}_{\text{QCT}}$, as $g \rightarrow \infty$. Since feedback proportional to $\hat{x}$ is commonly used in experiments, we also consider the Hamiltonian of the form $H_{\text{FB}}^{\hat{x}} = \hbar g(\langle x \rangle_c + \langle p \rangle_c) \hat{x}$ (blue dashed curve), but it does not allow the system to reach the information-theoretic bound of Eq. (14). Moreover, $\dot{W}_{\text{ext}}/T$ with feedback Hamiltonian $H_{\text{FB}}^{\hat{x},\hat{p}}$ (green curve) fails to asymptotically approach $\dot{I}_{\text{QCT}}$, indicating that the naive bound (15) without using the kinetic temperatures is not achievable.

In addition, we compared our measurement scheme with an alternative scheme that produces the same heating rate for the occupation number. We performed numerical simulations for both methods and demonstrated that our protocol has an advantage in cooling (see Appendix E).

C. Outline of the proof and more on quantum information flow

To outline the derivation of our main inequality (14) and to clarify the structure behind its equality condition, here we introduce yet another information quantity, named the quantum-classical (QC) information flow, which is slightly

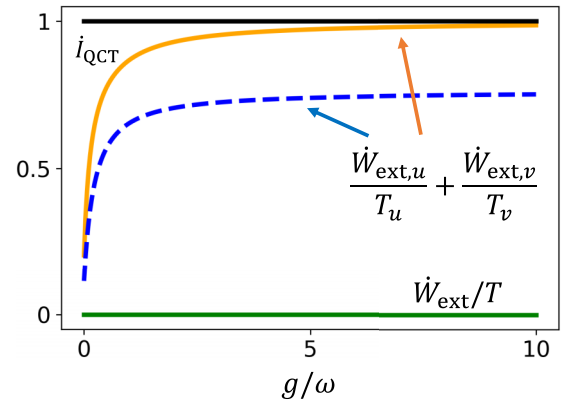


FIG. 3. The behavior of $\dot{W}_{\text{ext},u}/T_u + \dot{W}_{\text{ext},v}/T_v$ as a function of the feedback gain g in the steady state of Eq. (1). The feedback protocol is based on the quantum Kalman filter, where the feedback Hamiltonian is given by $H_{\text{FB}}^{\hat{x},\hat{p}} = \hbar g(-\langle x \rangle_c \hat{p} + \langle p \rangle_c \hat{x})$ (orange curve). In addition, we demonstrate another feedback Hamiltonian $H_{\text{FB}}^{\hat{x}} = \hbar g(\langle x \rangle_c + \langle p \rangle_c) \hat{x}$ (blue dashed curve) for the sake of comparison. The behavior of $\dot{W}_{\text{ext}}/T$ is also shown (green curve). The black line represents the QC-transfer entropy, the bound of Eq. (14), to which only the orange curve approaches in large g . The parameters are chosen as $k/\omega = 0.18$, $\eta = 0.34$, $T = 292$ K, and $\bar{n}\gamma/\omega = 0.0058$ [12]. Note that the kinetic temperatures T_u and T_v are orders of magnitude smaller than the bath temperature, and thus the green line becomes approximately equal to zero. The vertical axis is normalized by $\dot{I}_{\text{QCT}}$.

different from the QC-transfer entropy. The QC information flow is defined as

$$\dot{I}_{\text{QCI}}^S := \lim_{dt \rightarrow 0} \frac{1}{dt} [\chi(\hat{\rho}_{t+dt} : e_t) - \chi(\hat{\rho}_t : e_t)], \quad (17)$$

where $\chi(\rho : e) := S(\rho) - \mathbb{E}_e[S(\rho^e)]$ is the Holevo information [48]. Since the Holevo information is the measure of the correlation between a quantum state and a classical memory, the QC information flow $-\dot{I}_{\text{QCI}}^S$ quantifies the change in the system-memory correlation induced by the time evolution of the system during the feedback. We note that the QC information flow is a quantum version of the classical information flow [25], which is defined as the partial derivative of the classical mutual information.

Based on the QC information flow, we can derive a generalized second law that is also applicable to nonsteady states:

$$\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \leq -\dot{I}_{\text{QCI}}^S - \dot{S}_{\text{BA}}. \quad (18)$$

Here, $\dot{S}_{\text{BA}}$ represents the entropy change due to measurement backaction, defined as

$$\dot{S}_{\text{BA}} := \lim_{dt \rightarrow 0} \frac{1}{dt} \mathbb{E}_{e_t} [(S(\hat{\rho}_{t,\text{BA}}^{e_t}) - S(\hat{\rho}_t^{e_t}))], \quad (19)$$

where $\hat{\rho}_{t,\text{BA}}^{e_t} := \hat{\rho}_t^{e_t} + 2k\mathcal{D}_{\hat{x}}[\hat{\rho}_t^{e_t}]dt$ is the state evolved for a time interval dt under the contributions of measurement backaction. The term $-\dot{I}_{\text{QCI}}^S - \dot{S}_{\text{BA}}$ quantifies the information available for feedback control, because the unavoidable entropy change due to the measurement backaction is subtracted from the change in correlation originating from the dynamics of the system. The detailed derivation of Eq. (18) is provided

in Appendix B 1. A sufficient condition to achieve the equality of Eq. (18) is the quantum Kalman filter and a large feedback gain.

We next relate the two measures of quantum information flow: QC-transfer entropy and QC information flow. Again suppose that the system is in the steady state of Eq. (1). Then, the QC-transfer entropy provides an upper bound of the QC information flow:

$$-I_{\text{QCI}}^S - \dot{S}_{\text{BA}} \leq \dot{I}_{\text{QCT}}. \quad (20)$$

See Appendix B 2 for the derivation. A sufficient condition for the equality in Eq. (20) is the quantum Kalman filter. For the classical case of Eq. (20), the equality condition can be expressed in terms of sufficient statistics [49].

Now, by combining inequalities (18) and (20), our main result (14) is obtained. The equality condition for Eq. (14) directly follows from the equality conditions for inequalities (18) and (20).

IV. ACHIEVEMENT OF THE COOLING LIMIT BASED ON THE QUANTUM KALMAN FILTER

An example of experimental systems that obey the stochastic master equation (1) is the levitated nanoparticle system [12]. In nanoparticle systems, the degree of cooling is often evaluated by how small the occupation number of the particle, $\langle n \rangle = \langle a^\dagger a \rangle$, can be made. Using the variances of the state, this can be expressed as $\langle n \rangle = \frac{1}{2}(\sigma_{xx} + \sigma_{pp}) - 1/2$. Defining the occupation number corresponding to σ_c as $\langle n \rangle_{\min} := \frac{1}{2}(\sigma_{xx,c} + \sigma_{pp,c})$, we obtain

$$\langle n \rangle \geq \langle n \rangle_{\min} := \frac{1}{2}(\sigma_{xx,c}^2 + \sigma_{pp,c}^2) - \frac{1}{2}. \quad (21)$$

Here, $\langle n \rangle_{\min}$ represents the cooling limit. The equation of inequality (21) is fulfilled when $\sigma = \sigma_c$ holds. In the absence of a heat bath, the equality condition for Eq. (21) is known to be satisfied in the limit of strong feedback using the quantum Kalman filter estimates [5]. A similar sufficient condition can also be demonstrated in this case. At the same time, the equality condition of inequality (14) holds. This means that the information-thermodynamic bound obtained in this study and the cooling limit of nanoparticle systems are reached simultaneously. In what follows, we show the equality conditions of Eqs. (21) and (14).

We now briefly introduce the quantum Kalman filter, in which the estimated value $e_t = \{\langle \hat{x} \rangle_c, \langle \hat{p} \rangle_c\}$ is obtained. We assume that a feedback Hamiltonian $\hat{H}_{\text{FB}}(t) = -(a_x \langle x \rangle_c + a_p \langle p \rangle_c) \hat{p} + (b_x \langle x \rangle_c + b_p \langle p \rangle_c) \hat{x}$ is applied using this estimated value. Then, time evolutions of the expected values $\langle \hat{x} \rangle_c$ and $\langle \hat{p} \rangle_c$ governed by Eq. (1) are expressed as

$$d\langle \hat{x} \rangle_c = \left(\omega \langle \hat{p} \rangle_c - a_x \langle \hat{x} \rangle_c - a_p \langle \hat{p} \rangle_c - \frac{\gamma}{2} \langle \hat{x} \rangle_c \right) dt + 8k\eta \sigma_{xx,c} \cdot (dy - \langle x \rangle_c dt), \quad (22)$$

$$d\langle \hat{p} \rangle_c = \left(-\omega \langle \hat{x} \rangle_c - b_x \langle \hat{x} \rangle_c - b_p \langle \hat{p} \rangle_c - \frac{\gamma}{2} \langle \hat{p} \rangle_c \right) dt + 8k\eta \sigma_{xp,c} \cdot (dy - \langle x \rangle_c dt). \quad (23)$$

This stochastic differential equation allows us to determine the expected values of the position and momentum at the

next time step based on current expected values and the measurement outcome dy . In the quantum Kalman filter, this mechanism is used to sequentially obtain the estimate $e_t = \{\langle \hat{x} \rangle_c, \langle \hat{p} \rangle_c\}$. Here, the variance and covariance of the posterior state appearing in the filtering coefficients can be determined in advance because the differential equations of σ_c are closed with respect to σ_c [see Eqs. (B24)–(B26)].

