

Recursive analytical technique for the quantum analysis of a Clauser-Horne-Shimony-Holt parameter reduction

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An analytic technique for the investigation of the combined impacts of multiple photon pairs and a non-negligible joint spectral density on the measurement of a Clauser-Horne-Shimony-Holt (CHSH) parameter is introduced. We demonstrate this technique for an asymmetric Gaussian-normal joint spectral density of spontaneous parametric down-converted photons using two different techniques for generating polarization entanglement. We find that the correlation of the joint spectral density can improve the CHSH parameter for certain ranges of the squeezing parameter when the polarization-entangled state is generated using a beam-splitter technique. Additionally, we find that the impacts due to the correlation of the joint spectral density are made worse when asymmetry is present in the joint spectral density.

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I. INTRODUCTION

Entanglement [1–5] is a vital resource for robust quantum technologies. However, as with most quantum phenomena, entanglement is fragile and subject to system impairments. To gauge the status of the entanglement in a system, various metrics for entanglement quality are used [2–4,6–11]. Of these metrics, one that is particularly useful and easy to measure is the Clauser-Horne-Shimony-Holt (CHSH) parameter S [3].

The CHSH parameter may be measured using the setup shown in Fig. 1 when entanglement is encoded in the polarization degree of freedom of a system of photons [3,5]. When the S parameter is greater than 2, entanglement is confirmed to be present in the system [3]. Additionally, quantum mechanics allows for CHSH parameters up to $S = 2\sqrt{2}$ [12]. A CHSH parameter less than $2\sqrt{2}$ under optimal test conditions and alignment criteria indicates the presence of nonidealities in the system leading to a submaximally entangled quantum state.

It is important to understand the relationship between the CHSH S parameter and specific defects in a quantum system, so that they may be more adequately mitigated or otherwise accounted for. To this end, we previously investigated the impacts that unintentional additional photon pairs have on the CHSH parameter [13], as well as the impact of a nonideal biphoton state joint spectral density [14], and all other impairments with impacts that can be completely characterized in the logical qubit basis, which we refer to as qubit-preserving impairments [15]. However, we have not yet considered the

combined effects of qubit-preserving impairments and multiple photon pairs. In particular, we are interested in the combined effects of multiple photon pairs and a more realistic joint spectral density. This situation is important because, currently, the leading source of polarization-entangled photons is spontaneous parametric down-conversion (SPDC)—although quantum dots hold the promise of one day providing more robust sources of polarization-entangled photon states [16–21]. Additionally, bright sources—such as those that rely on SPDC—are required in protocols and experiments in satellite quantum key distribution (QKD) [22–24]. SPDC inherently has the potential to produce more than a single photon pair and also has a non-negligible joint spectral density.

In this paper, we will investigate this situation. This case is particularly challenging because of the high dimensionality of the problem. The joint spectral density is a continuous frequency distribution. We must also consider that each frequency in this continuum may have a different, countably infinite number of possible photons. Possible approximations include truncating the permissible number of photons or discretizing the number of frequencies that need to be considered [25]; however, we want to avoid making such approximations. Instead, we introduce a recursive, analytical technique that allows us to simultaneously treat both the infinite number of photon pairs and the full joint spectral density of the SPDC source. This technique is based on the standard quantum optical techniques discussed in Ref. [26]. We will then apply this recursive technique to the two different methods for generating polarization entanglement with an SPDC source discussed in Ref. [13]. These are the beam-splitter method [27,28] shown in Fig. 2 and the Sagnac method [29–33] shown in Fig. 3. While the Sagnac method is the standard arrangement for modern commercial polarization entanglement sources, the beam-splitter method is easier to align, and therefore, important to consider for simpler experimental setups as well. We will then employ the recursive technique

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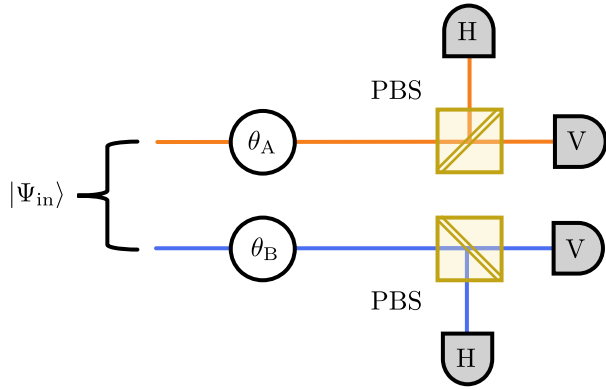


FIG. 1. The Clauser-Horne-Shimony-Holt experiment for entangled states encoded in the polarization degree of freedom of a system of photons. Each channel of the test state $|\Psi_{in}\rangle$ passes through wave plates in free space or polarization controllers in fibers that apply a rotation of the polarization state of each photon. These polarization rotations implement the Bell test angles needed to violate the CHSH inequality. To measure the polarization state of each photon, each channel passes through an ideal polarizing beam splitter (PBS). This converts the polarization degree of freedom into a spatial degree of freedom, which is read out by the detectors as shown.

using a simplified joint spectral density that will allow us to draw some conclusions about the fundamental physics of each of these setups. From this analysis, we observe significant corrections to our previous multipair CHSH results [13] when the spectral nature of the photons pairs is also incorporated, as we will see in the sections that follow.

This paper is structured as follows. In Sec. II, we review the CHSH experiment and introduce the theory that we will need for our technique. In Sec. III, we apply our technique to the beam-splitter method for generating entangled photon pairs in Sec. III A and to the Sagnac method in Sec. III B. In both of these sections, we will compute CHSH parameters for arbitrary joint spectral density and average number of photons. We then consider a simplified analytical joint spectral density in Sec. IV, first for a simplified symmetric case in Sec. IV A and then for the full asymmetric case in Sec. IV B. We will use this

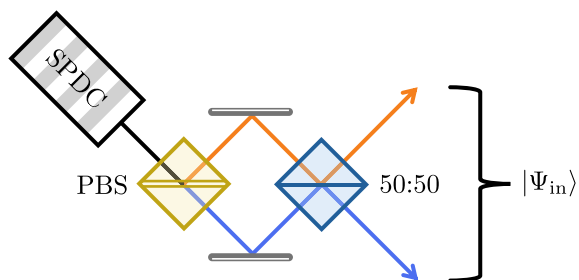


FIG. 2. A means for generating a $|\Psi^- \rangle$ Bell state using an SPDC source. The output state of this setup $|\Psi_{in}\rangle$ is the input state for Fig. 1. In the ideal case when a single photon pair is emitted from the type-II SPDC source, this setup will produce the $|\Psi^- \rangle$ Bell state encoded in the polarization degree of freedom of a system of photons 50% of the time. In practice, more than a single pair may be emitted by the source, and such pairs are spectrally distributed.

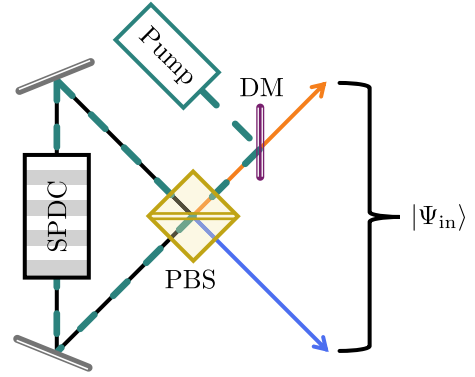


FIG. 3. A means for generating a $|\Psi^- \rangle$ Bell state using an SPDC source in a Sagnac-like configuration. A diagonally polarized pump is injected into the PBS using a dichroic mirror (DM) so that horizontal pump photons propagate one way around the loop and vertical pump photons propagate the other way. These two different propagation directions for the pump photons generate type-II SPDC pairs also propagating in both directions. If only a single pair is produced propagating in one direction or the other, a $|\Psi^- \rangle$ Bell state will be produced when this pair reaches the PBS. Unlike the beam-splitter method for generating entangled photon pairs, this state will be produced 100% of the time in the ideal case.

joint spectral density in the CHSH parameters that we derive in Sec. III to obtain analytical results in certain idealized situations in each case. Finally, we conclude in Sec. V.

II. THE CHSH EXPERIMENT

In this section, we consider the CHSH experiment of Fig. 1. In order to obtain the ideal CHSH parameter of $2\sqrt{2}$ from this setup, a maximally entangled single photon pair encoded in the polarization degree of freedom should be inserted as $|\Psi_{in}\rangle$. Furthermore, the members of the pair should not be entangled in any other degrees of freedom [14,15]. However, as outlined above, SPDC has the potential to produce multiple, spectrally dependent photon pairs. Not only do the additional photon pairs reduce the CHSH parameter [13], but also the spectral dependence can lead to spectrotemporal entanglement, which can naturally lead to a further reduced CHSH parameter [14,15]. We now must develop the fundamental theory of this experiment in order to be able to analyze it analytically.

To begin, we first consider a measurement of the CHSH parameter for a general, spectrally dependent, polarization-encoded, photonic state. In order to define the CHSH parameter, we must first define the Bell correlation [3]

$$E(\theta_A, \theta_B) \equiv \frac{R_{HH} - R_{HV} - R_{VH} + R_{VV}}{R_{HH} + R_{HV} + R_{VH} + R_{VV}}, \quad (1)$$

where R_{HH} , R_{HV} , R_{VH} , and R_{VV} are experimentally measured coincidence rates between detectors in the upper and lower paths of Fig. 1. For example, R_{XY} denotes the coincidence rate between the X detector in the upper path and the Y detector in the lower path, where X and Y may each individually be replaced with either H for the H detector in either path or V for the V detector. Here, H and V correspond to the horizontal and vertical linear polarization states, respectively. It is implied

that these are the photonic states output from the appropriate ports of the polarizing beam splitters shown in Fig. 1.

The coincidence rates R_{XY} implicitly depend on the test angles θ_A and θ_B . These test angles are physically implemented by the polarization rotations shown in Fig. 1. The CHSH parameter itself is measured by combining the results of a measurement of the correlation of Eq. (1) over four trials for four different combinations of the settings for the test angles θ_A and θ_B . This comes from considering two settings each for the two test angles. It is well known that choosing either 0 or $\pi/4$ for θ_A and either $\pi/8$ or $3\pi/8$ for θ_B yields the optimal value of the CHSH parameter for two of the four Bell states. Using these test angles in the Bell correlation of Eq. (1) and properly combining the trials gives the CHSH parameter as [3]

$$S \equiv E(0, \pi/8) - E(0, 3\pi/8) + E(\pi/4, \pi/8) + E(\pi/4, 3\pi/8), \quad (2)$$

which yields the optimal value $|S| = 2\sqrt{2}$ for the maximally entangled $|\Phi^+\rangle$ and $|\Psi^-\rangle$ Bell states. While Eqs. (1) and (2) demonstrate how to combine the measurable coincidence rates for each of the combinations of the test angles to compute an experimental result for the CHSH parameter, we now need to use these same expressions to compute the CHSH parameter using analytically derived coincidence rates that would result from using these test angles.

We will compute the CHSH parameter by evolving the observable operators corresponding to the coincidence rates back to the input state of Fig. 1 according to the Heisenberg picture. The coincidence rates may be analytically computed for a general, spectrally dependent state as

$$R_{HH} \equiv \int d\Omega d\Omega' \langle \tilde{A}_H^\dagger(\Omega) \tilde{A}_H(\Omega) \tilde{B}_H^\dagger(\Omega') \tilde{B}_H(\Omega') \rangle, \quad (3)$$

$$R_{HV} \equiv \int d\Omega d\Omega' \langle \tilde{A}_H^\dagger(\Omega) \tilde{A}_H(\Omega) \tilde{B}_V^\dagger(\Omega') \tilde{B}_V(\Omega') \rangle, \quad (4)$$

$$R_{VH} \equiv \int d\Omega d\Omega' \langle \tilde{A}_V^\dagger(\Omega) \tilde{A}_V(\Omega) \tilde{B}_H^\dagger(\Omega') \tilde{B}_H(\Omega') \rangle, \quad (5)$$

$$R_{VV} \equiv \int d\Omega d\Omega' \langle \tilde{A}_V^\dagger(\Omega) \tilde{A}_V(\Omega) \tilde{B}_V^\dagger(\Omega') \tilde{B}_V(\Omega') \rangle, \quad (6)$$

where the operators $\tilde{A}(\omega)$ and $\tilde{B}(\omega)$ are the spectrally dependent multimode annihilation operators for the two spatially distinct channels of Fig. 1 and the subscript denotes the polarization state. The PBSs of Fig. 1 are taken to be ideal, so we may ignore their effects. We now consider the polarization rotations that apply the test angles θ_A and θ_B . Analytically, we may let these angles be general to consider an arbitrary trial of the experiment and insert the appropriate values according to Eq. (2) to compute the final CHSH parameter at the end of the calculation. In the Heisenberg picture, the polarization rotations are treated using the following transformations to the field operators:

$$\hat{U}_A^\dagger \tilde{A}_H^\dagger(\omega) \hat{U}_A = \cos \theta_A \tilde{A}_H^\dagger(\omega) - \sin \theta_A \tilde{A}_V^\dagger(\omega), \quad (7)$$

$$\hat{U}_A^\dagger \tilde{A}_V^\dagger(\omega) \hat{U}_A = \cos \theta_A \tilde{A}_V^\dagger(\omega) + \sin \theta_A \tilde{A}_H^\dagger(\omega), \quad (8)$$

$$\hat{U}_B^\dagger \tilde{B}_H^\dagger(\omega) \hat{U}_B = \cos \theta_B \tilde{B}_H^\dagger(\omega) - \sin \theta_B \tilde{B}_V^\dagger(\omega), \quad (9)$$

$$\hat{U}_B^\dagger \tilde{B}_V^\dagger(\omega) \hat{U}_B = \cos \theta_B \tilde{B}_V^\dagger(\omega) + \sin \theta_B \tilde{B}_H^\dagger(\omega), \quad (10)$$

where $\hat{U}_A$ is the unitary evolution operator for the polarization rotation in path A and similarly for $\hat{U}_B$ in path B. Applying these transformations to Eqs. (3)–(6), and after a series of algebraic steps, we find that the denominator of Eq. (1) can be written as

$$\begin{aligned} R_{HH} + R_{HV} + R_{VH} + R_{VV} \\ = \int d\Omega d\Omega' \langle [\tilde{A}_H^\dagger(\Omega) \tilde{A}_H(\Omega) + \tilde{A}_V^\dagger(\Omega) \tilde{A}_V(\Omega)] \\ \times [\tilde{B}_H^\dagger(\Omega') \tilde{B}_H(\Omega') + \tilde{B}_V^\dagger(\Omega') \tilde{B}_V(\Omega')] \rangle. \end{aligned} \quad (11)$$

Similarly, the numerator becomes

$$\begin{aligned} R_{HH} - R_{HV} - R_{VH} + R_{VV} \\ = \int d\Omega d\Omega' \langle \{ \cos 2\theta_A [\tilde{A}_H^\dagger(\Omega) \tilde{A}_H(\Omega) - \tilde{A}_V^\dagger(\Omega) \tilde{A}_V(\Omega)] \\ - \sin 2\theta_A [\tilde{A}_H^\dagger(\Omega) \tilde{A}_V(\Omega) + \tilde{A}_V^\dagger(\Omega) \tilde{A}_H(\Omega)] \} \\ \times \{ \cos 2\theta_B [\tilde{B}_H^\dagger(\Omega') \tilde{B}_H(\Omega') - \tilde{B}_V^\dagger(\Omega') \tilde{B}_V(\Omega')] \\ - \sin 2\theta_B [\tilde{B}_H^\dagger(\Omega') \tilde{B}_V(\Omega') + \tilde{B}_V^\dagger(\Omega') \tilde{B}_H(\Omega')] \} \rangle. \end{aligned} \quad (12)$$

Note that Eq. (11) does not depend upon the test angles θ_A and θ_B . We now define the parameter

$$N \equiv R_{HH} + R_{HV} + R_{VH} + R_{VV}. \quad (13)$$

Inserting this parameter and Eq. (12) into Eq. (1) and then using the Bell test angles and Eq. (2) gives the CHSH parameter in the general case as

$$\begin{aligned} S = \frac{\sqrt{2}}{N} \int d\Omega d\Omega' \langle [\tilde{A}_H^\dagger(\Omega) \tilde{A}_H(\Omega) - \tilde{A}_V^\dagger(\Omega) \tilde{A}_V(\Omega)] \\ \times [\tilde{B}_H^\dagger(\Omega') \tilde{B}_H(\Omega') - \tilde{B}_V^\dagger(\Omega') \tilde{B}_V(\Omega')] \\ + [\tilde{A}_H^\dagger(\Omega) \tilde{A}_V(\Omega) + \tilde{A}_V^\dagger(\Omega) \tilde{A}_H(\Omega)] \\ \times [\tilde{B}_H^\dagger(\Omega') \tilde{B}_V(\Omega') + \tilde{B}_V^\dagger(\Omega') \tilde{B}_H(\Omega')] \rangle. \end{aligned} \quad (14)$$

Inspection of Eqs. (11) and (14) reveals that each term of S and N are of the form

$$S_{PXY} \equiv \int d\Omega d\Omega' \langle \tilde{A}_P^\dagger(\Omega) \tilde{A}_X(\Omega) \tilde{B}_Q^\dagger(\Omega') \tilde{B}_Y(\Omega') \rangle, \quad (15)$$

where P, Q, X, and Y are to be individually replaced with either H or V. Thus,

$$\begin{aligned} S = \frac{\sqrt{2}}{N} (S_{HHHH} - S_{HHVV} - S_{VVHH} + S_{VVVV} \\ + S_{HVHV} + S_{HVVH} + S_{VHHV} + S_{VHHV}), \end{aligned} \quad (16)$$

$$N = S_{HHHH} + S_{HHVV} + S_{VVHH} + S_{VVVV}. \quad (17)$$

This formalism will expedite the calculation of the CHSH parameter in the following sections of this paper.

