

Drainage front width in a three-dimensional random porous medium under gravitational and capillary effects

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A theoretical approach to estimating stable drainage front widths in three-dimensional (3D) random porous media under gravitational and capillary effects is presented here. Based on the frontier of the infinite cluster in gradient percolation, we propose an expression for the 3D front width dependent on the pore-network topology, the distribution of capillary-pressure thresholds for the pore throats, the stabilizing capillary-pressure gradient, the average pore size, and the correlation length critical exponent from percolation in three dimensions. Theoretical predictions are successfully compared to numerical results obtained with a bond invasion-percolation model for a wide range of drainage flow parameters.

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I. INTRODUCTION

The study of drainage in porous media is relevant to several subsurface transport processes and industrial applications [1–7]. It describes the displacement of a wetting fluid from the pore space by an immiscible nonwetting fluid, as their interface moves along a path of least resistance. At the pore scale, this movement is subject to the local capillary-pressure value P (defined as $P = P_{nw} - P_w$, where P_{nw} is the pressure in the nonwetting phase and P_w is the pressure in the wetting phase) overcoming the capillary-pressure threshold P_t of a pore body or pore throat. Depending on the spatial distribution of P_t values across the porous medium and variations of P along the invasion front, diverse drainage patterns can arise, ranging from compact to ramified and fractal [8–15]. Variations in P along the interface are commonly related to different pressure gradients developed within each phase during drainage, often caused by gravitational and viscous forces. Values of P_t are inversely proportional to the sizes of pore bodies and throats, and their spatial distribution in the medium can be homogeneous or heterogeneous [15–17].

Consider a porous medium where P_t values for the pore throats are randomly drawn from an arbitrary distribution $N(P_t)$. In a drainage process with constant capillary pressure along the front, the nonwetting-fluid invasion of such porous medium is characterized by capillary fingers [11], described well by the Invasion-Percolation (IP) method [18] (see Sec. III). Similarities between IP and ordinary percolation

allow the theoretical estimation of relevant quantities related to drainage processes, such as fluid saturations and trapped cluster size distributions [9,11,18]. By associating each pore throat in a porous medium with a bond in a lattice, the minimum capillary pressure for the percolation of a porous medium, P_{crit} , can also be estimated [13,19,20]. For this, each pore-throat capillary-pressure threshold P_t is associated with a bond occupation probability p , and P_{crit} is the capillary-pressure value that satisfies

$$p_c = \int_{-\infty}^{P_{crit}} N(P_t) dP_t, \quad (1)$$

where p_c is the bond-percolation threshold for the lattice representing the porous medium.

In the same isotropic random porous medium, gradients in the capillary pressure along the interface may stabilize or destabilize the invasion front, significantly affecting the efficiency of the drainage process [8,13,19,21–26]. Positive capillary-pressure gradients in the direction of the flow lead to the unbounded growth of thin drainage fingers, which bypass most of the defending phase [23]. Negative gradients limit the invasion front width to finite values, resulting in compact drainage patterns and a more efficient displacement of the defending phase. In this work, we focus our analysis on the drainage of a random three-dimensional (3D) porous medium under gravitational and capillary effects, where a less dense nonwetting fluid displaces a denser wetting fluid from the top. Under these conditions, the gradient of capillary pressures at the interface is $G = -\Delta\rho g$, where $\Delta\rho$ is the fluid density difference and g is the gravitational acceleration. In particular, we investigate the dependence of the stable drainage front width, η_{3D} , on the 3D porous medium structure and G .

A common approach to investigate gradient-stabilized drainage in porous media is to establish a parallel between the fluids' interface and the frontier of the infinite cluster in the gradient-percolation problem [27–29]. For two-dimensional

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(2D) systems, Sapoval *et al.* [27] demonstrated that this frontier takes place in a critical region, exhibiting a width η/l on the order of the correlation length ξ , where l is the length between two lattice sites. As further explained in Sec. II, the analogy between η/l and ξ leads to a drainage interface scaling of the type

$$\eta/a \propto F^{-\nu/(1+\nu)}, \quad (2)$$

where a is the average pore length, $\nu = 4/3$ is the percolation critical exponent for the correlation length in 2D, and F is the dimensionless fluctuation number, $F = -N(P_{\text{crit}})Ga$, defined in Ref. [19].

The theoretical prediction in Eq. (2) has been verified by numerous studies of drainage in 2D porous media, both numerically and experimentally [13,19,22,26,30–32]. In 3D porous media, however, the same interface scaling is not expected to be valid. By extending the gradient-percolation problem to 3D lattices, Gouyet *et al.* [29] demonstrated that the infinite-cluster frontier is no longer confined to a critical region but rather contains a critical region, termed the front tail. Only within the length of the front tail, η_t , does the front present a fractal structure and obey scaling laws obtained from percolation theory. In the context of drainage, Chaouche *et al.* [33] performed experiments with 3D porous media consisting of packed glass beads and verified that $\eta_t \propto \text{Bo}^{-\nu/(1+\nu)}$, where Bo is the Bond number. Similar scaling laws have been proposed by Wilkinson [34] for the maximum length of trapped wetting-fluid clusters, L_{max} , and for the residual wetting-fluid saturation, S_{res} . While both represent relevant drainage parameters, they are insufficient to disclose the full extent of the region spanned by the front.

In the next section, we build on the theoretical approach presented by Gouyet *et al.* [29] to propose an equation for η_{3D} in gradient-stabilized drainage in three-dimensional random porous media. The prediction is later verified with a modified IP model incorporating capillary and gravitational effects in Sec. III.

II. THEORY

In two dimensions, gradient percolation describes the behavior of $N \times N$ lattices, where the site or bond occupation probability p varies monotonically along a direction z . Let us assume a percolation problem in a square lattice, where p varies from $p(z=0) = 1$ to $p(z=N) = 0$. In this lattice, an infinite cluster exists, containing the occupied sites or bonds connected to the boundary where $p = 1$. The infinite cluster presents a fractal frontier of width η , centered at z_c , where $p(z=z_c) = p_c$. Considering that clusters formed in the frontier region should be on the order of the correlation length $\xi \propto |p - p_c|^{-\nu}$ (where ν is the percolation critical exponent for the correlation length ξ) and bounded by the frontier width, Sapoval *et al.* [27] reached the following scaling for η :

$$\eta \propto \left| \left(\frac{\partial p}{\partial z} \right)_{z=z_c} \right|^{-\nu/(1+\nu)}. \quad (3)$$

Analogous theoretical considerations can be used for the width η of a stable drainage front in 2D porous media, where

$\nabla P = G$ is a linear capillary-pressure gradient in the flow direction z . As proposed by Birovljev *et al.* [13], at some point z_c in the drainage front, the capillary pressure is P_{crit} , where the probability of invasion of a pore throat is p_c [see Eq. (1)]. Based on that, the probability of invasion p of a pore throat in the front at an arbitrary position $z = z_c + \Delta z$ can be calculated as

$$p = p_c + \int_{P_{\text{crit}}}^{P_{\text{crit}}+G\Delta z} N(P_t) dP_t. \quad (4)$$

Approximating the solution of Eq. (4) by the lowest-order term of the Taylor expansion of $N(P_t)$, we get $p \approx p_c + N(P_{\text{crit}})G\Delta z$. Therefore, the pore throats in the drainage front display a linear gradient in invasion probability near the critical value p_c , in the same way as the bonds in the frontier of the infinite cluster in gradient percolation. We can assume then that the drainage front width can be related to the correlation length as $\eta = \xi a$, where a is the average pore size. Substituting $|\Delta z|$ by η gives $|p - p_c| = N(P_{\text{crit}})G\xi a \propto \xi^{-1/\nu}$, which is equivalent to the scaling relation for the front width introduced by Eq. (2).

A fundamental aspect for the validity of the scaling presented in Eq. (3) is the percolation threshold of the lattice of interest [28]. In 2D lattices, p_c values often exceed 0.5. Considering that occupied sites or bonds in a lattice represent the nonwetting phase in pore bodies and throats during drainage, $p_c > 0.5$ implies that only one phase can percolate at a time. As the minimum capillary pressure required for the percolation of the nonwetting phase is achieved [see Eq. (1)], the drainage of a large fraction of pores is allowed, which traps the wetting phase in smaller clusters. For this reason, the drainage front does not extend over a large range of p , being restricted to the critical region around p_c , where Eq. (3) is valid.