We further analyze the fluctuations inherent in the estimates of the quantum Kalman filter $e_t = \{\langle \hat{x} \rangle_c, \langle \hat{p} \rangle_c\}$. The variance and covariance of the estimates are defined as $\sigma_{xx,m} := \mathbb{E}_{e_t}[\langle \hat{x} \rangle_c^2] - (\mathbb{E}_{e_t}[\langle \hat{x} \rangle_c])^2$, $\sigma_{pp,m} := \mathbb{E}_{e_t}[\langle \hat{p} \rangle_c^2] - (\mathbb{E}_{e_t}[\langle \hat{p} \rangle_c])^2$, and $\sigma_{xp,m} := \mathbb{E}_{e_t}[\langle \hat{x} \rangle_c \langle \hat{p} \rangle_c] - \mathbb{E}_{e_t}[\langle \hat{x} \rangle_c] \mathbb{E}_{e_t}[\langle \hat{p} \rangle_c]$. We define the covariance matrix of these as σ_m . The covariance matrix can be decomposed as $\sigma = \sigma_c + \sigma_m$. From Eqs. (22) and (23), the differential equations governing the time evolution of σ_m are obtained as

$$\begin{aligned} \frac{d}{dt} \sigma_{xx,m} &= 2\omega \sigma_{xp,m} - 2a_x \sigma_{xx,m} - 2a_p \sigma_{xp,m} \\ &\quad - \gamma \sigma_{xx,m} + 8k\eta \sigma_{xx,c}^2, \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{d}{dt} \sigma_{pp,m} &= -2\omega \sigma_{xp,m} - 2b_x \sigma_{xp,m} - 2b_p \sigma_{pp,m} \\ &\quad - \gamma \sigma_{pp,m} + 8k\eta \sigma_{xp,c}^2, \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{d}{dt} \sigma_{xp,m} &= -(\omega + b_x) \sigma_{xx,m} + (\omega - a_p) \sigma_{pp,m} \\ &\quad - (a_x + b_p + \gamma) \sigma_{xp,m} + 8k\eta \sigma_{xx,c} \sigma_{xp,c}. \end{aligned} \quad (26)$$

In these equations, terms involving a_x , a_p , b_x , and b_p represent the contributions from feedback control. We note that the steady-state value of σ_c does not depend on feedback gains [see Eqs. (B24)–(B26)]. Therefore, the reduction of $\langle n \rangle$ is solely determined by how much σ_m can be reduced via feedback control. For simplicity, let $a_x = b_p = g$ and $a_p = b_x = 0$, and assume that g is a constant. Under these conditions, it can be shown that, in the limit $g \rightarrow \infty$, each component of σ_m scales as $O(1/g)$. This implies that σ_m can be arbitrarily small in the limit of an infinite feedback rate. Since $\sigma \rightarrow \sigma_c$ holds in this regime, the cooling limit determined by Eq. (21) is achieved. In this case, $\hat{\rho}_t^{e_t} = \hat{\rho}_t$ also holds and the equality in Eq. (18) is achieved. Moreover, since Eq. (20) holds in the steady state with quantum Kalman filter, the equality in Eq. (14) is also achieved.

We now prove the asymptotic property $\sigma_m = O(1/g)$. The time evolution equations for σ_m , as described by Eqs. (24)–(26), can be rewritten in vector form as follows:

$$\frac{d}{dt} \mathbf{s}_m = \mathbf{A} \mathbf{s}_m + \mathbf{v}, \quad (27)$$

where

$$\mathbf{s}_m := [\sigma_{xx,m}, \sigma_{pp,m}, \sigma_{xp,m}]^T \quad (28)$$

represents the variances and covariances of σ_m . Here, $\mathbf{v} := [8k\eta \sigma_{xx,c}^2, 8k\eta \sigma_{xp,c}^2, 8k\eta \sigma_{xx,c} \sigma_{xp,c}]^T$ and the matrix $\mathbf{A}$ is given by

$$\mathbf{A} := \begin{pmatrix} -2a_x - \gamma & 0 & 2\omega \\ 0 & -2b_p - \gamma & -2\omega \\ -\omega & \omega & -a_x - b_p - \gamma \end{pmatrix}.$$

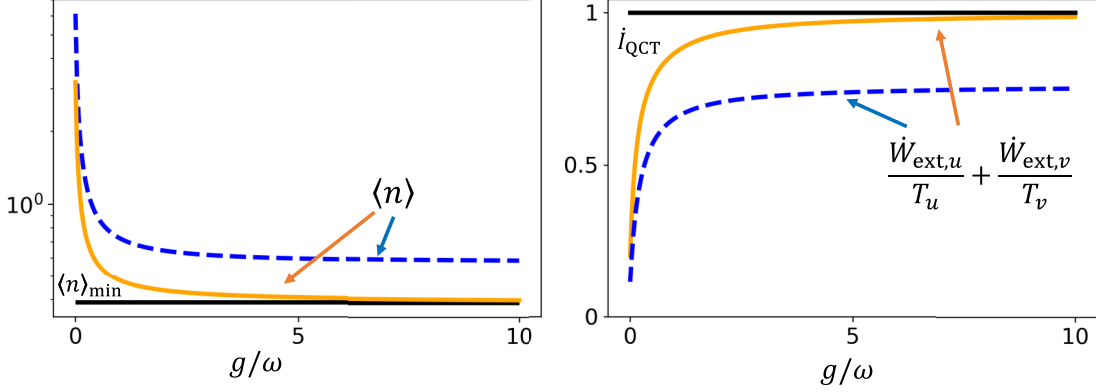


FIG. 4. The behavior of average occupation number $\langle n \rangle$ (left plot) and $\dot{W}_{\text{ext},u}/T_u + \dot{W}_{\text{ext},v}/T_v$ (right plot) as a function of the feedback gain g in the steady state of Eq. (1). The feedback protocol is based on the quantum Kalman filter, where the feedback Hamiltonian is given by $H_{\text{FB}}^{\hat{x},\hat{p}} = \hbar g(-\langle x \rangle_c \hat{p} + \langle p \rangle_c \hat{x})$ (orange curve). In addition, we demonstrate another feedback Hamiltonian $H_{\text{FB}}^{\hat{x}} = \hbar g(\langle x \rangle_c + \langle p \rangle_c) \hat{x}$ (blue dashed curve). In the left plot, the black line represents cooling limit $\langle n \rangle_{\text{min}}$. In the right plot, the black line represents the QC-transfer entropy, and the vertical axis is normalized by I_{QCT} . The parameters are chosen as $k/\omega = 0.18$, $\eta = 0.34$, $T = 292$ K, and $\hbar\gamma/\omega = 0.0058$ [12].

The eigenvalues of $\mathbf{A}$ are given by

$$\lambda_1 = -(a_x + b_p + \gamma), \quad (29)$$

$$\lambda_2 = -(a_x + b_p + \gamma) + \sqrt{(a_x - b_p)^2 - 4\omega^2}, \quad (30)$$

$$\lambda_3 = -(a_x + b_p + \gamma) - \sqrt{(a_x - b_p)^2 - 4\omega^2}. \quad (31)$$

Using a matrix $\mathbf{P}$, $\mathbf{A}$ can be diagonalized as $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. Defining $\tilde{\mathbf{s}} := \mathbf{P}\mathbf{s}_m$ and $\tilde{\mathbf{v}} := \mathbf{P}\mathbf{v}$, the components of $\tilde{\mathbf{s}}$ satisfy

$$\frac{d}{dt}\tilde{s}_i(t) = \lambda_i\tilde{s}_i(t) + \tilde{v}_i(t). \quad (32)$$

Assuming $\tilde{s}_i(0) = 0$, the solution is

$$\tilde{s}_i(t) = e^{\lambda_i t} \int_0^t d\tau e^{-\lambda_i \tau} \tilde{v}_i(\tau). \quad (33)$$

Applying the triangle inequality to Eq. (32), we obtain

$$|\tilde{s}_i(t)| \leq \frac{|\tilde{v}_{i,\text{max}}|}{|\lambda_i|} |1 - e^{\lambda_i t}|, \quad (34)$$

where $|\tilde{v}_{i,\text{max}}| = \max_{0 \leq \tau \leq t} |\tilde{v}_i(\tau)|$. For $a_x = b_p = g$, we have $1/\lambda_i = O(1/g)$ and $\mathbf{P} = O(1)$, so the right-hand side of Eq. (34) is $O(1/g)$. Therefore, $\tilde{\mathbf{s}}(t) = O(1/g)$. Since $\mathbf{s}_m = \mathbf{P}^{-1}\tilde{\mathbf{s}}$, we conclude that $\mathbf{s}_m = O(1/g)$. We present a numerical demonstration regarding the achievement of the cooling limit. Figure 4 illustrates the behavior of the average occupation number $\langle n \rangle$ as a function of the feedback gain in the steady state described by Eq. (1). We compute the behavior for the two Hamiltonians $H_{\text{FB}}^{\hat{x},\hat{p}} = g(-\langle x \rangle_c \hat{p} + \langle p \rangle_c \hat{x})$ and $H_{\text{FB}}^{\hat{x}} = g(\langle x \rangle_c + \langle p \rangle_c) \hat{x}$. It is evident that when using $H_{\text{FB}}^{\hat{x},\hat{p}}$, the cooling limit $\langle n \rangle_{\text{min}}$ is asymptotically achieved as the feedback gain g approaches infinity. In contrast, for $H_{\text{FB}}^{\hat{x}}$, the occupation number $\langle n \rangle$ asymptotically approaches a value larger than the cooling limit $\langle n \rangle_{\text{min}}$. In practical experiments, feedback proportional to $\hat{x}$ is commonly employed. However, such feedback does not allow the system to reach the cooling limit $\langle n \rangle_{\text{min}}$. A similar trend is observed in the relationship between $\dot{W}_{\text{ext},u}/T_u + \dot{W}_{\text{ext},v}/T_v$ and QC-transfer entropy.

V. CONCLUSION AND OUTLOOK

In this study, we established a fundamental limit of quantum feedback cooling in Gaussian systems under continuous measurement, by linking work extraction, kinetic temperatures, and QC-transfer entropy (14). Since the derived inequality is based on the kinetic temperatures rather than the bath temperature, it imposes a stricter constraint on cooling performance than known information-thermodynamic inequalities and can be saturated through a sufficiently large feedback gain with the quantum Kalman filter (Fig. 3). In addition, we discussed the relationship between QC-transfer entropy and yet another quantity called QC information flow, as characterized by inequality (20). These results elucidate to what extent information acquired from the measurement can enhance the extraction of the work and reduce the kinetic temperatures.

Our results can be experimentally verified using current or near-future experimental technologies. Low-frequency optomechanical systems, such as cantilever systems and nanoparticle systems, are suitable for implementing the quantum Kalman filter. In particular, feedback cooling based on the quantum Kalman filter has been realized [12], making experimental verification promising. However, current experiments typically utilize feedback proportional only to the position operator, whereas achieving the theoretical limit requires feedback proportional to the momentum operator as well. Realizing this optimal feedback is experimentally challenging, but is anticipated to lead to the verification of the fundamental connection between information and thermodynamics in feedback cooling.

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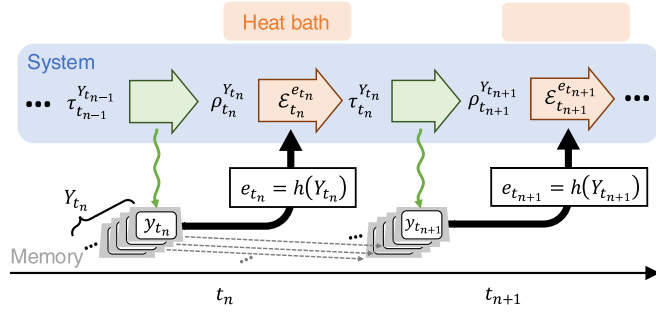


FIG. 5. Schematic for the multiple measurement and feedback setup. We consider the situation where the measurement and feedback are alternately performed. The n th measurement outcome is represented as y_{t_n} , and all the measurement outcomes until y_{t_n} are denoted as $Y_{t_n} := (y_{t_0}, y_{t_1}, \dots, y_{t_n})$. The states right before and after the $(n+1)$ th measurement are denoted as $\hat{\tau}_{t_n}^{Y_{t_n}}$ and $\hat{\rho}_{t_{n+1}}^{Y_{t_{n+1}}}$, respectively, whose superscript represents the corresponding measurement outcomes. The superoperator for n th feedback process is described as $\mathcal{E}_{t_n}^{e_{t_n}}$, where e_{t_n} represents the estimator determined by Y_{t_n} .

