

Operators with spectral density increasing as $\Omega(E) \sim e^{\text{Const.}E^2}$

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Motivated by a putative model of black holes as quantum objects, we consider what types of operators would have a corresponding spectrum. We find that there are indeed such operators, but of a rather unusual type, and for which the wave functions are only barely localized. We point out a tension between such almost delocalized states and black holes as compact objects.

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I. INTRODUCTION

According to Bekenstein and Hawking, a black hole has a (nondimensional) entropy $\frac{1}{4} \frac{A}{\ell_P^2}$, where A is the area of the black hole horizon and $\ell_P \approx 1.6 \times 10^{-35}$ m is the Planck length. For an electrically neutral spherically symmetric (Schwarzschild) black hole, this is also $4\pi \frac{M^2}{m_P^2}$, where M is the black hole mass and $m_P \approx 2.2 \times 10^{-8}$ kg is the Planck mass. If Boltzmann's formula holds for this entropy, for a solar mass black hole ($M \approx 2.0 \times 10^{30}$ kg), it would correspond to about $e^{10^{77}}$ activated degrees of freedom. It is a major unsolved problem of modern fundamental physics what these degrees of freedom are.

Let us note that the estimate of the black hole entropy can be arrived at in several ways. The most immediate is to use the classical thermodynamic formula $\frac{1}{T} = \frac{\partial S}{\partial E}$ in a setting where $E = Mc^2$ is the energy and T is the Hawking temperature, determined by Bogolyubov transformation between ingoing and outgoing states of elementary particles propagating in the curved spacetime around the black hole [1–4]. It makes no reference to anything happening inside the black hole. The second is to estimate the number of quantum states of matter that could collapse and give rise to a black hole with given macroscopic black hole parameters (mass, electric charge, angular momentum). Such an estimate was first carried out by Bekenstein in the very first paper on black hole entropy [5], and later refined by Mukhanov [6]. From the viewpoint of statistical mechanics, this is a coarse-grained entropy (coarse graining by the outside observer only knowing about mass, electric charge, and angular momentum). As it is based on

ignorance of what goes on inside, it also makes no statement on what the excited degrees of freedom of the black hole may be.

The number of hypotheses that have been put forward to describe quantum black holes is vast. Some of the approaches taken are holography [7–9], loop quantum gravity [10–12], fuzzballs [13,14], counting quasinormal mode states [15–18], D-branes [19,20], and graviton condensates [21,22]; see also well-known reviews for a comprehensive picture [23–27]. Recent high-profile contributions and gravitational path integrals and the “island model” include Refs. [28–36]; another branch of the recent literature is the connection to quantum codes, random matrix theory, and related topics [37–40].

For all these interpretations (and other) if they are based on quantum mechanics, a black hole with mass about M would, as a quantum object, have a number of states of approximately that mass, and the black hole entropy would be the logarithm of the density of states (ignoring the other two macroscopic parameters, electric charge and angular momentum).

Mukhanov and Bekenstein [41–43], see also Ref. [44], proposed a minimum model of this kind where the masses of black holes are quantized in multiples of the Planck mass (as $M_n \sim \sqrt{n} m_P$), each state being exponentially degenerate ($D_n \sim e^{an}$). Adjusting suitably the constants, one can thus arrive at an interpretation of black hole entropy as $S(M) = \log D_n$ where $n = \frac{M^2}{m_P^2}$. Since $E = Mc^2$, this translates to having a many-body quantum system whose spectral density $\Omega(E)$ (i.e., the number of microstates associated with energy E) scales superexponentially as $\sim e^{\text{Const.}E^2}$ for large E , motivating the title of this article. The physical models cited above aim for more than the Bekenstein-Mukhanov model, at the price of increased complexity. Holography and counting quasinormal mode states are perhaps the furthest away, as the horizon and its properties then enter prominently, while fuzzballs and graviton condensates perhaps are the least distant; we will later comment on the latter. Nevertheless, what has not been discussed previously in the literature is whether and when such spectra of operators can actually occur in reasonable mathematical models. To do so is our main goal here.

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II. A SIMPLE MODEL

We consider a quantum gas of N noninteracting bosons. As we will see, it will make no difference if the particles would be fermions. Since the bosons are noninteracting, this many-body system can be described in terms of the energy levels of a single particle and the associated occupation numbers. More precisely, let $\{\epsilon_i\}$ denote the single particle spectrum. For convenience, we choose $\epsilon_i \geq 0$ without any loss of generality. Then, a microscopic state of the system can be labeled by $\{n_i\}$, where $n_i \geq 0$ is the occupation number of the i th single particle level. The microcanonical partition function $\Omega(E)$, associated with a total energy E of the many-body system, can then be simply expressed as

$$\Omega(E) = \sum_{n_i \geq 0} \delta\left(E - \sum_i n_i \epsilon_i\right). \quad (1)$$