On the contrary, three-dimensional lattices can exhibit p_c values significantly lower than 0.5. This means that a phase can percolate the lattice-equivalent porous medium at a relatively low saturation, without restricting a second phase to isolated clusters. More precisely, there is a range of occupation probabilities $p_c \leq p \leq 1 - p_c$ over which drainage fronts can be extended, as both phases reach the percolation threshold. As a consequence, the capillary-pressure values at the 3D drainage front vary approximately from P_{crit} at its tip, where the invading phase saturation is low, to P_{res} at its end, where the invading phase saturation is high enough to trap the defending phase at the residual saturation S_{res} . Here, P_{res} is given by

$$1 - p_c = \int_{-\infty}^{P_{\text{res}}} N(P_t) dP_t. \quad (5)$$

In this scenario, as identified by Rosso *et al.* [28], the stretch of the front beyond the vicinity of p_c makes the front width scaling presented by Sapoval *et al.* [27] no longer valid. Instead, Gouyet *et al.* [29] demonstrated that only a part of the front, η_t , spanning the critical region near p_c , exhibits a fractal structure. Using a simple-cubic lattice and different linear gradients ∇p , they measured the subregion of the infinite-cluster frontier limited by $p \leq p_c$ and verified that $\eta_t \propto |\nabla p|^{-\nu/(1+\nu)}$. Following a similar reasoning, we can

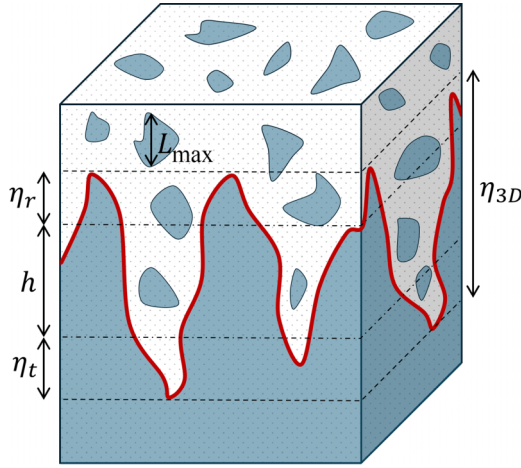


FIG. 1. Illustration of a 3D porous medium under drainage. The nonwetting phase is represented in white, the wetting phase in blue, and the drainage front, η_{3D} , in red. In this sketch, the quantities of interest of this study are highlighted: η_r , where the wetting phase is near its percolation threshold, h , the transition region, η_t , where the nonwetting phase is near its percolation threshold, and $L_{\max}$, the maximum length of trapped clusters.

assume that 3D stable drainage fronts also present a fractal tip region, η_t , with capillary-pressure values $P \leq P_{\text{crit}}$, which scales as $\eta_t/a \propto F^{-\nu/(1+\nu)}$.

Beyond what is proposed by Gouyet *et al.* [29], we suggest that within 3D stable drainage fronts in porous media, a second fractal region near $p = 1 - p_c$ exists, where capillary-pressure values are $P \geq P_{\text{res}}$. Analogous to the tip of the drainage front, where the nonwetting phase is near its percolation threshold, the end of the front should also exhibit a critical behavior, as the wetting phase is about to exit its percolation range. The width of this fractal region, termed η_r here, should be on the order of the maximum length of the trapped wetting-phase clusters $L_{\max}$ [27,29] and scale as $\eta_r/a \propto F_r^{-\nu/(1+\nu)}$ (where $F_r = -N(P_{\text{res}})Ga$ is a modified fluctuation number).

Similarly, Wilkinson [34] suggested that $L_{\max} \propto \text{Bo}^{-\nu/(1+\nu)}$, but no link between this scaling and the front width was established. Rather, a transition zone h was defined in that study, bounded by the planes where the invading phase occupation is p_c and $1 - p_c$, which was considered to scale as $h \propto \text{Bo}^{-1}$. We propose that this transition zone can be directly calculated as $h = (P_{\text{res}} - P_{\text{crit}})|G|^{-1}$ and corresponds to the distance between the two fractal regions, η_t and η_r . In this way, we can write a function for the full 3D front length equivalent to

$$\eta_{3D} = (P_{\text{res}} - P_{\text{crit}})|G|^{-1} + aC(F^{-\nu/(1+\nu)} + F_r^{-\nu/(1+\nu)}), \quad (6)$$

where C is on the order of unity and is related to ξ_0 in $\xi = \xi_0|p - p_c|^{-\nu}$, the correlation length scaling in percolation [35]. In 3D, $\nu = 0.88$. A sketch of drainage in a 3D porous medium, highlighting the drainage front, the maximum cluster length $L_{\max}$, and the partitioning of the drainage front into η_t , η_r , and h , is presented in Fig. 1.

III. INVASION-PERCOLATION MODELING

To test the theoretical assumptions presented in Sec. II, an IP model is used. Proposed in the 1980s by Wilkinson and Willemsen [18], IP modeling was conceived to represent porous-media displacement flows at a constant but infinitesimal flow rate, in which viscous forces are dominated by capillary forces. Since then, it has been intensively adopted in the porous-media literature, especially to represent drainage flows [14,15,32,36–39]. In IP models, a network of sites connected by bonds represents a porous medium of pore bodies connected by pore throats. To each site or bond, a capillary-pressure threshold P_t for invasion is assigned, which controls its likelihood of invasion in the pore network. Drainage is modeled as a bond invasion-percolation process, where bonds with low P_t are preferentially invaded. In imbibition, a site invasion-percolation process is adopted, where sites with high P_t are more likely to be invaded.

Briefly, our IP simulations represent drainage flows and start with a pore network fully occupied by the wetting phase. From sites defined as the pore-network inlet, the occupation of the nonwetting phase evolves as a growing cluster that sequentially incorporates the “easiest” available bond at its perimeter. During this process, wetting-phase clusters completely surrounded by the invading phase are considered trapped. Therefore, available bonds belong only to wetting-phase clusters connected to the network outlet. The bond invasion percolation progresses until the breakthrough—when the invading phase reaches the network outlet—and the drainage simulations end. For more details about the Invasion-Percolation method, we refer to Ref. [18].

A. Stable drainage under gravitational effects

Within drainage flows, we focus on the stable scenario in which a denser wetting fluid is slowly displaced by a less dense nonwetting fluid from the top. Thus, gravitational effects are included in the model by considering that the capillary pressure in the pore throats at the drainage front varies linearly, according to Eq. (7) [19,34]:

$$P(z) = P_0 - \Delta\rho gz, \quad (7)$$

where P_0 is a reference value of capillary pressure at $z = 0$ and the z axis points in the direction of the gravitational field.

As a consequence, our bond invasion-percolation model is controlled by both the capillary-pressure-threshold values and the position of the bond along the invasion front. In this way, at each invasion step, the “easiest” available bond at the invasion cluster perimeter corresponds to the one with the lowest value of $P_t^* = P_t - P(z)$.

B. Networks representing porous media

Networks with different topologies, capillary-pressure-threshold distributions $N(P_t)$, and pore sizes a are used to represent porous media in the IP drainage simulations presented in this work. With this, we aim to verify that the theoretical predictions presented in Sec. II are valid for pore networks with variable geometrical features.

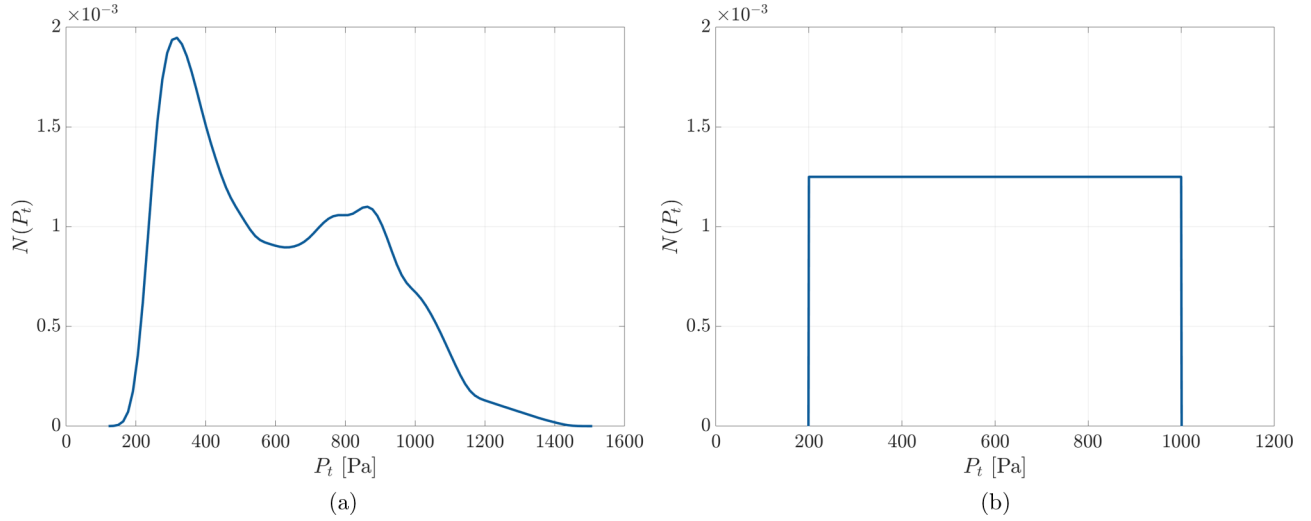


FIG. 2. Probability density functions of P_t values used to generate the pore networks. (a) Nonuniform $N(P_t)$ and (b) uniform $N(P_t)$.

1. Network topology

Two regular 3D structures are considered: a simple-cubic structure, in which each site has six nearest neighbors, and a diamond-cubic structure, in which each site has four nearest neighbors. Similar values of pore-network coordination numbers have been reported in the literature for relevant naturally occurring porous media, such as sandstones [40–42]. Due to their difference in connectivity, the evaluated structures display significantly different bond-percolation thresholds. Simple-cubic networks present a bond-percolation threshold $p_c \approx 0.25$, while this value is $p_c \approx 0.39$ for diamond-cubic networks. Based on that, important differences during the drainage of these structures are expected. With a lower p_c , porous media corresponding to a simple-cubic lattice can be percolated at a lower critical capillary pressure P_{crit} , present a wider range of occupations where both phases percolate ($p_c \leq p \leq 1 - p_c$), and require higher capillary-pressure values to trap the wetting phase, P_{res} .