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: QUANTUM INFORMATION FLOW IN GENERAL SETUP

In this section, we introduce the measures of quantum information flow and derive the generalized second law, in the setup of multiple quantum measurement and feedback, which includes the continuous feedback case, mainly addressed in this manuscript, as a special case. In Appendix A 1, we introduce QC information flow and derive the generalized second law. In Appendix A 2, we introduce QC-transfer entropy and discuss its relationship with QC information flow. In Appendix A 3, we demonstrate that the continuous feedback control setup discussed in the main text is a special case of the general multiple feedback setup.

1. Generalized second law with QC information flow

We here consider the general setup of multiple measurement and feedback, as shown in Fig. 5. The process starts at the time $t = t_0$, and the n th measurement and feedback are conducted during the time interval $[t_n, t_{n+1})$. The n th measurement outcome is represented as y_{t_n} , and all the measurement outcomes until y_{t_n} are denoted as $Y_{t_n} := (y_{t_0}, y_{t_1}, \dots, y_{t_n})$. The conditional density operators right before and after the $(n+1)$ th measurement are denoted as $\hat{\tau}_{t_n}^{Y_{t_n}}$ and $\hat{\rho}_{t_{n+1}}^{Y_{t_{n+1}}}$, respectively, whose superscript represents the corresponding measurement outcomes. The n th feedback process depends

on the n th estimator, which is determined as a function of Y_{t_n} as $e_{t_n} := h(Y_{t_n})$, and $\mathcal{E}_{t_n}^{e_{t_n}}$ is the superoperator representing the feedback process. The density operator conditioned on the estimator e_{t_n} is represented as $\hat{\rho}_{t_n}^{e_{t_n}}$, where the relationship $\sum_{h(Y_{t_n})=e_{t_n}} P(Y_{t_n}) \rho_{t_n}^{Y_{t_n}} = P(e_{t_n}) \hat{\rho}_{t_n}^{e_{t_n}}$ is satisfied. The feedback process consists of the heat-bath interaction and external driving, and therefore the superoperator can be decomposed as $\mathcal{E}_{t_n}^{e_{t_n}} = \mathcal{E}_{\text{drv}}^{e_{t_n}} \circ \mathcal{E}_{\text{bath}}^{e_{t_n}}$.

In such a setup, we derive the generalized second law for the time interval spanning from the $(n_i + 1)$ th to the n_f th measurement and feedback process. The QC information flow during this period can be described as

$$I_{\text{QCI}}^S := \sum_{n=n_i}^{n_f-1} \chi(\hat{\rho}_{t_{n+1}} : e_{t_n}) - \chi(\hat{\rho}_{t_n} : e_{t_n}), \quad (\text{A1})$$

where $\chi(\hat{\rho} : e) := S(\hat{\rho}) - \mathbb{E}_e[S(\hat{\rho}^e)]$ represents the Holevo information [48], $S(\hat{\rho}) := -\text{tr}[\hat{\rho} \ln \hat{\rho}]$ is the von Neumann entropy, $\mathbb{E}_X$ represents taking the expectation value over a random variable X , and $\hat{\rho}_{t_n} := \mathbb{E}_{e_{t_n}}[\hat{\rho}_{t_n}^{e_{t_n}}]$. Since the Holevo information quantifies the correlation between the quantum system and the (classical) estimator, $-I_{\text{QCI}}^S$ is the total information utilized in the multiple feedback. The effect of measurement backaction is quantified as

$$S_{\text{BA}} := \sum_{n=n_i}^{n_f-1} \mathbb{E}_{e_{t_n}}[S(\hat{\rho}_{t_{n+1}}^{e_{t_n}}) - S(\hat{\tau}_{t_n}^{e_{t_n}})]. \quad (\text{A2})$$

Since $\hat{\rho}_{t_{n+1}}^{e_{t_n}}$ is described as $\hat{\rho}_{t_{n+1}}^{e_{t_n}} = \mathcal{M}_{\text{BA}, t_n}(\hat{\tau}_{t_n}^{e_{t_n}})$, where $\mathcal{M}_{\text{BA}, t_n}$ is the superoperator representing the measurement backaction, S_{BA} is the accumulation of the entropy increase due to the measurement backaction.

By denoting the system Hamiltonian as H , the averaged heat transferred from the heat bath is defined as

$$Q := \sum_{n=n_i}^{n_f-1} \mathbb{E}_{e_{t_n}}[Q^{e_{t_n}}] \\ = \sum_{n=n_i}^{n_f-1} \mathbb{E}_{e_{t_n}}[\text{tr}[H(\mathcal{E}_{\text{bath}}^{e_{t_n}}(\hat{\rho}_{t_n}^{e_{t_n}}) - \hat{\rho}_{t_n}^{e_{t_n}})]], \quad (\text{A3})$$

which quantifies the total energy increase due to the interaction with the heat bath. The extracted work and free energy change can also be defined as $W_{\text{ext}} := -E(\hat{\rho}_{t_{n_f}}) + E(\hat{\rho}_{t_{n_i}}) + Q$ and $\Delta F := E(\hat{\rho}_{t_{n_f}}) - E(\hat{\rho}_{t_{n_i}}) - \beta^{-1}(S(\hat{\rho}_{t_{n_f}}) - S(\hat{\rho}_{t_{n_i}}))$, respectively, where $E(\hat{\rho}) := \text{tr}[H\hat{\rho}]$ represents the internal energy of the system.

By explicitly describing the expectation value of the function g , $\mathbb{E}_e[g(e_{t_n})]$, as the sum $\sum_{e_{t_n}} P[e_{t_n}]g(e_{t_n})$, we can derive the generalized second law as follows:

$$\beta(W_{\text{ext}} + \Delta F) + I_{\text{QCI}}^S + S_{\text{BA}} \\ = -S(\hat{\rho}_{t_{n_f}}) + S(\hat{\rho}_{t_{n_i}}) + \beta Q + I_{\text{QCI}}^S + S_{\text{BA}} \\ = -\sum_{n=n_i}^{n_f-1} \{\mathbb{E}_{e_{t_n}}[S(\hat{\tau}_{t_n}^{e_{t_n}}) - S(\hat{\rho}_{t_n}^{e_{t_n}}) - \beta Q^{e_{t_n}}]\}$$

$$\begin{aligned}
&= - \sum_{n=n_i}^{n_f-1} \{ \mathbb{E}_{e_n} [S(\mathcal{E}_{\text{bath}}^{e_n}(\hat{\rho}_{t_n}^{e_n})) - S(\hat{\rho}_{t_n}^{e_n})] \\
&\quad - \text{tr}[(\mathcal{E}_{\text{bath}}^{e_n}(\hat{\rho}_{t_n}^{e_n}) - \hat{\rho}_{t_n}^{e_n}) \ln \hat{\rho}^G] \} \\
&= - \sum_{n=n_i}^{n_f-1} \{ \mathbb{E}_{e_n} [S(\hat{\rho}_{t_n}^{e_n} \| \hat{\rho}^G) - S(\mathcal{E}_{\text{bath}}^{e_n}(\hat{\rho}_{t_n}^{e_n}) \| \hat{\rho}^G)] \} \\
&\leq 0,
\end{aligned} \tag{A4}$$

where $S(\hat{\rho} \| \hat{\sigma}) := \text{tr}[\hat{\rho}(\ln \hat{\rho} - \ln \hat{\sigma})]$ represents the quantum relative entropy and $\hat{\rho}^G \propto \exp(-\beta H)$ is a Gibbs state with the inverse temperature β . The equality in the third line follows from the fact that $\hat{\tau}_{t_n}^{e_n} = \mathcal{E}_{\text{drv}}^{e_n} \circ \mathcal{E}_{\text{bath}}^{e_n}(\hat{\rho}_{t_n}^{e_n})$ with the superoperator $\mathcal{E}_{\text{drv}}^{e_n}$ representing purely unitary driving, and therefore $S(\hat{\tau}_{t_n}^{e_n}) = S(\mathcal{E}_{\text{bath}}^{e_n}(\hat{\rho}_{t_n}^{e_n}))$. The inequality in the final line follows from the fact that $\hat{\rho}^G$ is the fixed point of the heat-bath interaction $\mathcal{E}_{\text{bath}}^{e_n}$, and the monotonicity of quantum relative entropy under a completely positive and trace-preserving (CPTP) map [50].

2. QC-transfer entropy

The QC-transfer entropy [29] for the time interval from the $(n_i + 1)$ th to n_f th process is described as

$$I_{\text{QCT}} := \sum_{n=n_i}^{n_f-1} \mathbb{E}_{Y_n} [I_{\text{QC}}(\hat{\tau}_{t_n}^{Y_n} : y_{t_{n+1}})], \tag{A5}$$

where I_{QC} represents the QC-mutual information [44,45], defined as $I_{\text{QC}}(\hat{\tau} : y) := S(\hat{\tau}) - \mathbb{E}_y[S(\hat{\rho}^y)]$, with the postselected state $\hat{\rho}^y$ for measurement outcome y . The QC-mutual information I_{QC} quantifies the information obtained by a single measurement, and the QC-transfer entropy I_{QCT} is the total information transferred through the multiple measurements.