This model of noninteracting bosons has appeared in many contexts, most recently in the context of cold-atom experiments. It has numerous applications in many domains of physics and mathematics, including even in number theory. For example, how many ways an integer E can be partitioned, i.e., expressed as a sum of smaller integers? This corresponds to the choice $\epsilon_i = i$ in Eq. (1). In this case, Hardy and Ramanujam showed that $\Omega(E) \sim \exp[\pi \sqrt{2/3} \sqrt{E}]$ for large E [45]. This problem and many of its generalizations concerning combinatorial aspects of the integer partition problem have been studied in the literature [46–49]. It turns out that this integer partition problem and its variations can be mapped into the model of noninteracting bosons, which helped explore many interesting questions in both physics and mathematics [50–55]. For example, using this mapping, a very interesting connection was found between the integer partition problem and the extreme value statistics [51].

To compute $\Omega(E)$ in Eq. (1), we first take its Laplace transform with respect to E and then sum over all $n_i \geq 0$. This gives

$$Z(\beta) = \int_0^\infty e^{-\beta E} \Omega(E) dE = \prod_i \frac{1}{1 - e^{-\beta \epsilon_i}}. \quad (2)$$

Inverting formally this Laplace transform, one gets

$$\Omega(E) = \int_\Gamma \frac{d\beta}{2\pi i} e^{\beta E - \sum_i \ln(1 - e^{-\beta \epsilon_i})} = \int_\Gamma \frac{d\beta}{2\pi i} e^{F(\beta)}, \quad (3)$$

where Γ denotes the Bromwich contour in the complex- β plane, and we denote

$$F(\beta) = \beta E - \sum_i \ln(1 - e^{-\beta \epsilon_i}). \quad (4)$$

To evaluate this integral in Eq. (3), we employ the standard saddle-point method by assuming $F(\beta)$ to be large (to be justified *a posteriori*). Minimizing $F(\beta)$ with respect to β , i.e., setting $F'(\beta) = 0$ at $\beta = \beta^*$, leads to the saddle-point equation

$$E = \sum_i \frac{\epsilon_i}{e^{\beta^* \epsilon_i} - 1}. \quad (5)$$

Note that $\langle n_i \rangle = 1/(e^{\beta^* \epsilon_i} - 1)$ in Eq. (5) is simply the Bose factor associated with the canonical partition function of the

gas at an inverse temperature β^* . For a given E , one needs to first solve the saddle-point equation (5) to express β^* in terms of E and then evaluate $F(\beta^*)$. Consequently, $\Omega(E)$ can be estimated (up to pre-exponential factors) as

$$\Omega(E) \approx e^{F(\beta^*(E))}. \quad (6)$$

For large E , one can approximate the discrete sum in Eq. (5) by an integral over the single particle energy ϵ

$$E \approx \int_0^E \frac{\epsilon \rho(\epsilon) d\epsilon}{e^{\beta^* \epsilon} - 1}, \quad (7)$$

where $\rho(\epsilon)$ denotes the density of energy states in the single particle problem. We have deliberately kept the upper limit E since one cannot occupy levels higher than E if the total energy is E . Clearly, if $\rho(\epsilon)$ grows slower than e^ϵ for large ϵ , to leading order for large E , one can replace the upper cutoff in the integral in Eq. (7) by ∞ . For example, if $\rho(\epsilon) \sim \epsilon^\alpha$ as $\epsilon \rightarrow \infty$ with $\alpha > 0$, one estimates

$$E \approx \int_0^\infty \frac{\epsilon^{\alpha+1} d\epsilon}{e^{\beta^* \epsilon} - 1} \sim [\beta^*]^{-(\alpha+2)}, \quad (8)$$

implying $\beta^*(E) \sim E^{-1/(\alpha+2)}$ for large E . This then leads to $F(\beta^*(E)) \sim E^{(\alpha+1)/(\alpha+2)}$, and hence, for large E ,

$$\Omega(E) \sim \exp[E^{(\alpha+1)/(\alpha+2)}], \quad \text{for } \rho(\epsilon) \sim \epsilon^\alpha. \quad (9)$$

For instance, for the integer partition problem, one can show that $\alpha = 0$ [51], leading to the Hardy-Ramanujam result $\Omega(E) \sim \exp[\sqrt{E}]$ [45]. Since the growth exponent $(\alpha + 1)/(\alpha + 2) < 1$, an algebraically growing $\rho(\epsilon)$ cannot give rise to an $\Omega(E)$ growing faster than exponential for large E , and in particular, it cannot give rise to an $\Omega \sim e^{C_0 E^2}$ as required for a black hole. This remains true for any $\rho(\epsilon)$ that grows slower than an exponential for large ϵ .