2. Capillary-pressure-threshold distributions

Both a uniform and a nonuniform $N(P_t)$ distribution are used in the simulations, as shown in Fig. 2. The distribution presented in Fig. 2(a) is based on values reported in Ref. [43], while the one presented in Fig. 2(b) is simply a uniform distribution within the range $P_t \in [200, 1000]$ Pa. From these distributions, values are assigned to each bond in the networks randomly, without establishing a spatial correlation between P_t values.

In Table I, the critical capillary pressures for percolation of the nonwetting phase [see Eq. (1)] and trapping of the wetting phase [see Eq. (5)] are shown. Also, the values of $N(P_{\text{crit}})$ and $N(P_{\text{res}})$ —which are used to calculate the fluctuation numbers at the critical capillary pressures—are calculated. For the uniform capillary-pressure-threshold distribution, we have a constant probability of occurrence for any P_t , equal to $N(P_t) = 1/W_t$, where W_t is the width of the distribution. For the nonuniform case, we calculate $N(P_t)$ with the numerical approximation of the derivative of the cumulative distribution

function of P_t at P_{crit} and P_{res} , similarly to the procedure presented in Res. [39].

3. Average pore size

Three different values of pore size $a = [0.1, 0.5, 1]$ cm are used in the drainage simulations. In this way, a wider range of invasion front widths, as shown in the next section, could be covered by this study. By varying the pore size, we can also examine how this variable affects the widths of the drainage front critical regions, while the transition zone remains the same, as proposed in Sec. II. It is important to notice that this size is not equivalent to the opening of the constrictions—or pore throats—in the media. While the constriction radius is related to the $N(P_t)$, shown in Fig. 2, a is related to the average distance between pore bodies projected in the direction z , which is aligned with the gravitational field g . Hence, this quantity can be linked to the length of pores.

4. Pore-network size and boundary conditions

For each pore-network topology, a different network size is used in the simulations. Diamond-cubic networks displayed an $n_x \times n_y \times n_z$ of $100 \times 100 \times 200$ sites, where n_x , n_y , and n_z are the number of sites in the x , y , and z directions, respectively, and z is aligned with the gravitational field. Simple-cubic networks had half the number of sites in each direction, namely, $50 \times 50 \times 100$, as their coordination number is higher and their bond-percolation threshold is

TABLE I. Critical capillary-pressure values and their probability in $N(P_t)$.

	Diamond cubic		Simple cubic	
	Nonuniform	Uniform	Nonuniform	Uniform
P_{crit} (Pa)	463.3	511.1	364.8	400
P_{res} (Pa)	693.5	688.9	830.8	800
$N(P_{\text{crit}})$ (Pa $^{-1}$)	1.13×10^{-3}	1.25×10^{-3}	1.77×10^{-3}	1.25×10^{-3}
$N(P_{\text{res}})$ (Pa $^{-1}$)	9.34×10^{-4}	1.25×10^{-3}	1.06×10^{-3}	1.25×10^{-3}

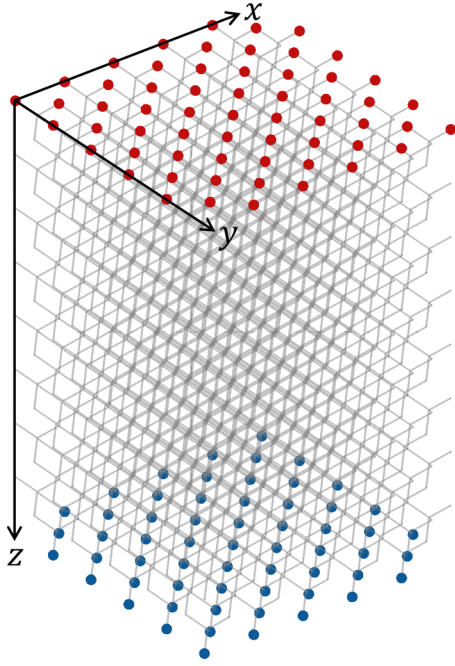


FIG. 3. Illustration of a diamond-cubic network with inlet sites highlighted in red and outlet sites highlighted in blue.

lower, which leads to more computationally expensive simulations.

As for the boundary conditions, all sites at $z = 0$ are assigned to the network inlet, while all sites at $z = an_z$ are assigned to the network outlet, as shown in Fig. 3. The network sides—corresponding to $x = 0$, $x = an_x$, $y = 0$, and $y = an_y$ —are considered closed.

C. Imposed gradient of capillary pressure at the drainage fronts

To obtain drainage fronts under different capillary-pressure gradients G , we vary the density difference between the phases in the IP drainage simulations. For each type of pore network used in this study, a different range of $\Delta\rho$ is defined.

At the low-density-contrast end of the range, drainage fronts are wide, and values of $\Delta\rho$ are chosen so that the obtained front fits the total size of the pore network. For this, some tests are conducted for each type of network until a minimum suitable value of $\Delta\rho$ is found. At the other end of the range, drainage fronts become narrow, and we seek to avoid nearly flat fronts. Thus, we set the maximum value of $\Delta\rho$ when front widths are $\approx 5a$. Therefore, in the results presented in Sec. IV, we include drainage cases with stable front widths belonging approximately to the interval $5a < \eta_{3D} < n_z a$.

IV. RESULTS

In this section, we use the bond invasion-percolation method detailed in Sec. III to simulate multiple slow drainage scenarios under stabilizing gravitational effects. The observed front characteristics are compared to the theoretical predictions from Sec. II. In particular, we aim to verify whether the estimate of the stable front width η_{3D} , proposed in Eq. (6), is valid.

To calculate the stable front width from each drainage simulation, we keep track of the pore-network sites and bonds belonging to the invasion front, from the beginning of the nonwetting phase invasion to the breakthrough. Figure 4 illustrates a simple-cubic network, midway through drainage, with the bonds of the invading cluster highlighted in dark blue, the bonds of defending clusters in gray, and the sites belonging to the invasion front in red.

Using the x, y, z coordinates of the bonds connected to the front sites, as well as their P_t values, we can estimate the transition zone width h and the width of the critical zones η_t and η_r throughout each drainage simulation. Values of these widths, averaged over the simulation time steps, are presented in Secs. IV A–IV C. Invasion fronts containing sites or bonds close to the inlet plane, namely, $0 \leq z \leq 0.25an_z$, are not considered in the analyses, as they are prone to distortion by boundary effects.

A. Transition zone

As proposed by Wilkinson [34], invasion-percolation stable fronts present a transition zone from the point where the occupation of the invading fluid is equivalent to the percolation threshold, to the point where the occupation of the defending fluid is at the percolation threshold. For any pore network with a defined p_c , the capillary-pressure values corresponding to these percolation thresholds can be estimated with Eqs. (1) and (5), as presented in Table I. Using these values, the limits of the transition zone within the stable fronts can be located, as explained next.

Let us consider a diamond-cubic pore network, with $a = 0.5$ cm and a uniform $N(P_t)$ from 200 to 1000 Pa [see Fig. 2(b)]. For the nonwetting phase to percolate this medium, the critical capillary pressure is estimated to be $P_{\text{crit}} = 511.1$ Pa. At $P_{\text{res}} = 688.9$ Pa, we expect the occupation of the wetting phase to be near its critical value, when it becomes confined to trapped clusters. In Fig. 5, we illustrate this network during drainage, when $\Delta\rho = 64$ kg/m³. As we aim to locate the points in the z axis delimiting the transition zone h , a view from the plane $x \times z$ is chosen. In Fig. 5(a), we show the invaded bonds in dark blue, the noninvaded bonds in gray, and the front sites in red. In Fig. 5(b), the sites belonging to the front are still shown in red, and only bonds connected to the front are displayed. Bonds within the transition zone are highlighted in blue, and the other network elements are gray. Figure 5(c) presents P_t of all invaded bonds at the front, according to their position in the z axis. It is clear that, as we move up in the front (in the negative direction of the z axis), the maximum capillary threshold value allowed for invasion increases. This results from the linear gradient in capillary pressures along the front, $G = -\Delta\rho g$. At the front tip, $z \approx 0.7$ m, the capillary pressure is relatively low, and only pore throats with P_t up to P_{crit} can be invaded. At the front end, near $z \approx 0.35$ m, the local capillary-pressure values are much larger, allowing the invasion of pore throats with $P_t \approx P_{\text{res}}$. Therefore, to define the limits of the transition zone h , we find the maximum value of P_t among the invaded bonds at the invasion front, for each value of z spanned by the front. At the points where these maxima correspond to P_{crit} and P_{res} , we locate the limits of h . It is important to note that