III. THE RECURSIVE TECHNIQUE FOR COMPUTING THE CHSH PARAMETER

A. The Beam-Splitter method for generating entangled photon pairs

Now that we have the equations needed to calculate the CHSH parameter, we calculate it for an entangled state generated by the SPDC process. We consider a $|\Psi^-\rangle$ Bell state generated using the beam-splitter method shown in Fig. 2. As in the previous section, we will again use the Heisenberg picture. To do this, we will need the unitary evolution operator for the beam splitter $\hat{U}_{\text{BS}}$ and the unitary evolution operator for the combined action of the SPDC source and the PBS. Let the latter evolution operator be represented by the two-mode squeezing operator [14,26]

$$\hat{U}_{\text{sq}} = \exp \left\{ -s \int d\omega d\omega' \tilde{f}(\omega, \omega') \times [\tilde{A}_H^\dagger(\omega) \tilde{B}_V^\dagger(\omega') - \tilde{A}_H(\omega) \tilde{B}_V(\omega')] \right\}, \quad (18)$$

where $\tilde{f}(\omega, \omega')$ is the joint spectral density of the photon pairs emitted from the SPDC source and s is the squeezing parameter directly related to the average number of pairs N_p generated as $N_p = \sinh^2 s$ [13].

Recall that the ideal CHSH experiment of Fig. 1 requires only a single photon pair. Therefore, since SPDC always has the potential for producing additional pairs, it is desirable to keep the average number of pairs low $N_p \lesssim 0.1$, for example. From the relation above, this corresponds to $s \lesssim 0.3$. However, s can be $\gtrsim 2$ [34,35]. Additionally, larger s values may be important for satellite QKD applications [22–24], so we will let s go up to 2 in our analysis with the understanding that less than about 0.3 are the typical values in the ideal case.

For the ideal beam splitter, it is well known that the operator $\hat{U}_{\text{BS}}$ performs the transformations

$$\hat{U}_{\text{BS}}^\dagger \tilde{A}_X^\dagger(\Omega) \hat{U}_{\text{BS}} = \frac{1}{\sqrt{2}} [\tilde{A}_X^\dagger(\Omega) - i \tilde{B}_X^\dagger(\Omega)], \quad (19)$$

$$\hat{U}_{\text{BS}}^\dagger \tilde{B}_X^\dagger(\Omega) \hat{U}_{\text{BS}} = \frac{1}{\sqrt{2}} [\tilde{B}_X^\dagger(\Omega) - i \tilde{A}_X^\dagger(\Omega)], \quad (20)$$

in the Heisenberg picture. In Eqs. (19) and (20), X is to be replaced with either H or V for the appropriate polarization state.

We now consider similar transformations to Eqs. (19) and (20) for the two-mode squeezing operator of Eq. (18). It is not difficult to see that

$$\hat{U}_{\text{sq}}^\dagger \tilde{A}_V^\dagger(\Omega) \hat{U}_{\text{sq}} = \tilde{A}_V^\dagger(\Omega), \quad (21)$$

$$\hat{U}_{\text{sq}}^\dagger \tilde{B}_H^\dagger(\Omega) \hat{U}_{\text{sq}} = \tilde{B}_H^\dagger(\Omega), \quad (22)$$

by inspection of Eq. (18). We now need to compute $\hat{U}_{\text{sq}}^\dagger \tilde{A}_H^\dagger(\Omega) \hat{U}_{\text{sq}}$ and $\hat{U}_{\text{sq}}^\dagger \tilde{B}_V^\dagger(\Omega) \hat{U}_{\text{sq}}$. As is typically done, we will use the Baker-Campbell-Hausdorff (BCH) lemma to compute these quantities. We introduce the notation

$$[\hat{X}, \hat{Y}]_0 \equiv \hat{Y}, \quad (23)$$

$$[\hat{X}, \hat{Y}]_1 \equiv [\hat{X}, \hat{Y}], \quad (24)$$

$$[\hat{X}, \hat{Y}]_n \equiv [\hat{X}, [\hat{X}, \hat{Y}]_{n-1}], \quad \text{for } n > 1, \quad (25)$$

so that the BCH lemma may be written as

$$e^{\hat{X}} \hat{Y} e^{-\hat{X}} = \sum_{j=0}^{\infty} \frac{1}{j!} [\hat{X}, \hat{Y}]_j. \quad (26)$$

Before continuing, however, we first define the following coupled, recursively defined distributions:

$$F_n(\Omega, \omega) \equiv \int d\nu G_{n-1}(\Omega, \nu) \tilde{f}(\omega, \nu), \quad (27)$$

$$G_n(\Omega, \omega) \equiv \int d\nu F_{n-1}(\Omega, \nu) \tilde{f}(\nu, \omega), \quad (28)$$

where $\tilde{f}(\omega, \omega')$ is the same joint spectral distribution of Eq. (18) and we have the base case

$$F_0(\Omega, \omega) = G_0(\Omega, \omega) \equiv \delta(\Omega - \omega). \quad (29)$$

With Eqs. (27)–(29), we may now use Eq. (26) to compute $\hat{U}_{\text{sq}}^\dagger \tilde{A}_H^\dagger(\Omega) \hat{U}_{\text{sq}}$ and $\hat{U}_{\text{sq}}^\dagger \tilde{B}_V^\dagger(\Omega) \hat{U}_{\text{sq}}$. Let us begin with $\hat{U}_{\text{sq}}^\dagger \tilde{A}_H^\dagger(\Omega) \hat{U}_{\text{sq}}$. Comparison of Eqs. (18) and (26) reveals that we identify

$$\hat{X} = s \int d\omega d\omega' \tilde{f}(\omega, \omega') [\tilde{A}_H^\dagger(\omega) \tilde{B}_V^\dagger(\omega') - \tilde{A}_H(\omega) \tilde{B}_V(\omega')], \quad (30)$$

$$\begin{aligned} \hat{Y} &= \tilde{A}_H^\dagger(\Omega) = \int d\omega \delta(\omega - \Omega) \tilde{A}_H^\dagger(\omega) \\ &= \int d\omega F_0(\Omega, \omega) \tilde{A}_H^\dagger(\omega). \end{aligned} \quad (31)$$

Computing the commutator in the $j = 1$ term of Eq. (26) yields

$$\begin{aligned} [\hat{X}, \hat{Y}]_1 &\equiv [\hat{X}, \hat{Y}] = -s \int d\omega d\omega' d\nu F_0(\Omega, \nu) \tilde{f}(\omega, \omega') \\ &\quad \times [\tilde{A}_H^\dagger(\omega), \tilde{A}_H^\dagger(\nu)] \tilde{B}_V(\omega') \\ &= -s \int d\omega d\omega' d\nu F_0(\Omega, \nu) \tilde{f}(\omega, \omega') \\ &\quad \times \delta(\omega - \nu) \tilde{B}_V(\omega') \\ &= -s \int d\omega d\nu F_0(\Omega, \nu) \tilde{f}(\nu, \omega) \tilde{B}_V(\omega), \end{aligned} \quad (32)$$

where we have performed the integral over ω and dropped the prime on the remaining ω' to obtain the final result. Using the definition of Eq. (28) in Eq. (32) yields

$$[\hat{X}, \hat{Y}]_1 = -s \int d\omega G_1(\Omega, \omega) \tilde{B}_V(\omega). \quad (33)$$

Similarly, using Eq. (27), it is not difficult to show

$$[\hat{X}, \hat{Y}]_2 \equiv [\hat{X}, [\hat{X}, \hat{Y}]] = s^2 \int d\omega F_2(\Omega, \omega) \tilde{A}_H^\dagger(\omega). \quad (34)$$

Given Eqs. (33) and (34), we are beginning to see the pattern

$$[\hat{X}, \hat{Y}]_j = \begin{cases} s^j \int d\omega F_j(\Omega, \omega) \tilde{A}_H^\dagger(\omega), & \text{for even } j, \\ -s^j \int d\omega G_j(\Omega, \omega) \tilde{B}_V(\omega), & \text{for odd } j. \end{cases} \quad (35)$$

Using Eq. (35), we may express the result of $\hat{U}_{\text{sq}}^\dagger \tilde{\mathbf{A}}_H^\dagger(\Omega) \hat{U}_{\text{sq}}$ as

$$\hat{U}_{\text{sq}}^\dagger \tilde{\mathbf{A}}_H^\dagger(\Omega) \hat{U}_{\text{sq}} = \int d\omega \sum_{j=0}^{\infty} s^{2j} \left[\frac{1}{(2j)!} F_{2j}(\Omega, \omega) \tilde{\mathbf{A}}_H^\dagger(\omega) - \frac{s}{(2j+1)!} G_{2j+1}(\Omega, \omega) \tilde{\mathbf{B}}_V(\omega) \right]. \quad (36)$$

For the sake of brevity, consider the following expressions:

$$\mathcal{C}^{(F)}(s, \Omega, \omega) \equiv \sum_{j=0}^{\infty} \frac{s^{2j}}{(2j)!} F_{2j}(\Omega, \omega), \quad (37)$$

$$\mathcal{S}^{(G)}(s, \Omega, \omega) \equiv \sum_{j=0}^{\infty} \frac{s^{2j+1}}{(2j+1)!} G_{2j+1}(\Omega, \omega). \quad (38)$$

Using Eqs. (37) and (38) in Eq. (36) simplifies $\hat{U}_{\text{sq}}^\dagger \tilde{\mathbf{A}}_H^\dagger(\Omega) \hat{U}_{\text{sq}}$ as

$$\hat{U}_{\text{sq}}^\dagger \tilde{\mathbf{A}}_H^\dagger(\Omega) \hat{U}_{\text{sq}} = \int d\omega [\mathcal{C}^{(F)}(s, \Omega, \omega) \tilde{\mathbf{A}}_H^\dagger(\omega) - \mathcal{S}^{(G)}(s, \Omega, \omega) \tilde{\mathbf{B}}_V(\omega)]. \quad (39)$$

Now that we have an expression for $\hat{U}_{\text{sq}}^\dagger \tilde{\mathbf{A}}_H^\dagger(\Omega) \hat{U}_{\text{sq}}$, we can compute $\hat{U}_{\text{sq}}^\dagger \tilde{\mathbf{B}}_V^\dagger(\Omega) \hat{U}_{\text{sq}}$ using the BCH lemma of Eq. (26). The operator $\hat{X}$ is the same as it was in Eq. (30), while $\hat{Y}$ is now

$$\begin{aligned} \hat{Y} &= \tilde{\mathbf{B}}_V(\Omega) = \int d\omega \delta(\omega - \Omega) \tilde{\mathbf{B}}_V^\dagger(\omega) \\ &= \int d\omega G_0(\Omega, \omega) \tilde{\mathbf{B}}_V^\dagger(\omega). \end{aligned} \quad (40)$$

The $j = 1$ term is computed as

$$\begin{aligned} \llbracket \hat{X}, \hat{Y} \rrbracket_1 &\equiv [\hat{X}, \hat{Y}] = -s \int d\omega d\omega' d\nu G_0(\Omega, \nu) \tilde{f}(\omega, \omega') \\ &\quad \times \tilde{\mathbf{A}}_H(\omega) [\tilde{\mathbf{B}}_V(\omega'), \tilde{\mathbf{B}}_V^\dagger(\nu)] \\ &= -s \int d\omega d\omega' d\nu G_0(\Omega, \nu) \tilde{f}(\omega, \omega') \\ &\quad \times \delta(\omega' - \nu) \tilde{\mathbf{A}}_H(\omega) \\ &= -s \int d\omega d\nu G_0(\Omega, \nu) \tilde{f}(\omega, \nu) \tilde{\mathbf{A}}_H(\omega) \\ &= -s \int d\omega F_1(\Omega, \omega) \tilde{\mathbf{A}}_H(\omega). \end{aligned} \quad (41)$$

Similarly, for the next term, we have

$$\llbracket \hat{X}, \hat{Y} \rrbracket_2 = s^2 \int d\omega G_2(\Omega, \omega) \tilde{\mathbf{B}}_V^\dagger(\omega). \quad (42)$$

We once again see a pattern given as

$$\llbracket \hat{X}, \hat{Y} \rrbracket_j = \begin{cases} s^j \int d\omega G_j(\Omega, \omega) \tilde{\mathbf{B}}_V^\dagger(\omega), & \text{for even } j, \\ -s^j \int d\omega F_j(\Omega, \omega) \tilde{\mathbf{A}}_H(\omega), & \text{for odd } j. \end{cases} \quad (43)$$

Using Eq. (43) yields $\hat{U}_{\text{sq}}^\dagger \tilde{\mathbf{B}}_V^\dagger(\Omega) \hat{U}_{\text{sq}}$ as

$$\hat{U}_{\text{sq}}^\dagger \tilde{\mathbf{B}}_V^\dagger(\Omega) \hat{U}_{\text{sq}} = \int d\omega \sum_{j=0}^{\infty} s^{2j} \left[\frac{1}{(2j)!} G_{2j}(\Omega, \omega) \tilde{\mathbf{B}}_V^\dagger(\omega) - \frac{s}{(2j+1)!} F_{2j+1}(\Omega, \omega) \tilde{\mathbf{A}}_H(\omega) \right]. \quad (44)$$

As in Eqs. (37) and (38), we now define

$$\mathcal{C}^{(G)}(s, \Omega, \omega) \equiv \sum_{j=0}^{\infty} \frac{s^{2j}}{(2j)!} G_{2j}(\Omega, \omega), \quad (45)$$

$$\mathcal{S}^{(F)}(s, \Omega, \omega) \equiv \sum_{j=0}^{\infty} \frac{s^{2j+1}}{(2j+1)!} F_{2j+1}(\Omega, \omega). \quad (46)$$

Using these expressions simplifies Eq. (44) as

$$\hat{U}_{\text{sq}}^\dagger \tilde{\mathbf{B}}_V^\dagger(\Omega) \hat{U}_{\text{sq}} = \int d\omega [\mathcal{C}^{(G)}(s, \Omega, \omega) \tilde{\mathbf{B}}_V^\dagger(\omega) - \mathcal{S}^{(F)}(s, \Omega, \omega) \tilde{\mathbf{A}}_H(\omega)]. \quad (47)$$