The QC-transfer entropy and the QC information flow satisfy the inequality

$$I_{\text{QCT}} - \Delta\chi \geq -I_{\text{QCI}}^S - S_{\text{BA}}, \tag{A6}$$

where $\Delta\chi := \chi(\hat{\rho}_{t_{n_f}} : Y_{t_{n_f}}) - \chi(\hat{\rho}_{t_{n_i}} : Y_{t_{n_i}})$ is the change in correlation from $n = n_i$ to $n = n_f$. By explicitly describing the expectation value $\mathbb{E}_{Y_n}[g(Y_n)]$ as the sum $\sum_{Y_n} P[Y_n]g(Y_n)$, we can derive this inequality as follows:

$$\begin{aligned}
&I_{\text{QCT}} - \Delta\chi + I_{\text{QCI}}^S + S_{\text{BA}} \\
&= \sum_{n=n_i}^{n_f-1} \mathbb{E}_{Y_n} [S(\hat{\rho}_{t_n}^{Y_n} \| \hat{\rho}_{t_n}^{e_n}) - S(\hat{\tau}_{t_n}^{Y_n} \| \hat{\tau}_{t_n}^{e_n})] \\
&\geq 0,
\end{aligned} \tag{A7}$$

where the first line follows from a straightforward rearrangement of terms and the second line follows from the equalities $\mathcal{E}_{t_n}^{e_n}(\hat{\rho}_{t_n}^{e_n}) = \hat{\tau}_{t_n}^{e_n}$ and $\mathcal{E}_{t_n}^{e_n}(\hat{\rho}_{t_n}^{Y_n}) = \hat{\tau}_{t_n}^{Y_n}$, along with the monotonicity of quantum relative entropy under a CPTP map. As can be seen from Eq. (A7), a simple equality condition for Eq. (A6) is that $\hat{\rho}_{t_n}^{e_n} = \hat{\rho}_{t_n}^{Y_n}$ holds for all Y_n . This condition implies that the feedback estimator $e_{t_n} = h(Y_{t_n})$ is optimally designed, which is known to be achieved by a Kalman filter.

3. Correspondence to the continuous feedback setup considered in the main text

We note that the definitions of thermodynamic quantities and information terms in the main text are consistent with those in Appendixes A 1 and A 2. In particular, the consistency of the heat income Q , work extraction W_{ext} , and QC information flow I_{QCI}^S can be directly verified by substituting the discrete time index t_n with the continuous time variable t as is demonstrated in Appendix B 1. While the correspondence is less straightforward for the measurement backaction term S_{BA} [Eq. (19)] and the QC-transfer entropy I_{QCT} [Eq. (4)], it can still be demonstrated by neglecting terms of order $O(dt)$. It is comprehensive to represent Eqs. (A2) and (A5) in the following forms taking the limit of $dt \rightarrow 0$:

$$\dot{S}_{\text{BA}} = \mathbb{E}_{e_t} \left[\lim_{dt \rightarrow 0} \frac{S(\mathcal{M}_{\text{BA},t}(\hat{\rho})) - S(\hat{\rho})}{dt} \Big|_{\hat{\rho}=\hat{\tau}_t^{e_t}} \right], \tag{A8}$$

$$\dot{I}_{\text{QCT}} = \mathbb{E}_{Y_t} \left[\lim_{dt \rightarrow 0} \frac{I_{\text{QC}}(\hat{\rho} : dy(t))}{dt} \Big|_{\hat{\rho}=\hat{\tau}_t^{Y_t}} \right]. \tag{A9}$$

From the equalities $\hat{\tau}_t^{e_t} = \hat{\rho}_t^{e_t} - \frac{i}{\hbar}[\hat{H}_{\text{HO}} + \hat{H}_{\text{FB}}(t), \hat{\rho}_t^{e_t}]dt + \mathcal{L}_{\text{bath}}[\hat{\rho}_t^{e_t}]dt$ and $\hat{\tau}_t^{Y_t} = \hat{\rho}_t^{Y_t} - \frac{i}{\hbar}[\hat{H}_{\text{HO}} + \hat{H}_{\text{FB}}(t), \hat{\rho}_t^{Y_t}]dt + \mathcal{L}_{\text{bath}}[\hat{\rho}_t^{Y_t}]dt$, and the fact that the terms in the square brackets are smooth at the respective points $\hat{\rho} = \hat{\tau}_t^{e_t}$ and $\hat{\rho} = \hat{\tau}_t^{Y_t}$, we can replace $\hat{\tau}_t^{e_t}$ and $\hat{\tau}_t^{Y_t}$ in these equations with $\hat{\rho}_t^{e_t}$ and $\hat{\rho}_t^{Y_t}$, respectively, introducing only negligible additional terms of order $O(dt)$. We also note that in the main text, we use the notations $\hat{\rho}_c(t) = \hat{\rho}_t^{Y_t}$ and $\hat{\rho}_{t,\text{BA}}^{e_t} = \mathcal{M}_{\text{BA},t}(\hat{\rho}_t^{e_t})$ for simplicity. In Sec. III C, we use expressions (A8) and (A9) to derive the main results presented in the main text.

APPENDIX B: PROOF OF THE RESULTS

In this section, we derive inequalities (16), (18), and (20). These inequalities are derived for a Gaussian system that obeys the following stochastic master equation (1). We note that the dynamics of the system described by Eq. (1) is a special case of the general setup discussed in Appendix A 1, where the explicit form of the superoperators for the heat-bath interaction, external driving, and measurement backaction are, respectively, given by

$$\mathcal{E}_{\text{bath}}^{e_t}(\hat{\rho}) = \hat{\rho} - \frac{i}{\hbar}[\hat{H}_{\text{HO}}, \hat{\rho}]dt + \mathcal{L}_{\text{bath}}[\hat{\rho}]dt, \tag{B1}$$

$$\mathcal{E}_{\text{drv}}^{e_t}(\hat{\rho}) = \hat{\rho} - \frac{i}{\hbar}[\hat{H}_{\text{FB}}(t), \hat{\rho}]dt, \tag{B2}$$

$$\mathcal{M}_{\text{BA},t}(\hat{\rho}) = \hat{\rho} + 2k\mathcal{D}_{\hat{x}}[\hat{\rho}]dt, \tag{B3}$$

and the time interval $dt := t_{n+1} - t_n$ is taken to the limit $dt \rightarrow 0$. Using these superoperators, the time evolution of the quantum state $\rho_t^{e_t}$ from t to $t + dt$ for given e_t reads

$$\begin{aligned}
\hat{\rho}_t^{e_t} &\rightarrow \mathcal{E}_{\text{bath}}^{e_t}(\hat{\rho}_t^{e_t}) =: \hat{\pi}_t^{e_t} \rightarrow \mathcal{E}_{\text{drv}}^{e_t}(\hat{\pi}_t^{e_t}) =: \hat{\tau}_t^{e_t} \\
&\rightarrow \mathcal{M}_{\text{BA},t}(\hat{\tau}_t^{e_t}) := \hat{\rho}_{t+dt}^{e_t}.
\end{aligned} \tag{B4}$$

We further assume that the feedback Hamiltonian is chosen so that, when the initial state satisfies $\langle \hat{x} \rangle(0) = \langle \hat{p} \rangle(0) = 0$, the expectation values at later times satisfy $\langle \hat{x} \rangle(t) = \langle \hat{p} \rangle(t) = 0$. Intuitively, such a condition is satisfied when the feedback

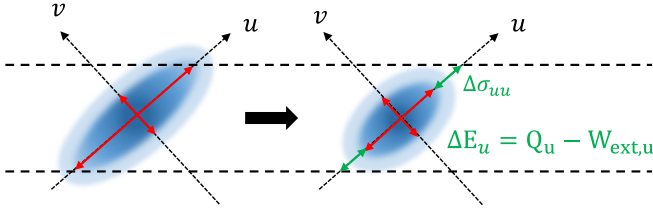


FIG. 6. Schematic of the change in variance along the eigenvalue directions. For simplicity, only the change along the u direction is illustrated in the figure. The work extracted in the eigenvalue direction is defined from the energy change associated with the variance change due to the feedback and backaction.

Hamiltonian is designed to reduce the mean position and momentum to the origin of the phase space. Note that the averaged state $\hat{\rho}$ converges to a well-defined steady state under this feedback Hamiltonian, and the condition $\langle \hat{x} \rangle = \langle \hat{p} \rangle = 0$ is satisfied in the steady state as well. On the other hand, the condition $\langle \hat{x} \rangle(t) = \langle \hat{p} \rangle(t) = 0$ is not satisfied when the feedback Hamiltonian realizes a constant displacement of the position $\hat{H}_{\text{FB}}(t) = a\hat{x}$, where a is a constant. In this case, the averaged state $\hat{\rho}$ also fails to reach a steady state.

1. Derivation of inequality (18)

In the following, we derive Eq. (18). We start by noting that from the definition of the heat (12), the heat rate reads

$$\dot{Q} := \mathbb{E}_{e_t} \left[\lim_{dt \rightarrow 0} \frac{1}{dt} \{ \text{tr}[(\hat{\pi}_t^{e_t} - \hat{\rho}_t^{e_t}) \hat{H}_{\text{HO}}] \} \right]. \quad (\text{B5})$$

Similarly, the work rate reads

$$\dot{W}_{\text{ext}} := -\mathbb{E}_{e_t} \left[\lim_{dt \rightarrow 0} \frac{1}{dt} \{ \text{tr}[(\hat{\rho}_{t+dt}^{e_t} - \hat{\pi}_t^{e_t}) \hat{H}_{\text{HO}}] \} \right]. \quad (\text{B6})$$

We then define the work rate along $\hat{u}$ (10) and $\hat{v}$ (11) as

$$\dot{W}_{\text{ext},u} := -\frac{\hbar\omega}{2} \mathbb{E}_{e_t} \left[\lim_{dt \rightarrow 0} \frac{1}{dt} \{ \text{tr}[(\hat{\rho}_{t+dt}^{e_t} - \hat{\pi}_t^{e_t}) \hat{u}^2] \} \right], \quad (\text{B7})$$

$$\dot{W}_{\text{ext},v} := -\frac{\hbar\omega}{2} \mathbb{E}_{e_t} \left[\lim_{dt \rightarrow 0} \frac{1}{dt} \{ \text{tr}[(\hat{\rho}_{t+dt}^{e_t} - \hat{\pi}_t^{e_t}) \hat{v}^2] \} \right]. \quad (\text{B8})$$

Here, $\dot{W}_{\text{ext},u}$ quantifies the energy extracted along the u axis by feedback and measurement backaction as the variance σ_{uu} is reduced (see Fig. 6); in this direction, the first law of thermodynamics is satisfied. We now combine with the expression (9) and find that the left-hand side of Eq. (18) is expressed as

$$\begin{aligned} \frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} &= \mathbb{E}_{e_t} \left[\lim_{dt \rightarrow 0} \frac{d}{dt} \{ \text{tr}[(\hat{\rho}_{t+dt}^{e_t} - \hat{\pi}_t^{e_t}) \ln \hat{\rho}_t] \} \right] \\ &= \lim_{dt \rightarrow 0} \frac{1}{dt} [\text{tr}[(\hat{\rho}_{t+dt} - \hat{\pi}_t) \ln \hat{\rho}_t]] \\ &= \lim_{dt \rightarrow 0} \frac{1}{dt} [-S(\hat{\rho}_{t+dt}) + S(\hat{\pi}_t)], \end{aligned} \quad (\text{B9})$$

where we used the condition $\langle \hat{x} \rangle(t) = \langle \hat{p} \rangle(t) = 0$ and the fact that $S(\hat{\rho}_{t+dt}) = -\text{tr}[\hat{\rho}_{t+dt} \ln \hat{\rho}_t] + O(dt^2)$, etc.