III. A HIGH-ENERGY CONDENSATION SCENARIO THAT LEADS TO THE RIGHT SPECTRUM

To reproduce an $\Omega(E)$ that grows faster than exponential for large E , we need a $\rho(\epsilon)$ that also grows faster than exponential for large ϵ . In that case, we need to keep the upper cutoff E in the integral in Eq. (7) (as otherwise the integral will be divergent), and the integral will be completely dominated by the contribution coming from the vicinity of this upper cutoff. Indeed, to leading order in E , one can then approximate the integral in Eq. (7) by

$$E \sim \frac{E \rho(E)}{e^{\beta^* E} - 1} \approx E \rho(E) e^{-\beta^* E}, \quad (10)$$

leading to

$$\beta^*(E) \sim \frac{\ln[\rho(E)]}{E}. \quad (11)$$

Consequently, from Eqs. (4) and (6) it follows that

$$\Omega(E) \sim \rho(E). \quad (12)$$

This result has a very simple and nice physical picture associated with it. This is an example of a “high energy condensation,” where one or few particles at a very high single particle level need to occupy a macroscopically large fraction of the total energy E . Thus, most of the energy is carried by

this high-energy condensate consisting of one or few particles. This is a counterpart to the standard Bose-Einstein condensation that occurs in Bose gases at low energy and low density. The particularity of the current model is that each mode is occupied by zero or one particle, and thus behaves like a fermion, although at the outset assumed a boson, while the number of modes in this set is large. Loosely speaking, the modes are so many at a given energy that each of them by chance is only picked once or not at all. Thus, this high-energy condensation is the physical mechanism that can produce an $\Omega(E)$ growing faster than an exponential for large E . According to this mechanism and the result in Eq. (12), if we want to reproduce the black hole result $\Omega(E) \sim \exp[C_0 E^2]$ for large E within this simple noninteracting quantum Bose gas model, we need to have a single particle quantum spectrum whose density of states $\rho(\epsilon)$ grows also as $\rho(\epsilon) \sim \exp[C_0 \epsilon^2]$ for large ϵ .

IV. THE QUANTUM POTENTIAL THAT LEADS TO $\rho(\epsilon) \sim e^{C_0 \epsilon^2}$

As a warm up, let us first discuss two well-known simple examples: (1) a d -dimensional harmonic oscillator with $V(r) = r^2$ and (2) a d -dimensional box of size L^d . In case (1), the single particle energies are given by $\epsilon = \sum_{j=1}^d m_j$ (up to a constant and in suitable units), where $m_j = 0, 1, 2, \dots$ are the quantum numbers. Consequently,

$$\rho(\epsilon) = \sum_{m_j=0}^{\infty} \delta\left(\epsilon - \sum_{j=1}^d m_j\right). \quad (13)$$

Taking a Laplace transform with respect to ϵ and carrying out the sums, one finds trivially

$$\tilde{\rho}(s) = \int_0^{\infty} \rho(\epsilon) e^{-s\epsilon} d\epsilon \sim s^{-d}, \quad \text{as } s \rightarrow 0, \quad (14)$$

leading to the well-known density of states for a d -dimensional harmonic oscillator

$$\rho(\epsilon) \sim \epsilon^{d-1}, \quad \text{as } \epsilon \rightarrow \infty. \quad (15)$$

Example (2), i.e., the d -dimensional box, can be solved in a similar way by noting that in this case $\epsilon = \sum_{j=1}^d m_j^2$ (in suitable units where $L = 1$). Proceeding as in case (1), it is easy to show that for the d -dimensional box

$$\rho(\epsilon) \sim \epsilon^{d/2-1}, \quad \text{as } \epsilon \rightarrow \infty. \quad (16)$$

In both examples, $\rho(\epsilon)$ grows algebraically for large ϵ .

To estimate $\rho(\epsilon)$ for a general confining central potential $V(r)$ in d dimensions, we can use the semiclassical Bohr-Sommerfeld quantization formula that gives an accurate estimate for large ϵ . To make this estimate, it is convenient to work with the cumulative density of states up to level ϵ , i.e., $\mathcal{N}(\epsilon) = \int_0^{\epsilon} \rho(\epsilon') d\epsilon'$. Then, the integrated density of states scales as $\mathcal{N}(\epsilon) \sim [n(\epsilon)]^d$ in d dimensions, where $n(\epsilon)$ is the quantum number (radial) associated with the energy level ϵ . The latter can be estimated from the Bohr-Sommerfeld quantization rule (again in appropriate units)

$$\int_0^{\infty} \sqrt{\epsilon - V(r)} dr \approx n(\epsilon) \sim [\mathcal{N}(\epsilon)]^{1/d}. \quad (17)$$