Now that we have Eqs. (19)–(22), (39), and (47), we can compute the parameters S_{PQXY} needed to compute the CHSH parameter of Eqs. (16) and (17). We only need to compute eight such parameters: S_{HHHH} , S_{HHVV} , S_{VVHH} , S_{VVVV} , S_{HVHV} , S_{HVHH} , S_{VHHV} , and S_{VHHH} . The results of this calculation may be summarized as

$$\begin{aligned} S_{\text{HHHH}} &= \frac{1}{4} \int d\Omega d\Omega' d\omega d\omega' \\ &\quad \times [|\mathcal{S}^{(G)}(s, \Omega, \omega)|^2 |\mathcal{S}^{(G)}(s, \Omega', \omega')|^2 \\ &\quad + \mathcal{C}^{(F)*}(s, \Omega, \omega) \mathcal{C}^{(F)}(s, \Omega', \omega) \\ &\quad \times \mathcal{S}^{(G)*}(s, \Omega', \omega') \mathcal{S}^{(G)}(s, \Omega, \omega')] \\ &\quad - \frac{1}{4} \int d\Omega d\omega |\mathcal{S}^{(G)}(s, \Omega, \omega)|^2, \end{aligned} \quad (48)$$

$$\begin{aligned} S_{\text{HHVV}} &= \frac{1}{4} \int d\Omega d\Omega' d\omega d\omega' \\ &\quad \times [|\mathcal{S}^{(G)}(s, \Omega, \omega)|^2 |\mathcal{S}^{(F)}(s, \Omega', \omega')|^2 \\ &\quad + \mathcal{C}^{(F)*}(s, \Omega, \omega) \mathcal{C}^{(G)}(s, \Omega', \omega') \\ &\quad \times \mathcal{S}^{(F)*}(s, \Omega', \omega) \mathcal{S}^{(G)}(s, \Omega, \omega')], \end{aligned} \quad (49)$$

$$\begin{aligned} S_{\text{VVHH}} &= \frac{1}{4} \int d\Omega d\Omega' d\omega d\omega' \\ &\quad \times [|\mathcal{S}^{(F)}(s, \Omega, \omega)|^2 |\mathcal{S}^{(G)}(s, \Omega', \omega')|^2 \\ &\quad + \mathcal{C}^{(G)*}(s, \Omega, \omega) \mathcal{C}^{(F)}(s, \Omega', \omega') \\ &\quad \times \mathcal{S}^{(G)*}(s, \Omega', \omega) \mathcal{S}^{(F)}(s, \Omega, \omega')], \end{aligned} \quad (50)$$

$$\begin{aligned} S_{\text{VVVV}} &= \frac{1}{4} \int d\Omega d\Omega' d\omega d\omega' \\ &\quad \times [|\mathcal{S}^{(F)}(s, \Omega, \omega)|^2 |\mathcal{S}^{(F)}(s, \Omega', \omega')|^2 \\ &\quad + \mathcal{C}^{(G)*}(s, \Omega, \omega) \mathcal{C}^{(G)}(s, \Omega', \omega) \end{aligned}$$

$$\begin{aligned} & \times \mathcal{S}^{(F)*}(s, \Omega', \omega') \mathcal{S}^{(F)}(s, \Omega, \omega') \\ & - \frac{1}{4} \int d\Omega d\omega |\mathcal{S}^{(F)}(s, \Omega, \omega)|^2, \end{aligned} \quad (51)$$

$$S_{\text{HVHV}} = 0, \quad (52)$$

$$\begin{aligned} S_{\text{HVVH}} &= \frac{1}{4} \int d\Omega d\omega |\mathcal{S}^{(G)}(s, \Omega, \omega)|^2 \\ & - \frac{1}{4} \int d\Omega d\Omega' d\omega d\omega' \\ & \times [\mathcal{C}^{(G)*}(s, \Omega, \omega) \mathcal{C}^{(G)}(s, \Omega', \omega) \\ & \times \mathcal{S}^{(G)*}(s, \Omega', \omega') \mathcal{S}^{(G)}(s, \Omega, \omega') \\ & + \mathcal{C}^{(G)*}(s, \Omega, \omega) \mathcal{C}^{(G)}(s, \Omega', \omega') \\ & \times \mathcal{S}^{(G)*}(s, \Omega', \omega) \mathcal{S}^{(G)}(s, \Omega, \omega')], \end{aligned} \quad (53)$$

$$\begin{aligned} S_{\text{VHHV}} &= \frac{1}{4} \int d\Omega d\omega |\mathcal{S}^{(F)}(s, \Omega, \omega)|^2 \\ & - \frac{1}{4} \int d\Omega d\Omega' d\omega d\omega' \\ & \times [\mathcal{C}^{(F)*}(s, \Omega, \omega) \mathcal{C}^{(F)}(s, \Omega', \omega) \\ & \times \mathcal{S}^{(F)*}(s, \Omega', \omega') \mathcal{S}^{(F)}(s, \Omega, \omega') \\ & + \mathcal{C}^{(F)*}(s, \Omega, \omega) \mathcal{C}^{(F)}(s, \Omega', \omega') \\ & \times \mathcal{S}^{(F)*}(s, \Omega', \omega) \mathcal{S}^{(F)}(s, \Omega, \omega')], \end{aligned} \quad (54)$$

$$S_{\text{VHVH}} = 0. \quad (55)$$

We can now combine each of these intermediate parameters to compute N and the CHSH parameter. Using Eqs. (48)–(55) in Eqs. (16) and (17) and after some algebraic steps and changes of variables, we obtain

$$\begin{aligned} N &= \frac{1}{4} \int d\Omega d\Omega' d\omega d\omega' \{ [|\mathcal{S}^{(F)}(s, \Omega, \omega)|^2 + |\mathcal{S}^{(G)}(s, \Omega, \omega)|^2] [|\mathcal{S}^{(F)}(s, \Omega', \omega')|^2 + |\mathcal{S}^{(G)}(s, \Omega', \omega')|^2] \\ & + [\mathcal{C}^{(F)*}(s, \Omega, \omega) \mathcal{S}^{(G)}(s, \Omega, \omega') + \mathcal{C}^{(G)*}(s, \Omega, \omega') \mathcal{S}^{(F)}(s, \Omega, \omega)] \\ & \times [\mathcal{S}^{(G)*}(s, \Omega', \omega') \mathcal{C}^{(F)}(s, \Omega', \omega) + \mathcal{S}^{(F)*}(s, \Omega', \omega) \mathcal{C}^{(G)}(s, \Omega', \omega')] \} \\ & - \int d\Omega d\omega [|\mathcal{S}^{(F)}(s, \Omega, \omega)|^2 + |\mathcal{S}^{(G)}(s, \Omega, \omega)|^2], \end{aligned} \quad (56)$$

$$\begin{aligned} S &= \frac{\sqrt{2}}{4N} \int d\Omega d\Omega' d\omega d\omega' \{ [|\mathcal{S}^{(F)}(s, \Omega, \omega)|^2 - |\mathcal{S}^{(G)}(s, \Omega, \omega)|^2] [|\mathcal{S}^{(F)}(s, \Omega', \omega')|^2 - |\mathcal{S}^{(G)}(s, \Omega', \omega')|^2] \\ & + [\mathcal{C}^{(F)*}(s, \Omega, \omega) \mathcal{S}^{(G)}(s, \Omega, \omega') - \mathcal{C}^{(G)*}(s, \Omega, \omega') \mathcal{S}^{(F)}(s, \Omega, \omega)] \\ & \times [\mathcal{S}^{(G)*}(s, \Omega', \omega') \mathcal{C}^{(F)}(s, \Omega', \omega) - \mathcal{S}^{(F)*}(s, \Omega', \omega) \mathcal{C}^{(G)}(s, \Omega', \omega')] \\ & - \mathcal{C}^{(F)*}(s, \Omega, \omega) \mathcal{S}^{(F)}(s, \Omega, \omega') [\mathcal{S}^{(F)*}(s, \Omega', \omega') \mathcal{C}^{(F)}(s, \Omega', \omega) + \mathcal{S}^{(F)*}(s, \Omega', \omega) \mathcal{C}^{(F)}(s, \Omega', \omega')] \\ & - \mathcal{C}^{(G)*}(s, \Omega, \omega) \mathcal{S}^{(G)}(s, \Omega, \omega') [\mathcal{S}^{(G)*}(s, \Omega', \omega') \mathcal{C}^{(G)}(s, \Omega', \omega) + \mathcal{S}^{(G)*}(s, \Omega', \omega) \mathcal{C}^{(G)}(s, \Omega', \omega')] \}. \end{aligned} \quad (57)$$

Equation (57), along with Eq. (56), is the main result of this section. It represents the CHSH parameter for a $|\Psi^-\rangle$ Bell state generated using an SPDC source in the beam-splitter configuration of Fig. 2 for an arbitrary joint spectral distribution. The calculation of these expressions relies on computing the sums of Eqs. (37), (38), (45), and (46), which in turn rely on the calculation of the recursively defined distributions of Eqs. (27)–(29). At first glance, it may seem impossible to compute all of the requisite distributions for these sums for an arbitrary distribution. However, we will later show in Sec. IV that it is possible to compute these distributions in closed form for a certain, simplified joint spectral density. Despite the simple nature of this distribution, it is still useful for drawing conclusions about the fundamental physics of this system.

B. The sagnac method for generating entangled photon pairs

In this section, we compute the CHSH parameter for entangled states generated using the Sagnac method of Fig. 3 irrespective of the joint spectral density of the source. That is,

we compute expressions analogous to Eqs. (56) and (57) for this setup.

Inspection of Fig. 3 reveals that we no longer need to concern ourselves with the operator for the beam splitter $\hat{U}_{\text{BS}}$ of the previous section. However, because we have pump photons propagating in both directions of the Sagnac loop, we now have two two-mode squeezing operators [13,26]

$$\begin{aligned} \hat{U}_{\text{sq1}} &= \exp \left\{ -s \int d\omega d\omega' \tilde{f}(\omega, \omega') \right. \\ & \quad \times [\tilde{\mathbf{A}}_{\text{H}}^\dagger(\omega) \tilde{\mathbf{B}}_{\text{V}}^\dagger(\omega') - \tilde{\mathbf{A}}_{\text{H}}(\omega) \tilde{\mathbf{B}}_{\text{V}}(\omega')] \}, \end{aligned} \quad (58)$$

$$\begin{aligned} \hat{U}_{\text{sq2}} &= \exp \left\{ s \int d\omega d\omega' \tilde{f}(\omega, \omega') \right. \\ & \quad \times [\tilde{\mathbf{A}}_{\text{V}}^\dagger(\omega') \tilde{\mathbf{B}}_{\text{H}}^\dagger(\omega) - \tilde{\mathbf{A}}_{\text{V}}(\omega') \tilde{\mathbf{B}}_{\text{H}}(\omega)] \}, \end{aligned} \quad (59)$$

where each operator treats the interaction of the SPDC source and the PBS. The first operator of Eq. (58) is analytically the same operator as Eq. (18), but now represents pairs being

generated from pump photons traveling in one direction about the Sagnac loop. The second operator represents pairs generated using pump photons traveling in the opposite direction. The presence of the PBS in the loop is the reason that both spatial paths are present in each operator. Furthermore, the absence of the minus sign in the beginning of the exponent of Eq. (59) is required for the unitarity of the PBS. Finally, note that we have assumed that the arguments of the joint spectral density are polarization dependent. This is why the vertical field operators in Eq. (59) are distributed according to the second argument of $\tilde{f}(\omega, \omega')$ even though they are path A field operators. The reason that they are now path A field operators is due to the fact that the corresponding pair now enters the other input port of the PBS.

We now consider the transformation to the field operators in the Heisenberg picture due to these unitary evolution operators. As in the previous section, we use the BCH lemma. However, note that Eq. (58) is analytically equivalent to the two-mode squeezing operator in Eq. (18) and represents the same transformation. This only transformed the operators $\hat{A}_H^\dagger(\omega)$ and $\hat{B}_V^\dagger(\omega)$, and left $\hat{A}_V^\dagger(\omega)$ and $\hat{B}_H^\dagger(\omega)$ unchanged as shown explicitly in Eqs. (21) and (22). Similarly, $\hat{U}_{sq2}$ of Eq. (59) will transform $\hat{A}_V^\dagger(\omega)$ and $\hat{B}_H^\dagger(\omega)$ while leaving $\hat{A}_H^\dagger(\omega)$ and $\hat{B}_V^\dagger(\omega)$ unchanged. We can compute these transformations using the same techniques that we used in the previous section. Given this, we can also consider the combined evolution operator $\hat{U}_{sq} \equiv \hat{U}_{sq1} \hat{U}_{sq2}$ that transforms all of the field operators in the setup of Fig. 3 at once. The transformations to $\hat{A}_H^\dagger(\omega)$ and $\hat{B}_V^\dagger(\omega)$ are again given by Eqs. (39) and (47) for the reasons outlined above, while the transformations to $\hat{A}_V^\dagger(\omega)$ and $\hat{B}_H^\dagger(\omega)$ are now given by

$$\hat{U}_{sq}^\dagger \hat{A}_V^\dagger(\Omega) \hat{U}_{sq} = \int d\omega [\mathcal{C}^{(G)}(s, \Omega, \omega) \tilde{A}_V^\dagger(\omega) + \mathcal{S}^{(F)}(s, \Omega, \omega) \tilde{B}_H(\omega)], \quad (60)$$

$$\hat{U}_{sq}^\dagger \hat{B}_H^\dagger(\Omega) \hat{U}_{sq} = \int d\omega [\mathcal{C}^{(F)}(s, \Omega, \omega) \tilde{B}_H^\dagger(\omega) + \mathcal{S}^{(G)}(s, \Omega, \omega) \tilde{A}_V(\omega)]. \quad (61)$$

Equations (60) and (61) were computed using the symmetry between Eqs. (58) and (59). The plus signs in Eqs. (60) and (61) result from the absence of the minus sign in Eq. (59).

Given Eqs. (39), (47), (60), and (61), we may now compute S_{PXQY} of Eq. (15) needed in Eqs. (16) and (17). The results of these calculations may be summarized as

$$S_{HHHH} = \int d\Omega d\Omega' d\omega d\omega' |\mathcal{S}^{(G)}(s, \Omega, \omega)|^2 |\mathcal{S}^{(G)}(s, \Omega', \omega')|^2, \quad (62)$$

$$S_{HHVV} = \int d\Omega d\Omega' d\omega d\omega' [|\mathcal{S}^{(G)}(s, \Omega, \omega)|^2 |\mathcal{S}^{(F)}(s, \Omega', \omega')|^2 + \mathcal{S}^{(F)*}(s, \Omega', \omega') \mathcal{C}^{(F)*}(s, \Omega, \omega) \times \mathcal{S}^{(G)}(s, \Omega, \omega) \mathcal{C}^{(G)}(s, \Omega', \omega)], \quad (63)$$

$$S_{VVHH} = \int d\Omega d\Omega' d\omega d\omega' [|\mathcal{S}^{(F)}(s, \Omega, \omega)|^2 |\mathcal{S}^{(G)}(s, \Omega', \omega')|^2 + \mathcal{S}^{(G)*}(s, \Omega', \omega') \mathcal{C}^{(G)*}(s, \Omega, \omega) \times \mathcal{S}^{(F)}(s, \Omega, \omega) \mathcal{C}^{(F)}(s, \Omega', \omega)], \quad (64)$$

$$S_{VVVV} = \int d\Omega d\Omega' d\omega d\omega' |\mathcal{S}^{(F)}(s, \Omega, \omega)|^2 |\mathcal{S}^{(F)}(s, \Omega', \omega')|^2, \quad (65)$$

$$S_{HVVH} = 0, \quad (66)$$

$$S_{HVVH} = - \int d\Omega d\Omega' d\omega d\omega' \mathcal{S}^{(G)*}(s, \Omega', \omega') \mathcal{C}^{(G)*}(s, \Omega, \omega) \times \mathcal{S}^{(G)}(s, \Omega, \omega) \mathcal{C}^{(G)}(s, \Omega', \omega), \quad (67)$$

$$S_{VHHV} = - \int d\Omega d\Omega' d\omega d\omega' \mathcal{S}^{(F)*}(s, \Omega', \omega') \mathcal{C}^{(F)*}(s, \Omega, \omega) \times \mathcal{S}^{(F)}(s, \Omega, \omega) \mathcal{C}^{(F)}(s, \Omega', \omega), \quad (68)$$

$$S_{VHHV} = 0, \quad (69)$$

We may now substitute Eqs. (62)–(69) into Eqs. (16) and (17) to obtain

$$N = \int d\Omega d\Omega' d\omega d\omega' \{ [|\mathcal{S}^{(F)}(s, \Omega, \omega)|^2 + |\mathcal{S}^{(G)}(s, \Omega, \omega)|^2] [|\mathcal{S}^{(F)}(s, \Omega', \omega')|^2 + |\mathcal{S}^{(G)}(s, \Omega', \omega')|^2] + \mathcal{S}^{(F)*}(s, \Omega', \omega') \mathcal{C}^{(F)*}(s, \Omega, \omega) \mathcal{S}^{(G)}(s, \Omega, \omega) \mathcal{C}^{(G)}(s, \Omega', \omega) + \mathcal{S}^{(G)*}(s, \Omega', \omega') \mathcal{C}^{(G)*}(s, \Omega, \omega) \mathcal{S}^{(F)}(s, \Omega, \omega) \mathcal{C}^{(F)}(s, \Omega', \omega) \}, \quad (70)$$

$$S = \frac{\sqrt{2}}{N} \int d\Omega d\Omega' d\omega d\omega' \{ [|\mathcal{S}^{(F)}(s, \Omega, \omega)|^2 + |\mathcal{S}^{(G)}(s, \Omega, \omega)|^2] [|\mathcal{S}^{(F)}(s, \Omega', \omega')|^2 + |\mathcal{S}^{(G)}(s, \Omega', \omega')|^2] - [\mathcal{S}^{(F)*}(s, \Omega', \omega') \mathcal{C}^{(F)*}(s, \Omega, \omega) + \mathcal{S}^{(G)*}(s, \Omega', \omega') \mathcal{C}^{(G)*}(s, \Omega, \omega)] \times [\mathcal{S}^{(F)}(s, \Omega, \omega) \mathcal{C}^{(F)}(s, \Omega', \omega) + \mathcal{S}^{(G)}(s, \Omega, \omega) \mathcal{C}^{(G)}(s, \Omega', \omega)] \}. \quad (71)$$

As with Eqs. (56) and (57), Eqs. (70) and (71) are the results for an arbitrary joint spectral density $\tilde{f}(\omega, \omega')$. We will specify a distribution in the following section.