We next discuss the QC information flow (17) for the time evolution given by Eq. (B4), which reads

$$\dot{I}_{\text{QCI}}^S := \lim_{dt \rightarrow 0} \frac{\chi(\hat{\rho}_{t+dt} : e_t) - \chi(\hat{\rho}_t : e_t)}{dt}. \quad (\text{B10})$$

We further decompose Eq. (B10) into each step in Eq. (B4), as

$$\dot{I}_{\text{QCI}}^S = \dot{I}_{\text{QCI},B}^S + \dot{I}_{\text{QCI},D}^S + \dot{I}_{\text{QCI},BA}^S, \quad (\text{B11})$$

where each term is defined as

$$\dot{I}_{\text{QCI},B}^S := \lim_{dt \rightarrow 0} \frac{\chi(\hat{\pi}_t : e_t) - \chi(\hat{\rho}_t : e_t)}{dt}, \quad (\text{B12})$$

$$\dot{I}_{\text{QCI},D}^S := \lim_{dt \rightarrow 0} \frac{\chi(\hat{\tau}_t : e_t) - \chi(\hat{\pi}_t : e_t)}{dt}, \quad (\text{B13})$$

$$\dot{I}_{\text{QCI},BA}^S := \lim_{dt \rightarrow 0} \frac{\chi(\hat{\rho}_{t+dt} : e_t) - \chi(\hat{\tau}_t : e_t)}{dt}. \quad (\text{B14})$$

Since the von Neumann entropy of the system is unchanged under the unitary time evolution (B2), i.e., $S(\hat{\pi}_t^{e_t}) = S(\hat{\tau}_t^{e_t})$, Eq. (B13) simplifies to

$$\dot{I}_{\text{QCI},D}^S = \lim_{dt \rightarrow 0} \frac{S(\hat{\tau}_t) - S(\hat{\pi}_t)}{dt}. \quad (\text{B15})$$

From Eqs. (B11) and (B15), we obtain the relation

$$\dot{I}_{\text{QCI}}^S = \dot{I}_{\text{QCI},B}^S - \dot{S}_{\text{BA}} + \lim_{dt \rightarrow 0} \frac{S(\hat{\rho}_{t+dt}) - S(\hat{\pi}_t)}{dt}, \quad (\text{B16})$$

where the entropy change of the system due to the measurement backaction $\dot{S}_{\text{BA}}$ is given by Eq. (A8). We now show the following relation:

$$\dot{I}_{\text{QCI},B}^S \leq 0, \quad (\text{B17})$$

which follows from

$$\begin{aligned} \chi(\hat{\pi}_t : e_t) - \chi(\hat{\rho}_t : e_t) &= S(\mathcal{E}_{\text{bath}}(\hat{\rho}_t)) - S(\hat{\rho}_t) - \mathbb{E}_{e_t} [S(\mathcal{E}_{\text{bath}}^{e_t}(\hat{\rho}_t^{e_t})) - S(\hat{\rho}_t^{e_t})] \\ &= \mathbb{E}_{e_t} [S(\mathcal{E}_{\text{bath}}^{e_t}(\hat{\rho}_t^{e_t}) \| \mathcal{E}_{\text{bath}}^{e_t}(\hat{\rho}_t)) - S(\hat{\rho}_t^{e_t} \| \hat{\rho}_t)] \\ &\leq 0. \end{aligned} \quad (\text{B18})$$

Here, we used the relation $\mathbb{E}_{e_t} [\mathcal{E}_{\text{bath}}^{e_t}(\hat{\rho}_t^{e_t})] = \mathcal{E}_{\text{bath}}^{e_t}(\mathbb{E}_{e_t}[\hat{\rho}_t^{e_t}])$ that directly follows from Eq. (B1) and the monotonicity of the quantum relative entropy, and obtained the last inequality in Eq. (B18). By combining Eqs. (B9), (B16), and (B18), we have

$$\begin{aligned} \frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} &= -\lim_{dt \rightarrow 0} \frac{S(\hat{\rho}_{t+dt}) - S(\hat{\pi}_t)}{dt} \\ &\leq -\dot{I}_{\text{QCI}}^S - \dot{S}_{\text{BA}}, \end{aligned} \quad (\text{B19})$$

which completes the proof of Eq. (18).

2. Derivation of inequality (20)

We shall now prove Eq. (20) by considering an infinitesimal time step shown in Eq. (B4) and the monotonicity of the quantum relative entropy:

$$\begin{aligned} \dot{I}_{\text{QCT}} - \dot{\chi} + \dot{I}_{\text{QCI}}^S + \dot{S}_{\text{BA}} &= \lim_{dt \rightarrow 0} \mathbb{E}_{Y_t} [S(\hat{\rho}_t^{Y_t} \| \hat{\rho}_t^{e_t}) - S(\hat{\tau}_t^{Y_t} \| \hat{\tau}_t^{e_t})] \geq 0, \end{aligned} \quad (\text{B20})$$

where the QC-transfer entropy $\dot{I}_{\text{QCT}}$ is given by Eq. (A9) and the infinitesimal variation of the Holevo information

reads

$$\dot{\chi} := \lim_{dt \rightarrow 0} \frac{1}{dt} [S(\hat{\rho}_{t+dt}) - \mathbb{E}_{Y_{t+dt}} [S(\hat{\rho}_{t+dt}^{Y_{t+dt}})] - S(\hat{\rho}_t) + \mathbb{E}_{Y_t} [S(\hat{\rho}_t^{Y_t})]]. \quad (\text{B21})$$

In what follows, we show that $\dot{\chi}$ vanishes in the steady state of the Gaussian time evolution described by Eq. (1), thereby proving Eq. (20). In doing so, we first note that the von Neumann entropy of a single-mode Gaussian state can be expressed using its covariance matrix σ as [47]

$$S(\hat{\rho}_t) = f(\sqrt{\det \sigma}), \quad (\text{B22})$$

where f is a single-variable function defined as

$$f(x) := (x + \frac{1}{2}) \ln(x + \frac{1}{2}) - (x - \frac{1}{2}) \ln(x - \frac{1}{2}). \quad (\text{B23})$$

In particular, the von Neumann entropy of the conditional state $\hat{\rho}_t^{Y_t}$ is solely determined by $\sigma_c(t) := [\sigma_{ij,c}(t)]_{i,j=x,p}$, where $\sigma_{xx,c}(t) := \text{tr}(\hat{x}^2 \hat{\rho}_t^{Y_t}) - [\text{tr}(\hat{x} \hat{\rho}_t^{Y_t})]^2$, and similar definition applies to $\sigma_{pp,c}(t)$ and $\sigma_{xp,c}(t)$. For Gaussian dynamics described by Eq. (1), we note that σ_c does not depend on the measurement outcome dy , as their time evolution explicitly reads

$$\frac{d}{dt} \sigma_{xx,c} = 2\omega \sigma_{xp,c} - \gamma \sigma_{xx,c} + \frac{\gamma}{2} (2\bar{n} + 1) - 8k\eta \sigma_{xx,c}^2, \quad (\text{B24})$$

$$\frac{d}{dt} \sigma_{pp,c} = -2\omega \sigma_{xp,c} - \gamma \sigma_{pp,c} + \frac{\gamma}{2} (2\bar{n} + 1) + 2k - 8k\eta \sigma_{xp,c}^2, \quad (\text{B25})$$

$$\frac{d}{dt} \sigma_{xp,c} = -\omega \sigma_{xx,c} + \omega \sigma_{pp,c} - \gamma \sigma_{xp,c} - 8k\eta \sigma_{xx,c} \sigma_{xp,c}. \quad (\text{B26})$$

Therefore, in the steady state, we have

$$\hat{\rho}_t = \hat{\rho}_{t+dt}, \quad \sigma_c(t) = \sigma_c(t + dt). \quad (\text{B27})$$

By combining Eqs. (B20), (B21), and (B27), we complete the proof of Eq. (20). As discussed in Appendix A 2, the equality in Eq. (20) is achieved by a quantum Kalman filter.

3. Derivation of inequality (16)

In this section, we prove inequality (16) in the steady state. Using the fact that the free energy change vanishes in the steady state, we obtain

$$\begin{aligned} & \frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} - \frac{\dot{W}_{\text{ext}}}{T} \\ &= \lim_{dt \rightarrow 0} \frac{1}{dt} \mathbb{E}_{e_t} [\text{tr}(\mathcal{E}_{\text{bath}}(\hat{\rho}_t^{e_t}) (\ln \hat{\rho}^G - \ln \mathcal{E}_{\text{bath}}(\hat{\rho}_t)) \\ & \quad - \hat{\rho}_t^{e_t} (\ln \hat{\rho}^G - \ln \hat{\rho}_t))] \\ &= \lim_{dt \rightarrow 0} \frac{1}{dt} \text{tr}(\mathcal{E}_{\text{bath}}(\hat{\rho}_t) (\ln \hat{\rho}^G - \ln \mathcal{E}_{\text{bath}}(\hat{\rho}_t)) \\ & \quad - \hat{\rho}_t (\ln \hat{\rho}^G - \ln \hat{\rho}_t)) \\ &= - \lim_{dt \rightarrow 0} \frac{1}{dt} (S(\mathcal{E}_{\text{bath}}(\hat{\rho}_t) \parallel \hat{\rho}^G) - S(\hat{\rho}_t \parallel \hat{\rho}^G)) \\ &\geq 0, \end{aligned} \quad (\text{B28})$$

where the final inequality follows from the monotonicity of the quantum relative entropy [50]. Note that this relation holds regardless of any ordering between T and T_u, T_v .

4. Technical notes

For a single-mode Gaussian state, the kinetic temperature introduced in this study plays the role of converting entropy into energy. For higher-dimensional systems or non-Gaussian states, it may be possible to generalize this relationship by extending the concept of kinetic temperature. However, a comprehensive method for defining kinetic temperatures in such scenarios remains elusive and requires further exploration. In particular, for extensions to multimodes, one can exploit the fact that any zero-mean multimode Gaussian state can be regarded as a Gibbs state subjected to generalized squeezing [51]; thus, one can define effective temperatures along each eigenvector direction in the same manner of this study, making such generalizations feasible.

In this work, we employ a linear feedback Hamiltonian. If one uses a nonlinear feedback Hamiltonian that contains quadratic or higher-order terms in $\hat{x}$ and $\hat{p}$, the averaged state $\hat{\rho}$ becomes non-Gaussian. Since the definition of kinetic temperatures relies on the Gaussianity of the state, extending the present framework to such non-Gaussian cases is an important future direction of research.