It is easy to check that this general result reproduces correctly the exact estimate in Eq. (15) for the harmonic oscillator case $V(r) = r^2$, as well as the result in Eq. (16) for the box case, where $V(r) = 0$. Now, we want $\rho(\epsilon) \sim e^{C_0 \epsilon^2}$. This means, up to pre-exponential factors, $\mathcal{N}(\epsilon) \sim e^{C_0 \epsilon^2}$. Substituting this on the right-hand side (rhs) of Eq. (17) gives for large ϵ

$$\int_0^{\infty} \sqrt{\epsilon - V(r)} dr \approx e^{C_0 \epsilon^2/d}. \quad (18)$$

A little inspection shows that a potential $V(r)$ that reproduces the rhs of Eq. (18) is of the form

$$V(r) \sim B \sqrt{\ln r}, \quad \text{as } r \rightarrow \infty, \quad (19)$$

where the prefactor B can be determined as follows. We substitute this ansatz on the left-hand side (lhs) of Eq. (18) and make a change of variable $B \sqrt{\ln r} = \epsilon y$. Consequently, the lhs of Eq. (18) reads

$$\int_0^{e^{\epsilon^2/B^2}} \sqrt{\epsilon - V(r)} dr = \frac{2\epsilon^{5/2}}{B^2} \int_0^1 \sqrt{1-y} y e^{\epsilon^2 y^2/B^2} dy. \quad (20)$$

For large ϵ , the dominant contribution to the integral over y comes from the vicinity of $y = 1$ and one can easily show that up to inconsequential factors

$$\int_0^{e^{\epsilon^2/B^2}} \sqrt{\epsilon - V(r)} dr \approx e^{\epsilon^2/B^2}, \quad \text{as } \epsilon \rightarrow \infty. \quad (21)$$

Comparing it to the rhs of Eq. (18) fixes the prefactor $B = \sqrt{d/C_0}$ uniquely. Hence, the required potential to produce $\Omega(E) \sim e^{C_0 E^2}$ for large E is given by

$$V(r) \approx \sqrt{\frac{d}{C_0} \ln r}, \quad \text{as } r \rightarrow \infty. \quad (22)$$

V. EIGENFUNCTIONS IN A POTENTIAL $\alpha \sqrt{\ln |x|}$

The potential in Eq. (22) grows extremely slowly with distance from the origin. It may be questioned if a potential increasing as slowly as the square root of the logarithm is even confining and hence has discrete eigenvalues. In other words, are the eigenfunctions sufficiently localized and square integrable? We now show that they are indeed localized, and for simplicity, we just show this in $d = 1$. Generalization to higher dimensions is straightforward. We then consider a single quantum particle in one dimension in a potential $V(x) = \alpha \sqrt{\ln |x|}$ with $\alpha > 0$. The eigenfunction $\psi_{\epsilon}(x)$ associated with an eigenvalue ϵ satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi_{\epsilon}(x)}{dx^2} + V(x) \psi_{\epsilon}(x) = \epsilon \psi_{\epsilon}(x), \quad (23)$$

where $V(x) = \alpha \sqrt{\ln |x|}$. According to the WKB approximation, a good estimate of the symmetric wave function for $x > 0$ can be expressed as

$$\psi_{\epsilon}(x) \approx C_+ \frac{\exp\left[\frac{i}{\hbar} \sqrt{2m} \int_0^x \sqrt{\epsilon - V(x')} dx'\right]}{\sqrt{2m(\epsilon - V(x))}}, \quad (24)$$

where C_+ is a constant. The real line is thus divided into a “classically allowed” region where $V(x) < \epsilon$ and a

“classically forbidden” region where $V(x) > \epsilon$. The two regions are separated by the “turning point” x_c given by $\epsilon = \alpha\sqrt{\ln|x_c|}$, i.e.,

$$x_c = \exp(\epsilon^2/\alpha^2). \quad (25)$$

We are interested in the large x regime, i.e., when $x > x_c$ (classically forbidden region) such that $V(x) > \epsilon$. If $V(x) \gg \epsilon$, we see from Eq. (24) that to leading order for large x , the wave function (up to inconsequential prefactors) decays as

$$\psi_\epsilon(x) \propto \exp\left[-\frac{\sqrt{2m}}{\hbar} \int_0^x \sqrt{V(x')} dx'\right]. \quad (26)$$

Substituting $V(x) = \alpha\sqrt{\ln|x|}$ in Eq. (26), we get for $x \gg x_c$

$$\psi_\epsilon(x) \propto \exp\left[-\frac{\sqrt{2m\alpha}}{\hbar} (\ln x)^{1/4} x\right], \quad (27)$$

which, of course, is square integrable. Hence, the potential $V(x) = \alpha\sqrt{\ln|x|}$ is indeed “confining” and has discrete eigenvalues.