IV. COMPUTING THE RECURSIVE DISTRIBUTIONS

A. The symmetric case

Now that Eqs. (57) and (71) give us expressions for the CHSH parameter for an arbitrary joint spectral density, we will consider a specific joint spectral density. We let the photon pairs emitted from the SPDC source be spectrally distributed according to a bivariate normal distribution. The joint spectral density $\tilde{f}(\omega, \omega')$ can then be written as

$$\tilde{f}(\omega, \omega') = \frac{1}{\sqrt{2\pi}\sigma(1-\tau^2)^{1/4}} \exp\left[-\frac{\omega^2 + \omega'^2 + 2\tau\omega\omega'}{4\sigma^2(1-\tau^2)}\right], \quad (72)$$

where σ is the spectral bandwidth, which is taken to be identical for both the horizontal and vertical photons in each of the pairs produced by the type-II SPDC source of Fig. 2, and τ is the correlation between the two photons of each pair. We will demonstrate how this distribution may be used in Eqs. (57) and (71) to compute these CHSH parameters numerically as well as analytically for certain specified values of the parameter τ .

The correlation τ should not be confused with the Bell correlation of Eq. (1). The correlation factor τ is a parameter that is bound between 0 and 1 by definition, and estimates how likely the measurements of the angular frequency of each photon of a given pair agree. A τ of 0 means that the measurements are completely uncorrelated, while a τ of 1 means perfect agreement between the two measurements. In realistic SPDC devices, this τ is close to unity [36]; however, spectral engineering [37] can be used to reduce this parameter to arbitrarily close to zero.

It should be noted that we are assuming it is safe to shift the joint spectral distribution to the origin using a suitable change of variables applied to the independent variables of the distribution. This assumption is necessary because we are considering a bivariate normal distribution with zero mean in both variables. Furthermore, we are assuming that the joint spectral density for each pair is normally distributed. This is not strictly true [36], but it is a valid approximation that will still allow us to comment on the underlying physics. Finally, we have also assumed that the distribution is symmetric upon the interchange of arguments. That is,

$$\tilde{f}(\omega, \omega') = \tilde{f}(\omega', \omega). \quad (73)$$

This is true when the pairs are first generated due to the fundamental bosonic nature of indistinguishable photons, but defects in the system can introduce distinguishability between these photons [38], rendering this approximation invalid. We will consider a more generalized asymmetric case in the following section.

Now that we have specified a joint spectral density, we need to compute the recursively defined distributions $F_n(\Omega, \omega)$ and $G_n(\Omega, \omega)$ of Eqs. (27)–(29). Comparison of these expressions and utilizing Eq. (73) reveals that $F_n(\Omega, \omega) = G_n(\Omega, \omega) \equiv \mathcal{F}_n(\Omega, \omega)$. This means that we only need to consider the single expression

$$\mathcal{F}_n(\Omega, \omega) = \int d\nu \mathcal{F}_{n-1}(\Omega, \nu) \tilde{f}(\nu, \omega), \quad (74)$$

where $\mathcal{F}_0(\Omega, \omega)$ has the same base case as in Eq. (29).

If we perform the operation of Eq. (74) directly a few times, we would find that $\mathcal{F}_1(\Omega, \omega) = \tilde{f}(\Omega, \omega)$ from the base case of Eq. (29), and for higher n , we obtain normal distributions each with a single spectral bandwidth common to both independent variables. This common spectral bandwidth along with the updated correlation coefficient in the resulting normal distribution will be different for each value of n , and we will likewise obtain an extra prefactor that results in the distribution's normalization being less than unity. Given all of this, let us assume that the form of the distribution $\mathcal{F}_n(\Omega, \omega)$ for a general n is given as

$$\mathcal{F}_n(\Omega, \omega) = \frac{c_n}{\sqrt{2\pi}a_n(1-b_n^2)^{1/4}} \times \exp\left[-\frac{\Omega^2 + \omega^2 - 2b_n\Omega\omega}{4a_n^2(1-b_n^2)}\right]. \quad (75)$$

We now need to find the parameters a_n , b_n , and c_n . If we insert Eqs. (72) and (75) into Eq. (74) and integrate over ν , we obtain

$$\begin{aligned} \mathcal{F}_n(\Omega, \omega) &= \frac{(1-\tau^2)^{1/4}(1-b_{n-1}^2)^{1/4}c_{n-1}}{\sqrt{\pi[a_{n-1}^2(1-b_{n-1})^2 + \sigma^2(1-\tau^2)]}} \\ &\times \exp\left\{-\frac{a_{n-1}^2 + \sigma^2(1-\tau^2)}{4a_{n-1}^2[a_{n-1}^2(1-b_{n-1})^2 + \sigma^2(1-\tau^2)]}\Omega^2\right. \\ &\quad - \frac{a_{n-1}^2(1-b_{n-1}^2) + \sigma^2}{4\sigma^2[a_{n-1}^2(1-b_{n-1})^2 + \sigma^2(1-\tau^2)]}\omega^2 \\ &\quad \left.- \frac{\tau b_{n-1}\Omega\omega}{2[a_{n-1}^2(1-b_{n-1})^2 + \sigma^2(1-\tau^2)]}\right\}. \end{aligned} \quad (76)$$

Inspection of Eq. (76) reveals that the prefactors to Ω^2 and ω^2 are different. We made the assumption that they were the same in the general form we assumed for $\mathcal{F}_n$ in Eq. (75). This was due to the common bandwidth that would be observed if Eq. (74) was integrated directly several times. We may now enforce the equality of the prefactors for Ω^2 and ω^2 by setting the prefactor coefficients equal to one another and solving for a_{n-1} and thus a_n . This gives

$$a_n = \frac{\sigma(1-\tau^2)^{1/4}}{(1-b_n^2)^{1/4}}. \quad (77)$$

Using Eq. (77) in Eq. (76) yields

$$\begin{aligned} \mathcal{F}_n(\Omega, \omega) &= \frac{1}{\sqrt{\pi}\sigma} \sqrt{\frac{\sqrt{1-b_{n-1}^2}}{\sqrt{1-b_{n-1}^2} + \sqrt{1-\tau^2}}} c_{n-1} \\ &\times \exp\left\{-\frac{1 + \sqrt{(1-\tau^2)(1-b_{n-1}^2)}}{4\sigma^2\sqrt{1-\tau^2}[\sqrt{1-b_{n-1}^2} + \sqrt{1-\tau^2}]}\Omega^2\right. \\ &\quad \left.- \frac{1 + \sqrt{(1-\tau^2)(1-b_{n-1}^2)}}{4\sigma^2\sqrt{1-\tau^2}[\sqrt{1-b_{n-1}^2} + \sqrt{1-\tau^2}]}\omega^2\right\} \end{aligned}$$

$$\left. \begin{aligned} & - \frac{1 + \sqrt{(1 - \tau^2)(1 - b_{n-1}^2)}}{4\sigma^2 \sqrt{1 - \tau^2} [\sqrt{1 - b_{n-1}^2} + \sqrt{1 - \tau^2}]} \omega^2 \\ & - \frac{\tau b_{n-1} \Omega \omega}{2\sigma^2 \sqrt{1 - \tau^2} [\sqrt{1 - b_{n-1}^2} - \sqrt{1 - \tau^2}]} \end{aligned} \right\}. \quad (78)$$

With Eq. (78), we can now isolate an expression for b_n . This is accomplished by equating the prefactors to the $\Omega\omega$ term in the exponentials of Eqs. (75) and (78). Doing this yields the relation

$$b_n = - \frac{\tau b_{n-1}}{1 + \sqrt{(1 - \tau^2)(1 - b_{n-1}^2)}}. \quad (79)$$

In the interest of working entirely with positive quantities, we define the following:

$$B_n \equiv (-1)^n b_n, \quad (80)$$

where it is trivial to demonstrate that $B_n^2 = b_n^2$. This gives

$$B_n = \frac{\tau B_{n-1}}{1 + \sqrt{(1 - \tau^2)(1 - b_{n-1}^2)}}. \quad (81)$$

Equation (81) is a recursive expression for the parameter B_n . To obtain a closed-form result for B_n , we make the following substitutions:

$$B_n \equiv \operatorname{sech} u_n, \quad (82)$$

$$\tau \equiv \operatorname{sech} v. \quad (83)$$

Substituting Eqs. (82) and (83) into Eq. (81) and using a series of hyperbolic trigonometric identities, we obtain

$$\operatorname{sech} u_n = \operatorname{sech}(u_{n-1} + v). \quad (84)$$

From this, we conclude that

$$u_n = u_{n-1} + v. \quad (85)$$

Equation (85) is a much simpler recursive expression to solve. We need a base case in order to solve it. Recall that $\mathcal{F}_1(\Omega, \omega) = \tilde{f}(\Omega, \omega)$. Comparison of Eqs. (72) and (75) shows that $b_1 = -\tau$. Using Eqs. (80), (82), and (83) gives

$$\begin{aligned} b_1 &= -\tau, \\ B_1 &= \tau, \\ \operatorname{sech} u_1 &= \operatorname{sech} v, \\ u_1 &= v. \end{aligned} \quad (86)$$

From this, we now write the closed-form solution for u_n of Eq. (85) as

$$u_n = nv. \quad (87)$$

This expression can be easily verified using a trivial proof by mathematical induction.

We now want to use Eq. (87) to obtain an expression for B_n . To do this, we need the identity

$$\operatorname{arcsech} x = \ln \left(\frac{1 + \sqrt{1 - x^2}}{x} \right). \quad (88)$$

Using Eqs. (83), (87), and (88) in Eq. (82) gives

$$B_n = \frac{2\tau^n (1 + \sqrt{1 - \tau^2})^n}{(1 + \sqrt{1 - \tau^2})^{2n} + \tau^{2n}}, \quad (89)$$

which, from Eq. (80), yields

$$b_n = (-1)^n \frac{2\tau^n (1 + \sqrt{1 - \tau^2})^n}{(1 + \sqrt{1 - \tau^2})^{2n} + \tau^{2n}}. \quad (90)$$

We can now insert Eq. (90) in Eq. (77) to obtain

$$a_n = \sigma (1 - \tau^2)^{1/4} \sqrt{\frac{(1 + \sqrt{1 - \tau^2})^{2n} + \tau^{2n}}{(1 + \sqrt{1 - \tau^2})^{2n} - \tau^{2n}}}. \quad (91)$$

Finally, we need to solve for c_n . Setting the prefactors of Eqs. (75) and (78) equal to one another and solving for c_n yields

$$c_n = \sqrt{\frac{2\sqrt{1 - b_{n-1}^2} \sqrt{1 - \tau^2}}{\sqrt{1 - b_{n-1}^2} + \sqrt{1 - \tau^2}}} c_{n-1}, \quad (92)$$

where we have used only Eq. (77) and left b_n unevaluated for the sake of simplicity. We now use Eqs. (80), (82), and (83) in Eq. (92). After making use of a series of hyperbolic trigonometric identities, we obtain

$$\begin{aligned} c_n^2 &= [\cosh(u_{n-1} + v) - \cosh(u_{n-1} - v)] \\ &\times \frac{c_{n-1}^2}{\sinh(u_{n-1} + v)}. \end{aligned} \quad (93)$$

We now use Eq. (85) for the argument in the hyperbolic sine function in the denominator of Eq. (93). For the numerator, we use the identity

$$\cosh x - \cosh y = 2 \sinh \left(\frac{x + y}{2} \right) \sinh \left(\frac{x - y}{2} \right). \quad (94)$$

This gives

$$\sinh u_n c_n^2 = \sinh v \sinh u_{n-1} c_{n-1}^2. \quad (95)$$

We now let

$$w_n \equiv c_n^2 \sinh u_n. \quad (96)$$

Using this in Eq. (95) yields the simpler recursion relation

$$w_n = 2 \sinh v w_{n-1}. \quad (97)$$

We need a base case for this recursion relation. Since $\mathcal{F}_1(\Omega, \omega) = \tilde{f}(\Omega, \omega)$, we have $c_1 = 1$. Thus,

$$w_1 = \sinh v, \quad (98)$$

where we have used Eq. (87). It is now straightforward to infer that

$$w_n = 2^{n-1} \sinh^n v. \quad (99)$$

Again, this may be trivially proven using the principle of mathematical induction.

Now that we have an expression for w_n , we can find an expression for c_n through Eq. (96). Once again, we employ

Eq. (88) to obtain

$$c_n = \sqrt{\frac{2^n(1 - \tau^2 + \sqrt{1 - \tau^2})^{2n}}{(1 + \sqrt{1 - \tau^2})^{2n} - \tau^{2n}}}. \quad (100)$$

We finally have all of the parameters needed to express Eq. (75) in closed form. We could now substitute Eqs. (90), (91), and (100) in Eq. (75) to express this distribution in terms of the physical parameters σ and τ alone. However, we find that it is easier to work with a_n , b_n , and c_n and to make such substitutions at the end of the calculation. This will be the procedure that we use in what follows.