APPENDIX C: DIRECT FEEDBACK

We consider the dynamics of direct feedback using the measurement outcome dy . In other words, dy itself is used as an estimate in this feedback scheme. The time evolution for the direct feedback is given by

$$\mathcal{E}_{\text{drv}}^{e_t}(\hat{\rho}_c(t)) = \hat{\rho}_c(t) - \frac{i}{\hbar} [-a_x \hat{p} + b_x \hat{x}, \hat{\rho}_c(t)] \circ dy(t - \tau_{\text{FB}}), \quad (\text{C1})$$

where a_x and b_x are the feedback gains, $\circ$ denotes the Stratonovich product [52], and τ_{FB} is the feedback delay time. The Stratonovich product is a stochastic differential defined with midpoint sampling, ensuring that the usual chain rule of ordinary calculus still applies. We consider the limit of $\tau_{\text{FB}} \rightarrow 0$ that corresponds to the case of Markovian feedback [1,4].

Figure 7 illustrates the behavior of the steady-state average occupation number $\langle n \rangle$ of the harmonic oscillator and the quantity $\dot{W}_{\text{ext},u}/T_u + \dot{W}_{\text{ext},v}/T_v$ as functions of the feedback gain g under direct feedback. Unlike the case where the quantum Kalman filter is used, the equality in Eq. (20) does not hold for direct feedback and the cooling limit is not achieved. Additionally, it is observed that the best cooling is achieved at a finite value of the feedback gain g , in contrast to the case of using the quantum Kalman filter, where the optimal cooling is achieved when the feedback gain becomes infinite.

APPENDIX D: CONTINUOUS POSITION MEASUREMENT

Here we review, following Ref. [2], how the continuous position measurement used in the main text [Eq. (1)] can be obtained as the infinitesimal time limit of a discrete position measurement. Let $|\psi(t)\rangle_c$ denote the state conditioned on all

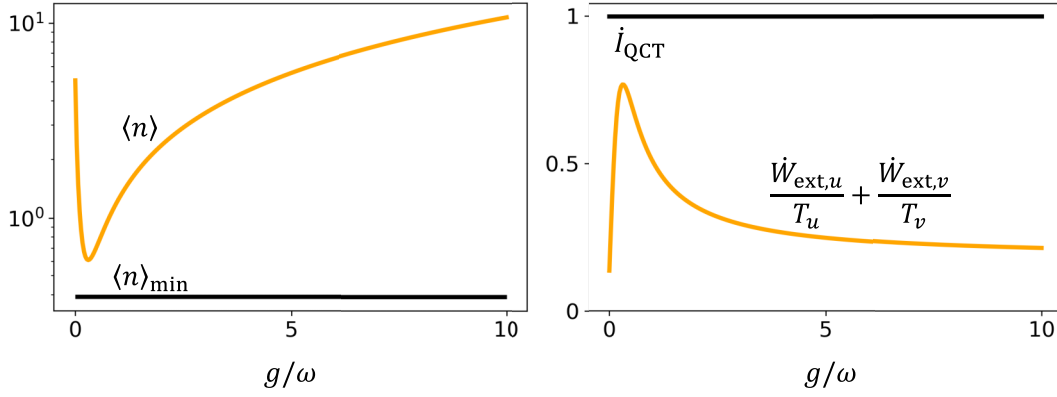


FIG. 7. Behavior of the steady-state average occupation number of the harmonic oscillator, $\langle n \rangle$ (left plot), and the quantity $\dot{W}_{\text{ext},u}/T_u + \dot{W}_{\text{ext},v}/T_v$ (right plot) as functions of the feedback gain g under direct feedback. The parameters are the same as in Fig. 3. Therefore, the values of $\langle n \rangle_{\min}$ and $\dot{I}_{\text{QCT}}$ are identical to those in Fig. 3.

measurement outcomes up to time t . First, suppose that over a time interval Δt the measurement is described by the operator

$$\hat{A}(\alpha) = \left(\frac{4k \Delta t}{\pi} \right)^{1/4} \int_{-\infty}^{\infty} dx e^{-2k \Delta t (x-\alpha)^2} |x\rangle \langle x|. \quad (\text{D1})$$

This corresponds to a measurement whose outcome α has a Gaussian error centered at α . Since $\hat{A}(\alpha)$ commutes with $\hat{x}$, it induces no backaction on the position. However, in the presence of the harmonic potential, the position operator in the Heisenberg picture $x(t)$ at time t incorporates the effects of backaction on the momentum. When Δt is small, the probability density for obtaining outcome α is approximately

$$P(\alpha) \approx \sqrt{\frac{4k \Delta t}{\pi}} \exp[-4k \Delta t (\alpha - \langle \hat{x} \rangle_c)^2]. \quad (\text{D2})$$

Hence, one may write the random outcome as

$$\alpha_s = \langle \hat{x} \rangle_c + \frac{\Delta W}{\sqrt{8k \Delta t}}. \quad (\text{D3})$$

Taking the $\Delta t \rightarrow 0$ limit, the conditioned pure state evolves as

$$|\psi(t + dt)\rangle_c \propto \{1 - [k\hat{x}^2 - 4k\hat{x}\langle \hat{x} \rangle_c]dt + \sqrt{2k}\hat{x}dW\} |\psi(t)\rangle_c. \quad (\text{D4})$$

Converting to the density operator yields the familiar continuous measurement form

$$d\hat{\rho}_c = -k[\hat{x}, [\hat{x}, \hat{\rho}_c]]dt + \sqrt{2k}(\hat{x}\hat{\rho}_c + \hat{\rho}_c\hat{x} - 2\langle \hat{x} \rangle_c \hat{\rho}_c)dW. \quad (\text{D5})$$

In the main text, we include an efficiency $\eta < 1$ multiplying the stochastic term, reflecting the assumption that finite measurement efficiency reduces the size of the state update. With efficiency η , the corresponding covariance dynamics become

$$\frac{d}{dt}\sigma_{xx,c} = -8k\eta\sigma_{xx,c}^2, \quad (\text{D6})$$

$$\frac{d}{dt}\sigma_{pp,c} = 2k - 8k\eta\sigma_{xp,c}^2, \quad (\text{D7})$$

$$\frac{d}{dt}\sigma_{xp,c} = -8k\eta\sigma_{xx,c}\sigma_{xp,c}. \quad (\text{D8})$$

In each equation, the term proportional to $-8k\eta$ corresponds to the reduction of fluctuations due to conditioning on the instantaneous position measurement. Note that only in Eq. (D7) does a backaction term of size $2k$ appear, representing the momentum kick from the position measurement.

APPENDIX E: CASE OF ANOTHER MEASUREMENT

So far, we have considered only cases involving position measurements, which induce no backaction on the position by the measurement itself. We here examine the effect of replacing the measurement with other measurements of the position that disturb the position and investigate how this change influences feedback cooling performance. First, we consider the position measurement related to the operator $\hat{a}$, which is given by [1]

$$\begin{aligned} d\hat{\rho}_c(t) = & -\frac{i}{\hbar}[\hat{H}_{\text{HO}} + \hat{H}_{\text{FB}}(t), \hat{\rho}_c(t)]dt \\ & + \mathcal{L}_{\text{bath}}[\hat{\rho}_c(t)]dt + 2k\mathcal{D}_{\hat{a}}[\hat{\rho}_c(t)]dt \\ & + \sqrt{2k\eta}(\hat{a}\hat{\rho}_c + \hat{\rho}_c\hat{a}^\dagger - \sqrt{2}\langle \hat{x} \rangle_c \hat{\rho}_c)dW. \end{aligned} \quad (\text{E1})$$

The measurement signal is given by $dy(t) = \langle \hat{x}(t) \rangle_c dt + dW/\sqrt{8k\eta}$. Assuming a feedback Hamiltonian of the form $\hat{H}_{\text{FB}}(t) = -(a_x \langle x \rangle_c + a_p \langle p \rangle_c)\hat{p} + (b_x \langle x \rangle_c + b_p \langle p \rangle_c)\hat{x}$, the time evolution of $\langle x \rangle_c$ and $\langle p \rangle_c$ is governed by the following stochastic differential equations:

$$\begin{aligned} d\langle \hat{x} \rangle_c = & \left(\omega \langle \hat{p} \rangle_c - a_x \langle \hat{x} \rangle_c - a_p \langle \hat{p} \rangle_c - \frac{\gamma}{2} \langle \hat{x} \rangle_c - k \langle \hat{x} \rangle_c \right) dt \\ & + 4\sqrt{2k\eta} \left(\sigma_{xx,c} - \frac{1}{2} \right) \cdot (dy - \langle x \rangle_c dt), \end{aligned} \quad (\text{E2})$$

$$\begin{aligned} d\langle \hat{p} \rangle_c = & \left(-\omega \langle \hat{x} \rangle_c - b_x \langle \hat{x} \rangle_c - b_p \langle \hat{p} \rangle_c - \frac{\gamma}{2} \langle \hat{p} \rangle_c - k \langle \hat{p} \rangle_c \right) dt \\ & + 4\sqrt{2k\eta} \sigma_{xp,c} \cdot (dy - \langle x \rangle_c dt). \end{aligned} \quad (\text{E3})$$

Assuming Gaussian states, the dynamics of covariance matrices σ_c and σ_m are given by

$$\begin{aligned} \frac{d}{dt}\sigma_{xx,c} = & 2\omega\sigma_{xp,c} - \gamma\sigma_{xx,c} + \frac{\gamma}{2}(2\bar{n} + 1) \\ & - 2k\sigma_{xx,c} + k - 4k\eta \left(\sigma_{xx,c} - \frac{1}{2} \right)^2, \end{aligned} \quad (\text{E4})$$

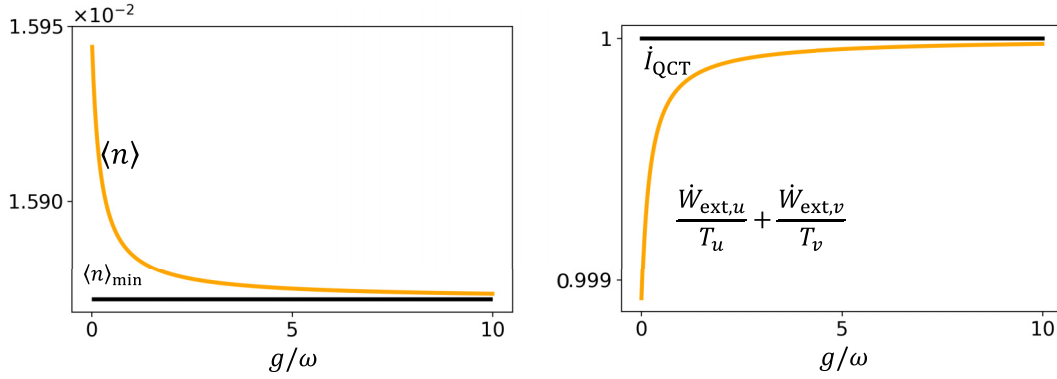


FIG. 8. Steady-state average occupation number $\langle n \rangle$ (left) and $\dot{W}_{\text{ext},u}/T_u + \dot{W}_{\text{ext},v}/T_v$ (right) vs feedback gain g , obtained from the dynamics in Eq. (E1). The parameters are the same as in Fig. 3.