VI. CONSIDERATIONS OF INTERACTIONS

So far, we have considered black holes as putative quantum objects and discussed which confining potentials could give rise to spectra that in turn would be similar to Bekenstein-Hawking entropy. We have shown that there exist very shallow potentials growing as $\sqrt{\ln r}$ with the distance that give rise to such spectra via a high-energy condensation mechanism. However, such shallow potentials lead to states with wave functions that, although square integrable, have a large spatial extension, which contradicts the idea of compact black holes. One way out could be to relax the ansatz of noninteracting bosons and to postulate that self-interactions could at the same time lead to the right spectral degeneracy and the right spatial extent, both considered as functions of black hole mass. Such a scenario was indeed suggested some time ago by Dvali and Gomez [21], see also Ref. [22], where the central quantity is N , the number of gravitons of wavelength approximately the Schwarzschild radius making up the black hole. This quantity N scales in the Dvali-Gomez theory with mass M like Bekenstein-Hawking entropy ($\sim \frac{M^2}{m_p^2}$) and is supposed to lead to e^N possible states when the gravitons are interacting. The argument is similar to the counting estimate first advanced in Ref. [41], and, like that one, not straightforwardly related to spectral degeneracy, which is our focus here. Also note that the conjectured graviton condensate is a low-temperature phenomenon; essentially, the black hole is at absolute zero. Within our approach, since the relevant states that contribute to the superexponential density of states via the high-energy condensation mechanism occur at very high energies, interactions are unlikely to play any role there. Similarly, we expect the same conclusion to hold even for models of fermions (with or without interactions). Thus, the tension between locality or compactness and reproducing superexponential density of states is rather robust, and not just restricted to the simple model of noninteracting bosons discussed above. The main point is that the physical mechanism responsible for the superexponential density of states is the high-energy condensation

as demonstrated above. At such high-energy scales, neither the interaction between the quantum particles nor their statistics will play a major role.

Finally, let us also point out another important general physical fact that emerged from the analysis of our toy model. One of the findings of our paper is that one needs shallow (and yet confining) quantum potential growing as $\sqrt{\ln r}$ for large r in order to have a superexponential high-energy spectra in a quantum many-body system, not necessarily limited to Bekenstein-Hawking black hole physics. We hope this will serve as a benchmark for future studies of many-body spectra in quantum systems, thus going beyond the black hole physics.

VII. MICROCANONICAL VERSUS CANONICAL ENSEMBLE

For black holes with Bekenstein-Hawking entropy such that $\Omega(E) \sim e^{E^2}$ for large E , canonical ensemble does not exist since the partition function $Z(\beta) = \int e^{-\beta E} \Omega(E) dE$ is always divergent. This means that if indeed one attempts to study a canonical ensemble, it will lead to a physical instability: if a black hole is in a bath at a higher temperature than its own, it will absorb energy from the bath, and yet become even cooler, in a runaway process. Hawking temperature can hence not be identified with a temperature of the bath as in standard thermodynamics, but rather it will correspond to the effective temperature associated with a microcanonical ensemble, as considered in our model. We note that such an effective temperature associated with a microcanonical ensemble and the breakdown of ensemble equivalence has been discussed in a number of different physical contexts [56–60]. Indeed, Bekenstein-Mukhanov model is inherently of this “microcanonical” type as it relates entropy to spectral degeneracy. The inverse relationship between the effective temperature and entropy ($T \sim 1/E$) and the negative heat capacity of a black hole are simple consequences of the thermodynamic definitions of these quantities and a dependence $S(R) \sim E^2$; the subtle point, however, is that such an entropy cannot be the standard entropy of a canonical ensemble.

VIII. THE SIZE OF INTERNAL SPACE

The above discussion supposes that the internal space of a black hole can be assimilated to an object like an atom, figuratively that the Schwarzschild radius of a black hole is analogous to the Bohr radius. This does not have to be the case, as the geometry of a black hole interior may be qualitatively different from the domain of ordinary space occluded by the black hole. Indeed, the mathematical structure of space-time inside a (classical) black hole is still an open problem and the focus of a substantial literature; see, e.g., Refs. [61–65]. Recently, the possibility of the (classical) gravitational dynamics inside a black hole being chaotic has been discussed [66–68] in specific models, in which case analytical solutions are likely to remain elusive. The (semiclassical) quantum structure of black hole interiors is also actively investigated; see, e.g., Refs. [69,70].

The internal volume of black hole might hence be large, often referred to as a “bag-of-gold” scenario [71], and, if

so, may accommodate many internal states without running into the conceptual problem outlined above. It is, however, not known why the internal volume should then always be of just the amount of vastness required to match Bekenstein-Hawking entropy, and in the recent literature, the issue is raised that it may even be too large; one then has to explain why black hole entropy is not even larger [72]. If a (classical) singularity is taken seriously as an end point of (classical) evolution in the black hole interior, one is further presumably not limited to internal spaces of finite dimension, but could also contemplate infinite-dimensional internal spaces, for instance, the locally treelike structures favored in disordered system theory [73,74], where the density of local ground states is known to be an exponentially increasing function of energy [75] [Eq. (20)]. In the context of Eq. (9), this corresponds to the same linear exponential scaling as in the limit $\alpha \rightarrow \infty$. Following Kac [76], a spectral density increasing as $e^{\text{Const.} \times E}$ combined with locality can in fact be conjectured to always require models in infinite dimensions. By analogy, one may hence hypothesize that localized models with spectral density growing even faster as $e^{\text{Const.} \times E^2}$ have to be super-infinite-dimensional, in a sense that is yet to be made mathematically precise.