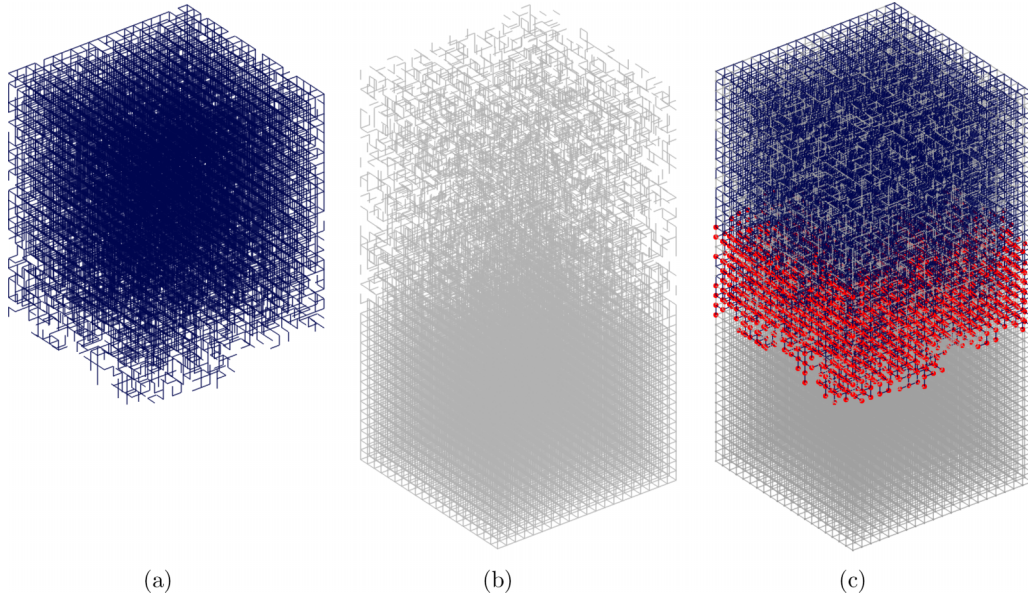


FIG. 4. Simple-cubic network midway through drainage, with invaded bonds in blue, noninvaded bonds in gray, and sites belonging to the invasion front in red. In this example, $a = 0.5$ cm and $\Delta\rho = 1000$ kg/m³. (a) Bonds occupied by the invading phase, (b) bonds occupied by the defending phase, and (c) sites at the invasion front.

a well-behaved gradient in the maximum P_t among invaded pore throats along the z axis, as seen in Fig. 5(c), may not be observed if the pore-network cross section ($n_x \times n_y$) is not large enough to contain a significant sample of $N(P_t)$.

Figure 6 presents the measured values of h using this procedure, for the two pore-network topologies, two distributions of capillary-pressure-threshold values, and three average pore sizes presented in Sec. III. Results from five random realizations of each different pore-network type are shown. Values of h are plotted against the difference in density between the phases, $\Delta\rho$, used in the drainage simulations. As discussed in Sec. II, we expect h to be equivalent to $(P_{\text{crit}} - P_{\text{res}})|G|^{-1}$. Therefore, in the case of gravity-stabilized drainage, the transition width between the front's critical regions should be inversely proportional to $\Delta\rho$.

In Figs. 6(a)–6(d), green diamond symbols represent $a = 1.0$ cm, light blue circles represent $a = 0.5$ cm, and dark blue squares represent $a = 0.1$ cm. The results show that the transition zone widths obtained with the IP model match the expected values for all tested pore-network types, if $\Delta\rho$ values are not too large. In this regard, we observe that curves from pore networks with larger a deviate from $h = (P_{\text{crit}} - P_{\text{res}})|G|^{-1}$ at lower $\Delta\rho$ values. This suggests that partitioning the front into a transition zone and critical regions may not be straightforward when the pressure gradients are too high and the front spans only a few pores.

For this reason, a brief investigation of the effect of high gradients on the proposed front scaling is presented in Appendixes A and B. Based on that, we verify that the theoretical framework presented in Sec. II may not be valid if the density difference between the phases is larger than $\Delta\rho^{\text{lim}} = |\Delta p|/(N(P_{\text{crit}})ga)$, where Δp is the invasion probability difference over the length of a bond at the front, with a suggested limit of $|\Delta p|^{\text{lim}} = 0.075$. To make this point clear, results obtained with $\Delta\rho < \Delta\rho^{\text{lim}}$ are represented with

big hollow symbols, while smaller filled symbols are used otherwise. Although this criterion is empirically based on gradient-percolation results, it seems to predict the limit of $\Delta\rho$ where measured h values fit the theoretical prediction adequately. Therefore, we keep this symbol distinction in the results presented in the following sections.

B. Drainage front critical region near $P = P_{\text{crit}}$

In Figs. 5(b) and 5(c), it is noticeable that a significant fraction of the drainage front is located outside the transition zone h . At the front tip, where the capillary pressure is near P_{crit} , the extent of the front below h corresponds to the critical region η_t , defined in Sec. II. To evaluate the scaling of the width of this region, values of η_t are calculated as

$$\eta_t^2 = \sum_{i=1}^{N_t} \frac{(z_i - z_c)^2}{N_t}, \quad (8)$$

where N_t is the total number of invaded bonds connected to the front at $p < p_c$, z_i is the position of these bonds, and z_c is the position of the lower limit of the transition zone h , at which the maximum allowed P_t for invasion is P_{crit} and $p = p_c$ [see Fig. 5(c)].

Instead of directly measuring the total extent of the front where $p < p_c$, Eq. (8) provides a more statistically relevant quantification of how far the invaded bonds in the critical region spread from the position where $p = p_c$. Analogous to computing a standard deviation, this method was proposed by Gouyet *et al.* [29] to verify the scaling of the infinite-cluster front tail in gradient percolation. In Fig. 7, values of $\bar{\eta}_t/a$ (where $\bar{\eta}_t$ is the value of η_t averaged over the simulation time steps) are plotted against $F = -N(P_{\text{crit}})Ga$. Results from five random realizations of each type of pore network are shown.

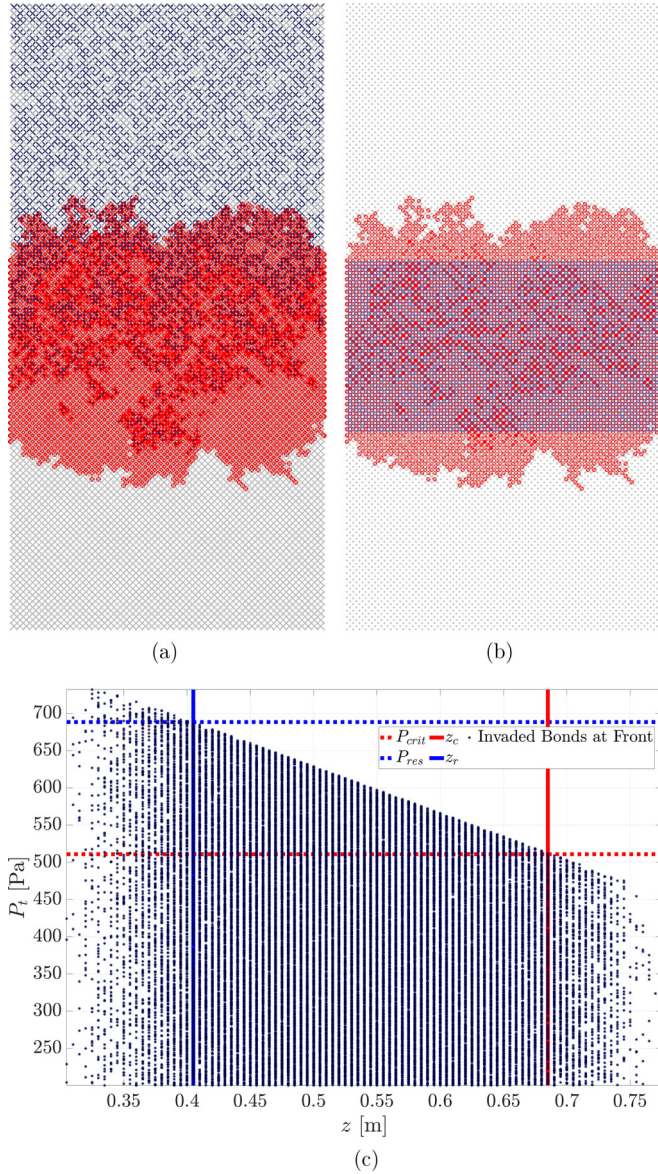


FIG. 5. Diamond-cubic network during drainage, viewed from the plane $x \times z$, with $a = 0.5$ cm and $\Delta\rho = 64$ kg/m³. (a) Invaded bonds are shown in dark blue, noninvaded bonds in gray, and front sites in red. (b) The transition zone is shown in blue, and the front sites in red. Only bonds connected to the front are shown. (c) P_t of the invaded bonds connected to the front vs their position in the z axis. z_c and z_r represent the lower and upper limits of the transition zone h , respectively.

The results from Fig. 7 clearly indicate that the values of η_t/a obtained with the bond IP simulations agree well with the theoretical scaling based on gradient percolation, shown in the black lines. As expected, we see that values of η_t depend on the average pore size a , unlike the transition zone h . Comparing the results obtained with the simple-cubic and diamond-cubic pore networks, we can also see that values of η_t differ even when F and a are the same. The observed differences are attributed to the scaling prefactor, which likely depends on the pore-network topology. In Appendix C, we propose a simple way to estimate the prefactor values, based on the front-tail extent in gradient percolation.

C. Drainage front critical region near $P = P_{res}$

Similarly to η_t , we propose that a second portion of the drainage front displays a critical behavior, as the occupation of the defending phase is near its percolation threshold. The extent of this region, termed η_r in Sec. II, is illustrated in Fig. 5(b) above the upper limit of the transition zone h . Here, we investigate the scaling of η_r with a procedure analogous to that presented in Sec. IV B. With Eq. (9), we calculate η_r during drainage in five random realizations of each pore-network type presented in Sec. III B:

$$\eta_r^2 = \sum_{i=1}^{N_r} \frac{(z_i - z_r)^2}{N_r}, \quad (9)$$

where N_r is the total number of noninvaded bonds connected to the front at $p > 1 - p_c$, z_i is the position of these bonds, and z_r is the position of the upper limit of the transition zone h , at which the maximum allowed P_t for invasion is P_{res} and $p = 1 - p_c$ [see Fig. 5(c)].