We now need to evaluate the quantities in Eqs. (37), (38), (45), and (46). Since $F_n(\Omega, \omega) = G_n(\Omega, \omega) \equiv \mathcal{F}_n(\Omega, \omega)$, we have

$$\mathcal{C}^{(F)}(s, \Omega, \omega) = \mathcal{C}^{(G)}(s, \Omega, \omega) \equiv \mathcal{C}(s, \Omega, \omega), \quad (101)$$

$$\mathcal{S}^{(F)}(s, \Omega, \omega) = \mathcal{S}^{(G)}(s, \Omega, \omega) \equiv \mathcal{S}(s, \Omega, \omega), \quad (102)$$

where

$$\mathcal{C}(s, \Omega, \omega) \equiv \sum_{j=0}^{\infty} \frac{s^{2j}}{(2j)!} \mathcal{F}_{2j}(\Omega, \omega), \quad (103)$$

$$\mathcal{S}(s, \Omega, \omega) \equiv \sum_{j=0}^{\infty} \frac{s^{2j+1}}{(2j+1)!} \mathcal{F}_{2j+1}(\Omega, \omega). \quad (104)$$

We now use Eqs. (103) and (104) to compute the CHSH parameter for the two different entangled-photon-pair generation techniques of Figs. 2 and 3 using Eqs. (56), (57), (70), and (71). We begin with the beam-splitter method of Fig. 2. Using Eqs. (101) and (102), Eqs. (56) and (57) simplify to

$$\begin{aligned} N = & \frac{1}{2} \int d\Omega d\Omega' d\omega d\omega' \left\{ 2|\mathcal{S}(s, \Omega, \omega)|^2 |\mathcal{S}(s, \Omega', \omega')|^2 \right. \\ & + \mathcal{C}^*(s, \Omega, \omega) \mathcal{S}(s, \Omega, \omega) \left[\mathcal{S}(s, \Omega', \omega) \mathcal{C}(s, \Omega', \omega') \right. \\ & \left. \left. + \mathcal{S}^*(s, \Omega', \omega') \mathcal{C}(s, \Omega', \omega) \right] \right\} \\ & - \frac{1}{2} \int d\Omega d\omega |\mathcal{S}(s, \Omega, \omega)|^2 \end{aligned} \quad (105)$$

and

$$\begin{aligned} S = & -\frac{\sqrt{2}}{N} \int d\Omega d\Omega' d\omega d\omega' \mathcal{C}^*(s, \Omega, \omega) \\ & \times \mathcal{C}(s, \Omega', \omega') \mathcal{S}^*(s, \Omega', \omega) \mathcal{S}(s, \Omega, \omega'). \end{aligned} \quad (106)$$

Inspection of Eqs. (105) and (106) reveals that we only need to compute three integrals:

$$I_1 \equiv \int d\Omega d\omega |\mathcal{S}(s, \Omega, \omega)|^2, \quad (107)$$

$$\begin{aligned} I_2 \equiv & \int d\Omega d\Omega' d\omega d\omega' \mathcal{C}^*(s, \Omega, \omega) \mathcal{C}(s, \Omega', \omega') \\ & \times \mathcal{S}^*(s, \Omega', \omega) \mathcal{S}(s, \Omega, \omega'), \end{aligned} \quad (108)$$

$$\begin{aligned} I_3 \equiv & \int d\Omega d\Omega' d\omega d\omega' \mathcal{C}^*(s, \Omega, \omega) \mathcal{C}(s, \Omega', \omega) \\ & \times \mathcal{S}^*(s, \Omega', \omega') \mathcal{S}(s, \Omega, \omega'). \end{aligned} \quad (109)$$

Given these integrals, the CHSH parameter for the beam-splitter method can now be written as

$$S = -2\sqrt{2} \frac{I_2}{I_2 + I_3 - I_1(1 - 2I_1)}. \quad (110)$$

Using the definitions of Eqs. (103) and (104) in Eqs. (107)–(109) gives

$$\begin{aligned} I_1 = & \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{s^{2j+2k+2}}{(2j+1)!(2k+1)!} \\ & \times \int d\Omega d\omega \mathcal{F}_{2j+1}(\Omega, \omega) \mathcal{F}_{2k+1}(\Omega, \omega), \end{aligned} \quad (111)$$

$$\begin{aligned} I_2 = & \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \frac{s^{2j+2k+2\ell+2m+2}}{(2j)!(2k)!(2\ell+1)!(2m+1)!} \\ & \times \int d\Omega d\Omega' d\omega d\omega' \mathcal{F}_{2j}(\Omega, \omega) \mathcal{F}_{2k}(\Omega', \omega') \\ & \times \mathcal{F}_{2\ell+1}(\Omega', \omega) \mathcal{F}_{2m+1}(\Omega, \omega'), \end{aligned} \quad (112)$$

$$\begin{aligned} I_3 = & \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \frac{s^{2j+2k+2\ell+2m+2}}{(2j)!(2k)!(2\ell+1)!(2m+1)!} \\ & \times \int d\Omega d\Omega' d\omega d\omega' \mathcal{F}_{2j}(\Omega, \omega) \mathcal{F}_{2k}(\Omega', \omega) \\ & \times \mathcal{F}_{2\ell+1}(\Omega', \omega') \mathcal{F}_{2m+1}(\Omega, \omega'). \end{aligned} \quad (113)$$

Inspection of Eqs. (111)–(113) reveals that we now only need to evaluate two integrals:

$$J_1 \equiv \int d\Omega d\omega \mathcal{F}_n(\Omega, \omega) \mathcal{F}_m(\Omega, \omega), \quad (114)$$

$$\begin{aligned} J_2 \equiv & \int d\Omega d\Omega' d\omega d\omega' \mathcal{F}_j(\Omega, \omega) \mathcal{F}_k(\Omega', \omega') \\ & \times \mathcal{F}_\ell(\Omega', \omega) \mathcal{F}_m(\Omega, \omega'). \end{aligned} \quad (115)$$

We begin by computing J_1 of Eq. (114). When using the form of $\mathcal{F}_n(\Omega, \omega)$ from Eq. (75), we need to perform two Gaussian integrals. Doing this, we find that

$$\begin{aligned} J_1 = & \sqrt{2} c_n c_m \\ & \times \sqrt{\frac{a_n^2 a_m^2 \sqrt{1 - b_n^2} \sqrt{1 - b_m^2}}{a_n^4 (1 - b_n^2) + a_m^4 (1 - b_m^2) + 2a_n^2 a_m^2 (1 - b_n b_m)}}. \end{aligned} \quad (116)$$

Using Eqs. (90), (91), and (100) in Eq. (116) gives

$$J_1 = \frac{(\sqrt{2}\sqrt{1 - \tau^2} + \sqrt{1 - \tau^2})^{n+m}}{(1 + \sqrt{1 - \tau^2})^{n+m} - \tau^{n+m}}. \quad (117)$$

Using a similar procedure, we find that J_2 of Eq. (115) is

$$J_2 = \frac{(\sqrt{2}\sqrt{1 - \tau^2} + \sqrt{1 - \tau^2})^{j+k+\ell+m}}{(1 + \sqrt{1 - \tau^2})^{j+k+\ell+m} - \tau^{j+k+\ell+m}}. \quad (118)$$

We can now use Eqs. (117) and (118) to evaluate Eqs. (111)–(113). We find that $I_2 = I_3$, which simplifies Eq. (110) to

$$S = -2\sqrt{2} \frac{I_2}{2I_2 - I_1(1 - 2I_1)}, \quad (119)$$

where

$$I_1 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{s^{2j+2k+2}}{(2j+1)!(2k+1)!} \times \frac{(\sqrt{2}\sqrt{1-\tau^2} + \sqrt{1-\tau^2})^{2j+2k+2}}{(1 + \sqrt{1-\tau^2})^{2j+2k+2} - \tau^{2j+2k+2}}, \quad (120)$$

$$I_2 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \frac{s^{2j+2k+2\ell+2m+2}}{(2j)!(2k)!(2\ell+1)!(2m+1)!} \times \frac{(\sqrt{2}\sqrt{1-\tau^2} + \sqrt{1-\tau^2})^{2j+2k+2}}{(1 + \sqrt{1-\tau^2})^{2j+2k+2\ell+2m+2} - \tau^{2j+2k+2\ell+2m+2}}. \quad (121)$$

We now need to evaluate Eqs. (120) and (121). We will first evaluate Eq. (120). Let

$$x \equiv \frac{\tau}{1 + \sqrt{1-\tau^2}}. \quad (122)$$

Recall that τ is the correlation factor between the two photons in a photon pair and its value is restricted to be between 0 and 1. Inspection of Eq. (122) reveals that τ 's restricted range of values also limits the values of the parameter x to between 0 and 1. This will be important in what follows. We next use Eq. (122) to express Eq. (120) more compactly as

$$I_1 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(s\sqrt{1-x^2})^{2j+2k+2}}{(2j+1)!(2k+1)!} \frac{1}{1-x^{2j+2k+2}}. \quad (123)$$

Since x is bounded between 0 and 1, so too is $x^{2j+2k+2}$ for all non-negative j and k . Given this, we can now use the definition of the geometric series to obtain

$$I_1 = \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(s\sqrt{1-x^2})^{2j+2k+2}}{(2j+1)!(2k+1)!} (x^{2j+2k+2})^{\ell} = \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(s\sqrt{1-x^2}x^{\ell})^{2j+2k+2}}{(2j+1)!(2k+1)!} = \sum_{\ell=0}^{\infty} \sinh^2(s\sqrt{1-x^2}x^{\ell}), \quad (124)$$

where we have used the Taylor expansion of the hyperbolic sine function to obtain the final line. With the double-angle formula for the hyperbolic functions, we can simplify Eq. (124) even further as

$$I_1 = \frac{1}{2} \sum_{\ell=0}^{\infty} [\cosh(2s\sqrt{1-x^2}x^{\ell}) - 1] = \frac{1}{2} \sum_{\ell=0}^{\infty} \left[\sum_{k=0}^{\infty} \frac{(2s\sqrt{1-x^2}x^{\ell})^{2k}}{(2k)!} - 1 \right] = \frac{1}{2} \sum_{\ell=0}^{\infty} \left[\sum_{k=1}^{\infty} \frac{(2s\sqrt{1-x^2}x^{\ell})^{2k}}{(2k)!} \right], \quad (125)$$

where we have used the Taylor series expansion of the hyperbolic cosine function. Using the geometric series once more in Eq. (125), we can express Eq. (123) as a single sum

given as

$$I_1 = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(2s\sqrt{1-x^2})^{2k}}{(2k)!} \frac{1}{1-x^{2k}}. \quad (126)$$

Next, consider I_2 of Eq. (121). Using a similar procedure, we find

$$I_2 = \sum_{n=0}^{\infty} \sinh^2(s\sqrt{1-x^2}x^n) \cosh^2(s\sqrt{1-x^2}x^n). \quad (127)$$

Using the double-angle formula for both the hyperbolic sine and cosine functions yields

$$I_2 = \frac{1}{4} \sum_{n=0}^{\infty} \sinh^2(2s\sqrt{1-x^2}x^n) = \frac{1}{8} \sum_{n=0}^{\infty} [\cosh(4s\sqrt{1-x^2}x^n) - 1] = \frac{1}{8} \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} \frac{(4s\sqrt{1-x^2}x^n)^{2k}}{(2k)!} - 1 \right] = \frac{1}{8} \sum_{n=0}^{\infty} \left[\sum_{k=1}^{\infty} \frac{(4s\sqrt{1-x^2}x^n)^{2k}}{(2k)!} \right], \quad (128)$$

and so

$$I_2 = \frac{1}{8} \sum_{k=1}^{\infty} \frac{(4s\sqrt{1-x^2})^{2k}}{(2k)!} \frac{1}{1-x^{2k}}. \quad (129)$$

Comparison of Eqs. (126) and (129) reveals that we now only need to compute the sum

$$\kappa(x, y) \equiv \sum_{k=1}^{\infty} \frac{(y\sqrt{1-x^2})^{2k}}{(2k)!} \frac{1}{1-x^{2k}}, \quad (130)$$

for $0 \leq x \leq 1$ and $y \geq 0$. This gives I_1 and I_2 as

$$I_1 = \frac{1}{2} \kappa\left(\frac{\tau}{1 + \sqrt{1-\tau^2}}, 2s\right), \quad (131)$$

$$I_2 = \frac{1}{8} \kappa\left(\frac{\tau}{1 + \sqrt{1-\tau^2}}, 4s\right). \quad (132)$$

We could now numerically compute the sum of Eq. (130) using protocols such as Mathematica's NSum. We provide some examples of this in Figs. 4 and 5. Furthermore, we can use these numerical results in Eq. (119) to approximate the CHSH parameter as shown in Figs. 6 and 7. However, we can also consider specific examples of this sum analytically—as well as an important approximation for the case when $y \ll 1$. To begin, consider the case when $x = 0$. In this case, we have

$$\kappa(0, y) \equiv \sum_{k=1}^{\infty} \frac{y^{2k}}{(2k)!} = \cosh y - 1 = 2 \sinh^2(y/2) = 8 \sinh^2(y/4) \cosh^2(y/4). \quad (133)$$

If we use this in Eqs. (131) and (132) and then substitute the results into Eq. (119), we obtain

$$S = -\frac{2\sqrt{2} \cosh^2(s)}{2 \cosh(2s) - 1}. \quad (134)$$

This is the same expression that was obtained in our previous work [13]. $x = 0$ implies that $\tau = 0$, which is the uncorrelated

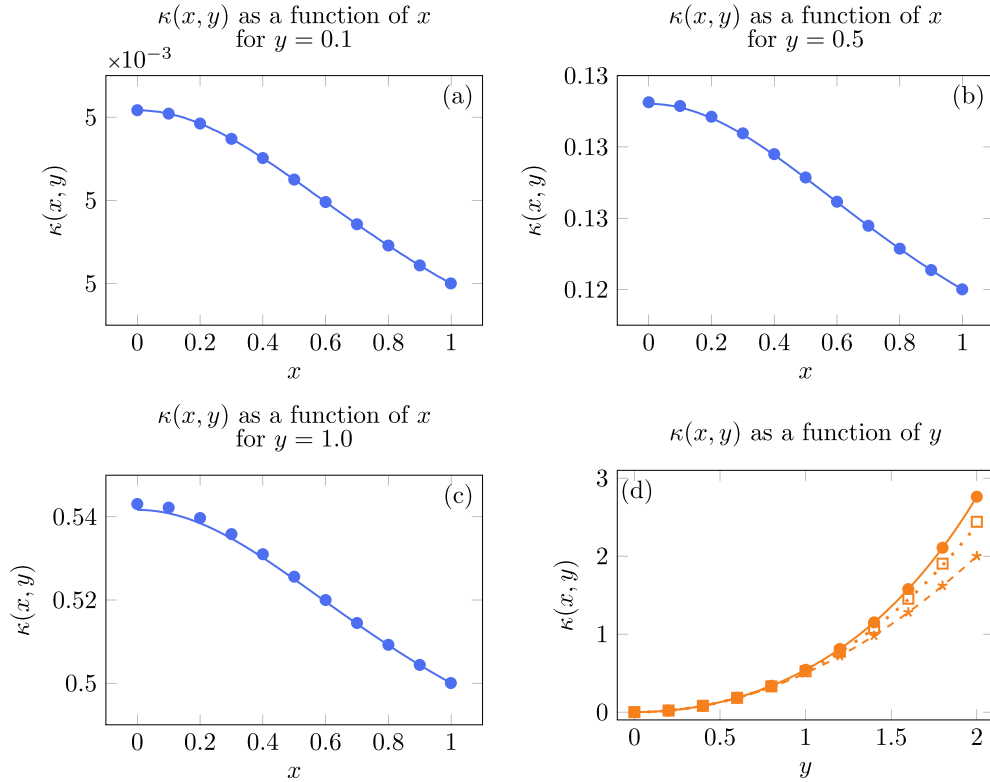


FIG. 4. Plots of the sum of Eq. (130) for various cases. Plots (a)–(c) show $\kappa(x, y)$ as a function of x for the specific values of y shown. The points are the exact numerical results from Mathematica’s NSum, while the curves result from the approximation made in Eq. (137). In panels (b) and (c), we see that the approximation fails to match the exact result for small values of x . This is more evident in panel (c) than in panel (b). In panel (d), we plot $\kappa(x, y)$ as a function of y for various values of x . The solid points are for $x = 0$, the empty squares are for $x = 0.5$, and the stars are for $x \approx 1$. Each point is numerically computed in an analogous fashion to those of panels (a)–(c). The solid curve is the analytical result for the uncorrelated case of Eq. (133), the dashed line is the analytical result in the perfectly correlated case of Eq. (135), and the dotted line is the approximate result of Eq. (137) for $x = 0.5$. We see that the uncorrelated and perfectly correlated curves agree with the appropriate exact numerical results, while the approximate result slightly disagrees with the corresponding exact numerical results for larger values of y .

case. Thus, we see that the general expression reduces to the CHSH parameter in the uncorrelated case as expected.