IX. CONCLUSION

Motivated by the spectral degeneracy of a black hole as a quantum object, we have considered the general question which Hermitian operators can have spectral density growing with energy E as $\sim \exp[\text{Const.} E^2]$. We have shown that typically such quickly increasing degeneracy and concomitant cumulant state counting function lead to the phenomenon of

high-energy condensation. We have further shown an explicit family of Hamiltonian operators in finite dimensions (including one dimension) with these properties. As we also show, these examples lead to very extended states, and therefore cannot be reasonable models of quantum black holes, as their support would extend (very) far outside the Schwarzschild radius. We have briefly considered these conclusions in the perspective of interacting bosons and have pointed out that to change the conclusions qualitatively the interactions would have to shrink the spatial extent of the relevant states in a major way. We have also commented on the possibility of different geometries inside black holes, which can correspond to these highly degenerate spectra. It is important to emphasize that in our toy model, there are no gravitational components such as curved spacetime, redshift, etc. that have important effects on black hole physics. Here, our goal was different: inspired by the Hawking-Bekenstein-Mukhanov entropy for black holes, we wanted to seek a simple model that illustrates the difficulty of reproducing a superexponential spectral density in a quantum many-body system. On a general level, our work serves to highlight what an unusual and indeed extreme mathematical object a quantum system must be to have the Bekenstein-Hawking-Mukhanov energy spectrum.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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- [1] S. W. Hawking, Black hole explosions? *Nature (London)* **248**, 30 (1974).
 - [2] S. W. Hawking, Particle creation by black holes, *Commun. Math. Phys.* **43**, 199 (1975).
 - [3] R. M. Wald, On particle creation by black holes, *Commun. Math. Phys.* **45**, 9 (1975).
 - [4] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago, Illinois, 1994).
 - [5] J. D. Bekenstein, Black holes and entropy, *Phys. Rev. D* **7**, 2333 (1973).
 - [6] V. F. Mukhanov, On the origin of black-hole entropy, *Found. Phys.* **33**, 271 (2003).
 - [7] G. 't Hooft, Dimensional reduction in quantum gravity, [arXiv:gr-qc/9310026](https://arxiv.org/abs/gr-qc/9310026).
 - [8] L. Susskind, The world as a hologram, *J. Math. Phys.* **36**, 6377 (1995).
 - [9] R. Bousso, The holographic principle, *Rev. Mod. Phys.* **74**, 825 (2002).
 - [10] C. Rovelli, Black hole entropy from loop quantum gravity, *Phys. Rev. Lett.* **77**, 3288 (1996).
 - [11] A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, Quantum geometry and black hole entropy, *Phys. Rev. Lett.* **80**, 904 (1998).
 - [12] K. A. Meissner, Black-hole entropy in loop quantum gravity, *Classical Quantum Gravity* **21**, 5245 (2004).
 - [13] S. D. Mathur, A proposal to resolve the black hole information paradox, *Int. J. Mod. Phys. D* **11**, 1537 (2002).
 - [14] K. Skenderis and M. Taylor, The fuzzball proposal for black holes, *Phys. Rep.* **467**, 117 (2008).
 - [15] M. Maggiore, Physical interpretation of the spectrum of black hole quasinormal modes, *Phys. Rev. Lett.* **100**, 141301 (2008).
 - [16] E. Berti, V. Cardoso, and A. O. Starinets, Quasinormal modes of black holes and black branes, *Classical Quantum Gravity* **26**, 163001 (2009).
 - [17] R. A. Konoplya and A. Zhidenko, Quasinormal modes of black holes: From astrophysics to string theory, *Rev. Mod. Phys.* **83**, 793 (2011).
 - [18] J. L. Jaramillo, R. P. Macedo, and L. A. Sheikh, Pseudospectrum and black hole quasinormal mode instability, *Phys. Rev. X* **11**, 031003 (2021).
 - [19] C. G. Callan and J. M. Maldacena, D-brane approach to black hole quantum mechanics, *Nucl. Phys. B* **472**, 591 (1996).
 - [20] J. Maldacena and L. Susskind, D-branes and fat black holes, *Nucl. Phys. B* **475**, 679 (1996).
 - [21] G. Dvali and C. Gomez, Black hole’s quantum N -portrait [arXiv:1112.3359](https://arxiv.org/abs/1112.3359).