In Fig. 8, values of $\bar{\eta}_r/a$ are plotted against $F_r = -N(P_{res})Ga$, defined as a modified fluctuation number. We observe that the extent of the drainage front where the defending phase becomes trapped in clusters scales as theoretically predicted, for all types of pore networks investigated. Along with the scaling of η_t and h , these results strongly indicate that the total width of 3D stable drainage fronts may be correctly estimated with Eq. (6).

D. Maximum length of trapped clusters

In two-dimensional gradient percolation, Sapoval *et al.* [27] proposed that the width of the infinite-cluster front should scale as the maximum length of clusters formed in its vicinity. The same relation was observed between the invasion front width and the maximum length of wetting-phase trapped clusters in gradient-stabilized drainage in 2D porous media [13,39]. In three-dimensional gradient percolation, however, Gouyet *et al.* [29] verified that the infinite-cluster front width is much larger than the clusters formed alongside it. Instead, they proposed that the length of these clusters is limited by the front-tail width, with $\eta_t \propto |\nabla p|^{-\nu/(1+\nu)}$. Using a 3D invasion-percolation model with buoyancy effects, Wilkinson [34] similarly verified that the maximum length of trapped defending-phase clusters scaled as $L_{max} \propto Bo^{-\nu/(1+\nu)}$, where Bo is the Bond number.

In Fig. 9, we present the maximum length of trapped clusters normalized by the average pore size at breakthrough in our IP drainage simulations. Results are plotted against F_r , as these clusters are formed in the critical region where the defending phase is at its critical occupation. Therefore, we expect L_{max} to scale as η_r .

Overall, the observed maximum cluster length scales reasonably well as $L_{max}/a \propto F_r^{-\nu/(1+\nu)}$. For high values of F_r , the tendency of L_{max} to fall short of the theoretically predicted values may be attributed to the high gradient of occupation probability, as discussed in Appendixes A and B. In fact, we notice that in these cases L_{max} corresponds to the length of only a few pores, which can negatively impact the expected scaling. For very low F_r values, we also identify that L_{max}

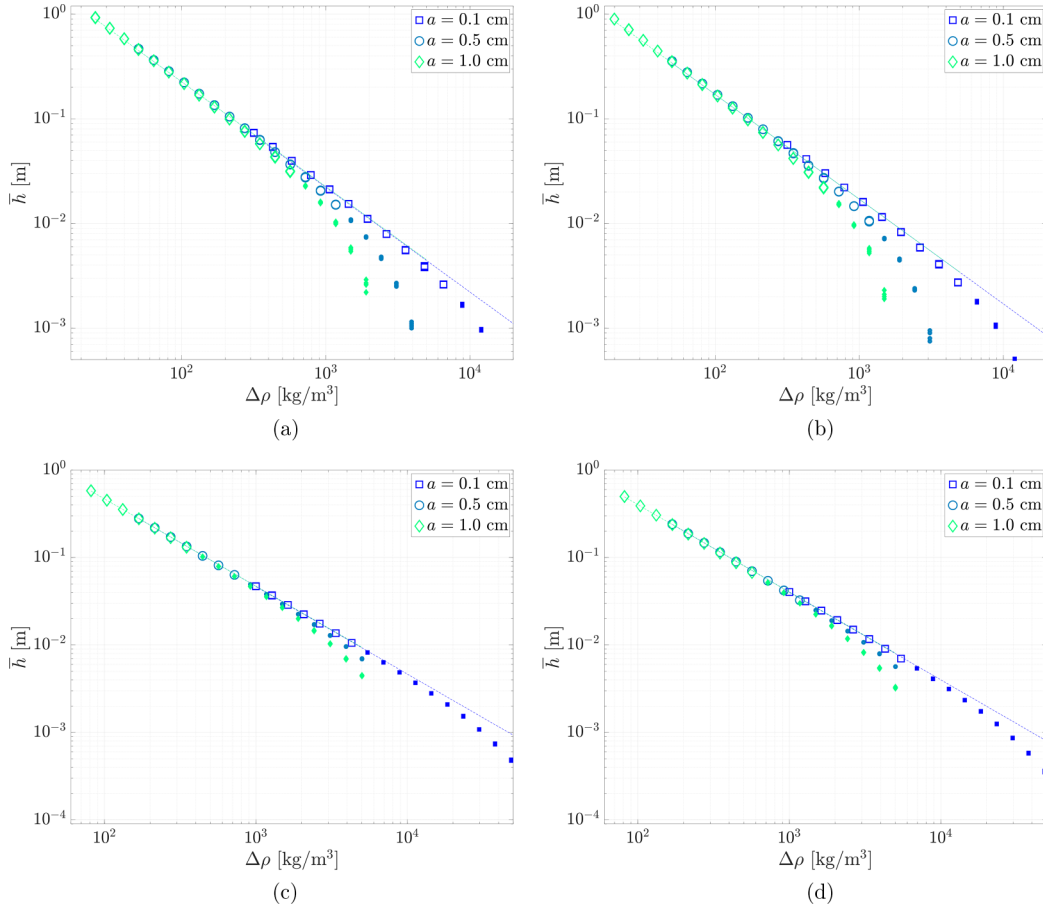


FIG. 6. Measured widths of the transition zone $\bar{h}$ in IP drainage simulations, for all tested pore networks. Big hollow symbols represent data within the range of simulation parameters; we expect the theoretical framework presented in Sec. II to be valid. Small filled symbols are shown otherwise. The dashed lines represent the expected values of $\bar{h} = (P_{\text{res}} - P_{\text{crit}})|G|^{-1}$. (a) Diamond lattice - nonuniform $N(P_t)$, (b) diamond lattice - uniform $N(P_t)$, (c) simple cubic lattice - nonuniform $N(P_t)$, and (d) simple cubic lattice - uniform $N(P_t)$.

deviates from the prediction, especially when the simple-cubic pore network is used. In this scenario, the length of the drainage front (see Sec. IV E) is very close to the total length

of the networks, and very few trapped clusters can be verified at breakthrough. Therefore, larger simulated domains may be required to estimate L_{max} when $\eta_{3D} \approx an_z$.

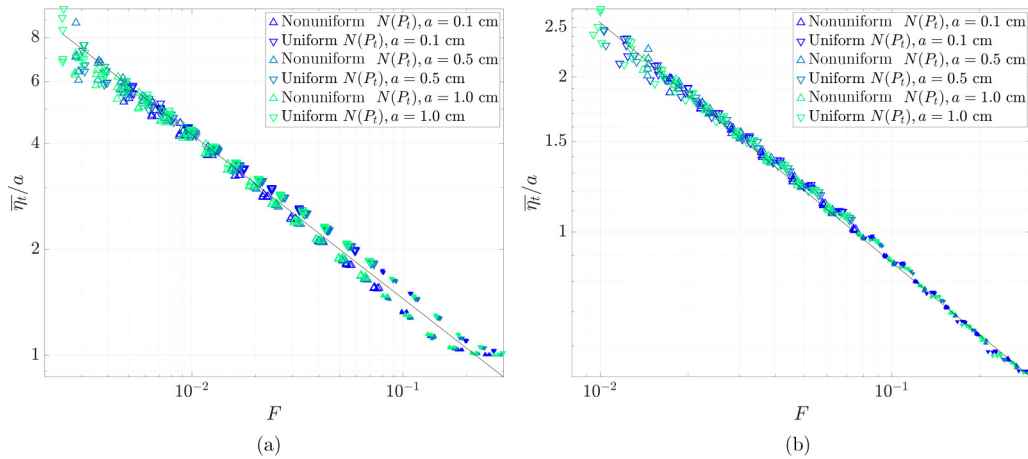


FIG. 7. Calculated values of $\bar{\eta}_t/a$ in IP drainage simulations using Eq. (8), for all tested pore networks. Big hollow symbols represent data within the range of simulation parameters; we expect the theoretical framework presented in Sec. II to be valid. Small filled symbols are shown otherwise. The black lines represent the expected scaling of $\bar{\eta}_t/a \propto F^{-\nu/(1+\nu)}$, where $\nu = 0.88$ from 3D percolation. For clarity, separate plots of $\bar{\eta}_t$ for each network type are included in the Supplemental Material [44]. (a) Diamond-cubic network and (b) simple-cubic network.