Next, we consider the case where $x = 1$, which corresponds to $\tau = 1$. $\tau = 1$ means that we have perfect correlation between the members of each of the pairs. This case can only be evaluated in the limit as $\tau \rightarrow 1$. Taking the limit as $x \rightarrow 1$ of Eq. (130) gives 0 in every term except the $k = 1$ term. Thus, the limit yields

$$\kappa(1, y) = \frac{y^2}{2}. \quad (135)$$

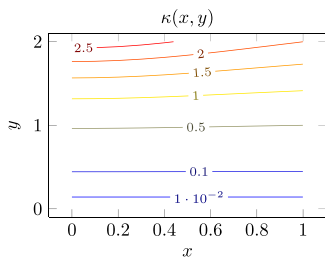


FIG. 5. Contour plot of $\kappa(x, y)$ generated using numerical data computed using Eq. (130) in Mathematica’s NSum function.

Using this in Eqs. (131) and (132) and then inserting the results in Eq. (119) yields

$$S = -\frac{2\sqrt{2}}{1 + 2s^2}. \quad (136)$$

Equation (136) is an exact analytical expression. We plot it alongside Eq. (134) as a function of the squeezing parameter s in Fig. 6(b). From this, we find the interesting result that the perfectly correlated case results in an improved CHSH parameter for a wide range of squeezing parameters compared to the uncorrelated case.

Finally, we consider an approximate analytical expression for the CHSH parameter. Typically, it is experimentally desirable to have a small squeezing parameter. This is because a small squeezing parameter corresponds to a small number of photon pairs and only one is produced in the ideal case. This corresponds to a small value of y in Eqs. (131) and (132). Therefore, we will now consider the case where $y \ll 1$. This allows us to truncate the sum in Eq. (130). For this truncated sum to have a dependence on x , we need to take at least two terms. Doing this yields

$$\kappa(x, y) \approx \frac{y^2}{2} \left(1 + \frac{y^2}{12} \frac{1 - x^2}{1 + x^2} \right). \quad (137)$$

Using this approximation in Eqs. (119), (131), and (132) gives

$$S \approx \frac{-2\sqrt{2}[27 - 48s^4(1 - \tau^2)]}{27 + 27s^2(2 + \sqrt{1 - \tau^2}) - 12s^4(7 - 7\tau^2 + 3\sqrt{1 - \tau^2}) - 42s^6(1 - \tau^2) - 8s^8(1 - \tau^2)^{3/2}}, \quad (138)$$

which is valid for $s \ll 1$.

Now that we have computed the CHSH parameter for the beam-splitter method for generating entangled photon pairs using the recursive distribution technique, we consider the Sagnac method of Fig. 3. Equations (101) and (102) simplify Eqs. (70) and (71) to

$$N = 2 \int d\Omega d\Omega' d\omega d\omega' [|\mathcal{S}(s, \Omega, \omega)|^2 |\mathcal{S}(s, \Omega', \omega')|^2 + \mathcal{S}^*(s, \Omega', \omega') \mathcal{C}^*(s, \Omega, \omega') \mathcal{S}(s, \Omega, \omega) \mathcal{C}(s, \Omega', \omega)], \quad (139)$$

$$S = -\frac{4\sqrt{2}}{N} \int d\Omega d\Omega' d\omega d\omega' \mathcal{S}^*(s, \Omega', \omega') \mathcal{C}^*(s, \Omega, \omega') \times \mathcal{S}(s, \Omega, \omega) \mathcal{C}(s, \Omega', \omega). \quad (140)$$

Inspection of Eqs. (139) and (140) reveals that they depend on the same integrals computed for the beam-splitter method. Therefore, we may now write the resulting CHSH parameter for the Sagnac method as

$$S = -2\sqrt{2} \frac{I_2}{2I_1 + I_2}, \quad (141)$$

We now plot Eqs. (134), (136), and (142) in Fig. 8, where Eq. (134) is the CHSH parameter of the beam-splitter method for generating photon pairs with uncorrelated pairs, Eq. (142) is the CHSH parameter of the Sagnac method with uncorrelated pairs, and Eq. (136) is the CHSH parameter of both methods with perfectly correlated pairs. From these curves, we see that the Sagnac method in the uncorrelated case has the largest CHSH parameter for all values of the squeezing parameter s . However, both setups outperform the beam-splitter method with uncorrelated pairs when they have perfectly correlated pairs for a wide range of squeezing parameters. From this, we see that the spectral correlation likely reduces the detrimental effects that using the beam-splitter arrangement itself has on the measurement of the CHSH parameter. Furthermore, it is an interesting result that the setups of Figs. 2 and 3 behave identically as a function of the squeezing parameter s , when the members of each pair emitted by the SPDC sources are perfectly correlated.

B. The full asymmetric case

In the previous section, we considered the CHSH parameters for the beam-splitter and Sagnac methods from Eqs. (57) and (71) using the symmetric joint spectral density of Eq. (72).

where I_1 and I_2 are again given by Eqs. (131) and (132), respectively, which, in turn, depend on the sum of Eq. (130).

Given Eq. (141), we may now consider the CHSH parameter in the same simplified situations considered for the beam-splitter method. For the uncorrelated ($\tau = 0$) case, we have

$$S = -2\sqrt{2} \frac{2 \cosh^2(s)}{3 \cosh(2s) - 1}, \quad (142)$$

which also agrees with our previous work [13].

We do, however, obtain an interesting result in the perfectly correlated case. Using $\tau = 1$ in Eqs. (130), (131), and (141) once again yields Eq. (136). That is, there is no difference between the beam-splitter method and the Sagnac method for generating entangled photon pairs when the photons of the pairs emitted by the SPDC source are perfectly correlated. This turns out to be unfortunate because, as we will see, Eq. (142) outperforms Eq. (136) for all values of s . This means that spectral correlation is a detriment to the Sagnac method for generating $|\Psi^-\rangle$, as we originally expected.

Finally, with the same approximation of small s that we used to obtain Eq. (137), we obtain for the Sagnac method

$$S \approx -2\sqrt{2} \frac{1 - 64s^4(1 - \tau^2)}{1 + 2s^2 - 8s^4(8 - 8\tau^2 + \sqrt{1 - \tau^2}) - 56s^6(1 - \tau^2) - 64s^8(1 - \tau^2)^{3/2}}. \quad (143)$$

However, impairments in the system can introduce asymmetry into the joint spectral density [38]. Therefore, we desire to simulate such asymmetry by introducing asymmetry into the idealized distribution of Eq. (72). In this section, we will compute the CHSH parameters of Eqs. (57) and (71) using this asymmetric distribution. From this, we can observe the impacts that asymmetry has on the CHSH parameter in the presence of multiple pairs.

Recall the joint spectral density function given by Eq. (72). In that equation, τ is the correlation of the signal and idler photons in each pair, and σ is the common spectral bandwidth. One simple way of making this distribution asymmetric is to not require that the signal and idler photons share a common spectral bandwidth. For example, we could let the spectral bandwidth of the signal photons be σ , while for the idler photons, it is σ' . This approach requires two independent variables to explore the asymmetry of the distribution. Instead, we define the parameter

$$\mu \equiv \frac{\sigma}{\sigma'}. \quad (144)$$

Clearly, μ is a parameter that can vary between 0 and ∞ and the spectral distribution becomes perfectly symmetric when $\mu = 1$. However, due the fundamental uncertainty of quantum mechanics, we should expect that it would not matter if we had

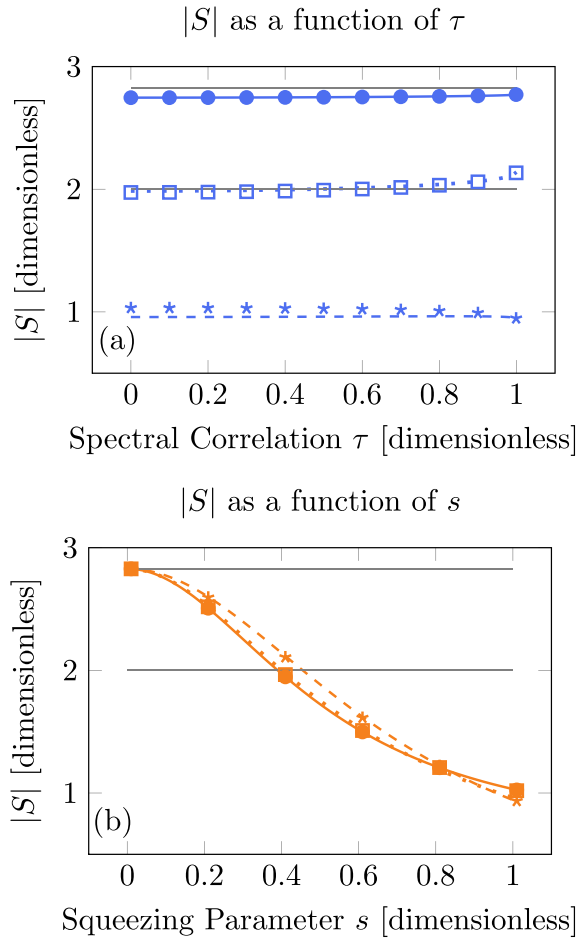


FIG. 6. Plots of the absolute value of the CHSH parameter of the beam-splitter method of Fig. 2 computed in Sec. IV A for various situations. The thin gray lines at $|S| = 2$ and $|S| = 2\sqrt{2}$ in panels (a) and (b) correspond to the upper bound for the CHSH inequality and Tsirelson's bound, respectively. In panel (a), we plot $|S|$ as a function of the spectral correlation τ . The solid points are for $s = 0.1$, the empty squares are for $s = 0.4$, and the stars are for $s = 1$. The points are numerically computed as in Fig. 4. The solid curve is the approximate result of Eq. (138) for $s = 0.1$, the dotted curve is the approximate result for $s = 0.4$, and the dashed curve is the approximate result for $s = 1$. We see that the approximate results fail to match the exact result when $s = 0.4$ and $s = 1$ for small values of τ . Furthermore, we also note that, when $s = 0.4$, as τ increases, the CHSH parameter goes from not violating the CHSH inequality to ultimately violating it. In panel (b), we plot the CHSH parameter as a function of the squeezing parameter s . The solid points are for $\tau = 0$, the empty squares are for $\tau = 0.5$, and the stars are for $\tau \approx 1$. As in panel (a), these points are numerically computed. The solid curve is the exact result in the uncorrelated case of Eq. (134), the dotted curve is the approximate result of Eq. (138) for $s = 0.5$, and the dashed curve is the exact result in the perfectly correlated case of Eq. (136). The exact results perfectly match their respective numerical results, while the approximate curve underestimates the corresponding numerical results for large values of s .

defined μ as the inverse of Eq. (144) as well. We will see later in this section that letting $\mu \rightarrow 1/\mu$ yields the same resulting CHSH parameter regardless of s and τ later in this section.

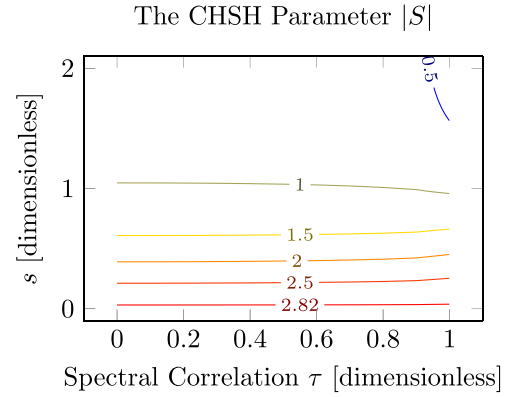


FIG. 7. Contour plot of the absolute value of the CHSH parameter $|S|$ in the beam-splitter method generated using the numerical data computed using Mathematica's NSum. From these contours, we see that as s increases, the CHSH parameter has a stronger dependence on the correlation τ . This is likely due to the increasing probability of receiving multiple photon pairs that compound the impacts of correlation of the joint spectral density.

Using Eq. (144) in Eq. (72) yields

$$\tilde{f}(\omega, \omega') = \sqrt{\frac{\mu}{2\pi}} \frac{1}{\sigma(1-\tau^2)^{1/4}} \times \exp\left[-\frac{\omega^2 + \mu^2\omega'^2 + 2\tau\mu\omega\omega'}{4\sigma^2(1-\tau^2)}\right]. \quad (145)$$

Contours of Eq. (145) for various values of μ are shown in Fig. 9. From these plots, we see how this distribution becomes asymmetric when μ deviates from unity.

We now compute the recursive distributions defined by Eqs. (27)–(29) for this asymmetric spectral density function. Let the original symmetric spectral density function of Eq. (72) be denoted as $\tilde{f}_0(\omega, \omega')$ to avoid confusion with the asymmetric distribution of Eq. (145).

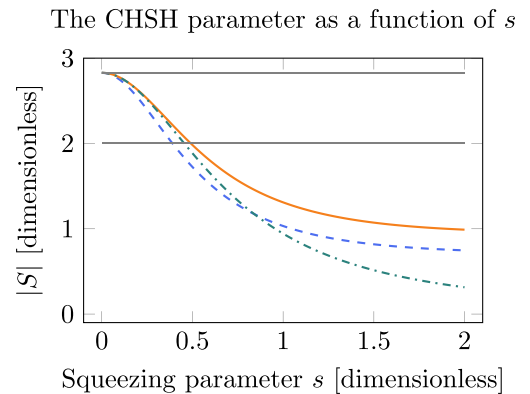


FIG. 8. Plots of the various theoretical CHSH parameters computed in Sec. IV A. The dashed curve is the CHSH parameter of the beam-splitter method with uncorrelated pairs of Eq. (134), the solid curve is the CHSH parameter of the Sagnac method with uncorrelated pairs of Eq. (142), and the dot-dashed curve is the CHSH parameter of both methods with perfectly correlated pairs given by Eq. (136).

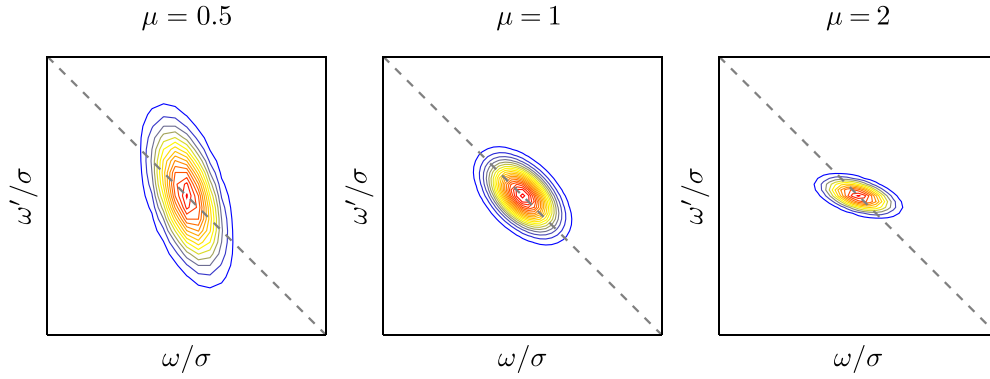


FIG. 9. Contour plots of the joint spectral density of Eq. (145) for different values of the asymmetry parameter μ . Each of these plots has a correlation coefficient of $\tau = 0.5$. As μ deviates from unity, the axis of the elliptical Gaussian deviates from the symmetric axis, which is depicted as a gray dashed line.

In the previous section, we were able to use the symmetry of Eq. (72) to simplify Eqs. (27) and (28) as Eq. (74). However, since Eq. (145) is not symmetric, we will now need to have separate F_n and G_n distributions. Nevertheless, we can still use the original recursive distribution parameters of Eqs. (75), (91), (90), and (100) to simplify the calculation of the revised F_n and G_n distributions.