$$\begin{aligned} \frac{d}{dt}\sigma_{pp,c} = & -2\omega\sigma_{xp,c} - \gamma\sigma_{pp,c} + \frac{\gamma}{2}(2\bar{n} + 1) \\ & - 2k\sigma_{pp,c} + k - 4k\eta\sigma_{xp,c}^2, \end{aligned} \quad (\text{E5})$$

$$\begin{aligned} \frac{d}{dt}\sigma_{xp,c} = & -\omega\sigma_{xx,c} + \omega\sigma_{pp,c} - \gamma\sigma_{xp,c} \\ & - 2k\sigma_{xp,c} - 4k\eta\left(\sigma_{xx,c} - \frac{1}{2}\right)\sigma_{xp,c} \end{aligned} \quad (\text{E6})$$

and

$$\begin{aligned} \frac{d}{dt}\sigma_{xx,m} = & 2\omega\sigma_{xp,m} - 2a_x\sigma_{xx,m} - 2a_p\sigma_{xp,m} - \gamma\sigma_{xx,m} \\ & - 2k\sigma_{xx,m} + 4k\eta\left(\sigma_{xx,c} - \frac{1}{2}\right)^2, \end{aligned} \quad (\text{E7})$$

$$\begin{aligned} \frac{d}{dt}\sigma_{pp,m} = & -2\omega\sigma_{xp,m} - 2b_x\sigma_{xp,m} - 2b_p\sigma_{pp,m} - \gamma\sigma_{pp,m} \\ & - 2k\sigma_{pp,m} + 4k\eta\sigma_{xp,c}^2, \end{aligned} \quad (\text{E8})$$

$$\begin{aligned} \frac{d}{dt}\sigma_{xp,m} = & -(\omega + b_x)\sigma_{xx,m} + (\omega - a_p)\sigma_{pp,m} \\ & - (a_x + b_p + \gamma + 2k)\sigma_{xp,m} \\ & + 4k\eta\left(\sigma_{xx,c} - \frac{1}{2}\right)\sigma_{xp,c}. \end{aligned} \quad (\text{E9})$$

From Eqs. (E4) and (E6), we observe that σ_c forms a closed set of differential equations. This allows us to construct the quantum Kalman filter estimate using Eqs. (E2) and (E3), in a manner analogous to the previous case. Thus, the feedback Hamiltonian adopted here is well justified. Moreover, an information-theoretic cooling limit can be derived using the same approach as in Appendix B 1. The bound given in Eq. (10) remains valid in this case.

A key distinction from our original scheme arises from the damping term $2k\mathcal{D}_{\hat{a}}[\hat{\rho}_c(t)]dt$ in Eq. (E1), which effectively describes coupling to a zero-temperature bath. Consequently, part of the observed cooling originates from this measurement-induced damping rather than from feedback alone, complicating direct comparisons with our original scheme. Figure 8 shows numerical results for the steady-state cooling performance under this dynamics. The minimum occupation number achieved is $\langle n \rangle_{\text{min}} = 0.016$, which is lower

than in the original case ($\langle n \rangle_{\text{min}} = 0.39$). This improvement is likely attributable to the additional damping provided by the measurement process itself. To compare the effect of the measurement method, it is desirable to eliminate the associated damping. To this end, we consider simultaneous detection of both the annihilation and creation operators. By construction, this setup cancels the damping terms while preserving the measurement backaction. Furthermore, we ensure that the resulting increase in $\langle n \rangle$ due to backaction matches that of the original case. The stochastic master equation for this situation is

$$\begin{aligned} d\hat{\rho}_c(t) = & -\frac{i}{\hbar}[\hat{H}_{\text{HO}} + \hat{H}_{\text{FB}}(t), \hat{\rho}_c(t)]dt + \mathcal{L}_{\text{bath}}[\hat{\rho}_c(t)]dt \\ & + k\mathcal{D}_{\hat{a}}[\hat{\rho}_c(t)]dt + \sqrt{\frac{k\eta}{2}}(\hat{a}\hat{\rho}_c + \hat{\rho}_c\hat{a}^\dagger - \sqrt{2}\langle x \rangle_c \hat{\rho}_c)dW_1 \\ & + k\mathcal{D}_{\hat{a}^\dagger}[\hat{\rho}_c(t)]dt + \sqrt{\frac{k\eta}{2}}(\hat{a}^\dagger\hat{\rho}_c + \hat{\rho}_c\hat{a} - \sqrt{2}\langle x \rangle_c \hat{\rho}_c)dW_2. \end{aligned} \quad (\text{E10})$$

Here, dW_1 and dW_2 are independent Wiener increments, and the corresponding signals are $dy_1 = \langle \hat{x} \rangle_c dt + dW/2\sqrt{k\eta}$ and $dy_2(t) = \langle \hat{x} \rangle_c dt + dW/2\sqrt{k\eta}$, which statistically yield the same information as that for the original case, i.e., $dy = \langle \hat{x} \rangle_c dt + dW/\sqrt{8k\eta}$. The corresponding dynamics of covariance matrices σ_c and σ_m under Eq. (E10) are given by

$$\begin{aligned} \frac{d}{dt}\sigma_{xx,c} = & 2\omega\sigma_{xp,c} - \gamma\sigma_{xx,c} + \frac{\gamma}{2}(2\bar{n} + 1) \\ & + k - 4k\eta\left(\sigma_{xx,c}^2 + \frac{1}{4}\right), \end{aligned} \quad (\text{E11})$$

$$\begin{aligned} \frac{d}{dt}\sigma_{pp,c} = & -2\omega\sigma_{xp,c} - \gamma\sigma_{pp,c} + \frac{\gamma}{2}(2\bar{n} + 1) \\ & + k - 4k\eta\sigma_{xp,c}^2, \end{aligned} \quad (\text{E12})$$

$$\begin{aligned} \frac{d}{dt}\sigma_{xp,c} = & -\omega\sigma_{xx,c} + \omega\sigma_{pp,c} - \gamma\sigma_{xp,c} \\ & - 4k\sigma_{xx,c}\sigma_{xp,c} \end{aligned} \quad (\text{E13})$$

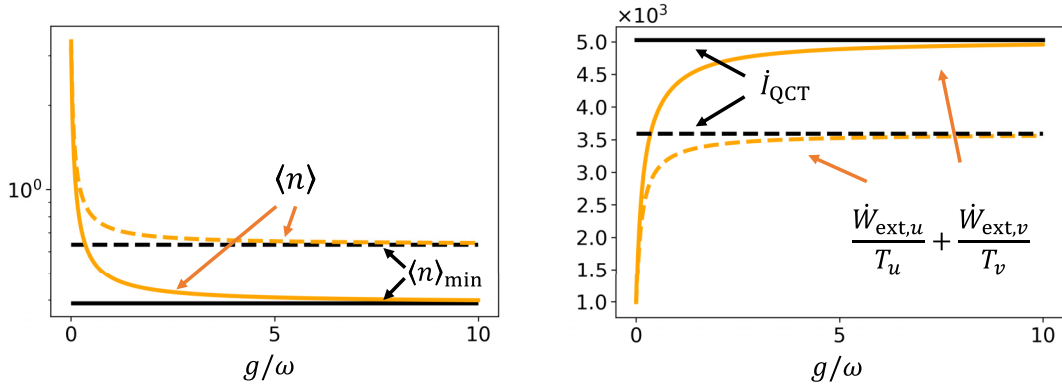


FIG. 9. Steady-state occupation number $\langle n \rangle$ (left) and $\dot{W}_{\text{ext},u}/T_u + \dot{W}_{\text{ext},v}/T_v$ (right) as functions of feedback gain g . Solid lines: Measurement without position disturbance by the measurement itself [Eq. (1)]. Dashed lines: Measurement on a and $a^\dagger$ without damping [Eq. (E10)]. Vertical axes are not normalized. The parameters are the same as in Fig. 3.

and

$$\frac{d}{dt}\sigma_{xx,m} = 2\omega\sigma_{xp,m} - 2a_x\sigma_{xx,m} - 2a_p\sigma_{xp,m} - \gamma\sigma_{xx,m} + 4k\eta\left(\sigma_{xx,c}^2 + \frac{1}{4}\right), \quad (\text{E14})$$

$$\frac{d}{dt}\sigma_{pp,m} = -2\omega\sigma_{xp,m} - 2b_x\sigma_{xp,m} - 2b_p\sigma_{pp,m} - \gamma\sigma_{pp,m} + 4k\eta\sigma_{xp,c}^2, \quad (\text{E15})$$

$$\frac{d}{dt}\sigma_{xp,m} = -(\omega + b_x)\sigma_{xx,m} + (\omega - a_p)\sigma_{pp,m} - (a_x + b_p + \gamma + 2k)\sigma_{xp,m} + 4k\eta\sigma_{xx,c}\sigma_{xp,c}. \quad (\text{E16})$$

These equations confirm that the simultaneous detection of $\hat{a}$ and $\hat{a}^\dagger$ eliminates the effective damping while retaining the measurement backaction. Compared to our original scheme, two key differences arise: (1) the backaction now affects both position and momentum, and (2) the terms governing the measurement-induced reduction of conditional variances are different. In particular, the final term on the right-hand side of

Eq. (E13) is proportional to $4k\eta\sigma_{xx,c}\sigma_{xp,c}$, while in the original case, the corresponding coefficient is $8k\eta\sigma_{xx,c}\sigma_{xp,c}$, resulting in a stronger suppression of uncertainty in the latter case. This highlights that our original scheme is more effective in reducing the conditional state variance.

Figure 9 presents numerical results comparing the cooling performance under measurement with no position disturbance by the measurement itself and measurement related to $\hat{a}$ and $\hat{a}^\dagger$ without damping. In the left panel, the minimum occupation number is lower in the original case ($\langle n \rangle_{\text{min}} = 0.39$) than in the latter case ($\langle n \rangle_{\text{min}} = 0.63$), indicating that original measurement is better in terms of feedback cooling. Since the rate of heating due to measurement backaction is the same in both scenarios, this difference in the performance is attributable to the greater reduction of conditional uncertainty achieved in the original scheme. The right panel compares $\dot{W}_{\text{ext},u}/T_u + \dot{W}_{\text{ext},v}/T_v$ and $\dot{I}_{\text{QCT}}$. Notably, $\dot{I}_{\text{QCT}}$ differs between the two cases, with original measurements yielding higher information gain. This supports the interpretation that original measurements allow obtaining larger amount of information that can be utilized for feedback, thereby achieving lower steady-state occupation numbers.