- [22] G. Dvali, Non-thermal corrections to Hawking radiation versus the information paradox, *Fortschr. Phys.* **64**, 106 (2016).
- [23] J. D. Bekenstein, Bekenstein-Hawking entropy, *Scholarpedia* **3**, 7375 (2008).
- [24] D. N. Page, Hawking radiation and black hole thermodynamics, *New J. Phys.* **7**, 203 (2005).
- [25] S. Hossenfelder and L. Smolin, Conservative solutions to the black hole information problem, *Phys. Rev. D* **81**, 064009 (2010).
- [26] D. Harlow, Jerusalem lectures on black holes and quantum information, *Rev. Mod. Phys.* **88**, 015002 (2016).
- [27] D. Marolf, The black hole information problem: Past, present and future, *Rep. Prog. Phys.* **80**, 092001 (2017).
- [28] A. Almheiri, N. Engelhardt, D. Marolf, and H. Maxfield, The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole, *J. High Energy Phys.* **12** (2019) 063.
- [29] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, Replica wormholes and the entropy of Hawking radiation, *J. High Energy Phys.* **05** (2020) 013.
- [30] G. Penington, Entanglement wedge reconstruction and the information paradox, *J. High Energy Phys.* **09** (2020) 002.
- [31] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, The entropy of Hawking radiation, *Rev. Mod. Phys.* **93**, 035002 (2021).
- [32] G. Penington, S. H. Shenker, D. Stanford, and Z. Yang, Replica wormholes and the black hole interior, *J. High Energy Phys.* **03** (2022) 205.
- [33] V. Balasubramanian, A. Lawrence, J. M. Magán, and M. Sasieta, Microscopic origin of the entropy of astrophysical black holes, *Phys. Rev. Lett.* **132**, 141501 (2024).
- [34] V. Balasubramanian, A. Lawrence, J. M. Magán, and M. Sasieta, Microscopic origin of the entropy of black holes in general relativity, *Phys. Rev. X* **14**, 011024 (2024).
- [35] R. Bousso and G. Penington, Islands far outside the horizon, *J. High Energy Phys.* **11** (2024) 164.
- [36] Y. Matsuo, Universal structure of islands in evaporating black holes, *J. High Energy Phys.* **03** (2025) 068.
- [37] A. Climent, R. Emparan, J. M. Magán, M. Sasieta, and A. Vilar López, Universal construction of black hole microstates, *Phys. Rev. D* **109**, 086024 (2024).
- [38] W. Mück, Black holes and Marchenko-Pastur distribution, *Phys. Rev. D* **109**, 126001 (2024).
- [39] C. Akers, N. Engelhardt, D. Harlow, G. Penington, and S. Vardhan, The black hole interior from non-isometric codes and complexity, *J. High Energy Phys.* **06** (2024) 155.
- [40] J. van der Heijden and E. Verlinde, An operator algebraic approach to black hole information, *J. High Energy Phys.* **02** (2025) 207.
- [41] V. Mukhanov, Are black holes quantised? *Pis. Eksp. Teor. Fiz.* **44**, 50 (1986).
- [42] J. D. Bekenstein and V. F. Mukhanov, Spectroscopy of the quantum black hole, *Phys. Lett. B* **360**, 7 (1995).
- [43] V. Mukhanov, Quantum black holes, *Jacob Bekenstein* (World Scientific, Singapore, 2018), pp. 99–108.
- [44] J. D. Bekenstein and A. Meisels, Einstein A and B coefficients for a black hole, *Phys. Rev. D* **15**, 2775 (1977).
- [45] G. H. Hardy and S. Ramanujan, Asymptotic formulæ in combinatory analysis, *Proc. London Math. Soc.* **s2-17**, 75 (1918).
- [46] P. Erdős and J. Lehner, The distribution of the number of summands in the partitions of a positive integer, *Duke Math. J.* **8**, 335 (1941).
- [47] F. C. Auluck and D. S. Kothari, Statistical mechanics and the partitions of numbers, *Math. Proc. Cambridge Philos. Soc.* **42**, 272 (1946).
- [48] B. K. Agarwala and F. C. Auluck, Statistical mechanics and partitions into non-integral powers of integers, *Math. Proc. Cambridge Philos. Soc.* **47**, 207 (1951).
- [49] C. B. Haselgrove and H. N. V. Temperley, Asymptotic formulæ in the theory of partitions, *Math. Proc. Cambridge Philos. Soc.* **50**, 225 (1954).
- [50] M. N. Tran, M. V. N. Murthy, and R. K. Bhaduri, On the quantum density of states and partitioning an integer, *Ann. Phys.* **311**, 204 (2004).
- [51] A. Comtet, P. Leboeuf, and S. N. Majumdar, Level density of a Bose gas and extreme value statistics, *Phys. Rev. Lett.* **98**, 070404 (2007).
- [52] A. Comtet, S. N. Majumdar, and S. Ouvry, Integer partitions and exclusion statistics, *J. Phys. A: Math. Theor.* **40**, 11255 (2007).
- [53] A. Comtet, S. N. Majumdar, S. Ouvry, and S. Sabhapandit, Integer partitions and exclusion statistics: Limit shapes and the largest parts of Young diagrams, *J. Stat. Mech.: Theory Exp.* (2007) P10001.
- [54] C. Echter, G. Maier, J.-D. Urbina, C. Lewenkopf, and K. Richter, Many-body density of states of bosonic and fermionic gases: A combinatorial approach, *J. Phys. A: Math. Theor.* **58**, 065002 (2025).
- [55] J. C. Peyen, L. V. Bogachev, and P. P. Martin, Boltzmann distribution on “short” integer partitions with power parts: Limit laws and sampling, *Adv. Appl. Math.* **159**, 102739 (2024).
- [56] H. H. Rugh, Dynamical approach to temperature, *Phys. Rev. Lett.* **78**, 772 (1997).
- [57] R. H. Swendsen and J.-S. Wang, Negative temperatures and the definition of entropy, *Physica A* **453**, 24 (2016).
- [58] P. Buonsante, R. Franzosi, and A. Smerzi, On the dispute between Boltzmann and Gibbs entropy, *Ann. Phys.* **375**, 414 (2016).
- [59] A. Puglisi, A. Sarracino, and A. Vulpiani, Temperature in and out of equilibrium: A review of concepts, tools and attempts, *Phys. Rep.* **709-710**, 1 (2017).
- [60] G. Gradenigo, S. Iubini, R. Livi, and S. N. Majumdar, Localization transition in the discrete nonlinear Schrödinger equation: Ensembles inequivalence and negative temperatures, *J. Stat. Mech.* (2021) 023201.
- [61] E. Poisson and W. Israel, Inner-horizon instability and mass inflation in black holes, *Phys. Rev. Lett.* **63**, 1663 (1989).
- [62] A. Bonanno, S. Droz, W. Israel, and S. M. Morsink, Structure of the charged spherical black hole interior, *Proc. R. Soc. London, Ser. A* **450**, 553 (1995).
- [63] D. Christodoulou, On the global initial value problem and the issue of singularities, *Classical Quantum Gravity* **16**, A23 (1999).
- [64] M. Dafermos, The interior of charged black holes and the problem of uniqueness in general relativity, *Commun. Pure Appl. Math.* **58**, 445 (2005).
- [65] M. Dafermos and J. Luk, The interior of dynamical vacuum black holes I: The c^0 -stability of the Kerr Cauchy horizon, *Ann. of Math.* **202**, 309 (2025).