In order to do this, we first note that Eqs. (72) and (145) are related to one another by

$$\tilde{f}(\omega, \omega') = \sqrt{\mu} \tilde{f}_0(\omega, \mu\omega'), \quad (146)$$

where $\tilde{f}_0(\omega, \omega')$ denotes the symmetric spectral density function of Eq. (72). Now, consider the distribution F_n from Eq. (27):

$$\begin{aligned} F_n(\Omega, \omega) &\equiv \int dv G_{n-1}(\Omega, v) \tilde{f}(\omega, v) \\ &= \int dv_1 dv_2 F_{n-2}(\Omega, v_1) \tilde{f}(v_1, v_2) \tilde{f}(\omega, v_2) \\ &= \mu \int dv_1 dv_2 F_{n-2}(\Omega, v) \tilde{f}_0(v_1, \mu v_2) \tilde{f}_0(\omega, \mu v_2), \end{aligned} \quad (147)$$

where we used Eq. (28) to obtain the second line and Eq. (146) to obtain the third. Consider the change of variables

$$v_2 \rightarrow \frac{v_2}{\mu}, \quad (148)$$

$$dv_2 \rightarrow \frac{dv_2}{\mu}. \quad (149)$$

After using Eqs. (148) and (149) in Eq. (147), we obtain

$$F_n(\Omega, v) = \int dv_1 dv_2 F_{n-2}(\Omega, v) \tilde{f}_0(v_1, v_2) \tilde{f}_0(\omega, v_2). \quad (150)$$

We have now successfully decoupled F_n from G_n and formulated the definition entirely in terms of the symmetric distribution of Eq. (72). However, each F_n is recursively defined using every other value of n instead of consecutive values. This means that we need to consider even and odd values for F_n separately.

Let us consider the even n values first. Since F_0 is the base case, let us consider F_2 :

$$\begin{aligned} F_2(\Omega, v) &= \int dv_1 dv_2 F_0(\Omega, v_1) \tilde{f}_0(v_1, v_2) \tilde{f}_0(\omega, v_2) \\ &= \int dv_1 dv_2 \delta(\Omega - v_1) \tilde{f}_0(v_1, v_2) \tilde{f}_0(\omega, v_2) \\ &= \int dv \tilde{f}_0(\Omega, v) \tilde{f}_0(\omega, v) \\ &= \int dv \mathcal{F}(\Omega, v) \tilde{f}_0(\omega, v) = \mathcal{F}_2(\Omega, \omega), \end{aligned} \quad (151)$$

where we have used Eq. (29) to obtain the second line, and Eqs. (29) and (74) to obtain the final line. Equation (151) suggests that

$$F_{2n}(\Omega, \omega) = \mathcal{F}_{2n}(\Omega, \omega). \quad (152)$$

We prove this using mathematical induction. We have already shown the base case in Eq. (151). We now show that if Eq. (152) is true for n , it is also true for $n + 1$. We do this by calculating

$$\begin{aligned} F_{2n+2}(\Omega, \omega) &= \int dv_1 dv_2 F_{2n}(\Omega, v_1) \tilde{f}_0(v_1, v_2) \tilde{f}_0(\omega, v_2) \\ &= \int dv_1 dv_2 \mathcal{F}_{2n}(\Omega, v_1) \tilde{f}_0(v_2, v_1) \tilde{f}_0(\omega, v_2) \\ &= \int dv \mathcal{F}_{2n+1}(\Omega, v) \tilde{f}_0(\omega, v) = \mathcal{F}_{2n+2}(\Omega, \omega), \end{aligned} \quad (153)$$

where we used the symmetry of Eq. (72) to obtain the second line and Eq. (74) to obtain the last equality. Equation (153) is clearly the $n + 1$ case of the inductive hypothesis of Eq. (152).

Next, we consider the odd n values of F_n . The first odd n value of F_n is F_1 , which is given as

$$\begin{aligned} F_1(\Omega, \omega) &= \int dv G_0(\Omega, v) \tilde{f}(\omega, v) \\ &= \int dv \delta(\Omega - v) \tilde{f}(\omega, v) \\ &= \tilde{f}(\omega, \Omega) = \sqrt{\mu} \tilde{f}_0(\omega, \mu\Omega) \\ &= \sqrt{\mu} \tilde{f}_0(\mu\Omega, \omega) = \sqrt{\mu} \mathcal{F}_1(\mu\Omega, \omega), \end{aligned} \quad (154)$$

where we used Eqs. (29) and (74) in the final line. Equation (154) suggests that

$$F_{2n+1}(\Omega, \nu) = \sqrt{\mu} \mathcal{F}_{2n+1}(\mu\Omega, \nu). \quad (155)$$

We again use mathematical induction to prove this. The base case was obtained in Eq. (154). We now assume that Eq. (155) is true for n and calculate

$$\begin{aligned} F_{2n+3}(\Omega, \omega) &= \int dv_1 dv_2 F_{2n+1}(\Omega, \nu_1) \tilde{f}_0(\nu_1, \nu_2) \tilde{f}_0(\omega, \nu_2) \\ &= \sqrt{\mu} \int dv_1 dv_2 \mathcal{F}_{2n+1}(\mu\Omega, \nu_1) \tilde{f}_0(\nu_1, \nu_2) \tilde{f}_0(\omega, \nu_2) \\ &= \sqrt{\mu} \int dv \mathcal{F}_{2n+2}(\mu\Omega, \nu) \tilde{f}(\omega, \nu) \\ &= \sqrt{\mu} \mathcal{F}_{2n+3}(\mu\Omega, \nu), \end{aligned} \quad (156)$$

which is the $n + 1$ case.

Now that we have considered both the even and odd n values of F_n , we next consider the distributions G_n . From Eqs. (27) and (28), we have

$$\begin{aligned} G_n(\Omega, \omega) &= \int dv F_{n-1}(\Omega, \nu) \tilde{f}(\nu, \omega) \\ &= \int dv_1 dv_2 G_{n-2}(\Omega, \nu_1) \tilde{f}(\nu_2, \nu_1) \tilde{f}(\nu_2, \omega) \\ &= \mu \int dv_1 dv_2 G_{n-2}(\Omega, \nu_1) \tilde{f}_0(\nu_2, \mu\nu_1) \tilde{f}_0(\nu_2, \mu\omega). \end{aligned} \quad (157)$$

In this case, it does not make sense to make the change of variables of Eqs. (148) and (149). Therefore, we let Eq. (157) be the decoupled recursive definition for G_n . Equation (157) may be compared with Eq. (76) for the distributions F_n .

As was the case for the F_n , we need to consider the even and odd n values for G_n separately. We first compute the even n values of G_n . Consider G_2 :

$$\begin{aligned} G_2(\Omega, \omega) &= \mu \int dv_1 dv_2 G_0(\Omega, \nu_1) \tilde{f}_0(\nu_2, \mu\nu_1) \tilde{f}_0(\nu_2, \mu\omega) \\ &= \mu \int dv_1 dv_2 \delta(\Omega - \nu_1) \tilde{f}_0(\nu_2, \mu\nu_1) \tilde{f}_0(\nu_2, \mu\omega) \\ &= \mu \int dv \tilde{f}_0(\nu, \mu\Omega) \tilde{f}_0(\nu, \mu\omega) \\ &= \mu \int dv \tilde{f}_0(\mu\Omega, \nu) \tilde{f}_0(\mu\omega, \nu) \\ &= \mu \int dv \mathcal{F}_1(\mu\Omega, \nu) \tilde{f}_0(\mu\omega, \nu) \\ &= \mu \mathcal{F}_2(\mu\Omega, \mu\omega). \end{aligned} \quad (158)$$

This seems to suggest that, for even n , we have

$$G_{2n}(\Omega, \omega) = \mu \mathcal{F}_{2n}(\mu\Omega, \mu\omega). \quad (159)$$

We again use mathematical induction to prove this. We have already demonstrated the base case in Eq. (158). We now

make the inductive hypothesis and calculate

$$\begin{aligned} G_{2n+2}(\Omega, \omega) &= \mu \int dv_1 dv_2 G_{2n}(\Omega, \nu_1) \tilde{f}_0(\nu_2, \mu\nu_1) \tilde{f}_0(\nu_2, \mu\omega) \\ &= \mu^2 \int dv_1 dv_2 \mathcal{F}_{2n}(\mu\Omega, \mu\nu_1) \tilde{f}_0(\nu_2, \mu\nu_1) \tilde{f}_0(\nu_2, \mu\omega). \end{aligned} \quad (160)$$

We now make changes of variables similar to Eqs. (148) and (149):

$$\nu_1 \rightarrow \frac{\nu_1}{\mu}, \quad (161)$$

$$dv_1 \rightarrow \frac{d\nu_1}{\mu}. \quad (162)$$

Using Eqs. (161) and (162) in Eq. (160) yields

$$\begin{aligned} G_{2n+2}(\Omega, \omega) &= \mu \int dv_1 dv_2 \mathcal{F}_{2n}(\mu\Omega, \nu_1) \tilde{f}_0(\nu_2, \nu_1) \tilde{f}_0(\nu_2, \mu\omega) \\ &= \mu \int dv \mathcal{F}_{2n+1}(\mu\Omega, \nu) \tilde{f}_0(\nu_2, \mu\omega) \\ &= \mu \mathcal{F}_{2n+2}(\mu\Omega, \mu\omega), \end{aligned} \quad (163)$$

which is the $n + 1$ case of Eq. (159).

We now compute the G_n for odd n values. We first compute G_1 :

$$\begin{aligned} G_1(\Omega, \omega) &= \int dv F_0(\Omega, \nu) \tilde{f}(\nu, \omega) \\ &= \int dv \delta(\Omega - \nu) \tilde{f}(\nu, \omega) \\ &= \tilde{f}(\Omega, \omega) = \sqrt{\mu} \tilde{f}_0(\Omega, \mu\omega) \\ &= \sqrt{\mu} \mathcal{F}_1(\Omega, \mu\omega). \end{aligned} \quad (164)$$

This suggests that

$$G_{2n+1}(\Omega, \omega) = \sqrt{\mu} \mathcal{F}_{2n+1}(\Omega, \mu\omega). \quad (165)$$

We have already obtained the base case needed for mathematical induction in Eq. (164). We now make the inductive hypothesis and calculate

$$\begin{aligned} G_{2n+3}(\Omega, \omega) &= \mu \int dv_1 dv_2 G_{2n+1}(\Omega, \nu_1) \tilde{f}_0(\nu_2, \mu\nu_1) \tilde{f}_0(\nu_2, \mu\omega) \\ &= \mu \sqrt{\mu} \int dv_1 dv_2 \mathcal{F}_{2n+1}(\Omega, \mu\nu_1) \tilde{f}_0(\nu_2, \mu\nu_1) \tilde{f}_0(\nu_2, \mu\omega). \end{aligned} \quad (166)$$

Using Eqs. (161) and (162) in Eq. (166) yields

$$\begin{aligned} G_{2n+3}(\Omega, \omega) &= \sqrt{\mu} \int dv_1 dv_2 \mathcal{F}_{2n+1}(\Omega, \nu_1) \tilde{f}_0(\nu_2, \nu_1) \tilde{f}_0(\nu_2, \mu\omega) \\ &= \sqrt{\mu} \int dv \mathcal{F}_{2n+2}(\Omega, \nu) \tilde{f}_0(\nu, \mu\omega) \\ &= \sqrt{\mu} \mathcal{F}_{2n+3}(\Omega, \mu\omega), \end{aligned} \quad (167)$$

which is the $n + 1$ case.

With Eqs. (152), (155), (159), and (165), we have now computed the distributions F_n and G_n from Eqs. (27)–(29). Note that we have different results for the even and odd n values for each distribution. This turns out to be useful in what follows.

Given the F_n and G_n distributions, we now want to calculate the hyperbolic-sinusoidal-like distributions of Eqs. (37), (38), (45), and (46). Before we begin, recall the simplified distributions of Eqs. (103) and (104). Using these expressions and the fact that the $\mathcal{C}$ distributions only depend on the even members of F and G while the $\mathcal{S}$ distributions only depend on the odd members allows us to compute the $\mathcal{C}^{(F)}$, $\mathcal{C}^{(G)}$, $\mathcal{S}^{(F)}$, and $\mathcal{S}^{(G)}$ of this section more simply as

$$\mathcal{C}^{(F)}(s, \Omega, \omega) = \mathcal{C}(s, \Omega, \omega), \quad (168)$$

$$\mathcal{C}^{(G)}(s, \Omega, \omega) = \mu \mathcal{C}(s, \mu\Omega, \mu\omega), \quad (169)$$

$$\mathcal{S}^{(F)}(s, \Omega, \omega) = \sqrt{\mu} \mathcal{S}(s, \mu\Omega, \omega), \quad (170)$$

$$\mathcal{S}^{(G)}(s, \Omega, \omega) = \sqrt{\mu} \mathcal{S}(s, \Omega, \mu\omega). \quad (171)$$

We will now use these expressions to compute the CHSH parameter for the beam-splitter and Sagnac methods for generating entangled photon pairs shown in Figs. 2 and 3 for the asymmetric distribution of Eq. (145). This is done by inserting Eqs. (168)–(171) in Eqs. (56) and (57) for the beam-splitter method and Eqs. (70) and (71) for the Sagnac method and then computing the resulting integrals.

We will begin with Eq. (56) for the beam-splitter method. Using Eqs. (168)–(171) in Eq. (56) yields Eq. (105) after changes of variables like those in Eqs. (148), (149), (161), and (162). Therefore, we may use the integrals of Eqs. (107)–(109) to express Eq. (105) more compactly as

$$N = \frac{1}{2}(2I_1^2 + 2I_2 - I_1), \quad (172)$$

which is the denominator of Eq. (119). This is not unreasonable to expect since the parameter N is related to the normalization of the system and the asymmetry introduced into Eq. (145) should not alter this normalization.

We now insert Eqs. (168)–(171) into Eq. (57) to compute the CHSH parameter for the beam-splitter method of Fig. 2. This gives

$$\begin{aligned} S = & -\frac{\sqrt{2}}{4\mu N} \int d\Omega d\Omega' d\omega d\omega' \\ & \times [\mu^2 \mathcal{C}^*(s, \Omega, \omega) \mathcal{C}(s, \Omega', \omega) \\ & \times \mathcal{S}^*(s, \mu\Omega', \omega') \mathcal{S}(s, \mu\Omega, \omega') \\ & + \mu^2 \mathcal{C}^*(s, \Omega, \omega) \mathcal{C}(s, \Omega', \omega') \\ & \times \mathcal{S}^*(s, \mu\Omega', \omega) \mathcal{S}(s, \mu\Omega, \omega') \\ & + \mathcal{C}^*(s, \Omega, \omega) \mathcal{C}(s, \Omega', \omega) \\ & \times \mathcal{S}^*(s, \mu^{-1}\Omega', \omega') \mathcal{S}(s, \mu^{-1}\Omega, \omega') \\ & + \mathcal{C}^*(s, \Omega, \omega) \mathcal{C}(s, \Omega', \omega') \\ & \times \mathcal{S}^*(s, \mu^{-1}\Omega', \omega) \mathcal{S}(s, \mu^{-1}\Omega, \omega')], \end{aligned} \quad (173)$$

where we have used several changes of variables, the integrals of Eqs. (107)–(109), and the fact that $I_2 = I_3$.

As we did in Eqs. (107)–(109), we now define the integrals

$$\begin{aligned} I_4(\mu) \equiv & \int d\Omega d\Omega' d\omega d\omega' \mathcal{C}^*(s, \Omega, \omega) \mathcal{C}(s, \Omega', \omega) \\ & \times \mathcal{S}^*(s, \mu\Omega', \omega') \mathcal{S}(s, \mu\Omega, \omega'), \end{aligned} \quad (174)$$

$$\begin{aligned} I_5(\mu) \equiv & \int d\Omega d\Omega' d\omega d\omega' \mathcal{C}^*(s, \Omega, \omega) \mathcal{C}(s, \Omega', \omega') \\ & \times \mathcal{S}^*(s, \mu\Omega', \omega) \mathcal{S}(s, \mu\Omega, \omega'). \end{aligned} \quad (175)$$

Using Eqs. (174) and (175) in Eq. (173) allows us to simplify the CHSH parameter to

$$S = -\frac{\sqrt{2}}{4\mu N} [\mu^2 I_4(\mu) + \mu^2 I_5(\mu) + I_4(\mu^{-1}) + I_5(\mu^{-1})]. \quad (176)$$

Inspection of Eq. (176) reveals that if we were to let $\mu \rightarrow 1/\mu$, we would obtain the same expression regardless of the actual results of I_4 and I_5 . This validates the claim we made following Eq. (144).