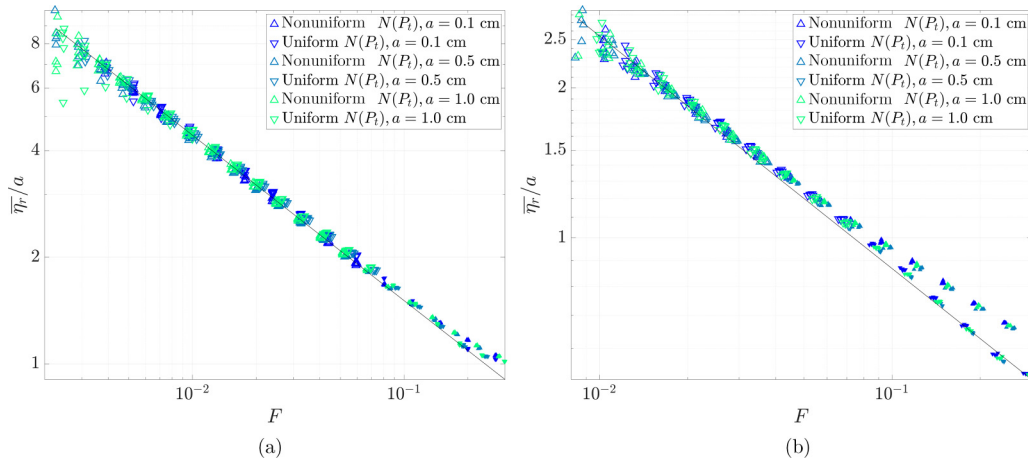


FIG. 8. Calculated values of $\bar{\eta}_r/a$ in IP drainage simulations using Eq. (9), for all tested pore networks. Big hollow symbols represent data within the range of simulation parameters; we expect the theoretical framework presented in Sec. II to be valid. Small filled symbols are shown otherwise. The black lines represent the expected scaling of $\eta_r/a \propto F_r^{-\nu/(1+\nu)}$, where $\nu = 0.88$ from 3D percolation. For clarity, separate plots of $\bar{\eta}_r$ for each network type are included in the Supplemental Material [44]. (a) Diamond-cubic network and (b) simple-cubic network.

E. Drainage front width

In Sec. II, we propose that pressure-stabilized drainage fronts in 3D random porous media can be partitioned into three sections: A critical region η_t where the invading phase is near its percolation threshold, a second critical region η_r where the defending phase is near percolation, and a transition zone h where both phases can percolate. In Secs. IV A–IV C, we verify that these three regions satisfactorily scale according to the theoretical predictions based on gradient percolation. Here, we compare the total drainage front predicted with Eq. (6) with our bond IP model results.

In Fig. 10, the total front width measured with the drainage simulations is shown as scattered data, while the predictions from Eq. (6) correspond to the continuous lines. To use this equation, we adopted $\mathcal{C} = 1.55$ for diamond-cubic pore net-

works and $\mathcal{C} = 0.9$ for simple-cubic pore networks. These values are obtained in Appendix C, based on gradient percolation, and seem to depend on the topology of the network only. The total front width is obtained from the simulations as simply $\eta_{3D} = z_{\eta}^{\max} - z_{\eta}^{\min}$, where $z_{\eta}^{\max}$ and $z_{\eta}^{\min}$ are the maximum and minimum values of z among the front bonds, respectively.

We observe that the proposed estimate for η_{3D} adequately fits the data obtained with numerical simulations, when values of $\Delta\rho$ are not too high. As presented in Appendixes A and B, we expect deviations to arise when the gradient of invasion probability along the front is large and the front spans only a few pores. In this scenario, it is verified in Sec. IV A that the prediction of the transition zone width $h = (P_{\text{res}} - P_{\text{crit}})|G|^{-1}$ also fails.

Conversely, when the total front width is larger than $\approx 10a$, we demonstrate that Eq. (6) provides a good

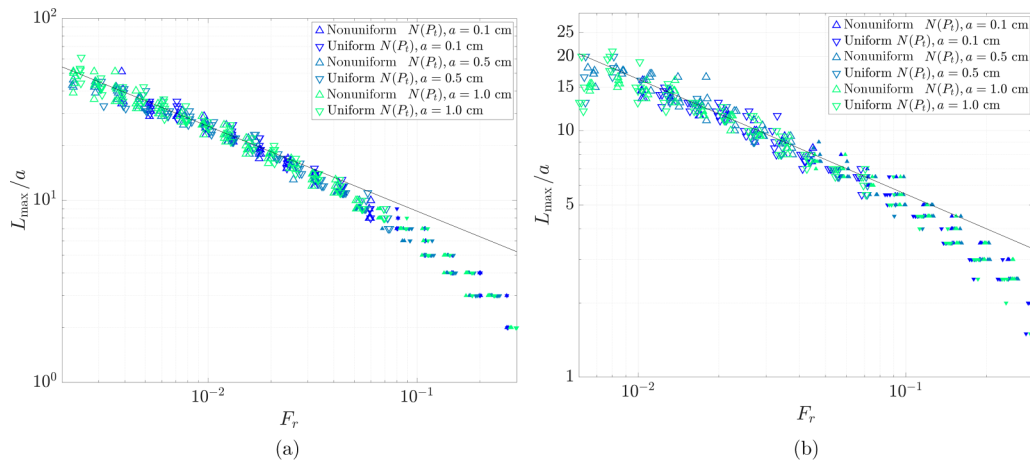


FIG. 9. Values of $L_{\max}/a$ measured in IP drainage simulations at breakthrough, for all tested pore networks. Big hollow symbols represent data within the range of simulation parameters; we expect the theoretical framework presented in Sec. II to be valid. Small filled symbols are shown otherwise. The black lines represent the expected scaling of $L_{\max}/a \propto F_r^{-\nu/(1+\nu)}$, where $\nu = 0.88$ from 3D percolation. For clarity, separate plots of $L_{\max}$ for each network type are included in the Supplemental Material [44]. (a) Diamond-cubic network and (b) simple-cubic network.

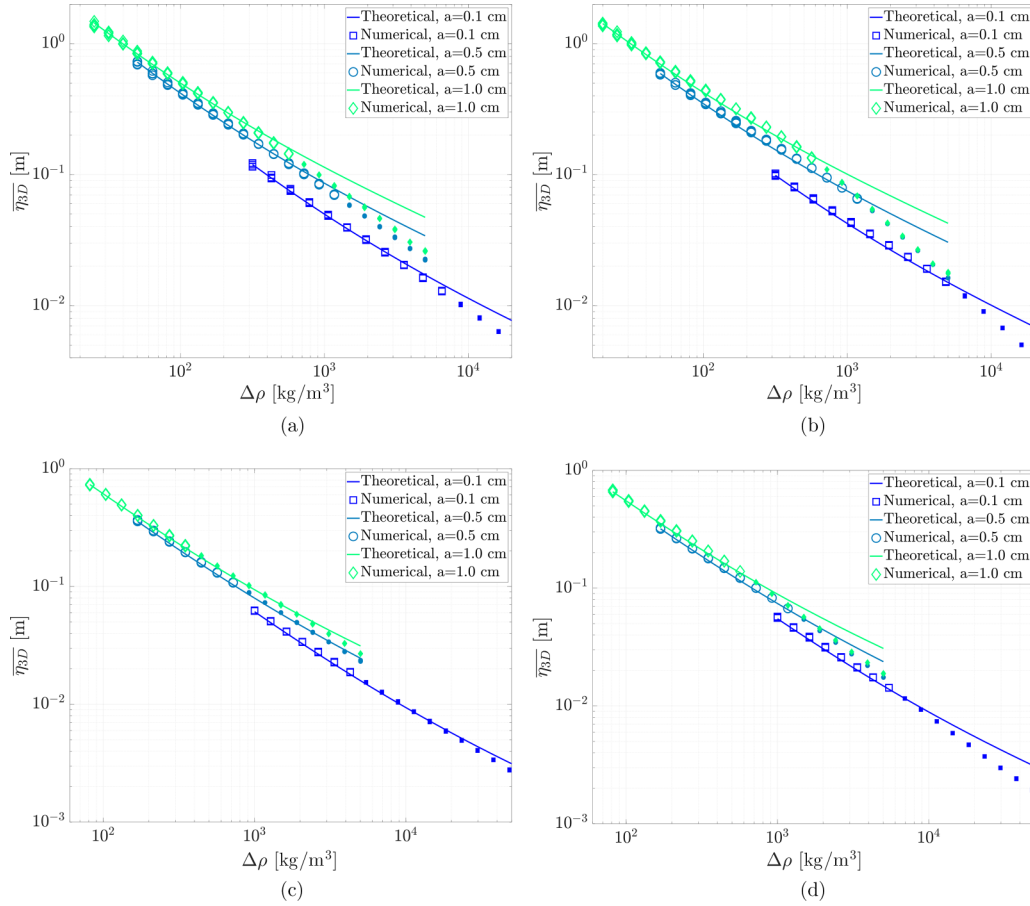


FIG. 10. Measured total front widths η_{3D} in IP drainage simulations, for all tested pore networks. Big hollow symbols represent data within the range of simulation parameters; we expect the theoretical framework presented in Sec. II to be valid. Small filled symbols are shown otherwise. The continuous lines represent the estimated values of η_{3D} , using Eq. (6). (a) Diamond lattice - nonuniform $N(P_I)$, (b) diamond lattice - uniform $N(P_I)$, (c) simple cubic lattice - nonuniform $N(P_I)$, and (d) simple cubic lattice - uniform $N(P_I)$.

estimate of the stabilized drainage front widths obtained with a bond IP model. The correspondence between numerically and theoretically obtained η_{3D} seems independent of the capillary-pressure-threshold distribution, average pore size, and network topology.

V. DISCUSSION

The results obtained with our bond invasion-percolation model with trapping suggest that gravity-stabilized drainage fronts in random porous media can be mapped onto the infinite-cluster frontier in gradient percolation. While both types of percolation do not belong to the same universality class [15,45], the capillary-pressure gradient along the direction of the flow leads to a gradient in invasion probability along the drainage front analogous to the gradient in occupation probability in gradient percolation.