- [66] M. De Clerck, S. A. Hartnoll, and J. E. Santos, Mixmaster chaos in an ADS black hole interior, [J. High Energy Phys. 07 \(2024\) 202](#).
- [67] P. Bueno, P. A. Cano, and R. A. Hennigar, Kasner epochs, eras and eons, [Phys. Rev. D **110**, L041503 \(2024\)](#).
- [68] E. Cáceres, Á. J. Murcia, A. K. Patra, and J. F. Pedraza, Kasner eons with matter: Holographic excursions to the black hole singularity, [J. High Energy Phys. 12 \(2024\) 077](#).
- [69] N. Zilberman, M. Casals, A. Ori, and A. C. Ottewill, Quantum fluxes at the inner horizon of a spinning black hole, [Phys. Rev. Lett. **129**, 261102 \(2022\)](#).
- [70] R. Carballo-Rubio, F. Di Filippo, S. Liberati, and M. Visser, Mass inflation without Cauchy horizons, [Phys. Rev. Lett. **133**, 181402 \(2024\)](#).
- [71] J. Wheeler, Geometrodynamics and the issue of the final state, in *1963 Houches Lectures*, edited by B. S. DeWitt and C. M. DeWitt (Gordon and Breach, London, UK, 1963).
- [72] J. Chakravarty, Overcounting of interior excitations: A resolution to the bags of gold paradox in ADS, [J. High Energy Phys. 02 \(2021\) 027](#).
- [73] M. Mézard and A. Montanari, *Information, Physics, and Computation* (Oxford University Press, Oxford, UK, 2009).
- [74] P. Charbonneau, E. Marinari, M. Mézard, G. Parisi, F. Ricci-Tersenghi, G. Sicuro, and F. Zamponi, *Spin Glass Theory and Far Beyond* (World scientific, Singapore, 2023).
- [75] M. Mézard and G. Parisi, The cavity method at zero temperature, [J. Stat. Phys. **111**, 1 \(2003\)](#).
- [76] M. Kac, Can one hear the shape of a drum? [Am. Math. Mon. **73**, 1 \(1966\)](#).