- [1] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, 2009).
- [2] K. Jacobs and D. A. Steck, A straightforward introduction to continuous quantum measurement, *Contemp. Phys.* **47**, 279 (2006).
- [3] H. M. Wiseman and G. J. Milburn, Quantum theory of optical feedback via homodyne detection, *Phys. Rev. Lett.* **70**, 548 (1993).
- [4] H. M. Wiseman, Quantum theory of continuous feedback, *Phys. Rev. A* **49**, 2133 (1994).
- [5] A. C. Doherty and K. Jacobs, Feedback control of quantum systems using continuous state estimation, *Phys. Rev. A* **60**, 2700 (1999).
- [6] P. Bushev, D. Rotter, A. Wilson, F. Dubin, C. Becher, J. Eschner, R. Blatt, V. Steixner, P. Rabl, and P. Zoller, Feedback cooling of a single trapped ion, *Phys. Rev. Lett.* **96**, 043003 (2006).

- [7] R. Vijay, C. Macklin, D. Slichter, S. Weber, K. Murch, R. Naik, and I. Siddiqi, Stabilizing Rabi oscillations in a superconducting qubit using quantum feedback, *Nature (London)* **490**, 77 (2012).
- [8] D. Ristè, C. Bultink, M. Tiggelman, R. Schouten, K. Lehnert, and L. Dicarlo, Millisecond charge-parity fluctuations and induced decoherence in a superconducting transmon qubit, *Nat. Commun.* **4**, 1913 (2013).
- [9] W. Livingston, M. Blok, E. Flurin, J. Dressel, A. Jordan, and I. Siddiqi, Experimental demonstration of continuous quantum error correction, *Nat. Commun.* **13**, 2307 (2022).
- [10] M. Rossi, D. Mason, J. Chen, Y. Tsaturyan, and A. Schliesser, Measurement-based quantum control of mechanical motion, *Nature (London)* **563**, 53 (2018).
- [11] J. Gieseler, B. Deutsch, R. Quidant, and L. Novotny, Subkelvin parametric feedback cooling of a laser-trapped nanoparticle, *Phys. Rev. Lett.* **109**, 103603 (2012).

- [12] L. Magrini, P. Rosenzweig, C. Bach, A. Deutschmann-Olek, S. G. Hofer, S. Hong, N. Kiesel, A. Kugi, and M. Aspelmeyer, Real-time optimal quantum control of mechanical motion at room temperature, *Nature (London)* **595**, 373 (2021).
- [13] M. Kamba, R. Shimizu, and K. Aikawa, Nanoscale feedback control of six degrees of freedom of a near-sphere, *Nat. Commun.* **14**, 7943 (2023).
- [14] C. Gonzalez-Ballester, M. Aspelmeyer, L. Novotny, R. Quidant, and O. Romero-Isart, Levitodynamics: Levitation and control of microscopic objects in vacuum, *Science* **374**, eabg3027 (2021).
- [15] V. P. Belavkin, Quantum filtering of Markov signals with white quantum noise, in *Quantum Communications and Measurement*, edited by V. P. Belavkin, O. Hirota, and R. L. Hudson (Springer US, Boston, MA, 1995), pp. 381–391.
- [16] J. M. Parrondo, J. Horowitz, and T. Sagawa, Thermodynamics of information, *Nat. Phys.* **11**, 131 (2015).
- [17] T. Sagawa and M. Ueda, Second law of thermodynamics with discrete quantum feedback control, *Phys. Rev. Lett.* **100**, 080403 (2008).
- [18] K. Jacobs, Second law of thermodynamics and quantum feedback control: Maxwell’s demon with weak measurements, *Phys. Rev. A* **80**, 012322 (2009).
- [19] T. Sagawa and M. Ueda, Generalized jarzynski equality under nonequilibrium feedback control, *Phys. Rev. Lett.* **104**, 090602 (2010).
- [20] J. M. Horowitz and S. Vaikuntanathan, Nonequilibrium detailed fluctuation theorem for repeated discrete feedback, *Phys. Rev. E* **82**, 061120 (2010).
- [21] K. Funo, Y. Watanabe, and M. Ueda, Integral quantum fluctuation theorems under measurement and feedback control, *Phys. Rev. E* **88**, 052121 (2013).
- [22] Z. Gong, Y. Ashida, and M. Ueda, Quantum-trajectory thermodynamics with discrete feedback control, *Phys. Rev. A* **94**, 012107 (2016).
- [23] T. Sagawa and M. Ueda, Nonequilibrium thermodynamics of feedback control, *Phys. Rev. E* **85**, 021104 (2012).
- [24] S. Ito and T. Sagawa, Information thermodynamics on causal networks, *Phys. Rev. Lett.* **111**, 180603 (2013).
- [25] J. M. Horowitz and M. Esposito, Thermodynamics with continuous information flow, *Phys. Rev. X* **4**, 031015 (2014).
- [26] N. Shiraishi and T. Sagawa, Fluctuation theorem for partially masked nonequilibrium dynamics, *Phys. Rev. E* **91**, 012130 (2015).
- [27] J. M. Horowitz, Multipartite information flow for multiple maxwell demons, *J. Stat. Mech.* (2015) P03006.
- [28] K. Ptaszyński and M. Esposito, Thermodynamics of quantum information flows, *Phys. Rev. Lett.* **122**, 150603 (2019).
- [29] T. Yada, N. Yoshioka, and T. Sagawa, Quantum fluctuation theorem under quantum jumps with continuous measurement and feedback, *Phys. Rev. Lett.* **128**, 170601 (2022).
- [30] S. Toyabe, T. Sagawa, M. Ueda, E. Muneyuki, and M. Sano, Experimental demonstration of information-to-energy conversion and validation of the generalized jarzynski equality, *Nat. Phys.* **6**, 988 (2010).
- [31] A. Bérut, A. Arakelyan, A. Petrosyan, S. Ciliberto, R. Dillenschneider, and E. Lutz, Experimental verification of Landauer’s principle linking information and thermodynamics, *Nature (London)* **483**, 187 (2012).
- [32] J. Koski, V. Maisi, J. Pekola, and D. Averin, Experimental realization of a Szilard engine with a single electron, *Proc. Natl. Acad. Sci. USA* **111**, 13786 (2014).
- [33] M. Ribezzi-Crivellari and F. Ritort, Large work extraction and the Landauer limit in a continuous Maxwell demon, *Nat. Phys.* **15**, 660 (2019).
- [34] M. Debiossac, D. Grass, J. J. Alonso, E. Lutz, and N. Kiesel, Thermodynamics of continuous non-Markovian feedback control, *Nat. Commun.* **11**, 1360 (2020).
- [35] M. Debiossac, M. L. Rosinberg, E. Lutz, and N. Kiesel, Non-Markovian feedback control and acausality: An experimental study, *Phys. Rev. Lett.* **128**, 200601 (2022).
- [36] P. A. Camati, J. P. S. Peterson, T. B. Batalhão, K. Micadei, A. M. Souza, R. S. Sarthour, I. S. Oliveira, and R. M. Serra, Experimental rectification of entropy production by Maxwell’s demon in a quantum system, *Phys. Rev. Lett.* **117**, 240502 (2016).
- [37] N. Cottet, S. Jezouin, L. Bretheau, P. Campagne-Ibarcq, Q. Ficheux, J. Anders, A. Auffèves, R. Azouit, P. Rouchon, and B. Huard, Observing a quantum Maxwell demon at work, *Proc. Natl. Acad. Sci. USA* **114**, 7561 (2017).
- [38] Y. Masuyama, K. Funo, Y. Murashita, A. Noguchi, S. Kono, Y. Tabuchi, R. Yamazaki, M. Ueda, and Y. Nakamura, Information-to-work conversion by Maxwell’s demon in a superconducting circuit quantum electrodynamical system, *Nat. Commun.* **9**, 1291 (2018).
- [39] M. Naghiloo, J. J. Alonso, A. Romito, E. Lutz, and K. W. Murch, Information gain and loss for a quantum Maxwell’s demon, *Phys. Rev. Lett.* **121**, 030604 (2018).
- [40] T. Yada, P.-J. Stas, A. Suleymanzade, E. N. Knall, N. Yoshioka, T. Sagawa, and M. D. Lukin, Experimentally probing entropy reduction via iterative quantum information transfer, *Phys. Rev. X* **15**, 031054 (2025).
- [41] J. M. Horowitz and H. Sandberg, Second-law-like inequalities with information and their interpretations, *New J. Phys.* **16**, 125007 (2014).
- [42] H. Sandberg, J.-C. Delvenne, N. J. Newton, and S. K. Mitter, Maximum work extraction and implementation costs for nonequilibrium Maxwell’s demons, *Phys. Rev. E* **90**, 042119 (2014).
- [43] D. Hartich, A. C. Barato, and U. Seifert, Stochastic thermodynamics of bipartite systems: Transfer entropy inequalities and a Maxwell’s demon interpretation, *J. Stat. Mech.* (2014) P02016.
- [44] H. J. Groenewold, A problem of information gain by quantal measurements, *Int. J. Theor. Phys.* **4**, 327 (1971).
- [45] M. Ozawa, On information gain by quantum measurements of continuous observables, *J. Math. Phys.* **27**, 759 (1986).
- [46] M. S. Kim, F. A. M. de Oliveira, and P. L. Knight, Properties of squeezed number states and squeezed thermal states, *Phys. Rev. A* **40**, 2494 (1989).
- [47] S. Olivares, Quantum optics in the phase space, *Eur. Phys. J.: Spec. Top.* **203**, 3 (2012).
- [48] A. S. Holevo, Some estimates of the information transmitted by quantum communication channels, *Probl. Inf. Transm.* **9**, 177 (1973).

- [49] T. Matsumoto and T. Sagawa, Role of sufficient statistics in stochastic thermodynamics and its implication to sensory adaptation, *Phys. Rev. E* **97**, 042103 (2018).
- [50] D. Petz, Monotonicity of quantum relative entropy revisited, *Rev. Math. Phys.* **15**, 79 (2003).
- [51] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, *Rev. Mod. Phys.* **84**, 621 (2012).
- [52] C. Gardiner, *Stochastic Methods: A Handbook for the Natural and Social Sciences*, Springer Series in Synergetics (Springer, Berlin, 2010).