We now need to evaluate the integrals of Eqs. (174) and (175). Using similar techniques to those of the previous section and after a lengthy series of algebraic steps, we find

$$\begin{aligned} I_4(\mu) = & \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \frac{s^{2j+2k+2\ell+2m+2}}{(2j)!(2k)!(2\ell+1)!(2m+1)!} \\ & \times \sqrt{\frac{4(1-x^2)^{2j+2k+2\ell+2m+2}}{(1+\mu^2)^2(1+x^{4j+4k+4\ell+4m+4}) + (1-\mu^2)^2(x^{4j+4\ell+2} + x^{4k+4m+2}) - 8\mu^2 x^{2j+2k+2\ell+2m+2}}}, \end{aligned} \quad (177)$$

$$\begin{aligned} I_5(\mu) = & \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \frac{s^{2j+2k+2\ell+2m+2}}{(2j)!(2k)!(2\ell+1)!(2m+1)!} \\ & \times \sqrt{\frac{4(1-x^2)^{2j+2k+2\ell+2m+2}}{(1+\mu^2)^2(1+x^{4j+4k+4\ell+4m+4}) - (1-\mu^2)^2(x^{4j+4k} + x^{4\ell+4m+4}) - 8\mu^2 x^{2j+2k+2\ell+2m+2}}}, \end{aligned} \quad (178)$$

where we have used the definition of Eq. (122) and Eqs. (130)–(132). We can now use Eqs. (130)–(132), (172), (177), and (178) to numerically compute values for the CHSH parameter of Eq. (176).

As with the symmetric distribution of the previous section, we can analytically compute the result in the case when $x = \tau = 0$ (the uncorrelated case) and when $x = \tau = 1$ (the perfectly correlated case). First, consider the uncorrelated case. Using $x = 0$ in Eqs. (177) and (178) and performing the sums results in

$$I_4(\mu) = \frac{2}{1+\mu^2} \cosh^2 s \sinh^2 s, \quad (179)$$

$$I_5(\mu) = \frac{2}{1+\mu^2} \sinh^4 s + \frac{1}{\mu} \sinh^2 s. \quad (180)$$

Using Eqs. (179) and (180) along with the uncorrelated case results from the previous section yields

$$S = -\frac{\sqrt{2}}{1+\mu^2} \frac{4\mu \cosh^2 s + (1-\mu)^2}{2 \cosh 2s - 1}. \quad (181)$$

Note that when $\mu = 1$, the distribution of Eq. (145) becomes the symmetric distribution of Eq. (72) and we recover the result we obtained previously for this case.

Next, we consider the perfectly correlated case. We now need to take the limit as $x \rightarrow 1$ in Eqs. (177) and (178). It is important to note that the limit as $x \rightarrow 1$ of $I_4(\mu)$ is highly sensitive to the value of μ . It is so sensitive that we need to define the limit using a piecewise distribution. The reason for this discontinuity is because the joint spectral density becomes an infinitely tall and infinitesimally narrow line spanning the entirety of the joint angular frequency space in the limit as $\tau \rightarrow 1$. $I_4(\mu)$ then becomes a measure of how much of the joint spectral density lies along the symmetric axis of this,

which would be zero whenever this highly singular joint spectral density is asymmetric, that is, when $\mu \neq 1$. Thus, taking the limit as $x \rightarrow 1$ in Eqs. (177) and (178) yields

$$I_4(\mu) = \begin{cases} s^2, & \mu = 1, \\ 0, & \mu \neq 1, \end{cases} \quad (182)$$

$$I_5(\mu) = \frac{s^2}{\mu}. \quad (183)$$

These give the CHSH parameter from Eq. (176) as

$$S = \begin{cases} -\frac{2\sqrt{2}}{1+2s^2}, & \mu = 1, \\ -\frac{\sqrt{2}}{1+2s^2}, & \mu \neq 1. \end{cases} \quad (184)$$

Once again, we have the result we obtained previously when $\mu = 1$. However, in this case, when μ is anything other than unity, the CHSH parameter is reduced by a factor of $1/2$. This is not unreasonable for the perfectly correlated case in particular. Recall that we cannot actually have the perfectly correlated case without the distribution of Eq. (72) collapsing to a delta function and this result is only true if $\tau = 1$.

We now consider the Sagnac method of Fig. 3. Using Eqs. (168)–(171) in Eq. (70) and using a series of suitable changes of variables yields

$$N = 2 \int d\Omega d\Omega' d\omega d\omega' [2|\mathcal{S}(s, \Omega, \omega)|^2 |\mathcal{S}(s, \Omega', \omega')|^2 + \mathcal{S}^*(s, \Omega', \omega') \mathcal{C}^*(s, \Omega, \omega') \mathcal{S}(s, \Omega, \omega) \mathcal{C}(s, \Omega', \omega)]. \quad (185)$$

Finally, using Eqs. (107)–(109) and the fact that $I_2 = I_3$ yields

$$N = 4I_1^2 + 2I_2, \quad (186)$$

which is proportional to the denominator of Eq. (141) as expected. Next, we use Eqs. (168)–(171) in Eq. (71) to compute the CHSH parameter as

$$S = -\frac{\sqrt{2}\mu}{N} \int d\Omega d\Omega' d\omega d\omega' [\mathcal{C}^*(s, \Omega, \omega') \mathcal{C}(s, \Omega', \omega) \mathcal{S}^*(s, \mu\Omega', \omega') \mathcal{S}(s, \mu\Omega, \omega) + \mu \mathcal{C}^*(s, \mu\Omega, \mu\omega') \mathcal{C}(s, \Omega', \omega) \\ \times \mathcal{S}^*(s, \Omega', \mu\omega') \mathcal{S}(s, \mu\Omega, \omega) + \mu \mathcal{C}^*(s, \Omega, \omega') \mathcal{C}(s, \mu\Omega', \mu\omega) \mathcal{S}^*(s, \mu\Omega', \omega') \mathcal{S}(s, \Omega, \mu\omega) \\ + \mu^2 \mathcal{C}^*(s, \mu\Omega, \mu\omega') \mathcal{C}(s, \mu\Omega', \mu\omega) \mathcal{S}^*(s, \Omega', \mu\omega') \mathcal{S}(s, \Omega, \mu\omega)]. \quad (187)$$

Now, using additional changes of variables and the integrals of Eqs. (107)–(109), (174), and (175), as well as the fact that $I_2 = I_3$, gives the CHSH parameter as

$$S = -\frac{\sqrt{2}}{N} \left\{ 2I_2 + \frac{1}{\mu} [\mu^2 I_4(\mu) + I_4(\mu^{-1})] \right\}. \quad (188)$$

As was the case in Eq. (176), if we let $\mu \rightarrow 1/\mu$ in Eq. (188), we obtain the same expression. Once again, we see that the result of the CHSH parameter is agnostic to the definition of Eq. (144). This will make plotting the CHSH parameter as a function of the asymmetry μ easier as we shall see.

As we did with the beam-splitter method, we now consider analytical expressions for the uncorrelated and perfectly cor-

related cases. In the uncorrelated ($\tau = x = 0$) case, Eqs. (186) and (188) become

$$S = -2\sqrt{2} \frac{(1+\mu)^2}{2(1+\mu^2)} \frac{2 \cosh^2 s}{3 \cosh 2s - 1}. \quad (189)$$

Note that the impairment due to the asymmetry is separable in this case. This is an interesting result. Furthermore, when $\mu = 1$, we recover the symmetric case of Eq. (142), as expected.

Finally, we consider the perfectly correlated ($\tau = x = 1$) case. Using Eq. (182) in Eq. (188) gives the same result we found in Eq. (184). Therefore, the beam-splitter method and Sagnac method will always yield the same CHSH parameter as a function of s in the perfectly correlated case, regardless of the symmetry of the joint spectral density.

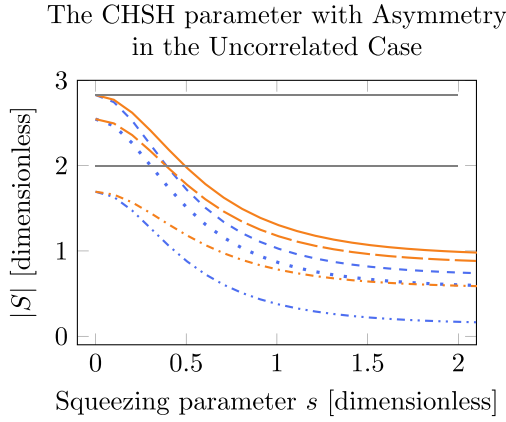


FIG. 10. Plots of the CHSH parameters in the uncorrelated case given by Eq. (181) for the beam-splitter method and Eq. (189) for the Sagnac method for different values of μ . The solid and the short-dashed curves are the CHSH parameters for the Sagnac method and beam-splitter method when $\mu = 1$, respectively. These two curves are equivalent to those shown in Fig. 8. The long-dashed and dotted curves are the CHSH parameters for the Sagnac method and beam-splitter method when $\mu = 2$ or $\mu = 1/2$, respectively. Similarly, the dot-dashed and dot-dot-dashed curves are the CHSH parameters for the Sagnac method and beam-splitter method when $\mu = 10$ or $\mu = 1/10$, respectively. In this case, the CHSH parameter never exceeds the classical limit of 2.

We now desire to plot these expressions. For the uncorrelated case, we plot Eq. (181) for the beam-splitter method and Eq. (189) for the Sagnac method. We plot these curves as a function of the squeezing parameter for different values of the asymmetry μ in Fig. 10. For the perfectly correlated case, we plot Eq. (184), which is the resulting CHSH parameter for both the beam-splitter and Sagnac methods in this case. Since the only dependence this expression has on μ is the infinitesimally narrow spike at $\mu = 1$, we only need to plot the CHSH parameter as a function of s when $\mu = 1$ and when $\mu \neq 1$. This is shown in Fig. 11. Note that, when $\mu \neq 1$, the CHSH parameter never exceeds the classical limit of $S = 2$ when the joint spectral density is asymmetric yet perfectly correlated.

Additionally, we will plot these expressions as well as the general results from Eqs. (176) and (188) as functions of the asymmetry μ . As we have pointed out above, both μ and $1/\mu$ result in the same CHSH parameter for both the beam-splitter and Sagnac methods regardless of the value of μ . Therefore, we will plot these expressions only when $\mu \geq 1$ for the sake of simplicity. These results are shown in Fig. 12. The curves are the theoretical results from the uncorrelated and perfectly correlated cases in Eqs. (181), (184), and (189), as described in the figure caption. The points are numerically computed for different correlation parameters from the integrals in Eqs. (177) and (178). As Mathematica's NSum is slow to converge for sums over four indices as required in this case, we instead use *Matlab* and truncate the sums at 21 terms for each index. From these results, we see how the CHSH parameter becomes more singular as τ approaches 1. Furthermore, we can clearly see that the combined effects

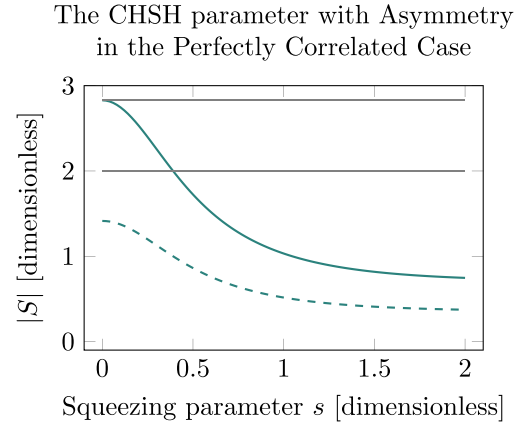


FIG. 11. Plots of the CHSH parameter in the perfectly correlated case computed in Sec. IV B. The solid curve is the CHSH parameter of both the beam-splitter method and the Sagnac method for generating entangled photon pairs of Eq. (184) when $\mu = 1$. Similarly, the dashed curve is the CHSH parameter of the both methods when $\mu \neq 1$. Note that the CHSH parameter never exceeds the classical limit of 2 in the latter case.

of spectral correlation and asymmetry are worse than either effect alone.

V. CONCLUSION

In this paper, we have introduced a recursive analytical technique for investigating the impacts that multiple photon pairs and an asymmetric joint spectral density have on a measurement of the CHSH parameter for a polarization-entangled quantum state. This technique relied on the definitions of the recursively defined distributions of Eqs. (27) and (28). Provided, we can compute these sets of distributions and the sums of these distributions in Eqs. (37), (38), (45), and (46), we can compute the CHSH parameter for polarization-entangled states generated by the beam-splitter method of Fig. 2 using Eqs. (56) and (57) and by the Sagnac method of Fig. 3 using Eqs. (70) and (71). We were able to compute these parameters using the symmetric bivariate normal distribution of Eq. (72). We were then able to generalize these results to the asymmetric distribution of Eq. (145).

While we used an approximate and ideal distribution for the joint spectral density of the photons produced by the SPDC sources shown in Figs. 2 and 3, we were still able to observe some interesting phenomena that have broader physical implications. For example, in the symmetric case, we observed that the beam-splitter method can be marginally improved for certain values of the squeezing parameter if the joint spectral density of the SPDC source is more correlated, while the Sagnac method is instead hindered by this correlation. Furthermore, when asymmetry is present in the joint spectral density due to experimental imperfections, we find that it is usually better for the distribution to be as uncorrelated as possible. Finally, we found that both the beam-splitter method and the Sagnac method provide identical distributions as a function of the squeezing parameter when the joint spectral density is perfectly correlated regardless of the symmetry of the joint spectral density. These results are interesting and demonstrate the importance of spectral engineering [37] when

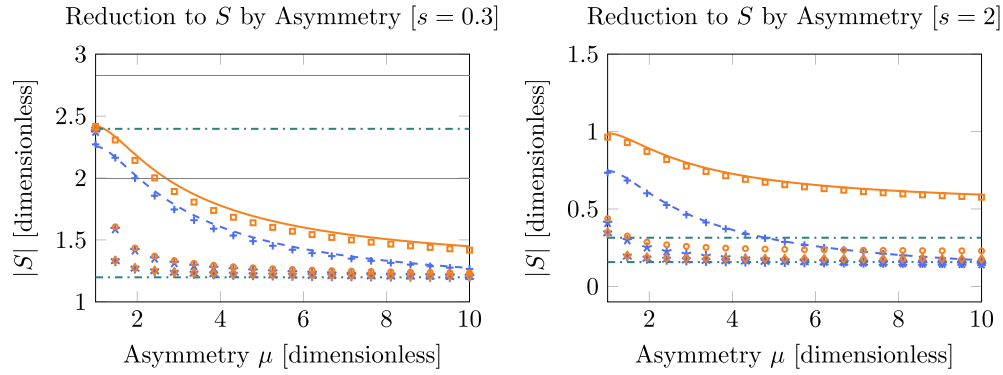


FIG. 12. Plots of the CHSH parameter $|S|$ as a function of the asymmetry parameter μ . Recall the CHSH parameter at μ is the same as the CHSH parameter at $1/\mu$ for both the beam-splitter and Sagnac methods regardless of the values of s and τ . The left plot is for a squeezing parameter of $s = 0.3$, while the right plot is for a squeezing parameter of $s = 2$. In both plots, the left edge corresponds to $\mu = 1$. The dashed curve is the theoretical curve for CHSH parameter of the beam-splitter method in the uncorrelated case from Eq. (181). Similarly, the solid curve is the CHSH parameter for the Sagnac method in the uncorrelated case from Eq. (189). The upper and lower dot-dashed curves correspond to the $\mu = 1$ and $\mu \neq 1$ cases of Eq. (184) for the perfectly correlated case of the beam-splitter and Sagnac methods, respectively. The points are numerically computed results in various situations. For the beam-splitter method, the pluses correspond to $\tau = 0.5$, the five-pointed stars correspond to $\tau = 0.99$, and the six-pointed stars (the asterisks) correspond to $\tau = 0.999$. Finally, for the Sagnac method, the squares correspond to $\tau = 0.5$, the open circles correspond to $\tau = 0.99$, and the diamonds correspond to $\tau = 0.999$.

high quality entanglement is desired for quantum informational protocols.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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