The partitioning of the front into two regions with critical behavior, η_t and η_r , separated by a transition zone, h , is supported by individually assessing the scaling of each segment. Additionally, η_{3D} predictions obtained with Eq. (6) are successfully compared with numerical results, when appropriate values of G and C are used. Based on that, we demonstrate how stable drainage front widths in 3D porous media may

have a more complex dependency on the stabilizing pressure gradients than in two dimensions.

Consider a random 3D porous medium undergoing stable drainage. At low $|G|$, the width of the front tends to be dominated by the width of the transition zone, which follows the scaling $h \propto |G|^{-1}$. At high $|G|$, as long as $|\Delta p|^{\text{lim}}$ is not exceeded, η_{3D} may become dominated by the width of the critical regions, which scale as $|G|^{-\nu/(1+\nu)}$, where $-\nu/(1+\nu) = -0.47$. Therefore, an attempt to simply establish a power-law relationship between η_{3D} and $|G|$ may lead to fitted exponents between -1 and -0.47 , depending on the evaluated range of $|G|$. Using a bond invasion-percolation model to represent gravity-stabilized drainage in a simple-cubic pore network, Breen *et al.* [46] reported a power-law fit equivalent to $\eta_{3D} \propto |G|^{-0.85}$, which is compatible with our observations.

The relative contribution of each segment to the front width also depends nontrivially on the pore structure. Highly connected networks exhibit low bond-percolation thresholds, leading to wider transition zones where both phases can percolate. In Fig. 6, we notice that values of h are larger in simple-cubic networks than in diamond-cubic networks with the same $N(P_I)$ and $\Delta\rho$. Besides the pore-network topology, the width and form of the capillary-pressure-threshold distribution have a complex influence on η_{3D} . For a given pore network, values of η_t and η_r are equivalent if $N(P_I)$ is uniform.

Using the nonuniform $N(P_t)$ illustrated in Fig. 2(a), η_r is larger than η_t , as shown in Figs. 7 and 8, especially with the simple-cubic network.

Understanding how the fluids and porous medium properties affect the invasion front during drainage can be relevant for estimating important parameters in displacement flow studies, such as the representative elementary volumes, the interfacial area, and the resulting residual saturation [47]. Similar arguments to those presented in this work could also be useful in studies of drying fronts in porous media, whose shape and extent can control evaporation rates [48].

Furthermore, the presented drainage front width analyses may be valid for a wide range of naturally occurring and engineered porous media, despite the suggested limitation in pressure gradient, when the characteristic pore size is on the order of micrometers. For example, consider the stable displacement of water by air in a Berea sandstone with properties presented in Ref. [49]. With pore-throat radii, r , varying from 5 to 80 μm , and an average pore length, a , of approximately 200 μm , a ballpark estimate of the variation in invasion probability along one pore at the front is $|\nabla p|a = N(P_{\text{crit}})\Delta\rho ga \approx 7.5 \times 10^{-5}$ (considering $g = 10 \text{ m/s}^2$, $\Delta\rho = 1000 \text{ kg/m}^3$, a uniform $N(P_t)$, and $P_t = 2\gamma/r$, where $\gamma = 72 \text{ mN/m}$). This value is three orders of magnitude smaller than the limit of 0.075 identified in our simulations.

Still, it is crucial for the theory presented in Sec. II to be validated experimentally. While IP models have been successfully used to predict drainage in two-dimensional porous media [13,15,23,30,38,39], their ability to represent flow in three-dimensional porous media may be more limited. Also, assuming that no spatial correlations exist among capillary-pressure thresholds in real 3D porous media may not be entirely realistic [50]. If that is the case, corrections for correlated disorder should be incorporated in the presented theoretical framework [16,17]. Another interesting limitation to our analyses may arise in poorly connected 3D porous media. As reported by Tran *et al.* [51], 3D lattices with three nearest neighbors exhibit bond-percolation thresholds $p_c \approx 0.55$. Then, porous media with topology equivalent to these lattices could be percolated by only one phase at a time, similar to two-dimensional porous media. As a result, drainage front widths in such porous media may follow the scaling presented in Eq. (2).

VI. CONCLUSION

In this work, we presented a theoretical approach to estimate stable drainage front widths in three-dimensional random porous media under capillary and gravitational effects. Based on the infinite-cluster frontier in gradient percolation, we suggest that the extent of the interface between wetting and nonwetting phases during drainage exhibits a significantly more complex dependency on the porous medium structure and stabilizing pressure gradients in 3D than in 2D. This difference in behavior stems from the fact that there is a range of saturations in which two phases can percolate three-dimensional porous media together. In two dimensions, the wetting phase is trapped in clusters as the nonwetting phase reaches its percolation threshold, leading to narrower stable drainage fronts.

The theoretical prediction of the front widths proposed in Eq. (6) was tested with a bond invasion-percolation model, which incorporates gravitational effects through a linear gradient of capillary pressure acting at the front. A satisfactory match between theoretical and numerical results was achieved when the gradient of invasion probability along the pore throats at the front was not too high. As a following step to these analyses, it is important for our results to be experimentally verified.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: EFFECT OF THE OCCUPATION GRADIENT MAGNITUDE IN GRADIENT PERCOLATION

The theoretical estimate of stable drainage front widths in Eq. (6) is based on characteristics of the infinite cluster in gradient percolation in 3D [28,29]. As proposed by Gouyet *et al.* [29], the extent of the infinite-cluster front at $p < p_c$ is termed the front tail, and should scale with the occupation gradient ∇p as $\eta_t \propto |\nabla p|^{-\nu/(1+\nu)}$, where $\nu = 0.88$ is the percolation critical exponent for the correlation length in 3D. In this Appendix, we analyze a series of gradient-percolation cases to verify whether the magnitude of ∇p affects this scaling.

Similarly to our drainage simulations, diamond-cubic and simple-cubic networks are used. To match the size of the networks used in our IP model, $n_x = n_y = 100$ for the diamond-cubic networks, while $n_x = n_y = 50$ for simple-cubic networks. n_z varied from 6 to 250 to obtain a large range of $\nabla p = 1/n_z$ values.

In Fig. 11, we present the values of η_t calculated with Eq. (8). The numerical results are the average from 5000 random realizations of each type of network and value of ∇p . A theoretical-scaling curve is also shown in each plot, corresponding to the power-law coefficient of $\beta = -\nu/(1+\nu)$. We can see that the numerically obtained values of η_t follow reasonably well the theoretical scaling in the range of ∇p evaluated. While higher values of ∇p impair the scaling of the front tails measured in the diamond-cubic networks, the calculated η_t values still oscillate near the predicted values.

In Fig. 12, we present results from the same gradient-percolation cases as in Fig. 11. However, now the front tail is measured directly as the total extent of the infinite-cluster front in the region where $p < p_c$. Since in Eq. (6) we propose

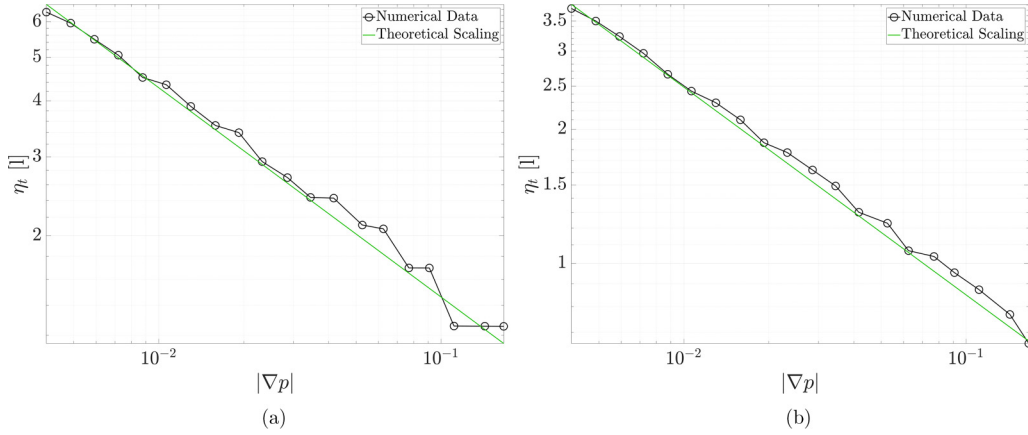


FIG. 11. Front-tail width scaling in gradient percolation. The presented numerical results are calculated with Eq. (8) and are the average values from 5000 realizations of random networks. The theoretical-scaling curve shows the expected exponent of $\beta = -\nu/(1+\nu)$, approximately equal to -0.47 in 3D. Widths are presented in bond units l . (a) Diamond-cubic network and (b) simple-cubic network.

an estimate for the total width of gradient-stabilized drainage fronts, we need to check whether the total extent of the tail front also satisfactorily scales as $|\nabla p|^{-\nu/(1+\nu)}$. For this, we simply calculate $\eta_t^* = z_{\eta_t}^{\max} - z_c$, where $z_{\eta_t}^{\max}$ is the maximum value of z among the front-tail bonds.

Unlike the front tail calculated with Eq. (8), the results in Fig. 12 suggest that the total extent of the front tails, η_t^* , scales with $|\nabla p|^{-\nu/(1+\nu)}$ only in low and moderate values of $|\nabla p|$. For both investigated network topologies, when $|\nabla p| \gtrsim 0.075$, directly measured values of the front tail fall way below the theoretical prediction. For this reason, in the results presented in Sec. IV, we suggest that our theoretical predictions of the front width are likely no longer valid above $|\nabla p|^{\text{lim}} \approx 0.075$.

In the following Appendix, we indicate how $|\nabla p|$ is estimated in our drainage IP simulations.

APPENDIX B: OCCUPATION GRADIENT MAGNITUDE IN GRADIENT-STABILIZED SLOW DRAINAGE IN POROUS MEDIA

In Appendix A, we suggest that for high occupation probability gradients in gradient percolation, the scaling

$|\nabla p|^{-\nu/(1+\nu)}$ for the total length of the infinite-cluster front tail may fail. Thus, an equivalent $|\nabla p|$ for our drainage simulations should be defined, so that we can estimate an appropriate range of simulation parameters for which our theoretical predictions are expected to be valid.

In the context of two-dimensional stable drainage, Eq. (4) was proposed by Birovljev *et al.* [13] to estimate the difference in invasion probability between two pore throats at the front separated by a distance Δz in the direction of the gradient. If we consider this distance to be equal to one pore size, the difference in invasion probability is equivalent to $\Delta p = \nabla p|_{p_c} a$ and given by

$$\Delta p = \int_{P_{\text{crit}}}^{P_{\text{crit}} + Ga} N(P_t) dP_t. \quad (\text{B1})$$

Approximating the solution of Eq. (B1) by the lowest-order term of the Taylor expansion and considering that in the case of stabilizing gravitational effects $G = -\Delta\rho g$, we get $\Delta p \approx -N(P_{\text{crit}})\Delta\rho ga$. As the variation in occupation probability in two-dimensional stable drainage fronts is narrow, we can consider this gradient to be approximately valid for the full front extent. However, 3D stable drainage fronts extend over

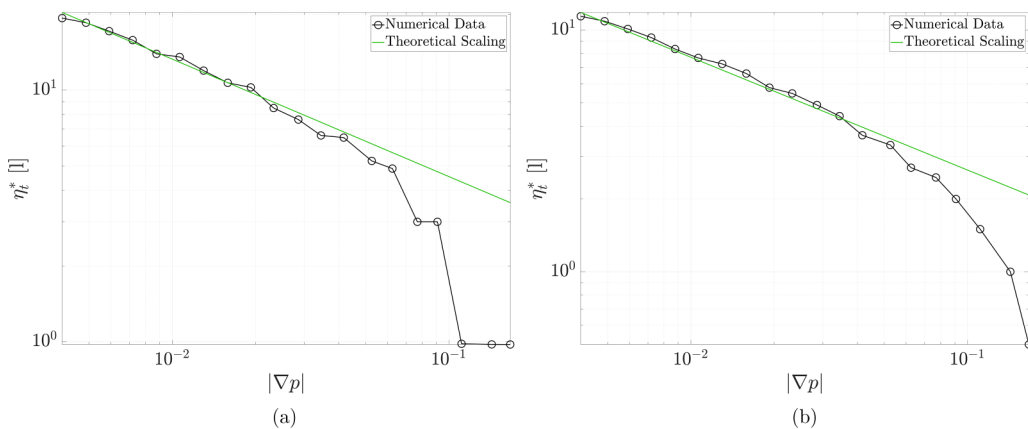


FIG. 12. Front-tail width in gradient percolation, directly measured as the portion of the front η_f where $p < p_c$: average values from 5000 realizations of random networks. Widths are presented in bond units l . (a) Diamond-cubic network and (b) simple-cubic network.

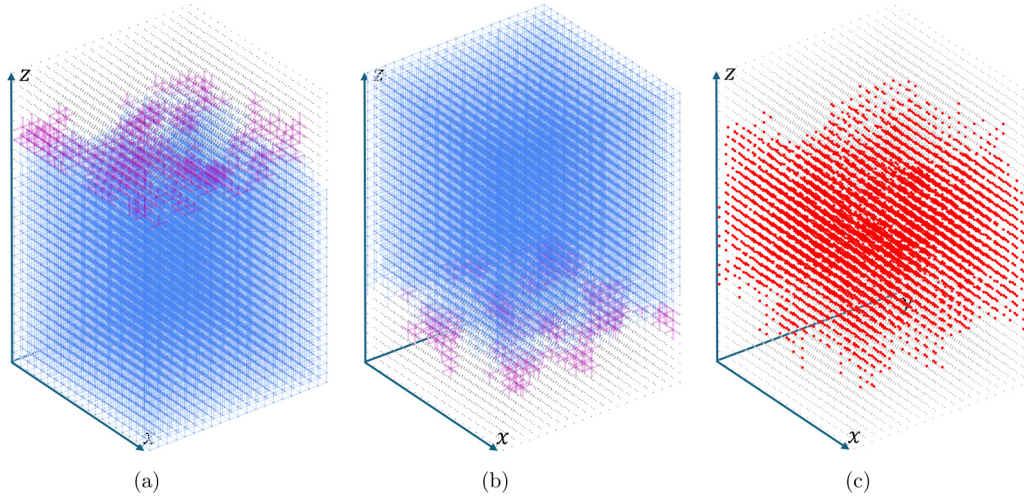


FIG. 13. Bond-percolation problem in a simple-cubic lattice ($p_c \approx 0.25$), where p varies linearly from 1 at $z = 0$ to 0 at $z = L$. (a) Infinite cluster of occupied bonds, with bonds at $z > (1 - p_c)L$ highlighted in purple, (b) “infinite cluster” of nonoccupied bonds, with bonds at $z < p_c L$ highlighted in purple, and (c) sites at the infinite-cluster frontier.

a large range of p , meaning that variation in ∇p may occur, if $N(P_t)$ is nonuniform. Still, since we simply aim to define an approximation for ∇p to estimate a validity range for drainage parameters, we adopt $|\Delta p| = N(P_{\text{crit}})\Delta\rho ga$.

Using this approximation, we can associate the limit gradient of occupation probability found in Appendix A, which represents the difference in p over one lattice unit, to the invasion probability difference over the length of a pore, leading to $|\Delta p|^{\text{lim}} = 0.075 = N(P_{\text{crit}})\Delta\rho ga$. Therefore, we suggest that results from drainage simulations with a density difference between the phases $\Delta\rho > 0.075/(N(P_{\text{crit}})ga)$ may not be comparable to the theoretical framework presented in Sec. II.

APPENDIX C: VALUES OF $\mathcal{C}$ FOR THE 3D STABLE DRAINAGE FRONT WIDTH ESTIMATE

In Appendix A, we investigate the effect of the magnitude of ∇p on the scaling of the front-tail total extent, η_t^* , in gradient percolation. There, we empirically define a limit for ∇p , below which our front-tail measurements follow $\eta_t^* \propto |\nabla p|^{-\nu/(1+\nu)}$. This limit is relevant to establish a range of drainage parameters within which our stable drainage front width estimate [see Eq. (6)] may be applicable.

To predict the front width, we take a step further and attempt to define a prefactor $\mathcal{C}$ that fulfills $\eta_t^* = \mathcal{C}|\nabla p|^{-\nu/(1+\nu)}$. While the front-tail scaling, based on percolation theory, is independent of the network topology, the results presented in Fig. 12 suggest that $\mathcal{C}$ should depend on it. Comparing the results of Figs. 12(a) and 12(b), we verify that the total extent of the front tail is larger using the diamond-cubic network than with the simple-cubic network.

Therefore, using the gradient-percolation results in Fig. 12 to fit the theoretical-scaling curves—shown as the green continuous lines—we estimate $\mathcal{C} = 0.90$ for the simple-cubic network and $\mathcal{C} = 1.55$ for the diamond-cubic lattice. This

range of values is in agreement with the correlation length scaling in percolation theory $\xi = \xi_0|p - p_c|^{-\nu}$, where ξ_0 should be on the order of unity [27]. Next, we present an argument as to why we adopt the same value of $\mathcal{C}$ as a prefactor for the scaling of both η_t and η_r .

As proposed in Sec. II, we use the frontier of the infinite cluster in gradient percolation as a proxy for the drainage front in porous media. Consider a 3D bond-percolation problem, in which the occupation of bonds in the network varies linearly from $p = 1$, at $z = 0$, to $p = 0$, at $z = L$. In this problem, there is an infinite cluster connected to $z = 0$, as illustrated in Fig. 13(a). In this image, the bonds at $z > (1 - p_c)L$ are highlighted in purple, indicating the region where the occupation is below the bond-percolation threshold. In the same problem, we can also imagine an infinite “cluster” of nonoccupied bonds, as illustrated in Fig. 13(b). In this image, the bonds at $z < p_c L$ are highlighted in purple, indicating the region where the fraction of nonoccupied bonds is below p_c . In a way, the clusters represented in Figs. 13(a) and 13(b) are the mirror image of each other, with similar structure and critical regions (highlighted in purple). In Fig. 13(c), we indicate in red the sites at the infinite-cluster frontier: The same frontier sites are obtained if we analyze the infinite cluster of occupied or nonoccupied bonds. In our analogy between drainage fronts and infinite-cluster frontiers in gradient percolation, these sites would represent the pores at the interface between the continuous wetting phase (occupied bonds) and the continuous nonwetting phase (nonoccupied bonds). Therefore, we expect both critical regions to be modeled with the same coefficients $\mathcal{C}$, as the structure of the “clusters” of occupied and nonoccupied bonds is interchangeable. Furthermore, the used values of $\mathcal{C}$ lead to a reasonable match between theoretical and numerical front widths, as presented in Sec. IV E. Still, a more rigorous way of defining $\mathcal{C}$ for Eq. (6) should be established in future studies.

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